Part II Sorting and Order Statistics
Part II Sorting and Order Statistics

- Chapter 6. Heapsort, the use of priority queue
- Chapter 7. Quicksort, probabilistic analysis, randomized algorithms
- Chapter 8. Sorting in linear time, lower bounds
- Chapter 9. Medians and order statistics
Chapter 6. Heapsort and the use of priority queue
Chapter 6. Heapsort

Chapter 6. Heapsort and the use of priority queue
Chapter 6. Heapsort

Chapter 6. Heapsort and the use of priority queue

- key(parent) ≥ key(leftChild), key(rightChild);

![Image of a heap tree and array representation](image-url)
Chapter 6. Heapsort

Chapter 6. Heapsort and the use of priority queue

- $\text{key}(\text{parent}) \geq \text{key}(\text{leftChild}), \text{key}(\text{rightChild})$;
- relationships are modeled with a complete binary tree
Chapter 6. Heapsort

Chapter 6. Heapsort and the use of priority queue

- key(parent) \geq\ key(leftChild), key(rightChild);
- relationships are modeled with a complete binary tree
- can be stored in arrays (indexes begin with 0),
Chapter 6. Heapsort

Chapter 6. Heapsort and the use of priority queue

• key(parent) \geq key(leftChild), key(rightChild);
• relationships are modeled with a complete binary tree
• can be stored in arrays (indexes begin with 0),
  index(leftChild) = 2 \times index(parent) + 1
Chapter 6. Heapsort

Chapter 6. Heapsort and the use of priority queue

- key(parent) ≥ key(leftChild), key(rightChild);
- relationships are modeled with a complete binary tree
- can be stored in arrays (indexes begin with 0),
  \[\text{index(leftChild)} = 2 \times \text{index(parent)} + 1\]
  \[\text{index(rightChild)} = 2 \times \text{index(parent)} + 2\]
Chapter 6. Heapsort

The heap sort algorithm consists of subroutines:
The heap sort algorithm consists of subroutines:

- \textbf{Build-Max-Heap}(A)

\textbf{Chapter 6. Heapsort}
The heap sort algorithm consists of subroutines:

- **BUILD-MAX-HEAP**(A)
- **MAX-HEAPIFY**(A, i)
Chapter 6. Heapsort

The heap sort algorithm consists of subroutines:

- **Build-Max-Heap**($A$)
- **Max-Heapify**($A, i$)
- **HeapSort**($A$)
Chapter 6. Heapsort

The heap sort algorithm consists of subroutines:

- `BUILD-MAX-HEAP(A)`
- `MAX-HEAPIFY(A, i)`
- `HEAP_SORT(A)`

heaps as priority queues
Chapter 6. Heapsort

The heap sort algorithm consists of subroutines:

- **Build-Max-Heap**$(A)$
- **Max-Heapify**$(A, i)$
- **HeapSort**$(A)$

Heaps as priority queues

- **Heap-Maximum**$(A)$
- **Heap-Extract-Max**$(A)$
- **Heap-Increase-Key**$(A, i, key)$
- **Max-Heap-Insert**$(A, key)$
Chapter 6. Heapsort

The heap sort algorithm consists of subroutines:

- \textbf{Build-Max-Heap}(A)
- \textbf{Max-Heapify}(A, i)
- \textbf{HeapSort}(A)

Heaps as priority queues

- \textbf{Heap-Maximum}(A)
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Chapter 6. Heapsort

The heap sort algorithm consists of subroutines:

- **Build-Max-Heap**(A)
- **Max-Heapify**(A, i)
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Heaps as priority queues

- **Heap-Maximum**(A)
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Chapter 6. Heapsort

The heap sort algorithm consists of subroutines:

- **Build-Max-Heap** \((A)\)
- **Max-Heapify** \((A, i)\)
- **HeapSort** \((A)\)

Heaps as priority queues

- **Heap-Maximum** \((A)\)
- **Heap-Extract-Max** \((A)\)
- **Heap-Increase-Key** \((A, I, key)\)
- **Max-Heap-Insert** \((A, key)\)
Chapter 6. Heapsort

Algorithm `HEAPSORT(A)`
Chapter 6. Heapsort

Algorithm \textsc{HeapSort}(A)

1. \textsc{Build-Max-Heap}(A)
Chapter 6. Heapsort

Algorithm `HEAPSORT(A)`

1. `BUILD-MAX-HEAP(A)`
2. `for i = length[A] downto 2`
Algorithm heapSort(A)

1. BUILD-MAX-HEAP(A)
2. for i = length[A] downto 2
Algorithm HeapSort\( (A) \)

1. **Build-Max-Heap\( (A) \)**
2. for \( i = \text{length}[A] \) downto 2
3. exchange \( A[1] \) \( \leftrightarrow \) \( A[i] \)
4. \( \text{heapsize}[A] = \text{heapsize}[A] - 1 \)
Chapter 6. Heapsort

Algorithm HeapSort\( (A) \)

1. Build-Max-Heap\( (A) \)
2. for \( i = \text{length}[A] \) downto 2
4. heapsize\[A\] = heapsize\[A\] - 1
5. Max-Heapify\( (A, 1) \)
Chapter 6. Heapsort

Algorithm **HEAPSORT**(*A*)

1. **BUILD-MAX-HEAP**(*A*)
2. **for** *i = length[A]* **downto** 2
5. **MAX-HEAPIFY**(*A, 1*)

\[ T_{HS}(n) = T_{BMH}(n) + (n - 1)T_{MH}(n, 1) \]
Chapter 6. Heapsort

Algorithm HeapSort(A)
1.  Build-Max-Heap(A)
2.  for $i = \text{length}[A]$ downto 2
4.  $\text{heapsize}[A] = \text{heapsize}[A] - 1$
5.  Max-Heapify(A, 1)

$T_{HS}(n) = T_{BMH}(n) + (n - 1)T_{MH}(n, 1)$

Subroutine Build-Max-Heap(A)
Algorithm **HEAPSORT**(\(A\))

1. **BUILD-MAX-HEAP**\((A)\)
2. for \(i = \text{length}[A]\) downto 2
3. exchange \(A[1] \leftrightarrow A[i]\)
4. \(\text{heapsize}[A] = \text{heapsize}[A] - 1\)
5. **MAX-HEAPIFY**\((A, 1)\)

\[
T_{HS}(n) = T_{BMH}(n) + (n - 1)T_{MH}(n, 1)
\]

Subroutine **BUILD-MAX-HEAP**\((A)\)

1. \(\text{heapsize}[A] = \text{length}[A]\)
Chapter 6. Heapsort

Algorithm HeapSort(A)

1. Build-Max-Heap(A)
2. for i = length[A] downto 2
4. heapsize[A] = heapsize[A] − 1
5. Max-Heapify(A, 1)

\[ T_{HS}(n) = T_{BMH}(n) + (n - 1)T_{MH}(n, 1) \]

Subroutine Build-Max-Heap(A)

1. heapsize[A] = length[A]
2. for i = \left\lfloor \frac{1}{2} \text{length}[A] \right\rfloor downto 1
Chapter 6. Heapsort

Algorithm HeapSort(A)
1. Build-Max-Heap(A)
2. for i = length[A] downto 2
4. heapsize[A] = heapsize[A] − 1
5. Max-Heapify(A, 1)

\[ T_{HS}(n) = T_{BMH}(n) + (n - 1)T_{MH}(n, 1) \]

Subroutine Build-Max-Heap(A)
1. heapsize[A] = length[A]
2. for i = \left\lceil \frac{1}{2} length[A] \right\rceil downto 1
3. Max-Heapify(A, i)
Chapter 6. Heapsort

Algorithm HeapSort(A)
1. Build-Max-Heap(A)
2. for \( i = \text{length}[A] \) downto 2
4. \( \text{heapsize}[A] = \text{heapsize}[A] - 1 \)
5. Max-Heapify(A, 1)

\[ T_{HS}(n) = T_{BMH}(n) + (n - 1)T_{MH}(n, 1) \]

Subroutine Build-Max-Heap(A)
1. \( \text{heapsize}[A] = \text{length}[A] \)
2. for \( i = \lfloor \frac{1}{2} \text{length}[A] \rfloor \) downto 1
3. Max-Heapify(A, i)

\[ T_{BMH}(n) = \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} T_{MH}(n, i) \]
Chapter 6. Heapsort

Subroutine \texttt{MAX-HEAPIFY}(A, i)
Chapter 6. Heapsort

Subroutine \textsc{Max-Heapify}(A, i)

1. \hspace{1cm} l = \textsc{Left}[i]

\[ T_{MH}(n,i) = c + T_{MH}(n,2i) \]

\[ T_{MH}(n,i) \leq c \log_2 n, \text{ for all } i = 1, 2, ..., n. \]

\[ T_{BMH}(n) = \left\lfloor \frac{n}{2} \right\rfloor \sum_{i=1}^{n} T_{MH}(n,i) \]

\[ T_{HS}(n) = T_{BMH}(n) + (n-1)T_{MH}(n,1) \]

\[ T_{HS}(n) \leq c n^2 \log_2 n + (n-1)c \log_2 n \leq O(n \log n) \]
Chapter 6. Heapsort

Subroutine MAX-HEAPIFY($A, i$)

1. $l = LEFT[i]$
2. $r = RIGHT[i]$

$\text{T}_{MH}(n,i) = c + \text{T}_{MH}(n,2i)$

$\text{T}_{MH}(n,i) \leq c \log_2 n$, for all $i = 1, 2, \ldots, n$.

$\text{T}_{BMH}(n) = \lfloor n/2 \rfloor \sum_{i=1}^{n} \text{T}_{MH}(n,i)$

$\text{T}_{BMH}(n) \leq cn^2 \log_2 n$.

$\text{T}_{HS}(n) = \text{T}_{BMH}(n) + (n-1) \text{T}_{MH}(n,1)$

$\text{T}_{HS}(n) \leq cn^2 \log_2 n + (n-1)c \log_2 n \leq O(n \log n)$. 
Chapter 6. Heapsort

Subroutine \textsc{Max-Heapify}(A, i)

1. \( l = \text{LEFT}[i] \)
2. \( r = \text{RIGHT}[i] \)
Chapter 6. Heapsort

Subroutine \textsc{Max-Heapify}(A, i)

1. \( l = LEFT[i] \)
2. \( r = RIGHT[i] \)
3. \( \text{if } (l \leq \text{heapsize}[A]) \text{ and } (A[l] > A[i]) \)
4. \( \text{then } largest = l \)
Chapter 6. Heapsort

Subroutine $\text{MAX-HEAPIFY}(A, i)$

1. $l = \text{LEFT}[i]$
2. $r = \text{RIGHT}[i]$
3. if $(l \leq \text{heapsize}[A])$ and $(A[l] > A[i])$
   4. then $\text{largest} = l$
   5. else $\text{largest} = i$

Chapter 6. Heapsort

Subroutine $\text{MAX-HEAPIFY}(A, i)$

1. $l = \text{LEFT}[i]$
2. $r = \text{RIGHT}[i]$
3. if $(l \leq \text{heapsize}[A]) \text{ and } (A[l] > A[i])$
4. then $\text{largest} = l$
5. else $\text{largest} = i$
6. if $(r \leq \text{heapsize}[A]) \text{ and } (A[r] > A[\text{largest}])$
Chapter 6. Heapsort

Subroutine \textsc{Max-Heapify}(A, i)

1. \( l = \text{LEFT}[i] \)
2. \( r = \text{RIGHT}[i] \)
3. \textbf{if} \((l \leq \text{heapsize}[A]) \textbf{ and } (A[l] > A[i])\)
4. \hspace{5em} \textbf{then} \( \text{largest} = l \)
5. \hspace{5em} \textbf{else} \( \text{largest} = i \)
6. \textbf{if} \((r \leq \text{heapsize}[A]) \textbf{ and } (A[r] > A[\text{largest}])\)
7. \hspace{5em} \textbf{then} \( \text{largest} = r \)
Subroutine \texttt{Max-Heapify}(A, i)

1. \( l = LEFT[i] \)
2. \( r = RIGHT[i] \)
3. \textbf{if} \((l \leq \text{heapsize}[A]) \text{ and } (A[l] > A[i])\) \textbf{then} \(\text{largest} = l\)
4. \textbf{else} \(\text{largest} = i\)
5. \textbf{if} \((r \leq \text{heapsize}[A]) \text{ and } (A[r] > A[\text{largest}])\) \textbf{then} \(\text{largest} = r\)
6. \textbf{if} \(\text{largest} \neq i\)
Subroutine `Max-Heapify(A, i)`

1. \( l = \text{LEFT}[i] \)
2. \( r = \text{RIGHT}[i] \)
3. if \( l \leq \text{heapsize}[A] \) and \( A[l] > A[i] \) then \( \text{largest} = l \)
4. else \( \text{largest} = i \)
5. if \( r \leq \text{heapsize}[A] \) and \( A[r] > A[\text{largest}] \) then \( \text{largest} = r \)
6. if \( \text{largest} \neq i \) then exchange \( A[i] \leftarrow A[\text{largest}] \)
Chapter 6. Heapsort

Subroutine **MAX-HEAPIFY**\( (A, i) \)

1. \( l = \text{LEFT}[i] \)
2. \( r = \text{RIGHT}[i] \)
3. if \( (l \leq \text{heapsize}[A]) \) and \( (A[l] > A[i]) \)
4. then \( \text{largest} = l \)
5. else \( \text{largest} = i \)
6. if \( (r \leq \text{heapsize}[A]) \) and \( (A[r] > A[\text{largest}] \)
7. then \( \text{largest} = r \)
8. if \( \text{largest} \neq i \)
9. then exchange \( A[i] \leftrightarrow A[\text{largest}] \)
10. **MAX-HEAPIFY**\( (A, \text{largest}) \)
Chapter 6. Heapsort

Subroutine \textsc{Max-Heapify}(A, i)

1. \( l = \text{LEFT}[i] \)
2. \( r = \text{RIGHT}[i] \)
3. \textbf{if} \((l \leq \text{heapsize}[A]) \text{ and } (A[l] > A[i])\) \textbf{then} \( \text{largest} = l \)
4. \( \text{else} \ \text{largest} = i \)
5. \textbf{if} \((r \leq \text{heapsize}[A]) \text{ and } (A[r] > A[\text{largest}])\) \textbf{then} \( \text{largest} = r \)
6. \textbf{if} \( \text{largest} \neq i \) \textbf{then} exchange \( A[i] \leftrightarrow A[\text{largest}] \)
7. \( \text{Max-Heapify}(A, \text{largest}) \)

\[ T_{MH}(n, i) = c + T_{MH}(n, 2i) \]
Chapter 6. Heapsort

Subroutine \textsc{Max-Heapify}(A, i)

1. \( l = \text{LEFT}[i] \)
2. \( r = \text{RIGHT}[i] \)
3. \textbf{if} \((l \leq \text{heapsize}[A]) \text{ and } (A[l] > A[i])\)
4. \hspace{1em} \textbf{then} \( \text{largest} = l \)
5. \hspace{1em} \textbf{else} \( \text{largest} = i \)
6. \textbf{if} \((r \leq \text{heapsize}[A]) \text{ and } (A[r] > A[\text{largest}])\)
7. \hspace{1em} \textbf{then} \( \text{largest} = r \)
8. \hspace{1em} \textbf{if} \( \text{largest} \neq i \)
9. \hspace{2em} \textbf{then} exchange \( A[i] \leftrightarrow A[\text{largest}] \)
10. \hspace{1em} \textsc{Max-Heapify}(A, \text{largest})

\( T_{MH}(n, i) = c + T_{MH}(n, 2i) \)

\( T_{MH}(n, i) \leq c \log_2 n, \text{ for all } i = 1, 2, \ldots, n. \)
Chapter 6. Heapsort

Subroutine MAX-HEAPIFY(A, i)

1. \( l = \text{LEFT}[i] \)
2. \( r = \text{RIGHT}[i] \)
3. \( \text{if } (l \leq \text{heapsize}[A]) \text{ and } (A[l] > A[i]) \)
4. \( \text{then } \text{largest} = l \)
5. \( \text{else } \text{largest} = i \)
6. \( \text{if } (r \leq \text{heapsize}[A]) \text{ and } (A[r] > A[\text{largest}]) \)
7. \( \text{then } \text{largest} = r \)
8. \( \text{if } \text{largest} \neq i \)
9. \( \text{then exchange } A[i] \leftrightarrow A[\text{largest}] \)
10. \( \text{MAX-HEAPIFY}(A, \text{largest}) \)

\[
T_{MH}(n, i) = c + T_{MH}(n, 2i)
\]

\[
T_{MH}(n, i) \leq c \log_2 n, \text{ for all } i = 1, 2, \ldots, n.
\]

\[
T_{BMH}(n) = \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} T_{MH}(n, i)
\]
Chapter 6. Heapsort

Subroutine **Max-Heapify**(*A, i*)

1. \( l = LEFT[i] \)
2. \( r = RIGHT[i] \)
3. if \( (l \leq heap_size[A]) \text{ and } (A[l] > A[i]) \) then \( largest = l \)
4. else \( largest = i \)
5. if \( (r \leq heap_size[A]) \text{ and } (A[r] > A[largest]) \) then \( largest = r \)
6. if \( largest \neq i \) then exchange \( A[i] \leftarrow A[largest] \)
7. **Max-Heapify**(*A, largest*)

\( T_{MH}(n, i) = c + T_{MH}(n, 2i) \)

\( T_{MH}(n, i) \leq c \log_2 n, \text{ for all } i = 1, 2, \ldots, n. \)

\( T_{BMH}(n) = \sum_{i=1}^{\lfloor n/2 \rfloor} T_{MH}(n, i) \leq c \frac{n}{2} \log_2 n \)
Chapter 6. Heapsort

Subroutine **MAX-HEAPIFY***(A, i)***

1. \( l = \text{LEFT}[i] \)
2. \( r = \text{RIGHT}[i] \)
3. if \((l \leq \text{heapsize}[A]) \text{ and } (A[l] > A[i])\) then \( \text{largest} = l \)
4. else \( \text{largest} = i \)
5. if \((r \leq \text{heapsize}[A]) \text{ and } (A[r] > A[\text{largest}])\) then \( \text{largest} = r \)
6. if \( \text{largest} \neq i \) then exchange \( A[i] \leftrightarrow A[\text{largest}] \)
7. \( \text{MAX-HEAPIFY}(A, \text{largest}) \)

\[
T_{MH}(n, i) = c + T_{MH}(n, 2i)
\]

\[
T_{MH}(n, i) \leq c \log_2 n, \text{ for all } i = 1, 2, \ldots , n.
\]

\[
T_{BMH}(n) = \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} T_{MH}(n, i) \leq c \frac{n}{2} \log_2 n
\]

\[
T_{HS}(n) = T_{BMH}(n) + (n - 1)T_{MH}(n, 1)
\]
Chapter 6. Heapsort

Subroutine MAX-HEAPIFY(A, i)

1. \( l = LEFT[i] \)
2. \( r = RIGHT[i] \)
3. \[ \text{if} \ (l \leq \text{heapsize}[A]) \text{ and } (A[l] > A[i]) \]
4. \[ \text{then} \ largest = l \]
5. \[ \text{else} \ largest = i \]
6. \[ \text{if} \ (r \leq \text{heapsize}[A]) \text{ and } (A[r] > A[largest]) \]
7. \[ \text{then} \ largest = r \]
8. \[ \text{if} \ largest \neq i \]
9. \[ \text{then} \ exchange \ A[i] \leftrightarrow A[largest] \]
10. MAX-HEAPIFY(A, largest)

\[ T_{MH}(n, i) = c + T_{MH}(n, 2i) \]

\[ T_{MH}(n, i) \leq c \log_2 n, \text{ for all } i = 1, 2, \ldots, n. \]

\[ T_{BMH}(n) = \sum_{i=1}^{\lfloor n/2 \rfloor} T_{MH}(n, i) \leq c \frac{n}{2} \log_2 n \]

\[ T_{HS}(n) = T_{BMH}(n) + (n - 1)T_{MH}(n, 1) \leq c \frac{n}{2} \log_2 n + (n - 1)c \log_2 n \]
Chapter 6. Heapsort

Subroutine $\text{MAX-HEAPIFY}(A, i)$

1. $l = \text{LEFT}[i]$
2. $r = \text{RIGHT}[i]$
3. if $(l \leq \text{heapsize}[A])$ and $(A[l] > A[i])$
4. then $\text{largest} = l$
5. else $\text{largest} = i$
6. if $(r \leq \text{heapsize}[A])$ and $(A[r] > A[\text{largest}])$
7. then $\text{largest} = r$
8. if $\text{largest} \neq i$
9. then exchange $A[i] \leftrightarrow A[\text{largest}]$
10. $\text{MAX-HEAPIFY}(A, \text{largest})$

$T_{MH}(n, i) = c + T_{MH}(n, 2i)$

$T_{MH}(n, i) \leq c \log_2 n$, for all $i = 1, 2, \ldots, n$.

$T_{BMH}(n) = \sum_{i=1}^{\lfloor n/2 \rfloor} T_{MH}(n, i) \leq c \frac{n}{2} \log_2 n$

$T_{HS}(n) = T_{BMH}(n) + (n - 1)T_{MH}(n, 1) \leq c \frac{n}{2} \log_2 n + (n - 1)c \log_2 n \leq O(n \log n)$
Chapter 6. Heapsort

Operations on heaps:
Chapter 6. Heapsort

Operations on heaps:

Function \texttt{HEAP-MAXIMUM}(A)
1. \hspace{1em} \textbf{return} \ (A[1])
Chapter 6. Heapsort

Operations on heaps:

Function **Heap-Maximum**$(A)$
1. **return** $(A[1])$

Function **Heap-Extract-Max**$(A)$
1. **if** $\text{heapsize}[A] < 1$
2. **then return** ("heap underflow")
3. $max = A[1]$
5. $\text{heapsize}[A] = \text{heapsize}[A] - 1$
6. **Max-Heapify**$(A, 1)$
7. **return** $(max)$

obtain the maximum

obtain and remove the maximum
Chapter 6. Heapsort

Operations on heaps:

Function **HEAP-MAXIMUM(A)**
1. return \( A[1] \) obtain the maximum

Function **HEAP-EXTRACT-MAX(A)**
1. if \( \text{heapsize}[A] < 1 \)
2. then return ("heap underflow")
3. \( \text{max} = A[1] \)
4. \( A[1] = A[\text{heapsize}[A]] \)
5. \( \text{heapsize}[A] = \text{heapsize}[A] - 1 \)
6. **Max-Heapify(A, 1)**
7. return \( \text{max} \) obtain and remove the maximum

Function **HEAP-INCREASE-KEY(A, i, key)**
1. if \( key < A[i] \)
2. then return ("new key is smaller than current key")
3. \( A[i] = key \)
4. while \( i > 1 \) and \( A[\text{PARENT}[i]] < A[i] \)
5. exchange \( A[i] \leftrightarrow A[\text{PARENT}[i]] \)
6. \( i = \text{PARENT}[i] \) replace a key with a larger value
Chapter 6. Heapsort

Operations on heaps:

Function **HEAP-MAXIMUM**(A) obtain the maximum
1. \textbf{return} \((A[1])\)

Function **HEAP-EXTRACT-MAX**(A) obtain and remove the maximum
1. \textbf{if} \(\text{heapsize}[A] < 1\)
2. \textbf{then return} ("heap underflow")
3. \textbf{max} = \(A[1]\)
5. \(\text{heapsize}[A] = \text{heapsize}[A] - 1\)
6. \textbf{MAX-HEAPIFY}(A, 1)
7. \textbf{return} \((\text{max})\)

Function **HEAP-INCREASE-KEY**(A, \(i, key\)) replace a key with a larger value
1. \textbf{if} \(key < A[i]\)
2. \textbf{then return} ("new key is smaller than current key")
3. \(A[i] = key\)
4. \textbf{while} \(i > 1\ \text{and} \ A[\text{PARENT}[i]] < A[i]\)
5. \textbf{exchange} \(A[i] \leftrightarrow A[\text{PARENT}[i]]\)
6. \(i = \text{PARENT}[i]\)

Function **MAX-HEAP-INSERT**(A, \(key\)) insert a new key to heap
1. \(\text{heapsize}[A] = \text{heapsize}[A] + 1\)
2. \(A[\text{heapsize}[A]] = -\infty\)
3. \textbf{HEAP-INCREASE-KEY}(A, \text{heapsize}[A], key)
Chapter 7. Quicksort and randomized algorithms

Idea of the Quicksort: divide-and-conquer

• divide: re-organize list $A[p, r]$ into two sublists $A[p, q - 1]$ and $A[q + 1, r]$ based on pivot $A[q]$, such that

(a) $A[i] \leq A[q]$ for all $i = p, \ldots, q - 1$

(b) $A[i] \geq A[q]$ for all $i = q + 1, \ldots, r$

• conquer: sort $A[p, q - 1]$ and $A[q + 1, r]$, recursively.
Chapter 7. Quicksort and randomized algorithms

Idea of the Quicksort: divide-and-conquer
Chapter 7. Quicksort and randomized algorithms

Idea of the Quicksort: divide-and-conquer

- divide: re-organize list \( A[p, r] \) into two sublists \( A[p, q - 1] \) and \( A[q + 1, r] \) based on pivot \( A[q] \), such that

(a) \( A[i] \leq A[q] \) for all \( i = p, \ldots, q - 1 \)

(b) \( A[i] \geq A[q] \) for all \( i = q + 1, \ldots, r \)
Chapter 7. Quicksort

Chapter 7. Quicksort and randomized algorithms

Idea of the Quicksort: divide-and-conquer

- divide: re-organize list $A[p, r]$ into two sublists $A[p, q - 1]$ and $A[q + 1, r]$ based on pivot $A[q]$, such that
  
  (a) $A[i] \leq A[q]$ for all $i = p, \cdots, q - 1$
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Chapter 7. Quicksort

Chapter 7. Quicksort and Randomized algorithms

Algorithm Quicksort \((A, p, r)\)

1. if \(p < r\) then
2. \(q = \text{Partition} (A, p, r)\)
3. \(\text{QuickSort} (A, p, q-1)\)
4. \(\text{QuickSort} (A, q+1, r)\)

How the pivot \(A[q]\) is identified is crucial to the performance of Quicksort.

- Assume \(A[q]\) partitions list \(A, p, r\) evenly, then \(T(n) \leq 2T(n/2) + cn = O(n \log_2 n)\)

- Assume \(A[q]\) partitions the list 20% vs 80%, then \(T(n) \leq T(n/5) + T(4n/5) + cn = O(n \log_2 n)\)

- Assume \(A[q]\) partitions the list 1% vs 99%, then \(T(n) \leq T(n/100) + T(99n/100) + cn = O(n \log_2 n)\)

How can we identify such a pivot?
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Algorithm QUICKSORT \((A, p, r)\)
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Algorithm QUICKSORT \((A, p, r)\)
1. \textbf{if} \(p < r\)
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Algorithm QUICKSORT \((A, p, r)\)
1. \(\textbf{if} \ p < r\)
2. \(\textbf{then} \ q = \text{PARTITION}\(A, p, r)\)
3. \(\text{QUICKSORT}(A, p, q - 1)\)

How the pivot \(A[q]\) is identified is crucial to the performance of Quicksort.

- Assume \(A[q]\) partitions list \(A, p, r\) even, then \(T(n) \leq 2T(n/2) + cn = O(n \log_2 n)\)
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- Assume \(A[q]\) partitions the list 1% vs 99%, then \(T(n) \leq T(100n/101) + T(99n/100) + cn = O(n \log_2 n)\)
Chapter 7. Quicksort

Chapter 7. Quicksort and Randomized algorithms

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Chapter 7. Quicksort

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Algorithm \textsc{QuickSort} \((A, p, r)\)
1. \hspace{1em} \textbf{if} \(p < r\)
2. \hspace{2em} \textbf{then} \(q = \text{Partition}(A, p, r)\)
3. \hspace{2em} \textsc{QuickSort} \((A, p, q - 1)\)
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How can we identify such a pivot?
Chapter 7. Quicksort
PARTITION($A, p, r$)
1 $x \leftarrow A[r]$
2 $i \leftarrow p - 1$
3 for $j \leftarrow p$ to $r - 1$
4 do if $A[j] \leq x$
5 then $i \leftarrow i + 1$
6 exchange $A[i] \leftrightarrow A[j]$
7 exchange $A[i + 1] \leftrightarrow A[r]$
8 return $i + 1$
Partition may not guarantee to partition the list to two fractions of sizes $\epsilon n : (1 - \epsilon)n$, for a constant $\epsilon > 0$. 
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- skewed situation like $1 : n - 1$ partition may happen, resulting in running time $\geq cn^2$. 

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- however, chances for skewed cases like above are very small.
- that is, the cases other than the skewed ones occur much more often.

So the idea of Quicksort may work well on a majority of data.
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Assume that the equal likely chance for every number to be in the last position, what is the chance to partition the list into

\[ x\% \text{ vs } (100 - x)\% \]

fragments, for \(10 \leq x \leq 90\)?
Chapter 7. Quicksort

Assume that the equal likely chance for every number to be in the last position, what is the chance to partition the list into

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fragments, for \( 10 \leq x \leq 90 \)?

The chance is \( = 80\% \)
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What running time would it be if 10:90 partition is always guaranteed?
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\[ l_0: \]

\[ cn \]

\[ l_h: \]

\[ cn/10 \]

\[ l_k: \]

\[ cn/10 \]

\[ \ldots \]

\[ l_{h':} \]

\[ cn/10 \]

\[ l_{k':} \]

\[ cn/10 \]

\[ \ldots \]

\[ l_{l':} \]

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What running time would it be if 10:90 partition is always guaranteed?

\[ T(n) \leq T(n/10) + T(9n/10) + cn \]

Using the recursive-tree method, we have

\[
\begin{align*}
  l_0: & \quad cn \\
  l_1: & \quad cn/10 \\
  \vdots & \quad \vdots \\
  l_h: & \quad cn/10^h \\
  l_k: & \quad c9n/10^k
\end{align*}
\]

where \( \frac{9}{10} \) is the base, i.e., \( h = \log_{10} n \) and \( k = \log_{10} \frac{n}{9} \), and

\[ T(n) \leq cn \log_{10} \frac{n}{9} \leq cn \log_2 \frac{n}{9} = c'n \log_2 n = O(n \log_2 n) \]
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- \( l_1: \)
  - \( cn/10 \) \( 9cn/10 \)

- \( l_2: \)
  - \( cn/10^2 \) \( 9cn/10^2 \) \( 9cn/10^2 \) \( 9^2cn/10^2 \)

\[ T(n) \leq cn \log_{10} 9 \approx O(n \log_2 n) \]
Chapter 7. Quicksort

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\[ l_0: \quad \begin{array}{c} \text{cn} \\ \text{cn} \end{array} \]
\[ l_1: \quad \begin{array}{c} \text{cn}/10 \quad \text{9cn}/10 \\ \text{cn}/10^2 \quad \text{9cn}/10^2 \quad \text{9cn}/10^2 \quad \text{9}^2\text{cn}/10^2 \end{array} \]
\[ l_2: \quad \begin{array}{c} \text{......} \\ \text{......} \end{array} \]

\[ \text{where} \quad c' = \frac{c}{\log_{10} 9} \]
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l_2: \quad cn/10^2 & \quad 9cn/10^2 & \quad 9^2 cn/10^2 \\
l_h: \quad cn/10^h & \quad \ldots \ldots & \quad c9^h n/10^h \\
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Chapter 7. Quicksort

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\[ \vdots \]
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  l_0: & & \quad \text{cn} & & \quad \text{cn} \\
  l_1: & & \frac{\text{cn}}{10} & & \frac{9\text{cn}}{10} & & \frac{9\text{cn}}{10} & & \frac{9^2\text{cn}}{10^2} \\
  l_2: & & \frac{\text{cn}}{10^2} & & \frac{9\text{cn}}{10^2} & & \frac{9\text{cn}}{10^2} & & \frac{9^2\text{cn}}{10^2} \\
  l_h: & & \frac{\text{cn}}{10^h} & & \cdots & & \frac{c9^h n}{10^h} & & \frac{cn}{10^h} \\
  l_k: & & \cdots & & \frac{c9^k n}{10^k} & & \frac{c9^k n}{10^k} & & \leq \frac{cn}{10^k}
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\[ l_h: \quad cn/10^h \quad \ldots \quad c9^h n/10^h \quad cn \]

\[ l_k: \quad \ldots \quad c9^k n/10^k \quad \leq cn \]

where \( \left( \frac{1}{10} \right)^h n = 1 \), i.e., \( h = \log_{10} n \)
Chapter 7. Quicksort

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    \vdots & \quad \vdots \\
    l_k: & \quad cn/10^k \\
\end{align*}
\]

where \((\frac{1}{10})^h n = 1\), i.e., \(h = \log_{10} n\) and \((\frac{9}{10})^k n = 1\), i.e., \(k = \log_{\frac{9}{10}} n\)
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\ell_h: & \quad cn/10^h \quad \cdots \\
\ell_k: & \quad \cdots \quad c9^hn/10^h \quad cn
\end{align*}
\]

where \((\frac{1}{10})^hn = 1\), i.e., \(h = \log_{10} n\)

\((\frac{9}{10})^kn = 1\), i.e., \(k = \log_{\frac{10}{9}} n\)

\[ cn \log_{10} n \leq T(n) \leq cn \log_{\frac{10}{9}} n \]
Chapter 7. Quicksort

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  & \quad \vdots \\
  l_k &: \quad \ldots \ldots \quad c9^k n/10^k \leq cn
\end{align*}
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Chapter 7. Quicksort

What running time would it be if 10:90 partition is always guaranteed?

\[ T(n) \leq T(n/10) + T(9n/10) + cn \]

Using the recursive-tree method, we have

\[
\begin{align*}
    l_0: & \quad cn \\
    l_1: & \quad cn/10 \\
    l_2: & \quad cn/10^2 \\
    \vdots & \\
    l_h: & \quad cn/10^h \\
    \vdots & \\
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Chapter 7. Quicksort

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<table>
<thead>
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where \( c' = c / \log_2 \frac{10}{9} \)
Chapter 7. Quicksort

Instead of analyzing \texttt{QUICKSORT} (with uniformly distributed for input) we design a randomized version of the algorithm and analyze it.
Chapter 7. Quicksort

Instead of analyzing \texttt{QUICKSORT} (with uniformly distributed for input) we design a randomized version of the algorithm and analyze it.

Algorithm \texttt{RANDOMIZED-PARTITION}(A, p, r)
1. \( i = \text{random}(p, r) \)
2. exchange \( A[r] \leftarrow A[i] \)
3. \textbf{return} (\texttt{PARTITION}(A, p, r))
Chapter 7. Quicksort

Instead of analyzing \textsc{QuickSort} (with uniformly distributed for input) we design a randomized version of the algorithm and analyze it.

Algorithm \textsc{Randomized-Partition}(A, p, r)
1. \(i = \text{random}(p, r)\)
2. exchange \(A[r] \leftrightarrow A[i]\)
3. \textbf{return} (\textsc{Partition}(A, p, r))

Algorithm \textsc{Randomized QuickSort} (A, p, r)
Chapter 7. Quicksort

Instead of analyzing Quicksort (with uniformly distributed for input) we design a randomized version of the algorithm and analyze it.

Algorithm RANDOMIZED-PARTITION($A, p, r$)
1. $i = \text{random}(p, r)$
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Algorithm RANDOMIZED QUICKSORT ($A, p, r$)
1. if $p < r$
Chapter 7. Quicksort

Instead of analyzing Quicksort (with uniformly distributed for input) we design a randomized version of the algorithm and analyze it.

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Algorithm Randomized Quicksort ($A, p, r$)
1. if $p < r$
2. then $q = \text{Randomized-Partition}(A, p, r)$
Instead of analyzing QuickSort (with uniformly distributed for input) we design a randomized version of the algorithm and analyze it.

Algorithm Randomized-Partition \((A, p, r)\)
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Algorithm Randomized QuickSort \((A, p, r)\)
1. if \(p < r\) then
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3. Randomized QuickSort \((A, p, q - 1)\)
Chapter 7. Quicksort

Instead of analyzing QuickSort (with uniformly distributed for input) we design a randomized version of the algorithm and analyze it.

Algorithm \texttt{Randomized-Partition}(A, p, r)
1. \(i = \text{random}(p, r)\)
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Algorithm \texttt{Randomized QuickSort} (A, p, r)
1. \textbf{if} \(p < r\)
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4. \texttt{Randomized QuickSort} (A, q + 1, r)
Chapter 7. Quicksort

Analysis of RANDOMIZED-QUICKSORT
Chapter 7. Quicksort

Analysis of **RANDOMIZED-QUICKSORT**

- count the expected number of comparisons between $x_i$ and $x_j$;
Analysis of Randomized-QuickSort

• count the expected number of comparisons between $x_i$ and $x_j$;

Observation 1: $x_i$ is compared with $x_j$ only when either is a pivot;
Analysis of Randomized-QuickSort

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• define random variable $X_{i,j}$ indicating
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Chapter 7. Quicksort

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Analysis of RandomIZED-QUICKSORT

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• let \( X = \sum_{i=1}^{n} \sum_{j=1, i<j}^{n} X_{i,j} \), written as \( X = \sum_{i<j} X_{i,j} \)
Chapter 7. Quicksort

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$$E(X) = E(\sum_{i<j} X_{i,j}) = \sum_{i<j} E(X_{i,j}) =$$
Chapter 7. Quicksort

Analysis of **Randomized-QuickSort**

- count the expected number of comparisons between $x_i$ and $x_j$;

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- the expected number of comparisons is

$$E(X) = E(\sum_{i<j} X_{i,j}) = \sum_{i<j} E(X_{i,j}) = \sum_{i<j} P(X_{i,j} = 1)$$
Analysis of \texttt{RANDOMIZED-QUICKSORT} (cont.)

\[ E(X) = E\left( \sum_{i<j} X_{i,j} \right) \]
Chapter 7. Quicksort

Analysis of **RANDOMIZED-QUICKSORT** (cont.)

\[ E(X) = E(\sum_{i<j} X_{i,j}) = \sum_{i<j} E(X_{i,j}) = \]

\[
(1) x_i, x_j \text{ are in the same sublist } L;
(2) \text{ either is chosen to be the pivot;}
\]

\[ P(X_{i,j} = 1) = \frac{2}{|L|}, \text{ where } |L| \text{ is the size of the sublist. why?} \]

but we do not know the size of the sublist \( L \)!

however, if \( x_i, x_j \) are so indexed in the final sorted list, then

\[ \text{size of the sublist (which } x_i, x_j \text{ belongs to)} |L| \geq (j - i + 1) \]

So

\[ P(X_{i,j} = 1) \leq \frac{2}{j - i + 1} \leq \frac{2}{j - i + 1} \]
Chapter 7. Quicksort

Analysis of **RANDOMIZED-QUICKSORT** (cont.)

\[
E(X) = E\left(\sum_{i<j} X_{i,j}\right) = \sum_{i<j} E(X_{i,j}) = \sum_{i<j} \text{Prob}(X_{i,j} = 1)
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Analysis of *RANDOMIZED-QUICKSORT* (cont.)

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\(X_{i,j} = 1\), i.e., comparison between \(x_i\) and \(x_j\) occurs only when
Chapter 7. Quicksort

Analysis of Randomized-QuickSort (cont.)

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Chapter 7. Quicksort

Analysis of **RANDOMIZED-QUICKSORT** (cont.)

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Analysis of Randomized-Quicksort (cont.)

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Analysis of \textsc{Randomized-QuickSort} (cont.)

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Analysis of **RANDOMIZED-QUICKSORT** (cont.)

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Chapter 7. Quicksort

Analysis of Randomized-QuickSort (cont.)

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Chapter 7. Quicksort

Analysis of \textsc{Randomized-QuickSort} (cont.)

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\(|L| \geq (j - i + 1)\)
Analysis of \textsc{Randomized-QuickSort} (cont.)

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|L| \geq (j - i + 1)
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So \(P(X_{i,j} = 1) \leq 2 \frac{1}{|L|} \leq 2 \frac{1}{j - i + 1}\)
Chapter 7. Quicksort

original unsorted list

| 5 | 23 | 10 |

sublist L containing elements 5 and 10
10 is a pivot

| 5 | 10 | ... |

| | | | |

|L|

L has to contain elements between 5 and 10
i.e., L has to contain elements 6, 7, 8, 9
|L| \geq j - i + 1 = 10 - 5 + 1 = 6

final sorted list

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |

\(x_5\) \(x_{10}\)
Chapter 7. Quicksort

Analysis of **Randomized-QuickSort** (cont.)

\[
E(X) = E\left(\sum_{i<j} X_{i,j}\right) = \sum_{i<j} E(X_{i,j}) = \sum_{i<j} \text{Prob}(X_{i,j} = 1)
\]

\[
\leq \sum_{i<j} 2 \frac{1}{j - i + 1}
\]
Chapter 7. Quicksort

Analysis of `RANDOMIZED-QUICKSORT` (cont.)

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E(X) = E(\sum_{i<j} X_{i,j}) = \sum_{i<j} E(X_{i,j}) = \sum_{i<j} \text{Prob}(X_{i,j} = 1)
\]

\[
\leq \sum_{i<j} 2 \frac{1}{j - i + 1} = \sum_{i=1}^{n-1} \sum_{j=2}^{n} 2 \frac{1}{j - i + 1}
\]

for some constant \(c > 0\).

So \(E(X) = O(n \log_2 n)\).
Chapter 7. Quicksort

Analysis of RANDOMIZED-QUICKSORT (cont.)

\[ E(X) = E\left(\sum_{i<j} X_{i,j}\right) = \sum_{i<j} E(X_{i,j}) = \sum_{i<j} \text{Prob}(X_{i,j} = 1) \]

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Analysis of \textsc{Randomized-QuickSort} (cont.)

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\[ \leq \sum_{i=1}^{n-1} 2 \sum_{k=1}^{n-1} \frac{1}{k+1} \leq \sum_{i=1}^{n-1} c \log_2 n \]

for some constant \( c > 0 \).

\[ E(X) = O(n \log_2 n) \]
Chapter 7. Quicksort

Analysis of RANDOMIZED-QUICKSORT (cont.)

\[ E(X) = E\left( \sum_{i<j} X_{i,j} \right) = \sum_{i<j} E(X_{i,j}) = \sum_{i<j} \text{Prob}(X_{i,j} = 1) \]

\[ \leq \sum_{i<j} 2 \cdot \frac{1}{j - i + 1} = \sum_{i=1}^{n-1} \sum_{j=2}^{n} 2 \cdot \frac{1}{j - i + 1} \]

\[ \leq \sum_{i=1}^{n-1} 2 \sum_{k=1}^{n-1} \frac{1}{k + 1} \leq \sum_{i=1}^{n-1} c \log_2 n \leq cn \log_2 n \]
Analysis of **Randomized-QuickSort** (cont.)

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E(X) = E\left(\sum_{i<j} X_{i,j}\right) = \sum_{i<j} E(X_{i,j}) = \sum_{i<j} \text{Prob}(X_{i,j} = 1)
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for some constant \(c > 0\).
Analysis of \textsc{Randomized-QuickSort} (cont.)

\[ E(X) = E\left(\sum_{i<j} X_{i,j}\right) = \sum_{i<j} E(X_{i,j}) = \sum_{i<j} \text{Prob}(X_{i,j} = 1) \]

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So \( E(X) = O(n \log_2 n) \).
Chapter 7. Quicksort

O(n log n) Sorting Algorithms

Graph showing the comparison of Heap, Merge, and Quick sorting algorithms in seconds for different values of n.
Chapter 8. Lower Bounds and Sorting in Linear Time

Chapter 8. Lower bounds and sorting in linear time
Chapter 8. Lower bounds and sorting in linear time

- We have used Big-$O$ for upper bounds.
Chapter 8. Lower bounds and sorting in linear time

- We have used Big-$O$ for upper bounds.
- We need another notation for lower bounds.

\[
\Omega(g(n)) = \{ f(n) : \exists c > 0, k > 0 \text{ such that } f(n) \geq cg(n) \text{ for all } n \geq k \}
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In other words, \( \lim_{n \to \infty} \frac{f(n)}{g(n)} = \text{constant} > 0 \) or \( \infty \).

For example, we have shown \( T(n) = \Omega(n^2) \) for Insertion Sort.

Proof techniques for Big-$\Omega$ are similar to those for Big-$O$. 
Chapter 8. Lower bounds and sorting in linear time

- We have used Big-$O$ for upper bounds.
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Define $\Omega(g(n))$ be the set of functions that have growth rates not slower than $cg(n)$ for any given constant $c > 0$.
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Important notes on lower bound and upper bound

- Insertion Sort: $O(n^2)$ → $\Omega(n^2)$
- Merge Sort: $O(n \log_2 n)$ → $\Omega(n \log_2 n)$

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- Insertion Sort runs in time $\Theta(n^2)$,
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meaning: the problem can be solved in time $O(n \log_2 n)$
and $\Omega(n \log_2 n)$ is necessary to solve the problem.
Chapter 8. Lower Bounds and Sorting in Linear Time

Deriving a lower bound for sorting

with decision tree as algorithm/computation model

Claim 1: total number of leaves is $\geq n!$.

Claim 2: the height of the tree at least $\geq \log n!$.

(The minimum of heights of all such trees!)
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**Theorem**: Sorting needs $\Omega(n \log n)$ comparisons on comparison-based computation models.
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**Prove.**
The longest path from the root to a leaf is $\Omega(\log n!)$. I.e., the number of comparisons needed in the worst case is $\Omega(\log n!)$. 

$n! = n(n-1)(n-2)\cdots(n-n^2)(n-n^2-1)\cdots2\times1$ 

$\geq (\frac{n}{2})^{\frac{n}{2}}$ 

or by Stirling's formula: 

$n! = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n (1 + O(1/n))$ 

$\Omega(\log(n!)) = \Omega(n \log n)$
Chapter 8. Lower Bounds and Sorting in Linear Time

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\[
\begin{align*}
n! &= n(n - 1)(n - 2) \cdots (n - \frac{n}{2})(n - \frac{n}{2} - 1) \cdots 2 \times 1 \\
&\geq \left(\frac{n}{2}\right)^{\frac{n}{2}} \times 2^{\frac{n}{2} - 1} \geq \frac{1}{2} n^{\frac{n}{2}}
\end{align*}
\]

or by Stirling’s formula:

\[
n! = \sqrt{2\pi n}(n/e)^n (1 + O(1/n))
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\[
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\]
Chapter 8. Lower Bounds and Sorting in Linear Time

Sorting algorithms with worst case linear time
Chapter 8. Lower Bounds and Sorting in Linear Time

Sorting algorithms with worst case linear time

- count sort
- radix sort
- bucket sort
Chapter 8. Lower Bounds and Sorting in Linear Time

Count sort

Algorithm Counting-Sort \((A, B, k)\)

1. for \(i = 0\) to \(k\)
2. \(C[i] = 0\)
3. for \(j = 1\) to length \([A]\)
4. \(C[A[j]] = C[A[j]] + 1\)
5. \(C[i]\) contains the number of elements whose values = \(i\)
6. for \(i = 1\) to \(k\)
7. \(C[i] = C[i] + C[i - 1]\)
8. \(C[i]\) contains the number of elements whose values ≤ \(i\)
9. for \(j = \text{length} [A]\) down to \(1\)
10. \(B[C[A[j]]] = A[j]\)
11. \(C[A[j]] = C[A[j]] - 1\)

Example: \(A: 2 5 3 0 2 3 0 3, k = 5, C: 2 0 2 3 0 1\)

analysis: \(T(n) = O(k + n)\)
Count sort

Algorithm \textsc{Counting-Sort} \((A, B, k)\) \hspace{1cm} \{\textit{A contains} \(n\) \textit{integers;} \(k\) \textit{is the max}\}

1. \textbf{for} \(i = 0\) \textbf{to} \(k\)
2. \(C[i] = 0\)
3. \textbf{for} \(j = 1\) \textbf{to} \(\text{length}[A]\)
4. \(C[A[j]] = C[A[j]] + 1\)
5. \{\(C[i]\) \textit{contains the number of elements whose values} = \(i\)\}
6. \textbf{for} \(i = 1\) \textbf{to} \(k\)
7. \(C[i] = C[i] + C[i-1]\)
8. \{\(C[i]\) \textit{contains the number of elements whose values} \(\leq i\)\}
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Example: \(A\): 2 5 3 0 2 3 0 3, \(k\) = 5, \(C\): 2 0 2 3 0 1

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Chapter 8. Lower Bounds and Sorting in Linear Time

Count sort

Algorithm Counting-Sort \((A, B, k)\) \(\{A\ contains\ n\ integers;\ k\ is\ the\ max\}\)

1. \(\text{for } i = 0 \text{ to } k\)
Count sort

Algorithm `COUNTING-SORT (A, B, k)` \{\textit{A contains} $n$ \textit{integers;} $k$ \textit{is the max}\}

1. \textbf{for} $i = 0$ \textbf{to} $k$
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6. \textbf{for} \(i = 1\) \textbf{to} \(k\)

Example: \(A: 2\ 5\ 3\ 0\ 2\ 3\ 0\ 3\), \(k = 5\), \(C: 2\ 0\ 2\ 3\ 0\ 1\)

Analysis: \(T(n) = O(k + n)\)
Count sort

Algorithm Counting-Sort \((A, B, k)\) \{ \(A\) contains \(n\) integers; \(k\) is the max\}

1. \textbf{for} \(i = 0\) to \(k\)
2. \hspace{1em} \(C[i] = 0\)
3. \textbf{for} \(j = 1\) to \(\text{length}[A]\)
4. \hspace{1em} \(C[A[j]] = C[A[j]] + 1\)
5. \hspace{1em} \{\(C[i]\) contains the number of elements whose values = \(i\)\}
6. \textbf{for} \(i = 1\) to \(k\)
7. \hspace{1em} \(C[i] = C[i] + C[i - 1]\)
Count sort

Algorithm \textsc{Counting-Sort} \((A, B, k)\) \space \space \{A contains \(n\) integers; \(k\) is the max\}

1. \textbf{for} \(i = 0\) to \(k\)
2. \hspace{1em} \(C[i] = 0\)
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6. \textbf{for} \(i = 1\) to \(k\)
7. \hspace{1em} \(C[i] = C[i] + C[i - 1]\)
8. \hspace{1em} \{\(C[i]\) contains the number of elements whose values \(\leq i\)\}

Example: \(A: 2\ 5\ 3\ 0\ 2\ 3\ 0\ 3\), \(k = 5\), \(C: 2\ 0\ 2\ 3\ 0\ 1\)

analysis: \(T(n) = O(k + n)\)
**Count sort**

Algorithm **COUNTING-SORT** $(A, B, k)$ \{ $A$ contains $n$ integers; $k$ is the max\}

1. `for i = 0 to k` 
2. $\quad C[i] = 0$
3. `for j = 1 to length[A]` 
4. $\quad C[A[j]] = C[A[j]] + 1$
5. $\quad \{C[i] contains the number of elements whose values = i\}$
6. `for i = 1 to k` 
7. $\quad C[i] = C[i] + C[i - 1]$
8. $\quad \{C[i] contains the number of elements whose values $\leq i\}$
9. `for j = length[A] downto 1`
Count sort

Algorithm Counting-Sort $(A, B, k)$ \hspace{1cm} \{A contains $n$ integers; $k$ is the max\}
1. for $i = 0$ to $k$
2. \hspace{1cm} $C[i] = 0$
3. for $j = 1$ to $\text{length}[A]$
4. \hspace{1cm} $C[A[j]] = C[A[j]] + 1$
5. \hspace{1cm} \{$C[i]$ contains the number of elements whose values = $i$\}
6. for $i = 1$ to $k$
7. \hspace{1cm} $C[i] = C[i] + C[i-1]$
8. \hspace{1cm} \{$C[i]$ contains the number of elements whose values $\leq i$\}
9. for $j = \text{length}[A]$ downto 1
10. \hspace{1cm} $B[C[A[j]]] = A[j]$

Example: $A$: 2 5 3 0 2 3 0 3, $k$ = 5, $C$: 2 0 2 3 0 1
Chapter 8. Lower Bounds and Sorting in Linear Time

Count sort

Algorithm COUNTING-SORT \((A, B, k)\) \(\{A\) contains \(n\) integers; \(k\) is the max\}
1. \textbf{for} \(i = 0\) \textbf{to} \(k\)
2. \hspace{1em} \(C[i] = 0\)
3. \textbf{for} \(j = 1\) \textbf{to} length\([A]\)
4. \hspace{1em} \(C[A[j]] = C[A[j]] + 1\)
5. \hspace{1em} \{\(C[i]\) contains the number of elements whose values = \(i\)\}
6. \textbf{for} \(i = 1\) \textbf{to} \(k\)
7. \hspace{1em} \(C[i] = C[i] + C[i - 1]\)
8. \hspace{1em} \{\(C[i]\) contains the number of elements whose values \(\leq i\)\}
9. \textbf{for} \(j = \text{length}[A]\) \textbf{downto} 1
10. \hspace{1em} \(B[C[A[j]]] = A[j]\)
11. \hspace{1em} \(C[A[j]] = C[A[j]] - 1\)

Example: \(A: 2\ 5\ 3\ 0\ 2\ 3\ 0\ 3\) \(k = 5\) \(C: 2\ 0\ 2\ 3\ 0\ 1\)

\(T(n) = O(k + n)\)
Chapter 8. Lower Bounds and Sorting in Linear Time

Count sort

Algorithm COUNTING-SORT (A, B, k)  \(\{A \text{ contains } n \text{ integers; } k \text{ is the max}\}\)
1. for \(i = 0\) to \(k\)
2. \(C[i] = 0\)
3. for \(j = 1\) to \(\text{length}[A]\)
4. \(C[A[j]] = C[A[j]] + 1\)
5. \(\{C[i] \text{ contains the number of elements whose values } = i\}\)
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7. \(C[i] = C[i] + C[i - 1]\)
8. \(\{C[i] \text{ contains the number of elements whose values } \leq i\}\)
9. for \(j = \text{length}[A]\) down to 1
10. \(B[C[A[j]]] = A[j]\)
11. \(C[A[j]] = C[A[j]] - 1\)

Example: A: 2 5 3 0 2 3 0 3, \(k = 5\),
Count sort

Algorithm COUNTING-SORT \((A, B, k)\) \{\(A\) contains \(n\) integers; \(k\) is the max\}
1. \(\text{for } i = 0 \text{ to } k\)
2. \(C[i] = 0\)
3. \(\text{for } j = 1 \text{ to } \text{length}[A]\)
4. \(C[A[j]] = C[A[j]] + 1\)
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10. \(B[C[A[j]]] = A[j]\)
11. \(C[A[j]] = C[A[j]] - 1\)

Example: \(A: 2 \ 5 \ 3 \ 0 \ 2 \ 3 \ 0 \ 3, \ \ k = 5, \ \ C: 2 \ 0 \ 2 \ 3 \ 0 \ 1\)
Chapter 8. Lower Bounds and Sorting in Linear Time

Count sort

Algorithm Counting-Sort \((A, B, k)\) \{ \(A\) contains \(n\) integers; \(k\) is the max\}

1. \textbf{for} \(i = 0\) \textbf{to} \(k\)
2. \hspace{1em} \(C[i] = 0\)
3. \textbf{for} \(j = 1\) \textbf{to} \text{length}[A]
4. \hspace{1em} \(C[A[j]] = C[A[j]] + 1\)
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11. \hspace{1em} \(C[A[j]] = C[A[j]] - 1\)

Example: \(A: 2\ 5\ 3\ 0\ 2\ 3\ 0\ 3,\ \ k = 5,\ \ C: 2\ 0\ 2\ 3\ 0\ 1\)

analysis:
Count sort

Algorithm **COUNTING-SORT** \((A, B, k)\) \{\(A\) contains \(n\) integers; \(k\) is the max\}

1. \textbf{for} \(i = 0\) \textbf{to} \(k\)
2. \hspace{1em} \(C[i] = 0\)
3. \textbf{for} \(j = 1\) \textbf{to} \text{length}[\(A\)]
4. \hspace{1em} \(C[A[j]] = C[A[j]] + 1\)
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Example: \(A: 2\ 5\ 3\ 0\ 2\ 3\ 0\ 3,\ \ k = 5,\ \ \ C: 2\ 0\ 2\ 3\ 0\ 1\)

analysis: \(T(n) = O(k + n)\)
Chapter 8. Lower Bounds and Sorting in Linear Time

Radix Sort:

Algorithm Radix-Sort \((A, d)\)
1. for \(i = 1\) to \(d\)
2. sort \(A\) on the \(i\)th digit

Lemma. Given \(n\) \(b\)-bit binary numbers and any positive \(r \leq b\).

Radix-Sort uses \(\Theta(\lceil b/r \rceil (n + 2r))\) time.
## Radix Sort:

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Chapter 8. Lower Bounds and Sorting in Linear Time

Radix Sort:

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Algorithm \textsc{Radix-Sort}(A, d)
Radix Sort:

329  720  720  329
457  355  329  355
657  436  436  436
839  457  839  457
436  657  355  657
720  329  457  720
355  839  657  839

Algorithm Radix-Sort\( (A, d) \)

1. \textbf{for } \( i = 1 \) \textbf{ to } \( d \)
Chapter 8. Lower Bounds and Sorting in Linear Time

Radix Sort:

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Algorithm Radix-Sort\((A, d)\)

1. for \(i = 1\) to \(d\)
2. sort \(A\) on the \(ith\) digit
## Chapter 8. Lower Bounds and Sorting in Linear Time

**Radix Sort:**

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Algorithm **Radix-Sort**($A, d$)

1. for $i = 1$ to $d$
2. sort $A$ on the $i$th digit

**Lemma.** Given $n$ $b$-bit binary numbers and any positive $r \leq b$. **Radix-Sort** uses $\Theta(\lceil b/r \rceil (n + 2^r))$ time.
Lemma. Given $n$ $b$-bit binary numbers and any positive $r \leq b$. \textsc{Radix-Sort} uses $\Theta(\lceil b/r \rceil (n + 2^r))$ time.
Lemma. Given $n$ $b$-bit binary numbers and any positive $r \leq b$, \textsc{Radix-Sort} uses $\Theta([b/r](n + 2^r))$ time.

Proof. Each $b$-digit binary number can be regarded as $[b/r]$ $r$-digit binary numbers. These $r$-digit binary numbers are of integer values in the range of $\{0, 1, \ldots, 2^r - 1\}$. 
Lemma. Given $n$ $b$-bit binary numbers and any positive $r \leq b$. Radix-Sort uses $\Theta(\lceil b/r \rceil (n + 2^r))$ time.

Proof. Each $b$-digit binary number can be regarded as $\lceil b/r \rceil$ $r$-digit binary numbers. These $r$-digit binary numbers are of integer values in the range of $\{0, 1, \ldots, 2^r - 1\}$.

Run Radix-Sort on the original binary numbers assumed to be $\lceil b/r \rceil$ columns.
Lemma. Given \( n \) \( b \)-bit binary numbers and any positive \( r \leq b \). \texttt{Radix-Sort} uses \( \Theta(\lceil b/r \rceil (n + 2^r)) \) time.

Proof. Each \( b \)-digit binary number can be regarded as \( \lceil b/r \rceil \) \( r \)-digit binary numbers. These \( r \)-digit binary numbers are of integer values in the range of \( \{0, 1, \ldots, 2^r - 1\} \).

Run \texttt{Radix-Sort} on the original binary numbers assumed to be \( \lceil b/r \rceil \) columns.

For every column, sorting by \texttt{Counting-Sort} with \( 2^r - 1 \) being the maximum.
Lemma. Given $n$ $b$-bit binary numbers and any positive $r \leq b$. 
Radix-Sort uses $\Theta([b/r](n + 2^r))$ time.

Proof. Each $b$-digit binary number can be regarded as $[b/r]$ $r$-digit binary numbers. These $r$-digit binary numbers are of integer values in the range of \{0, 1, \ldots, 2^r - 1\}. 

Run Radix-Sort on the original binary numbers assumed to be $[b/r]$ columns. 
For every column, sorting by Counting-Sort with $2^r - 1$ being the maximum. 
The total time is $O([b/r](n + 2^r))$, where $(n + 2^r)$ is time for Counting-Sort.
Lemma. Given \( n \) \( b \)-bit binary numbers and any positive \( r \leq b \). 
\( \text{Radix-Sort} \) uses \( \Theta([b/r](n+2^r)) \) time.

Proof. Each \( b \)-digit binary number can be regarded as \([b/r]\) \( r \)-digit binary numbers. These \( r \)-digit binary numbers are of integer values in the range of \( \{0, 1, \ldots, 2^r - 1\} \).

Run \( \text{Radix-Sort} \) on the original binary numbers assumed to be \([b/r]\) columns.

For every column, sorting by \( \text{Counting-Sort} \) with \( 2^r - 1 \) being the maximum.

The total time is \( O([b/r](n+2^r)) \), where \( (n+2^r) \) is time for \( \text{Counting-Sort} \).

Since all steps in the two algorithms are mandatory, the total time is also \( \Omega([b/r](n+2^r)) \), thus \( \Theta([b/r](n+2^r)) \).
Lemma. Given $n$ $b$-bit binary numbers and any positive $r \leq b$. Radix-Sort uses $\Theta(\lceil b/r \rceil (n + 2^r))$ time.

Proof. Each $b$-digit binary number can be regarded as $\lceil b/r \rceil$ $r$-digit binary numbers. These $r$-digit binary numbers are of integer values in the range of $\{0, 1, \ldots, 2^r - 1\}$.

Run Radix-Sort on the original binary numbers assumed to be $\lceil b/r \rceil$ columns.

For every column, sorting by Counting-Sort with $2^r - 1$ being the maximum.

The total time is $O(\lceil b/r \rceil (n + 2^r))$, where $(n + 2^r)$ is time for Counting-Sort.

Since all steps in the two algorithms are mandatory, the total time is also $\Omega(\lceil b/r \rceil (n + 2^r))$, thus $\Theta(\lceil b/r \rceil (n + 2^r))$.

Once $b$ and $n$ are given, we can choose $r$ to minimize the quantity $\lceil b/r \rceil (n + 2^r)$. 
Chapter 8. Lower Bounds and Sorting in Linear Time

**Bucket Sort** (assuming uniform distribution of inputs)
Chapter 8. Lower Bounds and Sorting in Linear Time

**Bucket Sort** (assuming uniform distribution of inputs)

*Algorithm* **B**UCKET-**S**ORT(*A*)
Bucket Sort (assuming uniform distribution of inputs)

Algorithm Bucket-Sort(A)
1. \( n = \text{length}[A] \)
Chapter 8. Lower Bounds and Sorting in Linear Time

**Bucket Sort** (assuming uniform distribution of inputs)

Algorithm \texttt{BUCKET-SORT}(A)
1. \( n = \textit{length}[A] \)
2. \texttt{for } \( i = 1 \) \texttt{ to } \( n \)
**Bucket Sort** (assuming uniform distribution of inputs)

Algorithm **BUCKET-SORT**\((A)\)
1. \(n = length[A]\)
2. for \(i = 1\) to \(n\)
3. insert \(A[i]\) into list \(B[[nA[i]]]\)
Chapter 8. Lower Bounds and Sorting in Linear Time

**Bucket Sort** (assuming uniform distribution of inputs)

Algorithm **Bucket-Sort**($A$)
1. $n = length[A]$
2. **for** $i = 1$ **to** $n$
3. insert $A[i]$ into list $B[\lfloor nA[i] \rfloor]$
4. **for** $i = 0$ **to** $n - 1$

A: .78 .17 .39 .26 .72 .94 .21 .12 .23 .68
B: 0 / → .12 → .17 → .21 → .23 → .26 / 3 / 4 / 5 / 6 → .68 → .72 → .78 → .94
Chapter 8. Lower Bounds and Sorting in Linear Time

**Bucket Sort** (assuming uniform distribution of inputs)

**Algorithm** BUCKET-SORT($A$)
1. $n = length[A]$
2. for $i = 1$ to $n$
3.   insert $A[i]$ into list $B[\lfloor nA[i] \rfloor]$
4. for $i = 0$ to $n - 1$
5.   sort list $B[i]$ with **Insertion Sort**
Bucket Sort (assuming uniform distribution of inputs)

Algorithm Bucket-Sort(A)
1. \( n = \text{length}[A] \)
2. for \( i = 1 \) to \( n \)
3. insert \( A[i] \) into list \( B[\lfloor nA[i] \rfloor] \)
4. for \( i = 0 \) to \( n - 1 \)
5. sort list \( B[i] \) with Insertion Sort
6. concatenate the lists \( B[0], B[1], ..., B[n - 1] \)
Chapter 8. Lower Bounds and Sorting in Linear Time

**Bucket Sort** (assuming uniform distribution of inputs)

**Algorithm** \textsc{Bucket-Sort}(A)

1. \( n = \text{length}[A] \)
2. \textbf{for} \( i = 1 \) \textbf{to} \( n \)
3. \hspace{1em} insert \( A[i] \) into list \( B[\lfloor nA[i] \rfloor] \)
4. \textbf{for} \( i = 0 \) \textbf{to} \( n - 1 \)
5. \hspace{1em} sort list \( B[i] \) with \textsc{Insertion Sort}
6. \hspace{1em} concatenate the lists \( B[0], B[1], \ldots, B[n - 1] \)

\begin{itemize}
  \item A: \hspace{1em} .78  .17  .39  .26  .72  .94  .21  .12  .23  .68
\end{itemize}
Bucket Sort (assuming uniform distribution of inputs)

Algorithm Bucket-Sort$(A)$
1. $n = length[A]$
2. for $i = 1$ to $n$
3. insert $A[i]$ into list $B[\lfloor nA[i] \rfloor]$
4. for $i = 0$ to $n - 1$
5. sort list $B[i]$ with Insertion Sort
6. concatenate the lists $B[0], B[1], ..., B[n - 1]$

A: \begin{align*} &.78 .17 .39 .26 .72 .94 .21 .12 .23 .68 \\
B: &0 / \end{align*}
Chapter 8. Lower Bounds and Sorting in Linear Time

**Bucket Sort** (assuming uniform distribution of inputs)

Algorithm **Bucket-Sort**(A)
1. \( n = length[A] \)
2. \( \text{for } i = 1 \text{ to } n \)
3. insert \( A[i] \) into list \( B[\lfloor n A[i] \rfloor] \)
4. \( \text{for } i = 0 \text{ to } n - 1 \)
5. sort list \( B[i] \) with **Insertion Sort**
6. concatenate the lists \( B[0], B[1], ..., B[n - 1] \)

A: 0.78 0.17 0.39 0.26 0.72 0.94 0.21 0.12 0.23 0.68

B: 1 → 0.12 → 0.17
Chapter 8. Lower Bounds and Sorting in Linear Time

Bucket Sort (assuming uniform distribution of inputs)

Algorithm Bucket-Sort(A)
1. \( n = \text{length}[A] \)
2. \( \text{for} \ i = 1 \ \text{to} \ n \)
3. \( \text{insert} \ A[i] \ \text{into list} \ B[\lfloor nA[i] \rfloor] \)
4. \( \text{for} \ i = 0 \ \text{to} \ n - 1 \)
5. \( \text{sort list} \ B[i] \ \text{with Insertion Sort} \)
6. \( \text{concatenate the lists} \ B[0], B[1], ..., B[n - 1] \)

A: \(.78 \ .17 \ .39 \ .26 \ .72 \ .94 \ .21 \ .12 \ .23 \ .68\)

B: \(0 / \)
1 \(\rightarrow \ .12 \ \rightarrow \ .17 \)
2 \(\rightarrow \ .21 \ \rightarrow \ .23 \ \rightarrow \ .26 \)
Chapter 8. Lower Bounds and Sorting in Linear Time

**Bucket Sort** (assuming uniform distribution of inputs)

Algorithm Bucket-Sort($A$)
1. $n = length[A]$
2. for $i = 1$ to $n$
3. insert $A[i]$ into list $B[\lfloor nA[i] \rfloor]$
4. for $i = 0$ to $n - 1$
5. sort list $B[i]$ with **Insertion Sort**
6. concatenate the lists $B[0], B[1], ..., B[n - 1]$

A: 0.78 0.17 0.39 0.26 0.72 0.94 0.21 0.12 0.23 0.68

B: 0 /
   1 → .12 → .17
   2 → .21 → .23 → .26
   3 → .39
Bucket Sort (assuming uniform distribution of inputs)

Algorithm Bucket-Sort($A$)
1. $n = \text{length}[A]$
2. for $i = 1$ to $n$
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6. concatenate the lists $B[0], B[1], ..., B[n - 1]$

A: $0.78 \ 0.17 \ 0.39 \ 0.26 \ 0.72 \ 0.94 \ 0.21 \ 0.12 \ 0.23 \ 0.68$

B: 0 /
   1 → $0.12$ → $0.17$
   2 → $0.21$ → $0.23$ → $0.26$
   3 → $0.39$
   4 /
**Bucket Sort** (assuming uniform distribution of inputs)

Algorithm **Bucket-Sort**($A$)
1. $n = length[A]$
2. \textbf{for} $i = 1$ to $n$
3. \hspace{1em} insert $A[i]$ into list $B[\lfloor nA[i] \rfloor]$
4. \textbf{for} $i = 0$ to $n - 1$
5. \hspace{1em} sort list $B[i]$ with \textsc{Insertion Sort}
6. \hspace{1em} concatenate the lists $B[0], B[1], ..., B[n - 1]$

**Example:**

A: \ .78 \ .17 \ .39 \ .26 \ .72 \ .94 \ .21 \ .12 \ .23 \ .68 \\
B: 0 / \\
1 \rightarrow \ .12 \rightarrow \ .17 \\
2 \rightarrow \ .21 \rightarrow \ .23 \rightarrow \ .26 \\
3 \rightarrow \ .39 \\
4 / \\
5 /
Chapter 8. Lower Bounds and Sorting in Linear Time

**Bucket Sort** (assuming uniform distribution of inputs)

Algorithm **Bucket-Sort**(A)

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2. for \( i = 1 \) to \( n \)
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4. for \( i = 0 \) to \( n - 1 \)
5. sort list \( B[i] \) with **Insertion Sort**
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A: \(.78 .17 .39 .26 .72 .94 .21 .12 .23 .68\)

B: \(0 /
1 \rightarrow .12 \rightarrow .17
2 \rightarrow .21 \rightarrow .23 \rightarrow .26
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Bucket Sort (assuming uniform distribution of inputs)

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Bucket Sort (assuming uniform distribution of inputs)

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Bucket Sort (assuming uniform distribution of inputs)

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Chapter 8. Lower Bounds and Sorting in Linear Time

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Chapter 9. Medians and Order Statistics

**Chapter 9. Medians and order statistics**
Chapter 9. Medians and Order Statistics

Chapter 9. Medians and order statistics

- find the maximum: linear time
Chapter 9. Medians and Order Statistics

- find the maximum: linear time
- find the minimum: linear time
- find the median (i.e., the \( \frac{n}{2} \)th smallest element) ? The problem has upper bound \( O(n \log_2 n) \).

Why?

Can we do better?
Chapter 9. Medians and Order Statistics

Chapter 9. Medians and order statistics

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Chapter 9. Medians and Order Statistics

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Chapter 9. Medians and Order Statistics

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Chapter 9. Medians and Order Statistics

Chapter 9. Medians and order statistics

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Chapter 9. Medians and Order Statistics

Selection problem

Input: a list $A$ of elements, an integer $i$;

Output: the $i$th smallest element in $A$;

There are algorithms solving it in linear time.

Two types of algorithms:

• Selection in expected linear time (but worst case $\Theta(n^2)$)

• Selection in worst case linear time
Chapter 9. Medians and Order Statistics

Selection problem
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Chapter 9. Medians and Order Statistics

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Idea of the algorithm:

- randomly identify a pivot $x$ and partition the list $A$ into two sublists $A_l$ and $A_u$; assume the **rank** of $x$ is $k$;

- if $i = k$, done, return ($x$);
- else if $k > i$, recursively do for $A_l$ with $i$;
- else recursively do for $A_u$ with $i - k$;
Selection in expected linear time

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Chapter 9. Medians and Order Statistics

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Chapter 9. Medians and Order Statistics

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Chapter 9. Medians and Order Statistics

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Chapter 9. Medians and Order Statistics

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Chapter 9. Medians and Order Statistics

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Chapter 9. Medians and Order Statistics

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time \(T(n)\) would have the recurrence

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assuming \(r \geq 2\),

\[
T(n) \leq cn\left(\frac{r-1}{r}\right) + cn\left(\frac{r-1}{r}\right)^2 + cn\left(\frac{r-1}{r}\right)^3 + \ldots cn\left(\frac{r-1}{r}\right)^m = O(n)
\]

where \(\left(\frac{r-1}{r}\right)^m n = 1\), \(m = \log_{\frac{r-1}{r}} n\)
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Performance analysis

\[ \Theta(n^2) \]

- on sublist \( A[p..r] \), assume \( n = r - p + 1 \);
- the algorithm identifies a pivot and recursively computes on sublist \( A[p..q] \) (or \( A[q+1..r] \));
- the pivot is chosen with probability \( \frac{1}{n} \).
Performance analysis

The worst case: running time $\Theta(n^2)$. 
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• so the expected time $E[T(n)]$ needs to include the average time of recursion on the case when sublist $A[p..q]$ possibly has lengths $k = 0, 1, 2, \ldots, n - 1$
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- thus the expected time $E[T(n)]$ is computed as

$$E[T(n)] \leq \sum_{k=1}^{n} \frac{1}{n} \cdot E[T(\max\{k-1, n-k\})] + an, \text{ for some constant } a > 0$$
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because $\max\{k-1, n-k\} = k-1$ if $k > n/2$ and $\max\{k-1, n-k\} = n-k$ if $k \leq n/2$

$$E[T(n)] \leq 2/n\sum_{k=n/2}^{n-1} E[T(k)] + an$$
We conclude that
\[ E[T(n)] \leq \frac{2}{n} \sum_{k=1}^{n-1} k = \frac{n}{2} \]

Theorem.

\[ E[T(n)] = O(n). \]

Proof (by substitution method).

We will prove that \( E[T(n)] \leq cn \) for some \( c > 0 \).

• Base case: \( n = 2 \), we will decide later;
• Assumption: for all \( k \leq n-1 \), 
  \[ E[T(k)] \leq ck; \]
• Induction: 
  \[ E[T(k)] \leq \frac{2}{n} \sum_{k=1}^{n-1} k = \frac{n}{2} E[T(k)] + an \leq 2c n \]

That is when \( n \geq \frac{2c}{c-4a} \).

• Base case: 
  \[ T(n) \leq cn, \text{ for } n < \frac{2c}{c-4a}; \]

How to prove?
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We conclude that $E[T(n)] \leq \frac{2}{n} \sum_{k=n/2}^{n-1} E[T(k)] + an$
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Chapter 9. Medians and Order Statistics

We conclude that \( E[T(n)] \leq \frac{2}{n} \sum_{k=n/2}^{n-1} E[T(k)] + an \)

**Theorem.** \( E[T(n)] = O(n) \).

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We conclude that $E[T(n)] \leq 2/n\sum_{k=n/2}^{n-1} E[T(k)] + an$.

**Theorem.** $E[T(n)] = O(n)$.

**Proof (by substitution method).** We will prove that $E[T(n)] \leq cn$ for some $c > 0$. 

$$E[T(n)] \leq 2/n\sum_{k=n/2}^{n-1} E[T(k)] + an \leq 2/n\sum_{k=n/2}^{n-1} \frac{n}{2} E[T(k)] + an \leq 2c n\left(\sum_{k=n/2}^{n-1} \frac{1}{2}ight) + an \leq \cdots \leq 3cn/4 + c/2 + an \leq cn$$

When $(cn/4 - c/2 - an) \geq 0$, we have $n \geq 2c/\left(c - 4a\right)$.
Chapter 9. Medians and Order Statistics

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Chapter 9. Medians and Order Statistics

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**Theorem.** $E[T(n)] = O(n)$.

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Chapter 9. Medians and Order Statistics

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That is when

$$n \geq 2c/(c - 4a)$$

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- **Induction:**

$$E[T(k)] \leq 2/n\sum_{k=n/2}^{n-1} E[T(k)] + an \leq 2/n\sum_{k=n/2}^{n-1} ck + an$$

$$= \frac{2c}{n} \left[ \sum_{k=1}^{n-1} k - \sum_{k=1}^{n/2-1} k \right] + an$$

That is when

$$n \geq \frac{2c}{(c - 4a)}.$$
We conclude that $E[T(n)] \leq 2/n\sum_{k=n/2}^{n-1} E[T(k)] + an$

**Theorem.** $E[T(n)] = O(n)$.

**Proof** (by substitution method). We will prove that $E[T(n)] \leq cn$ for some $c > 0$.

- Base case: $n = ?$, we will decide later;
- Assumption: for all $k \leq n - 1$, $E[T(k)] \leq ck$;
- Induction:

$$E[T(k)] \leq 2/n\sum_{k=n/2}^{n-1} E[T(k)] + an \leq 2/n\sum_{k=n/2}^{n-1} ck + an$$

$$= \frac{2c}{n} \left[ \sum_{k=1}^{n-1} k - \sum_{k=1}^{n/2-1} k \right] + an = \frac{2c}{n} \left[ \frac{n-1}{2} (n) - \frac{n/2-1}{2} (n/2) \right] + an$$
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Chapter 9. Medians and Order Statistics

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\]

\[
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\]

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when $(cn/4 - c/2 - an) \geq 0$. 
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We conclude that $E[T(n)] \leq 2/n \sum_{k=n/2}^{n-1} E[T(k)] + an$

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**Proof (by substitution method).** We will prove that $E[T(n)] \leq cn$ for some $c > 0$.

- Base case: $n = ?$, we will decide later;
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- Base case: $T(n) \leq cn$, for $n < 2c/(c - 4a)$,
We conclude that \( E[T(n)] \leq 2/n \sum_{k=n/2}^{n-1} E[T(k)] + an \)

**Theorem.** \( E[T(n)] = O(n) \).

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- **Base case:** \( T(n) \leq cn \), for \( n < 2c/(c - 4a) \), How to prove?
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Selection in worst case linear time

Input: set \( S \) of \( n \) elements and \( i \);
Output: the \( i \)th smallest element in \( S \);

Main idea:
• find a pivot \( x \) to partition the list \( S \) into two sublists \( S_1 \) and \( S_2 \), such that \( \forall y \in S_1 \ y < x \) and \( \forall z \in S_2 \ z > x \);
• both \( S_1 \) and \( S_2 \) are guaranteed only a fraction of \( S \);
• the \( i \)th smallest element is either \( x \), or in \( S_1 \) or in \( S_2 \) (but not both);
• in either of the latter two cases, the algorithm is applied recursively.

\[ T(n) \leq T(\beta n) + cn \] where \( 0 < \beta < 1 \), such that
\[ T(n) \leq cn + c\beta n + c\beta^2 n + \ldots + c\beta^m n \leq c \left( \frac{1}{1-\beta} \right) n = O(n) \]
Selection in worst case linear time
Chapter 9. Medians and Order Statistics

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$T(n) \leq T(\beta n) + cn$ where $0 < \beta < 1$, such that $T(n) \leq cn + c\beta n + c\beta^2 n + \ldots + c\beta^m n \leq c(1 - \beta^n) = O(n)$.
Chapter 9. Medians and Order Statistics

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Chapter 9. Medians and Order Statistics

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Chapter 9. Medians and Order Statistics

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Chapter 9. Medians and Order Statistics

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Chapter 9. Medians and Order Statistics

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Chapter 9. Medians and Order Statistics

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Chapter 9. Medians and Order Statistics

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Chapter 9. Medians and Order Statistics

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How to find such a pivot?

• the very selection algorithm is recursively called for finding the pivot
• the size of the sublist to find the pivot is also a fraction $\alpha n$ of the original list $S$, $|S| = n$;
• the total time actually is $T(n) \leq T(\alpha n) + T(\beta n) + cn$ where $\alpha + \beta < 1$. 
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How to find such a pivot?
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How to find such a pivot?

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$$T(n) \leq T(\alpha n) + T(\beta n) + cn$$

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Algorithm $\text{SELECT} \ (S, i); \ \{ \text{where } S \text{ contains } n \text{ distinct elements} \}$

1. divide $S$ into $\lceil n/5 \rceil$ groups of 5 elements
2. sort each group (of 5) and find the median of each group; let $M$ contain all these medians; where $|M| = \lceil n/5 \rceil$
3. recursively call $\text{SELECT} \ (M, \lceil n/10 \rceil)$; let the result be $x$ and let the rank of $x$ be $k$ in $S$
4. if $i = k$ return $(x)$
5. else use $x$ as the pivot to partition $S$ resulting in $S_1$ and $S_2$, such that $\forall y \in S_1 y < x$ and $\forall z \in S_2 z > x$
6. if $i < k$ recursively call $\text{SELECT} \ (S_1, i)$ else recursively call $\text{SELECT} \ (S_2, i - k)$
Algorithm \textsc{Select} \((S, i); \{ \text{where } S \text{ contains } n \text{ distinct elements}\}

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Chapter 9. Medians and Order Statistics

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Chapter 9. Medians and Order Statistics

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Algorithm \texttt{Select} \((S, i)\); \{ where \(S\) contains \(n\) distinct elements\}
(1) divide \(S\) into \(\lceil n/5 \rceil\) groups of 5 elements
(2) sort each group (of 5) and find the median of each group;
    let \(M\) contain all these medians; where \(|M| = \lceil n/5 \rceil\)
(3) \textbf{recursively call} \texttt{Select}(\(M, \lceil n/10 \rceil\));
    let the result be \(x\) and let the rank of \(x\) be \(k\) in \(S\)
(4) \textbf{if} \(i = k\) \textbf{return} \((x)\)
(5) \textbf{else} use \(x\) as the pivot to partition \(S\) resulting in \(S_1\) and \(S_2\),
Algorithm \texttt{Select} \((S, i)\); \{ \textit{where} \(S\) \textit{contains} \(n\) \textit{distinct elements} \}

(1) divide \(S\) into \([n/5]\) groups of 5 elements
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(3) \textbf{recursively call} \texttt{Select}(M, [n/10]);
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Algorithm \texttt{Select} \((S, i); \) \{ where \( S \) contains \( n \) distinct elements\}

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3. \textbf{recursively call} \texttt{Select}(\( M, \lceil n/10 \rceil \));
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6. \textbf{if} \( i < k \) \textbf{recursively call} \texttt{Select}(\( S_1, i)\)
   \textbf{else recursively call} \texttt{Select}(\( S_2, i - k)\)
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Note: the number of elements \( \leq x \) is at least:

\[ |S_1| \geq 3 \left\lceil \frac{n}{5} \right\rceil^2 \geq \frac{3}{10}n \]

\[ \Rightarrow |S_2| < n - \frac{3}{10}n = \frac{7}{10}n \]

Similarly, the number of elements \( \geq x \) is at least:

\[ |S_2| \geq 3 \left\lceil \frac{n}{5} \right\rceil^2 - 2 \geq \frac{3}{10}n - 6 \geq \frac{3}{10}n \]

\[ \Rightarrow |S_1| < n - \frac{3}{10}n + 6 = \frac{7}{10}n + 6 \]

So a time upper bound for \( \text{Select} \) is:

\[ T(n) = T_{\text{mom}} + T_{\text{sub}} + O(n) \]
Note: the number of elements \( \leq x \) is at least:

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So a time upper bound for $\text{SELECT}$ is $T(n) = T_{mom} + T_{sub} + O(n)$
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So a time upper bound for $\text{SELECT}$ is $T(n) = T_{mom} + T_{sub} + O(n)$

$$T(n) \leq T(\lceil n/5 \rceil) + T(\lceil 7n/10 + 6 \rceil) + O(n)$$

when $n \geq 140$
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Summary of Algorithm Analysis Scenarios
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Given an algorithm, carry out the following in order:
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- prove the correctness of the bound.

For example, given Insertion Sort:

- we first analyzed the algorithm and obtained
  
  $T(n) = c_1 n + c_2 (n - 1) + c_4 (n - 1) + c_5 \sum_{j=2}^{n} t_j + c_6 \sum_{j=2}^{n} (t_j - 1) + c_7 \sum_{j=2}^{n} (t_j - 1) + c_8 (n - 1)$

- we guessed upper bound $T(n) = O(n^2)$, i.e., $T(n) \leq cn^2$;
- and finally proved that it was indeed the case.
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Summary of Algorithm Analysis Scenarios

For recursive algorithms, for example, given the Binary Search algorithm, we first analyze the time $T(n)$ of the algorithm and obtained a recurrence for $T(n)$:

$$T(n) \leq T\left(\left\lfloor \frac{n}{2} \right\rfloor \right) + c$$

we guess upper bound $T(n) = O(\log_2 n)$, i.e., $T(n) \leq c \log_2 n$;

we prove the guessed bound.

(1) we can use the recursive tree method by unfolding the time function;

or

(2) we can use the substitution method by the principle of induction. But we need the recurrence to apply induction.

Using the recurrence: $T(n) \leq T\left(\left\lfloor \frac{n}{2} \right\rfloor \right) + c$ to prove $T(n) \leq c \log_2 n$.

See previous lecture notes.
Summary of Algorithm Analysis Scenarios

For recursive algorithms
   For example, given Binary Search algorithm,
Summary of Algorithm Analysis Scenarios

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see previous lecture notes
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