Part IV. Advanced Design and Analysis Techniques
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- Chapter 14.9 Exhaustive search
- Chapter 15. Dynamic programming
- Chapter 16. Greedy algorithms
Chapter 14.9. Exhaustive Search

Chapter 14.9. Exhaustive Search (not in the text!)
Chapter 14.9. Exhaustive Search

To enumerate all possible solutions to the problem instance (not in the text!)
Chapter 14.9. Exhaustive Search

To enumerate all possible solutions to the problem instance

How?
Chapter 14.9. Exhaustive Search

To enumerate all possible solutions to the problem instance

How?

- systematic examining all solutions
Chapter 14.9. Exhaustive Search

To enumerate all possible solutions to the problem instance

How?

• systematic examining all solutions

• without repeating solutions that have been examined
Chapter 14.9. Exhaustive Search

To enumerate all possible solutions to the problem instance

How?

- systematic examining all solutions
- without repeating solutions that have been examined
- stop when a satisfactory solution is found
First, we need to be able to count total number of “things” to be enumerated.
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• without over-counting (efficiency)
Chapter 14.9. Exhaustive Search

First, we need to be able to count total number of “things” to be enumerated.

- without missing one (correctness)
- without over-counting (efficiency)
- A sophisticated counting often has recursive solution.
Chapter 14.9. Exhaustive Search

Examples of counting:

1. Total number of permutations of \(1, 2, \ldots, n\) is \(P(n) = n \times P(n-1)\) with base case \(P(1) = 1\).

2. Total number of ways to choose \(k\) from \(n\) items is \(\binom{n}{k} = \frac{n!}{(n-k)!\cdot k!}\) or, alternatively, \(\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}\) with base cases: \(\binom{n}{0} = \binom{n}{n} = 1\).
Chapter 14.9. Exhaustive Search

Examples of counting:

(1) total number of permutations of \((1, 2, \ldots, n)\) is

\[
P(n) =
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Chapter 14.9. Exhaustive Search

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(2) total number of ways to choose \(k\) from \(n\) items is

\[ \binom{n}{k} = \frac{n(n-1) \ldots (n-k+1)}{k!} = \frac{n!}{(n-k)!k!} \\
\text{or, alternatively,} \\
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with base cases: (?)
Chapter 14.9. Exhaustive Search

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Chapter 14.9. Exhaustive Search

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Chapter 14.9. Exhaustive Search

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Chapter 14.9. Exhaustive Search

Example: Boolean Formula Satisfiability problem (SAT)
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**Input:** boolean formula \( f(x_1, x_2, \ldots, x_n) \),

**Output:** ”yes” if and only if \( f(x_1, x_2, \ldots, x_n) \) is satisfiable.
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\( f(x_1, x_2, \ldots, x_n) \) is **satisfiable** if there is an **assignment** to boolean variables

\[ x_i \in \{T, F\}, \ i = 1, 2, \ldots, n, \]
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Chapter 14.9. Exhaustive Search

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$f(x_1, x_2, x_3) = (x_1 \lor x_2 \lor \neg x_3) \land (\neg x_1 \lor x_3)$ is **satisfiable**
Chapter 14.9. Exhaustive Search

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\[
\begin{align*}
    f(x_1, x_2, x_3) &= (x_1 \lor x_2 \lor \neg x_3) \land (\neg x_1 \lor x_3) \text{ is satisfiable} \\
    g(x_1, x_2) &= (x_1 \lor \neg x_2) \land (\neg x_1 \lor x_2) \land (x_1 \lor x_2) \land (\neg x_1 \lor \neg x_2) \text{ is not!}
\end{align*}
\]
Chapter 14.9. Exhaustive Search

Use exhaustive search to solve the SAT problem.

Input: boolean formula $f(x_1, x_2, ..., x_n)$

Output: "yes" if and only if $f(x_1, x_2, ..., x_n)$ is satisfiable.

How?

What will you exhaustively search on?

• Enumerate all combinations of $T$ and $F$ for $x_1, ..., x_n$.

• Can you solve it with a recursive algorithm?

• Can you solve it with an iterative algorithm?
Chapter 14.9. Exhaustive Search

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Chapter 14.9. Exhaustive Search

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Chapter 14.9. Exhaustive Search

Solve SAT problem with a recursive algorithm:
Chapter 14.9. Exhaustive Search

Solve SAT problem with a recursive algorithm:

- what data will the recursion be applied to?
Solve SAT problem with a recursive algorithm:

- what data will the recursion be applied to?
  
  boolean formula $f(x_1, \ldots, x_n)$
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Solve SAT problem with a recursive algorithm:

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Solve SAT problem with a recursive algorithm:

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  \( n=0, \) formula without variables
Chapter 14.9. Exhaustive Search

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- what is the terminating (base) case?
  
  \( n=0 \), formula without variables

- what is the recursive case?
  
  \[
  f(x_1, \ldots, x_{n-1}, x_n) =
  \]

  \[
  f(x_1, \ldots, x_{n-1}, T) \lor f(x_1, \ldots, x_{n-1}, F) \Rightarrow g(x_1, \ldots, x_{n-1}) =
  \]

  \[
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  $$f(x_1, \ldots, x_{n-1}, x_n) = f(x_1, \ldots, x_{n-1}, T) \lor f(x_1, \ldots, x_{n-1}, F)$$
Chapter 14.9. Exhaustive Search

Solve SAT problem with a recursive algorithm:

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Algorithm \textsc{SAT Solver}(f(x_1, \ldots, x_{n-1}, x_n))
Algorithm SAT Solver($f(x_1,\ldots,x_{n-1},x_n)$)

1. if $n = 0$, return $(f)$;
Chapter 14.9. Exhaustive Search

Algorithm SAT Solver\((f(x_1, \ldots, x_{n-1}, x_n))\)
1. \textbf{if} \(n = 0\), \textbf{return} \((f)\);
2. \textbf{else} \(g(x_1, \ldots, x_{n-1}) = f(x_1, \ldots, x_{n-1}, \text{T})\)

Does this algorithm exhaustively search all assignments to the variables?
• draw a search tree based on the algorithm.
• what does the tree look like?
• what does each path mean?
• how many paths?
• time? \(T(n) = 2T(n-1) + cn\), \(T(0) = c\), \(=\) \(\Rightarrow T(n) = \Theta(2^n)\)
Chapter 14.9. Exhaustive Search

Algorithm SAT Solver($f(x_1, \ldots, x_{n-1}, x_n)$)

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3. $h(x_1, \ldots, x_{n-1}) = f(x_1, \ldots, x_{n-1}, F)$
4. return (SAT Solver($g(x_1, \ldots, x_{n-1})$) $\lor$ SAT Solver($h(x_1, \ldots, x_{n-1})$))
Algorithm SAT SOLVER\((f(x_1, \ldots, x_{n-1}, x_n))\)

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Chapter 14.9. Exhaustive Search

Algorithm \( \text{SAT Solver}(f(x_1, \ldots, x_{n-1}, x_n)) \)

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- draw a search tree based on the algorithm.
- what does the tree look like?

\[ T(n) = 2T(n-1) + cn, \quad T(0) = c \Rightarrow T(n) = \Theta(2^n) \]
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1. if $n = 0$, return ($f$);
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3. $h(x_1, \ldots, x_{n-1}) = f(x_1, \ldots, x_{n-1}, F)$
4. return (SAT SOLVER($g(x_1, \ldots, x_{n-1})$) $\lor$
   SAT SOLVER($h(x_1, \ldots, x_{n-1})$))

Does this algorithm exhaustively search all assignments to the variables?

- draw a search tree based on the algorithm.

- what does the tree look like?

- what does each path mean?
Algorithm \textsc{SAT Solver} \((f(x_1, \ldots, x_{n-1}, x_n))\)

1. \textbf{if} \(n = 0\), \textbf{return} \(f\);
2. \textbf{else} \(g(x_1, \ldots, x_{n-1}) = f(x_1, \ldots, x_{n-1}, \text{T})\)
3. \(h(x_1, \ldots, x_{n-1}) = f(x_1, \ldots, x_{n-1}, \text{F})\)
4. \textbf{return} \((\text{SAT Solver}(g(x_1, \ldots, x_{n-1})) \lor \text{SAT Solver}(h(x_1, \ldots, x_{n-1})))\)

Does this algorithm exhaustively search all assignments to the variables?

- draw a \textit{search tree} based on the algorithm.
- what does the tree look like?
- what does each path mean? how many paths?
Chapter 14.9. Exhaustive Search

Algorithm SAT Solver\((f(x_1, \ldots, x_{n-1}, x_n))\)

1. if \(n = 0\), return \((f)\);
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Chapter 14.9. Exhaustive Search

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Does this algorithm exhaustively search all assignments to the variables?

- draw a search tree based on the algorithm.
- what does the tree look like?
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- time? \(T(n) = 2T(n - 1) + cn, T(0) = c,\)
Chapter 14.9. Exhaustive Search

Algorithm SAT Solver($f(x_1, \ldots, x_{n-1}, x_n)$)

1. if $n = 0$, return ($f$);
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Does this algorithm exhaustively search all assignments to the variables?

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- what does the tree look like?
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- time? $T(n) = 2T(n - 1) + cn$, $T(0) = c$, $\Rightarrow T(n) = \Theta(2^n)$
Chapter 14.9. Exhaustive Search

Solve SAT problem with iterative algorithms
Solve SAT problem with iterative algorithms

• How? what to iterate on?
Chapter 14.9. Exhaustive Search

Solve SAT problem with iterative algorithms

- How? what to iterate on?

 assignments
Chapter 14.9. Exhaustive Search

Solve SAT problem with iterative algorithms

• How? what to iterate on?

  assignments

• what is the initial value?
Chapter 14.9. Exhaustive Search

Solve SAT problem with iterative algorithms

- How? what to iterate on?
  
  assignments

- what is the initial value?

  \[ x_1 = F, x_2 = F, \ldots, x_n = F, \]
Chapter 14.9. Exhaustive Search

Solve SAT problem with iterative algorithms

• How? what to iterate on?
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• what is the initial value?
  \[ x_1 = F, x_2 = F, \ldots, x_n = F, \text{ or simply } (F, F, \ldots, F) \]
Solve SAT problem with iterative algorithms

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Chapter 14.9. Exhaustive Search

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- How? what to iterate on?
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- what is the initial value?
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  \[ (\ldots, F, T, \ldots, T) \]
Chapter 14.9. Exhaustive Search

Solve SAT problem with iterative algorithms

• How? what to iterate on?
  assignments

• what is the initial value?
  \[ x_1 = F, x_2 = F, \ldots, x_n = F, \text{ or simply } (F, F, \ldots, F) \]

• what to increment
  \( (\ldots, F, T, \ldots, T) \longrightarrow (\ldots, T, F, \ldots, F) \)

Chapter 14.9. Exhaustive Search

Solve SAT problem with iterative algorithms

- How? what to iterate on?
  - assignments

- what is the initial value?
  - $x_1 = F, x_2 = F, \ldots, x_n = F$, or simply $(F, F, \ldots, F)$

- what to increment
  - $(\ldots, F, T, \ldots, T) \rightarrow (\ldots, T, F, \ldots, F)$
  - always flip the last bit.
Chapter 14.9. Exhaustive Search

Algorithm SAT Solver-Enum

1. for $\langle x_1, \ldots, x_{n-1}, x_n \rangle = \langle F, \ldots, F \rangle$ to $\langle T, \ldots, T \rangle$

2. $V = \text{Evaluate}(f, x_1, \ldots, x_n)$

3. if $V = T$, return $\langle T \rangle$

4. return $\langle F \rangle$

• for loop can be implemented by encoding vectors $\langle F, \ldots, F \rangle$, $\ldots$, $\langle T, \ldots, T \rangle$ with binary numbers then further with integers

• a decoding process is needed to converting integers back to vectors
Chapter 14.9. Exhaustive Search

Algorithm SAT Solver-Enum\((f(x_1, \ldots, x_{n-1}, x_n))\)
Algorithm SAT SOLVER-ENUM(\(f(x_1, \ldots, x_{n-1}, x_n)\))

1. for \(\langle x_1, \ldots, x_n \rangle = \langle F, \ldots, F \rangle\) to \(\langle T, \ldots, T \rangle\)
Chapter 14.9. Exhaustive Search

Algorithm \textsc{SAT Solver-Enum}(f(x_1, \ldots, x_{n-1}, x_n))

1. \textbf{for} \langle x_1, \ldots, x_n \rangle = \langle F, \ldots, F \rangle \textbf{ to } \langle T, \ldots, T \rangle
2. \hspace{1em} V = \textbf{Evaluate}(f, x_1, \ldots, x_n)
Algorithm SAT Solver-Enum($f(x_1, \ldots, x_{n-1}, x_n)$)

1. for $\langle x_1, \ldots, x_n \rangle = \langle F, \ldots, F \rangle$ to $\langle T, \ldots, T \rangle$
2. \hspace{0.5cm} $V = \text{Evaluate}(f, x_1, \ldots, x_n)$
3. \hspace{0.5cm} if $V = T$, return (T)
Chapter 14.9. Exhaustive Search

Algorithm SAT SOLVER-ENUM\((f(x_1, \ldots, x_{n-1}, x_n))\)

1. for \(\langle x_1, \ldots, x_n \rangle = \langle F, \ldots, F \rangle\) to \(\langle T, \ldots, T \rangle\)
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Chapter 14.9. Exhaustive Search

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Chapter 14.9. Exhaustive Search

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Chapter 14.9. Exhaustive Search

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Chapter 14.9. Exhaustive Search

Iterative exhaustive search seems to be more convenient.

Another example: Travel Salesman Problem (TSP)

Related problem: Hamiltonian Cycle

Input: a graph $G = (V, E)$

Output: yes if and only if $G$ contains a Hamiltonian cycle (Hamiltonian path is a cycle going through every vertex exactly once.)

How to enumerate all cycles and validate?

- enumerate all permutations of $(1, 2, \ldots, n)$
- how to encode these permutations as integers?
Chapter 14.9. Exhaustive Search

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Chapter 14.9. Exhaustive Search

Exhaustive search could be non-trivial

Maximum Independent Set
Input: a graph $G = (V,E)$
Output: a subset $I \subseteq V$ such that
(1) $\forall u,v \in I, (u,v) \not\in E$, and
(2) $|I|$ is the maximum.

• trivial exhaustive search: check every subset of $V$ and verify
• non-trivial: use a search tree, achieving a better time upper bound.

taking advantage of the independent set
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- non-trivial: use a search tree, achieving a better time upper bound.
  - taking advantage of the independent set
Chapter 14.9. Exhaustive Search

The algorithm follows a logical search tree

• given a graph $G$, it picks an arbitrary vertex $v$ from $G$;
• exhaustively, there are two cases to consider:
  (1) to include $v$ in the independent set;
  (2) to exclude $v$ from the independent set;
• resulting in two subgraphs $G_1$ and $G_2$ to be recursively considered,
  (1) $G_1$ is the result of $G$ after $v$ and all its neighbors are removed;
  (2) $G_2$ is the result of $G$ after $v$ is removed.
• the algorithm terminates when the considered graph is empty.
Chapter 14.9. Exhaustive Search

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Chapter 14.9. Exhaustive Search

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Chapter 14.9. Exhaustive Search

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Chapter 14.9. Exhaustive Search

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Chapter 14.9. Exhaustive Search

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Chapter 14.9. Exhaustive Search

Algorithm MaxIndSet \((G)\)
Chapter 14.9. Exhaustive Search

Algorithm \textsc{MaxIndSet} \((G)\)

1. \textbf{if} \(G = \emptyset\) \textbf{return} \((\emptyset)\)
Chapter 14.9. Exhaustive Search

Algorithm \textsc{MaxIndSet} \((G)\)

1. \textbf{if} \(G = \emptyset\) \textbf{return} \((\emptyset)\)
2. \textbf{else} pick an arbitrary vertex \(v\) in \(G\)
Algorithm \texttt{MaxIndSet} \((G)\)

1. \textbf{if} \; \(G = \emptyset\) \textbf{return} \((\emptyset)\)
2. \textbf{else} \; pick an arbitrary vertex \(v\) in \(G\)
3. \; let \(G_1\) be \(G\) with \(v\) and all its neighbors removed

\textbullet\; the algorithm is a search tree
\textbullet\; the time complexity:
\[T(n) = T(n-1-m) + T(n-1) + cn^2\]

where \(m\) is the number of neighbors of \(v\)’s, how to guarantee a large \(m\)?
Algorithm \texttt{MaxIndSet} ($G$)

1. \textbf{if} $G = \emptyset$ \textbf{return} ($\emptyset$)
2. \textbf{else} pick an arbitrary vertex $v$ in $G$
3. \hspace{1em} let $G_1$ be $G$ with $v$ and all its neighbors removed
4. \hspace{1em} $I_1 = \{v\} \cup \text{MaxIndSet} (G_1)$

\begin{align*}
T(n) &= T(n - 1 - m) + T(n - 1) + cn^2 \\
T(n) &= T(n - 1 - m) + T(n - 1) + cn^2
\end{align*}

where $m$ is the number of neighbors of $v$'s, how to guarantee a large $m$?
Algorithm `MaxIndSet (G)`

1. if $G = \emptyset$ return $(\emptyset)$
2. else pick an arbitrary vertex $v$ in $G$
3. let $G_1$ be $G$ with $v$ and all its neighbors removed
4. $I_1 = \{v\} \cup \text{MaxIndSet } (G_1)$
5. let $G_2$ be $G$ with $v$ removed

The algorithm is a search tree

The time complexity:

$$T(n) = T(n-1-m) + T(n-1) + cn^2$$

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Algorithm \texttt{MaxIndSet} \((G)\)

1. \textbf{if} \quad G = \emptyset \quad \textbf{return} \quad (\emptyset)
2. \textbf{else} \quad \text{pick an arbitrary vertex} \ v \ \text{in} \ G
3. \quad \text{let} \ G_1 \ \text{be} \ G \ \text{with} \ v \ \text{and all its neighbors removed}
4. \quad I_1 = \{v\} \cup \text{MaxIndSet} \ (G_1)
5. \quad \text{let} \ G_2 \ \text{be} \ G \ \text{with} \ v \ \text{removed}
6. \quad I_2 = \text{MaxIndSet} \ (G_2)
Algorithm **MaxIndSet** \((G)\)

1. **if** \(G = \emptyset\) **return** \((\emptyset)\)
2. **else** pick an arbitrary vertex \(v\) in \(G\)
3. let \(G_1\) be \(G\) with \(v\) and all its neighbors removed
4. \(I_1 = \{v\} \cup \text{MaxIndSet} \,(G_1)\)
5. let \(G_2\) be \(G\) with \(v\) removed
6. \(I_2 = \text{MaxIndSet} \,(G_2)\)
7. **if** \(|I_1| \geq |I_2|\) **return** \((I_1)\)
Chapter 14.9. Exhaustive Search

Algorithm \texttt{MaxIndSet} \((G)\)

1. \textbf{if} \(G = \emptyset\) \textbf{return} \((\emptyset)\)
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3. let \(G_1\) be \(G\) with \(v\) and all its neighbors removed
4. \(I_1 = \{v\} \cup \text{MaxIndSet} \((G_1)\)\)
5. let \(G_2\) be \(G\) with \(v\) removed
6. \(I_2 = \text{MaxIndSet} \((G_2)\)\)
7. \textbf{if} \(|I_1| \geq |I_2|\) \textbf{return} \((I_1)\)
8. \textbf{else} \textbf{return} \((I_2)\)
Chapter 14.9. Exhaustive Search

Algorithm $\text{MaxIndSet} \ (G)$

1. if $G = \emptyset$ return ($\emptyset$)
2. else pick an arbitrary vertex $v$ in $G$
3. let $G_1$ be $G$ with $v$ and all its neighbors removed
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5. let $G_2$ be $G$ with $v$ removed
6. $I_2 = \text{MaxIndSet} \ (G_2)$
7. if $|I_1| \geq |I_2|$ return ($I_1$)
8. else return ($I_2$)

- the algorithm is a search tree
Chapter 14.9. Exhaustive Search

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5. let $G_2$ be $G$ with $v$ removed
6. $I_2 = \text{MaxIndSet} (G_2)$
7. if $|I_1| \geq |I_2|$ return ($I_1$)
8. else return ($I_2$)

- the algorithm is a search tree
- the time complexity: $T(|G|) = cn^2 + T(|G_1|) + T(|G_2|)$
Algorithm $\text{MaxIndSet} \ (G)$

1. if $G = \emptyset$ return $(\emptyset)$
2. else pick an arbitrary vertex $v$ in $G$
3. let $G_1$ be $G$ with $v$ and all its neighbors removed
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- the algorithm is a search tree
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  $T(n) = T(n - 1 - m) + T(n - 1) + cn^2$

  where $m$ is the number of neighbors of $v$'s,
Chapter 14.9. Exhaustive Search

Algorithm MaxIndSet \((G)\)

1. **if** \(G = \emptyset\) **return** \((\emptyset)\)
2. **else** pick an arbitrary vertex \(v\) in \(G\)
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5. let \(G_2\) be \(G\) with \(v\) removed
6. \(I_2 = \text{MaxIndSet}(G_2)\)
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8. **else** **return** \((I_2)\)

- the algorithm is a search tree
- the time complexity: \(T(|G|) = cn^2 + T(|G_1|) + T(|G_2|)\)

\[ T(n) = T(n - 1 - m) + T(n - 1) + cn^2 \]

where \(m\) is the number of neighbors of \(v\)'s,
how to guarantee a large \(m\)?
Chapter 14.9. Exhaustive Search

\[ T(n) = T(n - 1 - m) + T(n - 1) + cn^2 \]
Chapter 14.9. Exhaustive Search

\[ T(n) = T(n - 1 - m) + T(n - 1) + cn^2 \]

- \( m \geq 0 \),
Chapter 14.9. Exhaustive Search

\[ T(n) = T(n - 1 - m) + T(n - 1) + cn^2 \]

- \( m \geq 0 \), \( T(n) \leq T(n - 1) + T(n - 1) + cn^2 \),
Chapter 14.9. Exhaustive Search

\[ T(n) = T(n - 1 - m) + T(n - 1) + cn^2 \]

- \( m \geq 0, \ T(n) \leq T(n - 1) + T(n - 1) + cn^2, \implies T(n) = O(2^n) \)
Chapter 14.9. Exhaustive Search

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- \( m \geq 0, \ T(n) \leq T(n - 1) + T(n - 1) + cn^2, \rightarrow T(n) = O(2^n) \)

- how to guarantee \( m \geq 1? \) if we can,
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Chapter 14.9. Exhaustive Search

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Chapter 14.9. Exhaustive Search

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Chapter 14.9. Exhaustive Search

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Chapter 14.9. Exhaustive Search

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  \[ T(n) \leq T(n - 2) + T(n - 1) + cn^2, \implies T(n) = O(1.6181^n) \]
- how to guarantee \( m \geq 2? \) if we can,
  \[ T(n) \leq T(n - 3) + T(n - 1) + cn^2, \implies T(n) = O(1.5^n) \]
Chapter 14.9. Exhaustive Search

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- can we guarantee \( m \geq 3 \)?
Chapter 14.9. Exhaustive Search

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use the substitution method to prove \( T(n) = O(1.5^n) \).

- can we guarantee \( m \geq 3 \)? possible but a little more complicated.
Chapter 14.9. Exhaustive Search

- Algorithms for SAT and $\text{MAXINDS}$ set run in exponential time $O(2^n)$ or $O(\gamma^n)$ for $1 < \gamma < 2$
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• search tree (solution search space) is large, inherently large
Chapter 14.9. Exhaustive Search

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- search tree does not have obvious overlapping subproblems,
Chapter 14.9. Exhaustive Search

- Algorithms for SAT and \texttt{MaxIndSet} run in exponential time $O(2^n)$ or $O(\gamma^n)$ for $1 < \gamma < 2$

- search tree (solution search space) is large, \textit{inherently} large

- search tree does not have obvious overlapping subproblems, which otherwise would incur dynamic programming approaches.
Chapter 15. Dynamic Programming

• We will deal with optimization problems, where the output solution satisfies certain desired optimality conditions.
• Such problems can be solved with divide-and-conquer approaches, and direct, top-down recursive approaches would incur high time complexity.
• There are overlapping subproblems, and bottom-up approaches avoid re-computation for these subproblems.
Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

Will discuss the following examples:

• assembly-line scheduling
• casino dice problem and decoding algorithm
• probabilistic finite automata, Viterbi’s decoding algorithm
• matrix-chain multiplication
• longest common subsequence
• molecular sequence comparison/alignment
• molecular structure prediction
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Chapter 15. Dynamic Programming

Revisit the problem of computing the $n$th element of Fibonacci sequence

$F(n) = F(n-1) + F(n-2)$, $F(1) = F(2) = 1$.
Chapter 15. Dynamic Programming

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- bottom-up computation is much faster

compute table $T[1..n]$ from left to right,  
by looking up values that have already been computed
Assembly-line scheduling

There are $2^n$ possible ways.
Chapter 15. Dynamic Programming

Assembly-line scheduling

- $S_{i,j}$: the $j$th station on line $i$, $i = 1, 2$, $j = 1, 2, \ldots, n$. 
Chapter 15. Dynamic Programming

Assembly-line scheduling

- $S_{i,j}$: the $j$th station on line $i$, $i = 1, 2$, $j = 1, 2, \ldots, n$.
- $a_{i,j}$ is the assembly time at station $S_{i,j}$.
Chapter 15. Dynamic Programming

Assembly-line scheduling

- \( S_{i,j} \): the \( j \)th station on line \( i \), \( i = 1, 2, j = 1, 2, \cdots , n \).
- \( a_{i,j} \) is the assembly time at station \( S_{i,j} \)
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Chapter 15. Dynamic Programming

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- $e_1$ and $e_2$ are entering time costs
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To determine the fastest way through a factory
Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

Analyzing the problem:
Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

For general $j, 1 \leq j \leq n$, the fastest way through $S_{1,j}$ is faster between
Chapter 15. Dynamic Programming

For general $j, 1 \leq j \leq n$, the fastest way through $S_{1,j}$ is faster between

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Chapter 15. Dynamic Programming

For general $j, 1 \leq j \leq n$, the fastest way through $S_{1,j}$ is faster between

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![Diagram of stations and paths]
Chapter 15. Dynamic Programming

For general $j, 1 \leq j \leq n$, the fastest way through $S_{1,j}$ is faster between

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Chapter 15. Dynamic Programming

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- the fastest way through $S_{2,j-1}$ then $S_{2,j-1} \rightarrow S_{2,j}$.
A dynamic programming approach:

\[ f_i(j) = \min \{ f_{i-1}(j) + a_{i,j}, f_{i-1}(j-1) + t_{i-1,j-1} + a_{i,j} \} \]

\[ f_i(j) = \min \{ f_{i-1}(j) + a_{i,j}, f_{i-1}(j-1) + t_{i-1,j-1} + a_{i,j} \} \]
A dynamic programming approach:

**Step 1:** the above analysis
Chapter 15. Dynamic Programming

A dynamic programming approach:

**step 1**: the above analysis

**step 2**: formulate a recursive solution to the problem

\[
define f_i(j) \text{ to be the minimum time through station } S_{i,j}, \text{ for } i = 1, 2, \ldots, n.
\]

- the problem is to compute
  \[
  \min \{ f_1(n) + x_1, f_2(n) + x_2 \}
  \]
  where
  \[
  f_1() \text{ and } f_2()
  \]
  are defined recursively:
  
  \[
  (1) \quad f_1(j) = \min \{ f_1(j-1) + a_{1,j}, f_2(j-1) + t_{2,j-1} + a_{1,j} \}
  \]
  
  \[
  (2) \quad f_2(j) = \min \{ f_2(j-1) + a_{2,j}, f_1(j-1) + t_{1,j-1} + a_{2,j} \}
  \]

- base cases 
  
  \[
  f_1(1) = e_1 + a_{1,1}
  \]
  
  \[
  f_2(1) = e_2 + a_{2,1}
  \]
Chapter 15. Dynamic Programming

A dynamic programming approach:

**step 1:** the above analysis

**step 2:** formulate a recursive solution to the problem

define $f_i(j)$ to be the minimum time through station $S_{i,j}$, for $i = 1, 2$ and $j = 1, \ldots, n$. 

• the problem is to compute $\min\{f_1(n) + \pi_1, f_2(n) + \pi_2\}$

• where $f_1$ and $f_2$ are defined recursively:

  1. $f_1(j) = \min\{f_1(j-1) + a_{1,j}, f_2(j-1) + t_2, j-1 + a_{1,j}\}$

  2. $f_2(j) = \min\{f_2(j-1) + a_{2,j}, f_1(j-1) + t_1, j-1 + a_{2,j}\}$

• base cases $f_1(1) = e_{1,1} + a_{1,1}$ and $f_2(1) = e_{2,1} + a_{2,1}$.
A dynamic programming approach:

**step 1**: the above analysis

**step 2**: formulate a recursive solution to the problem

- define $f_i(j)$ to be the minimum time through station $S_{i,j}$, for $i = 1, 2$ and $j = 1, \ldots, n$.
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Chapter 15. Dynamic Programming

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- the problem is to compute \( \min\{f_1(n) + x_1, f_2(n) + x_2\} \)
- where \( f_1() \) and \( f_2() \) are defined recursively:

\[
\begin{align*}
(1) \quad f_1(j) &= \min\{f_1(j - 1) + a_{1,j}, f_2(j - 1) + t_{2,j-1} + a_{1,j}\} \\
(2) \quad f_2(j) &= \min\{f_2(j - 1) + a_{2,j}, f_1(j - 1) + t_{1,j-1} + a_{2,j}\}
\end{align*}
\]
Chapter 15. Dynamic Programming

A dynamic programming approach:

**step 1:** the above analysis

**step 2:** formulate a recursive solution to the problem

Define $f_i(j)$ to be the minimum time through station $S_{i,j}$, for $i = 1, 2$ and $j = 1, \ldots, n$.

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  (1) $f_1(j) = \min\{f_1(j - 1) + a_{1,j}, f_2(j - 1) + t_{2,j-1} + a_{1,j}\}$
  (2) $f_2(j) = \min\{f_2(j - 1) + a_{2,j}, f_1(j - 1) + t_{1,j-1} + a_{2,j}\}$

- base cases $f_1(1) = e_1 + a_{1,1}$ and $f_2(1) = e_2 + a_{2,1}$
Chapter 15. Dynamic Programming

Data dependency:
Chapter 15. Dynamic Programming

step 3: compute $f_1(j)$ and $f_2(j)$, for all $j = 1, 2, \ldots, n$. 
Chapter 15. Dynamic Programming

**step 3**: compute $f_1(j)$ and $f_2(j)$, for all $j = 1, 2, \ldots, n$. to fill out a $2 \times n$ table, from left to right

<table>
<thead>
<tr>
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• using the recurrences to compute; and
Chapter 15. Dynamic Programming

**step 3:** compute $f_1(j)$ and $f_2(j)$, for all $j = 1, 2, \ldots, n$.

to fill out a $2 \times n$ table, from left to right

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- using the recurrences to compute; and
- looking up already-computed values, where
Chapter 15. Dynamic Programming

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$(0)$ base cases $f_1(1) = e_1 + a_{1,1}$ and $f_2(1) = e_2 + a_{2,1}$
step 3: compute \( f_1(j) \) and \( f_2(j) \), for all \( j = 1, 2, \ldots, n \).

To fill out a \( 2 \times n \) table, from left to right

\[
\begin{array}{ccccccc}
  f_1(1) & f_1(2) & f_1(3) & \ldots & f_1(n) \\
  f_2(1) & f_2(2) & f_2(3) & \ldots & f_2(n)
\end{array}
\]

- using the recurrences to compute; and
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\[
\text{(0) base cases } f_1(1) = e_1 + a_{1,1} \text{ and } f_2(1) = e_2 + a_{2,1} \\
\text{(1) } f_1(j) = \min\{f_1(j - 1) + a_{1,j}, f_2(j - 1) + t_{2,j-1} + a_{1,j}\}
\]
Chapter 15. Dynamic Programming

**step 3**: compute $f_1(j)$ and $f_2(j)$, for all $j = 1, 2, \ldots, n$.

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- using the recurrences to compute; and
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2. $f_1(j) = \min\{f_1(j - 1) + a_{1,j}, f_2(j - 1) + t_{2,j-1} + a_{1,j}\}$
3. $f_2(j) = \min\{f_2(j - 1) + a_{2,j}, f_1(j - 1) + t_{1,j-1} + a_{2,j}\}$

The fastest time is $f^* = \min\{f_1(n) + x_1, f_2(n) + x_2\}$.
step 3: compute \( f_1(j) \) and \( f_2(j) \), for all \( j = 1, 2, \ldots, n \). to fill out a \( 2 \times n \) table, from left to right

\[
\begin{array}{cccccc}
  f_1(1) & f_1(2) & f_1(3) & \ldots & f_1(n) \\
  f_2(1) & f_2(2) & f_2(3) & \ldots & f_2(n)
\end{array}
\]

• using the recurrences to compute; and
• looking up already-computed values, where

(0) base cases \( f_1(1) = e_1 + a_{1,1} \) and \( f_2(1) = e_2 + a_{2,1} \)
(1) \( f_1(j) = \min\{f_1(j-1) + a_{1,j}, f_2(j-1) + t_{2,j-1} + a_{1,j}\} \)
(2) \( f_2(j) = \min\{f_2(j-1) + a_{2,j}, f_1(j-1) + t_{1,j-1} + a_{2,j}\} \)

• the fastest time is \( f^* = \min\{f_1(n) + x_1, f_2(n) + x_2\} \)
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Example
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Example

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<tr>
<th></th>
<th>1</th>
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<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
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<tr>
<td>$f_1$</td>
<td>9</td>
<td>18</td>
<td>20</td>
<td>24</td>
<td>32</td>
<td>36</td>
</tr>
<tr>
<td>$f_2$</td>
<td>12</td>
<td>16</td>
<td>22</td>
<td>26</td>
<td>31</td>
<td>38</td>
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Min finish time $f^* = 36 + 3 = 39$
Algorithm ASSEMBLYLINE\((a, t, e, x, n)\)

1. \(M_{1,1} = e_{1} + a_{1,1}\)
2. \(M_{2,1} = e_{2} + a_{2,1}\)
3. for \(j = 2\) to \(n\)
4. if \(M_{1,j-1} + a_{1,j} \leq M_{2,j-1} + t_{2,j-1} + a_{1,j}\)
   then \(M_{1,j} = M_{1,j-1} + a_{1,j}\)
   \(P_{1,j} = 1\)
5. else \(M_{1,j} = M_{2,j-1} + t_{2,j-1} + a_{1,j}\)
   \(P_{1,j} = 2\)
6. if \(M_{2,j-1} + a_{2,j} \leq M_{1,j-1} + t_{1,j-1} + a_{2,j}\)
   then \(M_{2,j} = M_{2,j-1} + a_{2,j}\)
   \(P_{2,j} = 2\)
7. else \(M_{2,j} = M_{1,j-1} + t_{1,j-1} + a_{2,j}\)
   \(P_{2,j} = 1\)
8. return \(\min\{f_{1}(n) + x_{1}, f_{2}(n) + x_{2}\}\)
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Algorithm ASSEMBLYLINE($a, t, e, x, n$)
1. $M[1, 1] = e_1 + a_{1,1}$
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Algorithm ASSEMBLYLINE\((a, t, e, x, n)\)
1. \(M[1, 1] = e_1 + a_{1,1}\)
2. \(M[2, 1] = e_2 + a_{2,1}\)
Algorithm \texttt{ASSEMBLYLINE}(a, t, e, x, n)

1. $M[1, 1] = e_1 + a_{1,1}$
2. $M[2, 1] = e_2 + a_{2,1}$
3. for j=2 to n
Algorithm \textsc{AssemblyLine}(a, t, e, x, n)
1. \hspace{1em} M[1, 1] = e_1 + a_{1,1}
2. \hspace{1em} M[2, 1] = e_2 + a_{2,1}
3. \hspace{1em} \textbf{for } j=2 \textbf{ to } n
4. \hspace{1em} \hspace{1em} \textbf{if } M[1, j - 1] + a_{1,j} \leq M[2, j - 1] + t_{2,j-1} + a_{1,j}
5. \hspace{1em} \hspace{1em} \hspace{1em} M[1, j] = M[1, j - 1] + a_{1,j}
6. \hspace{1em} \hspace{1em} \hspace{1em} P[1, j] = 1
7. \hspace{1em} \hspace{1em} \textbf{else}
8. \hspace{1em} \hspace{1em} \hspace{1em} M[1, j] = M[2, j - 1] + t_{2,j-1} + a_{1,j}
9. \hspace{1em} \hspace{1em} \hspace{1em} P[1, j] = 2
10. \hspace{1em} \hspace{1em} \textbf{if } M[2, j - 1] + a_{2,j} \leq M[1, j - 1] + t_{1,j-1} + a_{2,j}
11. \hspace{1em} \hspace{1em} \hspace{1em} M[2, j] = M[2, j - 1] + a_{2,j}
12. \hspace{1em} \hspace{1em} \hspace{1em} P[2, j] = 2
13. \hspace{1em} \hspace{1em} \textbf{else}
14. \hspace{1em} \hspace{1em} \hspace{1em} M[2, j] = M[1, j - 1] + t_{1,j-1} + a_{2,j}
15. \hspace{1em} \hspace{1em} \hspace{1em} P[2, j] = 1
16. \hspace{1em} \textbf{return } (\min\{ f_1(n) + x_1, f_2(n) + x_2 \})
Algorithm $\text{ASSEMBLYLINE}(a, t, e, x, n)$
1. $M[1, 1] = e_1 + a_{1,1}$
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5.         $M[1, j] = M[1, j - 1] + a_{1,j}$
6.         $P[1, j] = 1$

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Algorithm `ASSEMBLYLINE(a, t, e, x, n)`

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Algorithm \textsc{AssemblyLine}(a, t, e, x, n)
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4. \quad \textbf{if} \ M[1, j - 1] + a_{1,j} \leq M[2, j - 1] + t_{2,j-1} + a_{1,j} \ 
5. \quad M[1, j] = M[1, j - 1] + a_{1,j} \ 
6. \quad P[1, j] = 1 \ 
7. \quad \textbf{else} \ M[1, j] = M[2, j - 1] + t_{2,j-1} + a_{1,j} \ 
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4. \hspace{1em} \textbf{if} \(M[1, j - 1] + a_{1,j} \leq M[2, j - 1] + t_{2,j - 1} + a_{1,j}\)
5. \hspace{2em} \(M[1, j] = M[1, j - 1] + a_{1,j}\)
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\textbf{13. return } (\min \{ f_1(n) + x_1, f_2(n) + x_2 \})
Algorithm **ASSEMBLYLINE**(*a, t, e, x, n*)

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13. \(P[2, j] = 1\)
14. \(\text{return } (\min\{f_1(n) + x_1, f_2(n) + x_2\})\)
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step 4: get the fastest path, not just the faster time.
**Chapter 15. Dynamic Programming**

**step 4:** get the fastest path, not just the faster time.

Do we know the fastest path from the fastest time $f^*$?
**Chapter 15. Dynamic Programming**

**step 4:** get the fastest path, not just the faster time.

Do we know the fastest path from the fastest time \( f^* \)?

\[
\begin{array}{cccccc}
  f_1(1) & f_1(2) & f_1(3) & \ldots & f_1(n-1) & f_1(n) \\
  f_2(1) & f_2(2) & f_2(3) & \ldots & f_2(n-1) & f_2(n)
\end{array}
\]
**Chapter 15. Dynamic Programming**

**step 4:** get the fastest path, not just the faster time.

---

**Do we know the fastest path from the fastest time \( f^* \)?**

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<tr>
<td>( f_2(1) )</td>
<td>( f_2(2) )</td>
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\[ f^* = \min\{f_1(n) + x_1, f_2(n) + x_2\} \]

\[ f_1(j) = \min\{f_1(j-1) + a_{1,j}, f_2(j-1) + t_{2,j-1} + a_{1,j}\} \]

\[ f_2(j) = \min\{f_2(j-1) + a_{2,j}, f_1(j-1) + t_{1,j-1} + a_{2,j}\} \]
Algorithm PRINTPATHMAIN($M, P, f^*, x$)
1. \textbf{if} ($f^* - x_1$) = $M[1, n]$

```python

```

```
Algorithm PRINTPATHMAIN($M, P, f^*, x$)
1. \textbf{if} $(f^* - x_1) = M[1, n]$
2. \hspace{1em} $i = 1$

Algorithm PRINTPATH($P, i, j$)
1. \textbf{if} $j \geq 1$
2. \hspace{1em} PRINTPATH($P, P[i,j], j - 1$)
3. \hspace{1em} print ($i, j$)
Algorithm PRINTPathMain\((M, P, f^*, x)\)
1. \(\text{if } (f^* - x_1) = M[1, n]\)
2. \(i = 1\)
3. \(\text{else } i = 2\)
Algorithm PRINTPathMain($M, P, f^*, x$)
1. if $(f^* - x_1) = M[1, n]$
2. $i = 1$
3. else $i = 2$
4. PRINTPath $(P, i, n)$
Algorithm \texttt{PrintPathMain}(M, P, f^*, x)
1. \textbf{if } \ (f^* - x_1) = M[1, n] \\
2. \quad i = 1 \\
3. \quad \textbf{else} \ i = 2 \\
4. \quad \texttt{PrintPath} \ (P, i, n)

Algorithm \texttt{PrintPath} \ (P, i, j)
1. \textbf{if } j \geq 1
Chapter 15. Dynamic Programming

Algorithm \texttt{PrintPathMain}(M, P, f^*, x)
1. \textbf{if} \ (f^* - x_1) = M[1, n]
2. \hspace{1em} i = 1
3. \textbf{else} \ i = 2
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Algorithm \texttt{PrintPath} \ (P, i, j)
1. \textbf{if} \ j \geq 1
2. \hspace{1em} \texttt{PrintPath} \ (P, P[i, j], j - 1)
Algorithm **PrintPathMain** (*M*, *P*, *f*\(^\ast\), \(x\))

1. if \((f^\ast - x_1) = M[1, n]\)  
2. \(i = 1\)  
3. else \(i = 2\)  
4. **PrintPath** (*P*, *i*, *n*)

Algorithm **PrintPath** (*P*, *i*, *j*)

1. if \(j \geq 1\)  
2. **PrintPath** (*P*, *P*[\(i, j\]), \(j - 1\))  
3. print \((i, j)\)
Chapter 15. Dynamic Programming

Summary on steps for dynamic programming solving the assembly line problem
Chapter 15. Dynamic Programming

Summary on steps for dynamic programming solving the assembly line problem

(1) the structure of optimal solution
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Summary on steps for dynamic programming solving the assembly line problem

(1) the structure of optimal solution
(2) defining optimal cost recursively
Chapter 15. Dynamic Programming

Summary on steps for dynamic programming solving the assembly line problem

(1) the structure of optimal solution
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(3) computing optimal cost (bottom-up approach)
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Summary on steps for dynamic programming solving the assembly line problem

(1) the structure of optimal solution
(2) defining optimal cost recursively
(3) computing optimal cost (bottom-up approach)
(4) constructing optimal solution (traceback)
We consider a little more about the assembly line problem.
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Assume that each station $S_{i,j}$ produces a part $p_k$ with time cost $\tau_{i,j,k}$, where $k = 1, 2, 3$. 
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- can we still compute the fastest time? yes.
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  so can the fastest path and the sequence of parts be computed!
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- can we still compute the fastest time? yes.
  so can the fastest path and the sequence of parts be computed!

- Given a sequence of parts produced, can we know the path?
  No, but we may predict such a path with a confidence.
Chapter 15. Dynamic Programming

Example-2. Casino dice and decoding algorithm

The house rolls die \( n \) times: e.g., 5, 3, 2, 5, 6, 6, 1, ..., 1

- not honest, switch between dice: fair (F) and loaded (L)
- a small chance to switch is small, a large chance to stay on the same die

Given the sequence of numbers, e.g., 5, 3, 2, 5, 6, 6, 1, ..., 1, what is the sequence of dice used most likely?
Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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![Diagram of decision process](image-url)
Chapter 15. Dynamic Programming

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Given the sequence of numbers, e.g., 5, 3, 2, 5, 6, 6, 1, \ldots, 1, what is the sequence of dices used most likely?
We compute the most probable path of dices to roll a given sequence of numbers $S = d_1 \ldots d_n$, where $d_i \in \{1, 2, \ldots, 6\}$.
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i.e., we compute the highest probability of a path of dice to roll $d_1 \ldots d_n$.
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i.e., we compute the highest probability of a path of dice to roll $d_1 \ldots d_n$

As in AssemblyLine problem, we compute the most probable path to roll $d_1 \ldots d_i$: 
We compute the most probable path of dices to roll a given sequence of numbers \( S = d_1 \ldots d_n \), where \( d_i \in \{1, 2, \ldots, 6\} \)

i.e., we compute the highest probability of a path of dice to roll \( d_1 \ldots d_n \)

As in AssemblyLine problem, we compute the most probable path to roll \( d_1 \ldots d_i \):

- \( p(S, i, F) \) for which the fair die \( F \) is used in the last step
- \( p(S, i, L) \) for which the loaded die \( L \) is used in the last step
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Both $p(S, i, F)$ and $p(S, i, L)$ have recursive solutions.
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Both \( p(S, i, F) \) and \( p(S, i, L) \) have recursive solutions.

\[
p(S, i, F) = \max \left\{ p(S, i - 1, F) \times 0.95 \times \frac{1}{6}, \ p(S, i - 1, L) \times 0.1 \times \frac{1}{6} \right\}
\]
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\[
p(S, i, L) = \max\{p(S, i - 1, L) \times 0.9 \times q, \; p(S, i - 1, F) \times 0.05 \times q\}
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where $q = \frac{1}{10}$ if $1 \leq d_i \leq 5$; $q = \frac{1}{2}$ if $d_i = 6$
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Base cases: when $i =$
Chapter 15. Dynamic Programming

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Base cases: when \( i = 1 \)
Both $p(S, i, F)$ and $p(S, i, L)$ have recursive solutions.

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Base cases: when $i = 1$

$p(S, 1, F) = 0.5 \times \frac{1}{6}$

$p(S, 1, L) = 0.5 \times q$
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- design dynamic programming algorithm to compute $\max\{P(S, n, F), P(S, n, L)\}$ given from $d_1 d_2 \ldots d_n$
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- design dynamic programming algorithm to compute $\max\{P(S, n, F), P(S, n, L)\}$ given from $d_1d_2\ldots d_n$

- design a traceback process to print out the sequence of dices used.
Chapter 15. Dynamic Programming

Consider a sequence

... 1 3 2 4 6 6 6 4 1 6 5 6 6 2 4 2 1 2 3 5 ...
Consider a sequence

\[ \ldots 1 \ 3 \ 2 \ 4 \ 6 \ 6 \ 6 \ 4 \ 1 \ 6 \ 5 \ 6 \ 6 \ 2 \ 4 \ 2 \ 1 \ 2 \ 3 \ 5 \ldots \]
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Consider a sequence

\[ \ldots \ 1 \ 3 \ 2 \ 4 \ 6 \ 6 \ 6 \ 4 \ 1 \ 6 \ 5 \ 6 \ 6 \ 2 \ 4 \ 2 \ 1 \ 2 \ 3 \ 5 \ldots \]

By computing the maximum probability to produce the sequence, we decode the dices used

\[ \ldots \ 1 \ 3 \ 2 \ 4 \ 6 \ 6 \ 6 \ 4 \ 1 \ 6 \ 5 \ 6 \ 6 \ 2 \ 4 \ 2 \ 1 \ 2 \ 3 \ 5 \ldots \]
Consider a sequence
\[ \ldots 1 3 2 4 6 6 6 4 1 6 5 6 6 2 4 2 1 2 3 5 \ldots \]

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\[ \ldots F F F F L L L L L L L L F F F F F F F \ldots \]
Chapter 15. Dynamic Programming

Consider a sequence

\[ \ldots 1 \, 3 \, 2 \, 4 \, 6 \, 6 \, 6 \, 6 \, 4 \, 1 \, 6 \, 5 \, 6 \, 6 \, 2 \, 4 \, 2 \, 1 \, 2 \, 3 \, 5 \ldots \]

By computing the maximum probability to produce the sequence, we decode the dices used

\[ \ldots 1 \, 3 \, 2 \, 4 \, 6 \, 6 \, 6 \, 6 \, 4 \, 1 \, 6 \, 5 \, 6 \, 6 \, 2 \, 4 \, 2 \, 1 \, 2 \, 3 \, 5 \ldots \]
\[ \ldots F \, F \, F \, F \, L \, L \, L \, L \, L \, L \, L \, F \, F \, F \, F \, F \, F \, F \ldots \]

But besides the casino interest, is such algorithm meaningful in other applications?
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Consider a sequence

\[ \ldots \quad 1 \quad 3 \quad 2 \quad 4 \quad 6 \quad 6 \quad 6 \quad 4 \quad 1 \quad 6 \quad 5 \quad 6 \quad 6 \quad 2 \quad 4 \quad 2 \quad 1 \quad 2 \quad 3 \quad 5 \ldots \]

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\[ \ldots \quad F \quad F \quad F \quad F \quad L \quad L \quad L \quad L \quad L \quad L \quad L \quad L \quad F \quad F \quad F \quad F \quad F \quad F \quad F \ldots \]

But besides the casino interest, is such algorithm meaningful in other applications?

The answer is definitely YES!
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A segment of DNA sequence, is it meaningful?
if the highlighted segment is not random, it may be a gene.
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Technically, the “linear model” is unfolded of a “more condensed model”.

A hidden Markov model (HMM)
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Chapter 15. Dynamic Programming
Knapsack Problem

Input: $n$ items of sizes $s_1, \ldots, s_n$ and values $v_1, \ldots, v_n$; and a knapsack size $B$.
Output: a subset $A \subseteq \{1, 2, \ldots, n\}$ that maximizes $\sum_{i \in A} v_i$ with constraint $\sum_{i \in A} s_i \leq B$.

Analysis: For $n$ items and knapsack size $B$, examine the last item $n$, then either put item $n$ into the knapsack or discard it; that is, (1) either increase value by $v_n$, reduce available space from $B$ to $B - s_n$, reduce the number of items by 1, or (2) or simply reduce the number of items by 1. Both situations reduce the problem size to a subproblem.
Chapter 15. Dynamic Programming

Knapsack Problem

**INPUT:** \( n \) items of sizes \( s_1, \ldots, s_n \) and values \( v_1, \ldots, v_n \); and a knapsack size \( B \)
Knapsack Problem

**INPUT:** $n$ items of sizes $s_1, \ldots, s_n$ and values $v_1, \ldots, v_n$; and a knapsack size $B$

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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For $n$ items and knapsack size $B$; examine the last item $n$, then either put item $n$ into the knapsack or discard it; That is

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(1) either **increase** value by \( v_n \), **reduce** available space from \( B \) to \( B-s_n \),
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(2) or simply **reduce** the number of items by 1
Chapter 15. Dynamic Programming

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(2) or simply *reduce* the number of items by 1

Both situations reduce the problem size to a subproblem.
In general, let \( \{1, 2, \ldots, k\} \) be the first \( k \) items, and \( X \) be the available space in the knapsack;
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If we define \( A(k, X) \) to be the subset of items drawn from \( \{1, 2, \ldots, k\} \) such that \( A \) maximizes the total value while fits into the space \( X \)
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Then

\[
A(k, X) = \begin{cases}
\text{either } & A(k - 1, X - s_k) \cup \{k\} & \text{value gained by } v_k \\
\text{or } & A(k - 1, X) & \text{value not gained}
\end{cases}
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In general, let \( \{1, 2, \ldots, k\} \) be the first \( k \) items, and \( X \) be the available space in the knapsack;

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\end{cases}
\]

Define \( V(k, X) \) to be the maximum value of items drawn from \( \{1, 2, \ldots, k\} \) which fit into the space of \( X \), then

\[
V(k, X) = \max \left\{ V(k - 1, X - s_k) + v_k \quad X \geq s_k, \quad V(k - 1, X) \right\}
\]
• For every $k \leq n$ and every $X \leq B$.

\[
V(k, X) = \max \left\{ V(k - 1, X - s_k) + v_k, \quad X \geq s_k \\
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base case,
Chapter 15. Dynamic Programming

- For every $k \leq n$ and every $X \leq B$.

\[
V(k, X) = \max \left\{ \begin{array}{ll}
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V(k - 1, X) & \text{base case,}
\end{array} \right.
\]

\[
V(0, X) = 0,
\]
Chapter 15. Dynamic Programming

- For every $k \leq n$ and every $X \leq B$.

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$$V(0, X) = 0, \ V(k, 0) = 0$$
Chapter 15. Dynamic Programming

- For every $k \leq n$ and every $X \leq B$.

\[
V(k, X) = \max \begin{cases} 
V(k - 1, X - s_k) + v_k & X \geq s_k \\
V(k - 1, X) & 
\end{cases}
\]

base case,

\[
V(0, X) = 0, \quad V(k, 0) = 0
\]

- What does the table look like?
Chapter 15. Dynamic Programming

- For every $k \leq n$ and every $X \leq B$.

\[
V(k, X) = \max \begin{cases} 
V(k - 1, X - s_k) + v_k & \text{if } X \geq s_k \\
V(k - 1, X) & \text{else}
\end{cases}
\]

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- What does the table look like? How to fill it out?
Chapter 15. Dynamic Programming

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$$V(0, X) = 0, \ V(k, 0) = 0$$

- What does the table look like? How to fill it out?

- How to traceback for $A$, the optimal subset of items?
Chapter 15. Dynamic Programming

Example

The optimal knapsack should contain \( \{1,2\} = 7 \)

<table>
<thead>
<tr>
<th>( \backslash s )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
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<td>0</td>
<td>0</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>7</td>
</tr>
</tbody>
</table>

\[
V(k, X) = \max \begin{cases} 
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V(k - 1, X) & \text{otherwise}
\end{cases}
\]
Chapter 15. Dynamic Programming

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<td>5</td>
<td>7</td>
</tr>
</tbody>
</table>

\[ V(k, X) = \max \left\{ \begin{array}{l} V(k - 1, X - s_k) + v_k \quad X \geq s_k \\ V(k - 1, X) \end{array} \right. \]

- fill out the base case row and column;
- the order of cells to be filled;
- \(V(i, X) = V(i - 1, X)\) if \(X < s_i\).
- time complexity
Chapter 15. Dynamic Programming

Hidden Markov models and Viterbi’s algorithm
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Hidden Markov models and Viterbi’s algorithm
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Chapter 15. Dynamic Programming
Chapter 15. Dynamic Programming

(a)

(b) state sequence (hidden):

... 1 1 1 1 1 2 2 2 2 1 1 ...

transitions: ? 0.99 0.99 0.99 0.99 0.01 0.9 0.9 0.9 0.1 0.99

(c) symbol sequence (observable):

... A T C A A G G C G A T ...

emissions: 0.4 0.4 0.1 0.4 0.4 0.5 0.5 0.4 0.4 0.4 0.4
Viterbi Algorithm

- Input: an HMM $\mathcal{M}$, and data $D = d_1 d_2 \ldots d_n$
Chapter 15. Dynamic Programming

Viterbi Algorithm

- **Input:** an HMM $\mathcal{M}$, and data $D = d_1 d_2 \ldots d_n$
- **Output:** hidden state sequences $H = s_1 s_2 \ldots s_n$ that generate $D$; such that
Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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---

... 1 3 2 4 6 6 6 4 1 6 5 6 6 2 4 2 1 2 3 5 ...

... F F F F L L L L L L L L L L L L F F F F F F F ...

---
A hidden Markov model (HMM) consists of \((S, T, e, t)\)
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  \[ e(F, d) = \frac{1}{6} \text{ for } d = 1, 2, \ldots, 6; \]
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  \[
e(F, d) = \frac{1}{6} \text{ for } d = 1, 2, \ldots, 6;
  e(L, 6) = \frac{1}{2} \text{ and } e(L, d) = \frac{1}{10} \text{ for } d = 1, 2, \ldots, 5;
  \]
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Chapter 15. Dynamic Programming

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  e(L,6) = \frac{1}{2} \text{ and } e(L,d) = \frac{1}{10} \text{ for } d = 1, 2, \ldots, 5;
  
- transition probability distribution \(t\)
  \[
  t(F,F) = 0.95, t(F,L) = 0.05, t(L,F) = 0.1, t(L,L) = 0.90;
  
\]
Chapter 15. Dynamic Programming

The goal is to identify hidden states $s_1 s_2 \ldots s_n$ that generates $d_1 d_2 \ldots d_n$, to maximize the associated probability.

The diagram shows the transition probabilities and states for a fair and a loaded die. The hidden states and their probabilities are shown as follows:

**Fair**
- 1: $1/6$
- 2: $1/6$
- 3: $1/6$
- 4: $1/6$
- 5: $1/6$
- 6: $1/6$

**Loaded**
- 1: $1/10$
- 2: $1/10$
- 3: $1/10$
- 4: $1/10$
- 5: $1/10$
- 6: $1/2$

The transitions are as follows:
- From Fair to Fair: 0.95
- From Fair to Loaded: 0.05
- From Loaded to Fair: 0.1
- From Loaded to Loaded: 0.9

The goal is to identify the hidden states that maximize the associated probability.
Chapter 15. Dynamic Programming

The goal is to identify hidden states \( s_1 s_2 \ldots s_n \) that generates \( d_1 d_2 \ldots d_n \), to maximize the associated probability

\[
P(s_1 s_2 \ldots s_n, d_1 d_2 \ldots d_n)
\]
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$$P(s_1s_2\ldots s_n, d_1d_2\ldots d_n)$$

e.g., $d_1 = 3$, $d_2 = 6$, there are 4 possible ways to generate them:
Chapter 15. Dynamic Programming

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- $FF$, $FL$, $LL$, $LF$,
Chapter 15. Dynamic Programming

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$$P(s_1 s_2 \ldots s_n, d_1 d_2 \ldots d_n)$$

e.g., $d_1 = 3$, $d_2 = 6$, there are 4 possible ways to generate them:

- $FF$, $FL$, $LL$, $LF$, but with different probabilities
Chapter 15. Dynamic Programming

- $P(FF, 36)$
Chapter 15. Dynamic Programming

\[ P(FF, 36) = e(F, 3) \times t(F, F) \times e(F, 6) \]
Chapter 15. Dynamic Programming

- $P(FF, 36) = e(F, 3) \times t(F, F) \times e(F, 6) = \frac{1}{6} \times 0.95 \times \frac{1}{6}$
Chapter 15. Dynamic Programming

- \( P(FF, 36) = e(F, 3) \times t(F, F) \times e(F, 6) = \frac{1}{6} \times 0.95 \times \frac{1}{6} = 0.02639 \)
Chapter 15. Dynamic Programming

- $P(FF, 3 6) = e(F, 3) \times t(F, F) \times e(F, 6) = \frac{1}{6} \times 0.95 \times \frac{1}{6} = 0.02639$
- $P(FL, 3 6)$
Chapter 15. Dynamic Programming

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- \[ P(FL, 36) = e(F, 3) \times t(F, L) \times e(L, 6) \]
Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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• $P(LL, 36)$
Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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but for 10 digits, how many cases to consider?
Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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but for 10 digits, how many cases to consider?
Chapter 15. Dynamic Programming

Take a dynamic programming approach:
Take a dynamic programming approach:

For prefix $d_1 \ldots d_k$, define:

- $P(F, k)$ to be the maximum probability for states $F$ to generate $d_1 \ldots d_k$.
- $P(L, k)$ to be the maximum probability for states $L$ to generate $d_1 \ldots d_k$. 
Take a dynamic programming approach:

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Chapter 15. Dynamic Programming

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Then recursively,
Then recursively,

\[
P(F, k) = \max \left\{ \begin{array}{l}
P(F, k - 1) \times t(F, F) \times e(F, d_k) \\
P(L, k - 1) \times t(L, F') \times e(F, d_k)
\end{array} \right. 
\]
Chapter 15. Dynamic Programming

Then recursively,

\[ P(F, k) = \max \left\{ P(F, k - 1) \times t(F, F) \times e(F, d_k) \right\} \]

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\[ P(F, 1) = \frac{1}{6} \]
Chapter 15. Dynamic Programming

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\[ P(F, 1) = \frac{1}{6} \quad P(L, 1) = \begin{cases} \frac{1}{10} & 1 \leq d_1 \leq 5 \\ \frac{1}{2} & d_1 = 6 \end{cases} \]
The outlined method is **Viterbi Algorithm**.
Chapter 15. Dynamic Programming

The outlined method is Viterbi Algorithm.

- **Start** and **Finish** states can be added into the model, but they do not emit digits (called **silent states**);
The outlined method is \textit{Viterbi Algorithm}. 

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- The state in phase \( k - 1 \) determines the state in phase \( k \);
The outlined method is Viterbi Algorithm.

- **Start** and **Finish** states can be added into the model, but they do not emit digits (called **silent states**);
- The state in phase $k - 1$ determines the state in phase $k$;
- Given a state in current phase $k$, the previous state in phase $k - 1$ can only be one of those having transition to the current state.
Chapter 15. Dynamic Programming

A hidden Markov model (HMM) consists of \((S, T, e, t)\), where

- \(S\) is a set of states: \(s_0, s_1, \ldots, s_m\); where \(s_0\) is the begin state;
- transitions \(T \subseteq S \times S\);
- each state \(s_i\) is associated with a (emission) probability distribution \(e(i, a)\) for \(a \in \Sigma\), such that \(\sum_{a \in \Sigma} e(i, a) = 1\) for every \(i, 1 \leq i < m\);
- every \((s_i, s_j)\) \(\in T\) is associated with a probability distribution \(t(i, j)\), such that \(\sum_{1 \leq j \leq m} t(i, j) = 1\) for every \(i, 0 \leq i < m\).
Chapter 15. Dynamic Programming

A hidden Markov model (HMM) consists of \((S, T, e, t)\), where

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\[
\sum_{a \in \Sigma} e(i, a) = 1
\]

\[
\sum_{1 \leq j \leq m} t(i, j) = 1
\]

\[
\sum_{0 \leq i < m} t(i, j) = 1
\]
Chapter 15. Dynamic Programming

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- each state \(s_i\) is associated with a (emission) probability distribution \(e(i, a)\) for \(a \in \Sigma\), such that

\[
\sum_{a \in \Sigma} e(i, a) = 1 \quad \text{for every } i, 1 \leq i < m
\]
Chapter 15. Dynamic Programming

A hidden Markov model (HMM) consists of \((S, T, e, t)\), where

- \(S\) is a set of states: \(s_0, s_1, \ldots, s_m\); where \(s_0\) is the begin state;
- transitions \(T \subseteq S \times S\);
- each state \(s_i\) is associated with a (emission) probability distribution \(e(i, a)\) for \(a \in \Sigma\), such that
  \[
  \sum_{a \in \Sigma} e(i, a) = 1 \quad \text{for every } i, \ 1 \leq i < m
  \]
- every \((s_i, s_j) \in T\) is associated with a probability distribution \(t(i, j)\), such that
  \[
  \sum_{1 \leq j \leq m} t(i, j) = 1 \quad \text{for every } i, \ 0 \leq i < m
  \]
Chapter 15. Dynamic Programming

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  \[\sum_{1 \leq j \leq m} t(i, j) = 1 \quad \text{for every } i, 0 \leq i < m\]
Chapter 15. Dynamic Programming

HMM decoding problem:
Input: HMM $M$, sequence of symbols $x \in \Sigma^*$
Output: $y^*$, a sequence of states such that $y^* = \arg \max_y \{ \text{Prob} (y, x| M) \}$ where $y$ begins from $s_0$.
Chapter 15. Dynamic Programming

HMM decoding problem:

\[ M \]

Input: HMM \( M \), sequence of symbols \( x \in \Sigma^* \)

Output: \( y^* \), a sequence of states such that

\[ y^* = \arg \max_y \{ \text{Prob}(y, x|M) \} \]

where \( y \) begins from \( s_0 \).

\[ e(i, a) = 0.2 \]
\[ e(i, b) = 0.8 \]

\[ t(i, k) = 0.6 \]
\[ t(i, j) = 0.4 \]

\[ \Sigma = \{a, b\} \]
Chapter 15. Dynamic Programming

HMM decoding problem:

**Input**: HMM $\mathcal{M}$, sequence of symbols $x \in \Sigma^*$

**Output**: $y^*$, a sequence of states such that

$$y^* = \arg \max_y \{ \text{Prob}(y, x | \mathcal{M}) \}$$
Chapter 15. Dynamic Programming

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The idea of Viterbi algorithm:
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For any prefix $x_1 \ldots x_k, \ k \geq 0,$
Chapter 15. Dynamic Programming

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For any prefix $x_1 \ldots x_k$, $k \geq 0$, and state $s_j \in S$;
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For any prefix $x_1 \ldots x_k$, $k \geq 0$, and state $s_j \in S$;

$$\max_{y=s_0 \ldots s_j} \{\text{Prob}(y, x_1 \ldots x_k | M)\}$$

the maximum probability of a path $s_0 \ldots s_j$ to generate $x_1 \ldots x_k$;
The idea of Viterbi algorithm:

For any prefix $x_1 \ldots x_k$, $k \geq 0$, and state $s_j \in S$;

$$\max_{y = s_0 \ldots s_j} \{ \text{Prob}(y, x_1 \ldots x_k | \mathcal{M}) \}$$

the maximum probability of a path $s_0 \ldots s_j$ to generate $x_1 \ldots x_k$;
Chapter 15. Dynamic Programming

Define $f(j,k) = \max_{y=s_0 \ldots s_j} \{\text{Prob}(y,x_1 \ldots x_k)\}$

Recursively,

$f(j,k) = \max_i \{f(i,k-1) \times t(i,j) \times e(j,x_k)\}$

Base case:

$f(0,0) = 1, f(i,0) = 0$ for all $i \geq 1$. 

The diagram illustrates the relationship between the states $s_0$ and $s_j$, with $y$ representing a sequence that generates $x_1x_2 \ldots x_{k-1}$ and $s_j$ generating $x_k$. The transitions are marked with arrows indicating the progression from $s_0$ to $s_j$.
Define \( f(j, k) = \max_{y=s_0...s_j} \{Prob(y, x_1 ... x_k)\} \)
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Define \( f(j, k) = \max_{y = s_0 \ldots s_j} \{ Prob(y, x_1 \ldots x_k) \} \)

Recursively, \( f(j, k) = \max_i \{ f(i, k - 1) \times t(i, j) \times e(j, x_k) \} \)

Base case: \( f(0, 0) = 1 \), \( f(i, 0) = 0 \) for all \( i \geq 1 \).
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Define \( f(j,k) = \max_{y=s_0 \ldots s_j} \{ P(y, x_1 \ldots x_k) \} \)

Recursively, \( f(j,k) = \)
Chapter 15. Dynamic Programming

Define \( f(j, k) = \max_{y = s_0 \ldots s_j} \{Prob(y, x_1 \ldots x_k)\} \)

Recursively, \( f(j, k) = \max_{i} \{ f(i, k - 1) \times t(i, j) \times e(j, x_k) \} \)

Base case: \( f(0, 0) = 1 \), \( f(i, 0) = 0 \) for all \( i \geq 1 \).
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Recursively, $f(j, k) = \max_i \{f(i, k - 1)\}$
Chapter 15. Dynamic Programming

Define \( f(j, k) = \max_{y=s_0 \ldots s_j} \{ \text{Prob}(y, x_1 \ldots x_k) \} \)

Recursively, \( f(j, k) = \max_i \{ f(i, k - 1) \times t(i, j) \} \)
Chapter 15. Dynamic Programming

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Recursively,

\[ f(j, k) = \max_i \{ f(i, k - 1) \times t(i, j) \times e(j, x_k) \} \]
Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Recursively, \( f(j, k) = \max_i \{ f(i, k - 1) \times t(i, j) \times e(j, x_k) \} \)

Base case: \( f(0, 0) = \)
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Define \( f(j, k) = \max_{y=s_0 \ldots s_j} \{ \text{Prob}(y, x_1 \ldots x_k) \} \)

Recursively, \( f(j, k) = \max_i \{ f(i, k - 1) \times t(i, j) \times e(j, x_k) \} \)

Base case: \( f(0, 0) = 1, \)
Chapter 15. Dynamic Programming

Define \( f(j, k) = \max_{y=s_0 \ldots s_j} \{Prob(y, x_1 \ldots x_k)\} \)

Recursively, \( f(j, k) = \max_i \{f(i, k - 1) \times t(i, j) \times e(j, x_k)\} \)

Base case: \( f(0, 0) = 1, \quad f(i, 0) \)
Chapter 15. Dynamic Programming

Define $f(j, k) = \max_{y=s_0 \ldots s_j} \{ \text{Prob}(y, x_1 \ldots x_k) \}$

Recursively, $f(j, k) = \max_i \{ f(i, k-1) \times t(i, j) \times e(j, x_k) \}$

Base case: $f(0, 0) = 1$, $f(i, 0) = 0$ for all $i \geq 1$. 
Algorithm \textsc{Viterbi}(x, n, t, e, m)

0. \hspace{1em} T(0, 0) = 1; \hspace{0.5em} P(0, 0) = 0;
Algorithm VITERBI($x, n, t, e, m$)

0. \( T(0, 0) = 1; \ P(0, 0) = 0; \)
1. \textbf{for} \( j = 1 \) to \( m \)
2. \( T(j, 0) = 0; \)
Chapter 15. Dynamic Programming

Algorithm VITERBI \((x, n, t, e, m)\)

0. \(T(0, 0) = 1; \ P(0, 0) = 0;\)
1. \(\text{for } j = 1 \text{ to } m\)
2. \(T(j, 0) = 0;\)
3. \(\text{for } k = 1 \text{ to } n\)
4. \(\text{for } j = 1 \text{ to } m\) for every 'last state' \(s_j\)
5. \(\text{initialize the prefix probability}\)
6. \(\text{for } i = 0 \text{ to } m\) try every 'predecessor state' \(s_i\)
7. \(\text{if } \text{premax} < T(i, k-1) \times t(i, j) \times e(j, x_k)\)
8. \(\text{premax} = T(i, k-1) \times t(i, j) \times e(j, x_k)\) update probability
9. \(P(j, k) = i\) memorize the predecessor
10. \(T(j, k) = \text{premax}\)
11. \(\text{laststate} = 1; \ \text{max} = T(1, n)\)
12. \(\text{for } j = 2 \text{ to } m\) identify maximum probability
13. \(\text{if } \text{max} < T(j, n)\)
14. \(\text{laststate} = j; \ \text{max} = T(j, n)\)
15. \(\text{return } (T, P, \text{max}, \text{laststate})\)

Time complexity: \(O(nm^2)\)
Chapter 15. Dynamic Programming

Algorithm VITERBI\((x, n, t, e, m)\)

0. \(T(0, 0) = 1; \ P(0, 0) = 0;\)
1. \(\textbf{for} \ j = 1 \ \textbf{to} \ m\)
2. \(T(j, 0) = 0;\)
3. \(\textbf{for} \ k = 1 \ \textbf{to} \ n \quad \text{for every prefix upto } k\text{th symbol}\)
4. \(\textbf{for} \ j = 1 \ \textbf{to} \ m\)
5. \(\text{premax} = 0 \quad \text{initialize the prefix probability}\)
6. \(\textbf{for} \ i = 0 \ \textbf{to} \ m \quad \text{try every } \text{predecessor state}\)
7. \(\text{if } \text{premax} < T(i, k - 1) \times t(i, j) \times e(j, x_k)\)
8. \(\text{premax} = T(i, k - 1) \times t(i, j) \times e(j, x_k) \quad \text{update probability}\)
9. \(P(j, k) = i \quad \text{memorize the predecessor}\)
10. \(T(j, k) = \text{premax}\)
11. \(\text{laststate} = 1; \ \text{max} = T(1, n)\)
12. \(\textbf{for} \ j = 2 \ \textbf{to} \ m \quad \text{identify maximum probability}\)
13. \(\text{if } \text{max} < T(j, n)\)
14. \(\text{laststate} = j; \ \text{max} = T(j, n)\)
15. \(\textbf{return} \ (T, P, \text{max}, \text{laststate})\)

Time complexity: \(O(nm^2)\)
Algorithm VITERBI($x, n, t, e, m$)

0. $T(0, 0) = 1$; $P(0, 0) = 0$;
1. for $j = 1$ to $m$
2. $T(j, 0) = 0$;
3. for $k = 1$ to $n$  
   for every prefix upto $k$th symbol
4. for $j = 1$ to $m$
Algorithm \textsc{Viterbi}(x, n, t, e, m)

0. \( T(0, 0) = 1; \ P(0, 0) = 0; \)
1. \( \text{for } j = 1 \text{ to } m \)
2. \( T(j, 0) = 0; \)
3. \( \text{for } k = 1 \text{ to } n \quad \text{for every prefix upto } k\text{th symbol} \)
4. \( \text{for } j = 1 \text{ to } m \quad \text{for every ‘last state’ } s_j \)
Chapter 15. Dynamic Programming

Algorithm VITERBI($x, n, t, e, m$)

0. $T(0, 0) = 1; P(0, 0) = 0$;
1. for $j = 1$ to $m$
2. $T(j, 0) = 0$;
3. for $k = 1$ to $n$ for every prefix upto $k$th symbol
4. for $j = 1$ to $m$ for every ‘last state’ $s_j$
5. $premax = 0$
Algorithm VITERBI\((x, n, t, e, m)\)

0. \(T(0, 0) = 1; P(0, 0) = 0;\)
1. \(\text{for } j = 1 \text{ to } m\)
2. \(T(j, 0) = 0;\)
3. \(\text{for } k = 1 \text{ to } n\) for every prefix upto \(k\)th symbol
4. \(\text{for } j = 1 \text{ to } m\) for every ‘last state’ \(s_j\)
5. \(\text{premax} = 0\) initialize the prefix probability
Algorithm VITERBI($x, n, t, e, m$)

0. $T(0, 0) = 1; P(0, 0) = 0$;
1. for $j = 1$ to $m$ 
2. $T(j, 0) = 0$; 
3. for $k = 1$ to $n$ for every prefix upto $k$th symbol
4. for $j = 1$ to $m$ for every ‘last state’ $s_j$
5. premax = 0 initialize the prefix probability
6. for $i = 0$ to $m$ try every ‘predecessor state’ $s_i$

7. if premax < $T(i, k - 1) \times t(i, j) \times e(j, x_k)$ update probability
8. premax = $T(i, k - 1) \times t(i, j) \times e(j, x_k)$ 
9. $P(j, k) = i$ memorize the predecessor
10. $T(j, k) = \text{premax}$ 
11. laststate = 1 $\text{max} = T(1, n)$ 
12. for $j = 2$ to $m$ identify maximum probability
13. if $\text{max} < T(j, n)$ 
14. laststate = $j$ 
15. $\text{max} = T(j, n)$ 
16. return $(T, P, \text{max}, \text{laststate})$

Time complexity: $O(nm^2)$
Chapter 15. Dynamic Programming

Algorithm VITERBI($x, n, t, e, m$)

0. $T(0, 0) = 1; \ P(0, 0) = 0$;
1. for $j = 1$ to $m$
2. $T(j, 0) = 0$;
3. for $k = 1$ to $n$ for every prefix upto $k$th symbol
4. \ \ for $j = 1$ to $m$ for every ‘last state’ $s_j$
5. \ \ $\text{premax} = 0$ initialize the prefix probability
6. \ \ for $i = 0$ to $m$ try every ‘predecessor state’ $s_i$
7. \ \ if $\text{premax} < T(i, k - 1) \times t(i, j) \times e(j, x_k)$

Time complexity: $O(nm^2)$
Algorithm VITERBI\((x, n, t, e, m)\)

0. \(T(0, 0) = 1; P(0, 0) = 0;\)
1. for \(j = 1\) to \(m\)
2. \(T(j, 0) = 0;\)
3. for \(k = 1\) to \(n\)  
   for every prefix upto \(k\)th symbol
4. for \(j = 1\) to \(m\)  
   for every ‘last state’ \(s_j\)
5. \(\text{premax} = 0\)  
   initialize the prefix probability
6. for \(i = 0\) to \(m\)  
   try every ‘predecessor state’ \(s_i\)
7. if \(\text{premax} < T(i, k - 1) \times t(i, j) \times e(j, x_k)\)
8. \(\text{premax} = T(i, k - 1) \times t(i, j) \times e(j, x_k)\)
Algorithm \textsc{Viterbi}(x, n, t, e, m)

0. \( T(0, 0) = 1; \ P(0, 0) = 0; \)
1. \( \text{for } j = 1 \text{ to } m \)
2. \( T(j, 0) = 0; \)
3. \( \text{for } k = 1 \text{ to } n \quad \text{for every prefix upto } k\text{th symbol} \)
4. \( \text{for } j = 1 \text{ to } m \quad \text{for every 'last state' } s_j \)
5. \( \text{premax} = 0 \quad \text{initialize the prefix probability} \)
6. \( \text{for } i = 0 \text{ to } m \quad \text{try every 'predecessor state' } s_i \)
7. \( \text{if } \text{premax} < T(i, k - 1) \times t(i, j) \times e(j, x_k) \)
8. \( \text{premax} = T(i, k - 1) \times t(i, j) \times e(j, x_k) \quad \text{update probability} \)
Algorithm \textsc{Viterbi}(x, n, t, e, m)

0. \(T(0, 0) = 1; P(0, 0) = 0;\)
1. \textbf{for} \(j = 1\) to \(m\)
2. \(T(j, 0) = 0;\)
3. \textbf{for} \(k = 1\) to \(n\) \text{ for every prefix upto } k\text{th symbol}
4. \textbf{for} \(j = 1\) to \(m\) \text{ for every 'last state' } s_j
5. \(\text{premax} = 0\) \text{ initialize the prefix probability}
6. \textbf{for} \(i = 0\) to \(m\) \text{ try every 'predecessor state' } s_i
7. \textbf{if} \(\text{premax} < T(i, k - 1) \times t(i, j) \times e(j, x_k)\)
8. \(\text{premax} = T(i, k - 1) \times t(i, j) \times e(j, x_k)\) \text{ update probability}
9. \(P(j, k) = i\)

Time complexity: \(O(nm^2)\)
Chapter 15. Dynamic Programming

Algorithm VITERBI\((x,n,t,e,m)\)

0. \(T(0,0) = 1; P(0,0) = 0;\)
1. \(\text{for } j = 1 \text{ to } m\)
2. \(T(j,0) = 0;\)
3. \(\text{for } k = 1 \text{ to } n \text{ for every prefix upto } k\text{th symbol}\)
4. \(\text{for } j = 1 \text{ to } m \text{ for every ‘last state’ } s_j\)
5. \(\text{premax } = 0 \text{ initialize the prefix probability}\)
6. \(\text{for } i = 0 \text{ to } m \text{ try every ‘predecessor state’ } s_i\)
7. \(\text{if } \text{premax } < T(i,k - 1) \times t(i,j) \times e(j,x_k)\)
8. \(\text{premax } = T(i,k - 1) \times t(i,j) \times e(j,x_k) \text{ update probability}\)
9. \(P(j,k) = i \text{ memorize the predecessor}\)

Time complexity: \(O(nm^2)\)
Algorithm \textsc{Viterbi}(x, n, t, e, m)

0. \quad T(0, 0) = 1; \quad P(0, 0) = 0;
1. \quad \textbf{for } j = 1 \textbf{ to } m
2. \quad T(j, 0) = 0;
3. \quad \textbf{for } k = 1 \textbf{ to } n \quad \text{ for every prefix upto } k\text{th symbol}
4. \quad \textbf{for } j = 1 \textbf{ to } m \quad \text{ for every ’last state’ } s_j
5. \quad \text{premax} = 0 \quad \text{ initialize the prefix probability}
6. \quad \textbf{for } i = 0 \textbf{ to } m \quad \text{ try every ’predecessor state’ } s_i
7. \quad \textbf{if } \text{premax} < T(i, k - 1) \times t(i, j) \times e(j, x_k)
8. \quad \text{premax} = T(i, k - 1) \times t(i, j) \times e(j, x_k) \quad \text{ update probability}
9. \quad P(j, k) = i \quad \text{ memorize the predecessor}
10. \quad T(j, k) = \text{premax}
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Algorithm VITERBI\((x, n, t, e, m)\)

0. \(T(0, 0) = 1; P(0, 0) = 0;\)

1. \textbf{for} \(j = 1\) \textbf{to} \(m\)

2. \(T(j, 0) = 0;\)

3. \textbf{for} \(k = 1\) \textbf{to} \(n\) \textit{for every prefix upto \(k\)th symbol}

4. \textbf{for} \(j = 1\) \textbf{to} \(m\) \textit{for every ‘last state’ \(s_j\)}

5. \(\text{premax} = 0\) \textit{initialize the prefix probability}

6. \textbf{for} \(i = 0\) \textbf{to} \(m\) \textit{try every ‘predecessor state’ \(s_i\)}

7. \textbf{if} \(\text{premax} < T(i, k - 1) \times t(i, j) \times e(j, x_k)\)

8. \(\text{premax} = T(i, k - 1) \times t(i, j) \times e(j, x_k)\) \textit{update probability}

9. \(P(j, k) = i\) \textit{memorize the predecessor}

10. \(T(j, k) = \text{premax}\)

11. \(\text{laststate} = 1; \text{max} = T(1, n)\)

Time complexity: \(O(nm^2)\)
Algorithm VITERBI\((x, n, t, e, m)\)

0. \(T(0, 0) = 1; \ P(0, 0) = 0;\)
1. \textbf{for} \(j = 1 \text{ to } m\)
2. \(T(j, 0) = 0;\)
3. \textbf{for} \(k = 1 \text{ to } n\) \quad \text{for every prefix up to } k\text{th symbol}
4. \textbf{for} \(j = 1 \text{ to } m\) \quad \text{for every ‘last state’ } s_j
5. \(\text{premax} = 0\) \quad \text{initialize the prefix probability}
6. \textbf{for} \(i = 0 \text{ to } m\) \quad \text{try every ‘predecessor state’ } s_i
7. \quad \textbf{if} \ \text{premax} < T(i, k - 1) \times t(i, j) \times e(j, x_k)\)
8. \quad \text{premax} = T(i, k - 1) \times t(i, j) \times e(j, x_k) \quad \text{update probability}
9. \quad P(j, k) = i \quad \text{memorize the predecessor}
10. \(T(j, k) = \text{premax}\)
11. \(\text{laststate} = 1; \ \text{max} = T(1, n)\)
12. \textbf{for} \(j = 2 \text{ to } m\) \quad \text{identify maximum probability}

\text{Time complexity: } O\((nm^2)\)
Algorithm VITERBI($x, n, t, e, m$)

0. $T(0, 0) = 1$; $P(0, 0) = 0$;
1. for $j = 1$ to $m$
2.     $T(j, 0) = 0$;
3. for $k = 1$ to $n$ for every prefix upto $k$th symbol
4.     for $j = 1$ to $m$ for every ‘last state’ $s_j$
5.         premax = 0 initialize the prefix probability
6.     for $i = 0$ to $m$ try every ‘predecessor state’ $s_i$
7.         if premax $< T(i, k - 1) \times t(i, j) \times e(j, x_k)$
8.             premax $= T(i, k - 1) \times t(i, j) \times e(j, x_k)$ update probability
9.         $P(j, k) = i$ memorize the predecessor
10. $T(j, k) = premax$
11. laststate = 1; max = $T(1, n)$
12. for $j = 2$ to $m$ identify maximum probability
13. if max $< T(j, n)$
Algorithm VITERBI\( (x, n, t, e, m) \)

0. \( T(0,0) = 1; P(0,0) = 0; \)
1. for \( j = 1 \) to \( m \)
2. \( T(j,0) = 0; \)
3. for \( k = 1 \) to \( n \) for every prefix upto \( k \)th symbol
4. for \( j = 1 \) to \( m \) for every ‘last state’ \( s_j \)
5. \( \text{premax} = 0 \) initialize the prefix probability
6. for \( i = 0 \) to \( m \) try every ‘predecessor state’ \( s_i \)
7. if \( \text{premax} < T(i,k-1) \times t(i,j) \times e(j,x_k) \)
8. \( \text{premax} = T(i,k-1) \times t(i,j) \times e(j,x_k) \) update probability
9. \( P(j,k) = i \) memorize the predecessor
10. \( T(j,k) = \text{premax} \)
11. \( \text{laststate} = 1; \text{max} = T(1,n) \)
12. for \( j = 2 \) to \( m \) identify maximum probability
13. if \( \text{max} < T(j,n) \)
14. \( \text{laststate} = j; \)
Algorithm VITERBI\((x, n, t, e, m)\)

0. \(T(0, 0) = 1; P(0, 0) = 0;\)
1. \(\text{for } j = 1 \text{ to } m\)
2. \(T(j, 0) = 0;\)
3. \(\text{for } k = 1 \text{ to } n \text{ for every prefix upto } k\text{th symbol}\)
4. \(\text{for } j = 1 \text{ to } m \text{ for every 'last state' } s_j\)
5. \(\text{premax} = 0 \text{ initialize the prefix probability}\)
6. \(\text{for } i = 0 \text{ to } m \text{ try every 'predecessor state' } s_i\)
7. \(\text{if } \text{premax} < T(i, k - 1) \times t(i, j) \times e(j, x_k)\)
8. \(\text{premax} = T(i, k - 1) \times t(i, j) \times e(j, x_k) \text{ update probability}\)
9. \(P(j, k) = i \text{ memorize the predecessor}\)
10. \(T(j, k) = \text{premax}\)
11. \(\text{laststate} = 1; \text{max} = T(1, n)\)
12. \(\text{for } j = 2 \text{ to } m \text{ identify maximum probability}\)
13. \(\text{if } \text{max} < T(j, n)\)
14. \(\text{laststate} = j;\)
15. \(\text{max} = T(j, n)\)

Time complexity: \(O(nm^2)\)
Chapter 15. Dynamic Programming

Algorithm VITERBI\((x, n, t, e, m)\)

0. \(T(0, 0) = 1; P(0, 0) = 0;\)
1. for \(j = 1\) to \(m\)
2. \(T(j, 0) = 0;\)
3. for \(k = 1\) to \(n\) for every prefix upto \(k\)th symbol
4. for \(j = 1\) to \(m\) for every ‘last state’ \(s_j\)
5. \(\text{premax} = 0\) initialize the prefix probability
6. for \(i = 0\) to \(m\) for every ‘predecessor state’ \(s_i\)
7. if \(\text{premax} < T(i, k - 1) \times t(i, j) \times e(j, x_k)\)
8. \(\text{premax} = T(i, k - 1) \times t(i, j) \times e(j, x_k)\) update probability
9. \(P(j, k) = i\) memorize the predecessor
10. \(T(j, k) = \text{premax}\)
11. \(\text{laststate} = 1; \text{max} = T(1, n)\)
12. for \(j = 2\) to \(m\) identify maximum probability
13. if \(\text{max} < T(j, n)\)
14. \(\text{laststate} = j;\)
15. \(\text{max} = T(j, n)\)
16. return \((T, P, \text{max}, \text{laststate})\)

Time complexity: \(O(mn^2)\)
Algorithm VITERBI\((x, n, t, e, m)\)

0. \(T(0, 0) = 1; \ P(0, 0) = 0;\)
1. for \(j = 1\) to \(m\)
2. \(T(j, 0) = 0;\)
3. for \(k = 1\) to \(n\) for every prefix upto \(k\)th symbol
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5. \(\text{premax} = 0\) initialize the prefix probability
6. for \(i = 0\) to \(m\) try every ‘predecessor state’ \(s_i\)
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8. \(\text{premax} = T(i, k - 1) \times t(i, j) \times e(j, x_k)\) update probability
9. \(P(j, k) = i\) memorize the predecessor
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12. for \(j = 2\) to \(m\) identify maximum probability
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14. \(\text{laststate} = j;\)
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16. return \((T, P, \text{max}, \text{laststate})\)

Time complexity:
Algorithm \textsc{Viterbi}$(x, n, t, e, m)$

0. \hspace{0.5cm} $T(0, 0) = 1$; $P(0, 0) = 0$;
1. \hspace{0.5cm} for $j = 1$ to $m$
2. \hspace{1cm} $T(j, 0) = 0$;
3. \hspace{0.5cm} for $k = 1$ to $n$ \hspace{0.5cm} for every prefix upto $k$th symbol
4. \hspace{1cm} for $j = 1$ to $m$ \hspace{0.5cm} for every ‘last state’ $s_j$
5. \hspace{1.5cm} $\text{premax} = 0$ \hspace{0.5cm} initialize the prefix probability
6. \hspace{1cm} for $i = 0$ to $m$ \hspace{0.5cm} try every ‘predecessor state’ $s_i$
7. \hspace{1.5cm} if $\text{premax} < T(i, k - 1) \times t(i, j) \times e(j, x_k)$
8. \hspace{1.5cm} \hspace{0.5cm} $\text{premax} = T(i, k - 1) \times t(i, j) \times e(j, x_k)$ \hspace{0.5cm} update probability
9. \hspace{1cm} $P(j, k) = i$ \hspace{0.5cm} memorize the predecessor
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12. \hspace{0.5cm} for $j = 2$ to $m$ \hspace{0.5cm} identify maximum probability
13. \hspace{1cm} if $\text{max} < T(j, n)$
14. \hspace{1.5cm} $\text{laststate} = j$
15. \hspace{1.5cm} $\text{max} = T(j, n)$
16. \hspace{0.5cm} return $(T, P, \text{max}, \text{laststate})$

Time complexity: $O(nm^2)$
Use the computed table $P$ and $laststate$ to traceback the hidden states.
Use the computed table $P$ and laststate to traceback the hidden states

Algorithm $\text{TRACEHIDDENSTATES}(P, j, k, x)$

1. if $j > 0$
Use the computed table $P$ and $laststate$ to traceback the hidden states

Algorithm $\text{TraceHiddenStates}(P, j, k, x)$

1. if $j > 0$
2. $\text{TraceHiddenStates}(P, P[j, k], k - 1, x)$
Use the computed table $P$ and laststate to traceback the hidden states

Algorithm `TraceHiddenStates($P, j, k, x$)`

1. if $j > 0$
2. `TraceHiddenStates($P, P[j,k], k - 1, x$)`
3. `print ('State' $j$ 'emits symbol' $x_k$)`
Chapter 15. Dynamic Programming

Use the computed table $P$ and $laststate$ to traceback the hidden states

Algorithm $\text{TraceHiddenStates}(P, j, k, x)$

1. if $j > 0$
2. $\text{TraceHiddenStates}(P, P[j, k], k - 1, x)$
3. print ('State' $j$ 'emits symbol' $x_k$)

First, call $\text{TraceHiddenStates}(P, laststate, n, x)$
Chapter 15. Dynamic Programming

There are other DP algorithms associated with HMMs.

• Viterbi algorithm belongs to the type of forward algorithms because it computes for prefixes of the given symbol sequence. A DP algorithm can be designed so it computes for suffixes of the given symbols sequence, a backward algorithm.

Algorithm

How to find objective function?
Recalled for forward, we defined

\[ f(j,k) = \max_{y=s_0 \ldots s_j} \{ \text{Prob}(y,x_1 \ldots x_k) \} \]

\( f(j,k) \) is the maximum probability to generate prefix \( x_1 \ldots x_k \) beginning from state \( s_0 \) and ending with state \( s_j \);

For backward, we may define

\[ b(l,k) = \max_{y=s_l \ldots s_\omega} \{ \text{Prob}(y,x_k \ldots x_n) \} \]

\( b(l,k) \) is the maximum probability to generate suffix \( x_k \ldots x_n \) beginning from state \( s_l \) and ending with state \( s_\omega \);
Chapter 15. Dynamic Programming

There are other DP algorithms associated with HMMs.

- **Viterbi** algorithm belongs to the type of *forward* algorithms because it computes for *prefixes* of the given symbol sequence.

\[
f(j,k) = \max_{y=s_0 \ldots s_j} \left\{ \text{Prob}(y, x_1 \ldots x_k) \right\}
\]

- For backward, we may define

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b(l,k) = \max_{y=s_l \ldots s_\omega} \left\{ \text{Prob}(y, x_k \ldots x_n) \right\}
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\(f(j,k)\) is the maximum probability to generate prefix \(x_1 \ldots x_k\) beginning from state \(s_0\) and ending with state \(s_j\);

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Chapter 15. Dynamic Programming

There are other DP algorithms associated with HMMs.

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Chapter 15. Dynamic Programming

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**Algorithm IBRETIV**
Chapter 15. Dynamic Programming

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**Algorithm I B R E T I V**

How to find objective function?
Recalled for forward, we defined

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f(j, k) = \max_{y = s_0 \ldots s_j} \{\text{Prob}(y, x_1 \ldots x_k)\}
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\(f(j, k)\) is the maximum probability to generate prefix \(x_1 \ldots x_1\) beginning from state \(s_0\) and ending with state \(s_j\);
Chapter 15. Dynamic Programming

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**Algorithm I BRETIV**

How to find objective function?
Recalled for forward, we defined

$$f(j, k) = \max_{y = s_0 \ldots s_j} \{\text{Prob}(y, x_1 \ldots x_k)\}$$

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$$b(l, k) = \max_{y = s_l \ldots s_\omega} \{\text{Prob}(y, x_k \ldots x_n)\}$$

$b(l, k)$ is the maximum probability to generate suffix $x_k \ldots x_n$ beginning from state $s_l$ and ending with state $s_\omega$;
• We need to amend the HMM model with a silent state $s_\omega$;

$$
b(l, k) = \max_{y=s_l \ldots s_\omega} \{ \text{Prob}(y, x_k \ldots x_n) \}
$$
Chapter 15. Dynamic Programming

• We need to amend the HMM model with a silent state \( s_\omega \);

\[
b(l, k) = \max_{y=s_1...s_\omega} \{ \text{Prob}(y, x_k ... x_n) \}
\]

• Recurrence (draw a diagram)

\[
b(l, k) =
\]
Chapter 15. Dynamic Programming

- We need to amend the HMM model with a silent state $s_\omega$;

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\[
b(l, k) = \max_i
\]
Chapter 15. Dynamic Programming

- We need to amend the HMM model with a silent state $s_\omega$;

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$$b(l, k) = \max_{y=s_l \ldots s_\omega} \{ \text{Prob}(y, x_k \ldots x_n) \}$$

• Recurrence (draw a diagram)

$$b(l, k) = \max_i \{ b(i, k + 1) \times t(l, i) \}$$
Chapter 15. Dynamic Programming

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$$b(l, k) = \max_i \{b(i, k + 1) \times t(l, i) \times e(l, x_k)\}$$

the larger the $k$ value, the shortest the suffix is
Chapter 15. Dynamic Programming

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- Base cases:
Chapter 15. Dynamic Programming

- We need to amend the HMM model with a silent state $s_\omega$;

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the larger the $k$ value, the shortest the suffix is

- Base cases: $b(s_\omega, n+1) = 1$, $b(j, n+1) = 0$ for all $j \neq \omega$.

$k = n+1$ is the case when the suffix is empty.
Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

- We need to amend the HMM model with a silent state $s_ω$;

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Chapter 15. Dynamic Programming

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What is the relationship between $\max_j \{f(j, n)\}$ and $\max_l \{b(l, 1)\}$?
Chapter 15. Dynamic Programming

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What is the relationship between $\max_j f(j, n)$ and $\max_l b(l, 1)$? (draw diagram to show)
Chapter 15. Dynamic Programming

Computing the total probability (instead of the maximum prob)
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- often we are interested in knowing the chance for a given sequence of symbols to be produced by a model.
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  e.g., used in discrimination against background
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- given a HMM $M$, and sequence $x \in \Sigma^*$, what is the probability

$$Prob(x|M)$$
Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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• given a HMM $\mathcal{M}$, and sequence $x \in \Sigma^*$, what is the probability

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\[
p(j, k) \text{ to be the total probability for } \mathcal{M} \text{ to generate prefix } x_1 \ldots x_k \text{ ending at state } s_j.
\]
Chapter 15. Dynamic Programming

Define $p(j, k)$ to be the total probability for $M$ to generate prefix $x_1 \ldots x_k$ ending at state $s_j$. 

$$p(j, k) = \sum_i p(i, k-1) \times t(i, j) \times e(j, x_k)$$

with base cases:

- $p(0, 0) = 1$
- $p(j, 0) = 0$ for all $j \geq 1$. 

Chapter 15. Dynamic Programming

Define $p(j, k)$ to be the total probability for $\mathcal{M}$ to generate prefix $x_1 \ldots x_k$ ending at state $s_j$.

Then

$$p(j, k) = \sum_i p(i, k - 1) \times t(i, j) \times e(j, x_k)$$

with base cases: $p(0, 0) = 1$, and $p(j, 0) = 0$ for all $j \geq 1$. 
Chapter 15. Dynamic Programming

Implementation of the total probability algorithm

- screen a stream of signals (symbols) for occasional but desired pattern
- use HMM to model the desired pattern
- scanning window on the long stream; computes the total probability
- report segments with 'significant' probability values

but what does it mean by 'significant probability values'?
Chapter 15. Dynamic Programming

Implementation of the total probability algorithm

- almost the same as for Viterbi algorithm.
Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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but what does it mean by ’significant probability values’?
Necessary elements for problems solvable with DP

(1) it is an optimization problem with an objective function to optimize by solutions
   e.g., solution: a path of stations, objective function: time

(2) solution has optimal substructure
   solutions are recursively definable
   e.g., a fastest path consists of other fastest subpaths

(3) overlapping subproblems
   e.g., two or more paths share a subpath.
Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

Longest Common Subsequence (LCS)
Chapter 15. Dynamic Programming

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\[ x = \text{ACCGGTTCGAGTGCG} \]
\[ y = \text{GTCGTTCGGATGCCC} \]
Chapter 15. Dynamic Programming

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\[ x = \text{ACCGGTCGAGTGCG} \]
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To see how much the two sequences are related, we may examine how much the two are in common.
Chapter 15. Dynamic Programming

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E.g., a common subsequence for the two sequences
Chapter 15. Dynamic Programming

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\[ \text{ACCGGTTCAGGTGCG} \]
\[ \text{CGTCGATGC} \quad \leftarrow \text{common sequence} \]
\[ \text{GTCGTTTCGGATGCCC} \]
Chapter 15. Dynamic Programming

Longest Common Subsequence (LCS)

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To see how much the two sequences are related, we may examine how much the two are in common.

e.g., a common subsequence for the two sequences

\[
\begin{align*}
\text{ACCGGTTCGAGTGCG} \\
\text{CGTCGATGC} & \quad \leftrightarrow \quad \text{common sequence} \\
\text{GTCGTTTCGATGCCC}
\end{align*}
\]

significance of common sequence, viewed as:
Chapter 15. Dynamic Programming

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e.g., a common subsequence for the two sequences

\[ \text{ACCGGTCGAGTGCG} \]
\[
\begin{array}{c}
\text{CGTCGATGC} \\ \text{GTCGGTTGGATGCCC}
\end{array}
\]

significance of common sequence, viewed as:

\[ \text{ACCGGT} \quad \text{TCGAGT} \quad \text{CGCG} \]
\[
\begin{array}{c}
\text{CGTCGATGC} \quad \text{common sequence} \\
\text{GTCGGTTGGATGCCC}
\end{array}
\]

\[ \text{ACCGGTCG \quad A \quad GTGCG} \]
\[
\begin{array}{c}
\text{CG \quad TC \quad GA \quad TGC} \quad \text{common sequence} \\
\text{GTCGGTTGGA \quad TGCCC}
\end{array}
\]
Chapter 15. Dynamic Programming

Let $X = x_1 x_2 \cdots x_m$ be a sequence, $x_i \in \Sigma$. 

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$Z = z_1 z_2 \cdots z_k$ is a subsequence of $X$
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\[
z_j = x_{i_j} \quad \text{for } j = 1, \cdots, k.
\]
Chapter 15. Dynamic Programming

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\[
\begin{array}{cccc}
34 & 6789 & 11 & 12 \\
X = & ACGGT & C & GAGTGCG \\
\| & \| & \| & \|
\end{array}
\]

\[
\begin{array}{cccc}
12 & 3456 & 789 \\
Z = & CG & TCGA & TGC
\end{array}
\]
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$$z_j = x_{i_j} \text{ for } j = 1, \cdots, k.$$  

\begin{align*}
34 & \quad 6789 & \quad 11^{12} & \quad 13 \\
X & = & ACCG & & TCGAGTGCG \\
| & | & | & | & | & | & | \\
Z & = & CG & TCGA & TGC \\
12 & & 3456 & & 789
\end{align*}

the corresponding indexes in $X$ are:

$$3 < 4 < 6 < 7 < 8 < 9 < 11 < 12 < 13$$
$Z$ is a *common subsequence* of $X$ and $Y$
if $Z$ is a *subsequence* of $X$ and $Z$ is a *subsequence* of $Y$.
Chapter 15. Dynamic Programming

$Z$ is a *common subsequence* of $X$ and $Y$
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Chapter 15. Dynamic Programming

$Z$ is a common subsequence of $X$ and $Y$ if $Z$ is a subsequence of $X$ and $Z$ is a subsequence of $Y$.

\begin{align*}
\text{ACCGGTC} & \quad \text{GAGTGC}G \\
\vline & \quad \vline \quad \vline \quad \vline \quad \vline \\
\text{CG} & \quad \text{TC} & \quad \text{GA} & \quad \text{TGC} \quad \leftarrow \text{common sequence} \\
\vline & \quad \vline \quad \vline \quad \vline \quad \vline \\
\text{GTCGTTCGGA} & \quad \text{TGCC}C
\end{align*}

Longest Common Subsequence (LCS) problem:

**Input:** $X = x_1x_2 \cdots x_m$, $Y = y_1y_2 \cdots y_n$.

**Output:** $Z$, a common subsequence of $X$ and $Y$ such that $\text{length}(Z)$ is the maximum.
Step 1: Analyze the LCS problem to see if it has the desired features:
Chapter 15. Dynamic Programming

**Step 1:** Analyze the LCS problem to see if it has the desired features:

- optimal substructure
- overlapping subproblems
Chapter 15. Dynamic Programming

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We examine prefixes \( x_1 x_2 \ldots x_i, \ i \leq m \)
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Step 1: Analyze the LCS problem to see if it has the desired features:

- optimal substructure
- overlapping subproblems

We examine prefixes $x_1 x_2 \ldots x_i, i \leq m$
and $y_1 y_2 \ldots y_j, j \leq n$

- how is the LCS of $x_1 x_2 \ldots x_i$ and $y_1 y_2 \ldots y_j$ related to the LCS of shorter prefixes of $X$ and $Y$?
Chapter 15. Dynamic Programming

if $x_i = y_j$, include $x_i$ in the LCS, reducing to LCS for $x_1 \ldots x_{i-1}x_i$ and $y_1 \ldots y_{j-1}y_j$.

If $x_i \neq y_j$, should we abandon either?

Two options:

1. Keep $x_i$ but abandon $y_j$ resulting in prefixes: $x_1 \ldots x_{i-1}x_i$ and $y_1 \ldots y_{j-1}y_j$.

2. Keep $y_j$ but abandon $x_i$ resulting in prefixes: $x_1 \ldots x_{i-1}x_i$ and $y_1 \ldots y_{j-1}y_j$. 


if $x_i = y_j$, include $x_i$ in the LCS, reducing to LCS for $x_1 \ldots x_{i-1}$ and $y_1 \ldots y_{j-1}$.
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\[ x_1 \ldots x_{i-1}x_i \]
\[ y_1 \ldots y_{j-1}y_j \]

if \( x_i = y_j \), include \( x_i \) in the LCS, reducing to LCS for \( x_1 \ldots x_{i-1} \) and \( y_1 \ldots y_{j-1} \)

if \( x_i \neq y_j \), should we abandon either?
Chapter 15. Dynamic Programming

\[ x_1 \ldots x_{i-1}x_i \]
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if \( x_i \neq y_j \), should we abandon either? two options

(1) keep \( x_i \) but abandon \( y_j \)
    resulting in prefixes: \( x_1x_2 \ldots x_i \) and \( y_1y_2 \ldots y_{j-1} \), or
Chapter 15. Dynamic Programming

\[ x_1 \ldots x_{i-1} x_i \]
\[ y_1 \ldots y_{j-1} y_j \]

if \( x_i = y_j \), include \( x_i \) in the LCS, reducing to LCS for \( x_1 \ldots x_{i-1} \) and \( y_1 \ldots y_{j-1} \)

if \( x_i \neq y_j \), should we abandon either? two options

(1) keep \( x_i \) but abandon \( y_j \)
    resulting in prefixes: \( x_1 x_2 \ldots x_i \) and \( y_1 y_2 \ldots y_{j-1} \), or

(2) keep \( y_j \) but abandon \( x_i \)
    resulting in prefixes: \( x_1 x_2 \ldots x_{i-1} \) and \( y_1 y_2 \ldots y_j \)
Chapter 15. Dynamic Programming

Let $LCS(i, j)$ be the longest common sequence for $x_1 x_2 \ldots x_i$ and $y_1 y_2 \ldots y_j$, then

- if $x_i = y_j$, then $LCS(i, j) = \text{concat}(LCS(i-1, j-1), x_i)$
- if $x_i \neq y_j$, then $LCS(i, j) = \text{max}(LCS(i, j-1), LCS(i-1, j))$ (depending on which is of a larger length.)

We conclude: $LCS$ problem has the optimal substructure property;
$LCS$ problem has overlapping subproblems.
Let $LCS(i, j)$ be the longest common sequence for $x_1x_2 \ldots x_i$ and $y_1y_2 \ldots y_j$, then

- if $x_i = y_j$, then $LCS(i, j) = \text{concat}(LCS(i - 1, j - 1), x_i)$

- if $x_i \neq y_j$, then $LCS(i, j) = LCS(i, j - 1)$ OR $LCS(i, j) = LCS(i - 1, j)$ (depending on which is of a larger length.)

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Chapter 15. Dynamic Programming

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We conclude:

LCS problem has the optimal substructure property;
LCS problem has overlapping subproblems.
Step 2: define objective function and formulate recurrence

Define $l(i,j)$ to be the length of the LCS for prefixes $x_1...x_i$ and $y_1...y_j$.

$$l(i,j) = \begin{cases} l(i-1,j-1) + 1 & \text{if } x_i = y_j; \\ \max\{l(i,j-1), l(i-1,j), l(i-1,j-1)\} & \text{if } x_i \neq y_j. \end{cases}$$

Base cases:

- $l(i,0) = 0$, for $0 \leq i \leq m$, either prefix is empty $\rightarrow$ LCS is empty
- $l(0,j) = 0$, for $0 \leq j \leq n$
Step 2: define objective function and formulate recurrence

Define $l(i, j)$ to be the length of the LCS for prefixes $x_1 \ldots x_i$ and $y_1 \ldots y_j$. 

The value of $l(i, j)$ depends on values of $l(i-1, j-1)$, $l(i, j-1)$, and $l(i-1, j)$. 

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\begin{cases}
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  \end{cases}
\end{cases}
\]

base cases:

\[
l(i, 0) = 0, \text{ for } 0 \leq i \leq m, \text{ either prefix is empty};
\]

\[
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Chapter 15. Dynamic Programming

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$l(0, j) = 0$, for $0 \leq j \leq n$
Chapter 15. Dynamic Programming

**Step 3**: bottom-up table building for function $l(i, j)$.

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Chapter 15. Dynamic Programming

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**Chapter 15. Dynamic Programming**

**Step 3:** bottom-up table building for function $l(i, j)$.

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Chapter 15. Dynamic Programming

**Step 3**: bottom-up table building for function \( l(i, j) \).

LCS result between prefixes GCCCTAGC and GCG:

```
GCCCTAGC
G   C   G
```
Chapter 15. Dynamic Programming

Algorithm LCS($x, y$)
1. $m = \text{length}(x), \ n = \text{length}(y)$;
Chapter 15. Dynamic Programming

Algorithm LCS\((x, y)\)
1. \( m = \text{length}(x) \), \( n = \text{length}(y) \);
2. \textbf{for} \( i = 0 \) \textbf{to} \( m \)

Time complexity: \( O(mn) \)
Algorithm LCS($x, y$)
1. $m = \text{length}(x), \ n = \text{length}(y)$;
2. for $i = 0$ to $m$
3. \hspace{1em} $T[i, 0] = 0$
Chapter 15. Dynamic Programming

Algorithm LCS\((x, y)\)
1. \(m = \text{length}(x), n = \text{length}(y)\);
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3. \hspace{1em} \(T[i, 0] = 0\)
4. \textbf{for } j = 0 \textbf{ to } n

Algorithm LCS\((x, y)\)
1. \(m = \text{length}(x), n = \text{length}(y)\);
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3. \(T[i, 0] = 0\)
4. \textbf{for} \(j = 0 \text{ to } n\)
5. \(T[0, j] = 0\)
Algorithm LCS($x, y$)

1. $m = \text{length}(x)$, $n = \text{length}(y)$;
2. for $i = 0$ to $m$
3.     $T[i, 0] = 0$
4. for $j = 0$ to $n$
5.     $T[0, j] = 0$
6. for $i = 1$ to $m$
7.     for $j = 1$ to $n$
8.         if $x[i] = y[j]$
9.             $T[i, j] = T[i-1, j-1] + 1$
10.            $P[i, j] = \uparrow$
11.     else if $T[i, j-1] > T[i-1, j]$
12.             $T[i, j] = T[i, j-1]$
13.            $P[i, j] = \leftarrow$
14.     else
15.         $T[i, j] = T[i-1, j]$
16.            $P[i, j] = \uparrow$
17. return ($T, P$)

Time complexity: $O(mn)$
Chapter 15. Dynamic Programming

Algorithm LCS$(x, y)$
1. $m = \text{length}(x), \ n = \text{length}(y)$;
2. for $i = 0$ to $m$
3. \hspace{1em} $T[i, 0] = 0$
4. for $j = 0$ to $n$
5. \hspace{1em} $T[0, j] = 0$
6. for $i = 1$ to $m$
7. \hspace{1em} for $j = 1$ to $n$
8. if $x_i = y_j$
9. \hspace{2em} $T[i, j] = T[i-1, j-1] + 1$
\hspace{2em} $P[i, j] = \uparrow$
10. else if $T[i, j-1] > T[i-1, j]$
\hspace{2em} $T[i, j] = T[i, j-1]$
\hspace{2em} $P[i, j] = \leftarrow$
11. else
\hspace{2em} $T[i, j] = T[i-1, j]$
\hspace{2em} $P[i, j] = \uparrow$
12. return $(T, P)$

Time complexity: $O(mn)$
Algorithm LCS($x, y$)
1. \( m = \text{length}(x), \ n = \text{length}(y); \)
2. \textbf{for} \( i = 0 \) \textbf{to} \( m \)
3. \( T[i, 0] = 0 \)
4. \textbf{for} \( j = 0 \) \textbf{to} \( n \)
5. \( T[0, j] = 0 \)
6. \textbf{for} \( i = 1 \) \textbf{to} \( m \)
7. \textbf{for} \( j = 1 \) \textbf{to} \( n \)
8. \textbf{if} \( x_i = y_j \)
Chapter 15. Dynamic Programming

Algorithm LCS(x, y)
1. \( m = \text{length}(x), \ n = \text{length}(y); \)
2. \( \text{for } i = 0 \ \text{to } m \)
3. \( \quad T[i, 0] = 0 \)
4. \( \text{for } j = 0 \ \text{to } n \)
5. \( \quad T[0, j] = 0 \)
6. \( \text{for } i = 1 \ \text{to } m \)
7. \( \quad \text{for } j = 1 \ \text{to } n \)
8. \( \quad \text{if } x_i = y_j \)
9. \( \quad \quad T[i, j] = T[i - 1, j - 1] + 1; \quad P[i, j] = \text{↖} \)
10. \( \quad \text{else if } T[i, j - 1] > T[i - 1, j] \)
11. \( \quad T[i, j] = T[i, j - 1]; \quad P[i, j] = \text{←} \)
12. \( \quad \text{else } T[i, j] = T[i - 1, j]; \quad P[i, j] = \text{↑} \)
13. \( \text{return } (T, P) \)

Time complexity: \( O(mn) \)
Chapter 15. Dynamic Programming

Algorithm LCS($x, y$)
1. $m = \text{length}(x)$, $n = \text{length}(y)$;
2. for $i = 0$ to $m$
3.     $T[i, 0] = 0$
4. for $j = 0$ to $n$
5.     $T[0, j] = 0$
6. for $i = 1$ to $m$
7.     for $j = 1$ to $n$
8.         if $x_i = y_j$
9.             $T[i, j] = T[i - 1, j - 1] + 1$; $P[i, j] = '\↖'$
10. else if $T[i, j - 1] > T[i - 1, j]$
11. return $(T, P)$

Time complexity: $O(mn)$
Algorithm LCS($x, y$)
1. $m = \text{length}(x)$, $n = \text{length}(y)$;
2. for $i = 0$ to $m$
   3. $T[i, 0] = 0$
4. for $j = 0$ to $n$
   5. $T[0, j] = 0$
6. for $i = 1$ to $m$
   7. for $j = 1$ to $n$
      8. if $x_i = y_j$
         9. $T[i, j] = T[i - 1, j - 1] + 1$; $P[i, j] = \uparrow$
      10. else if $T[i, j - 1] > T[i - 1, j]$
         11. $T[i, j] = T[i, j - 1]$; $P[i, j] = \leftarrow$
12. else
         13. $T[i, j] = T[i - 1, j]$; $P[i, j] = \uparrow$
13. return $(T, P)$

Time complexity: $O(mn)$
Algorithm LCS($x, y$)
1. \( m = \text{length}(x), \ n = \text{length}(y); \)
2. \( \text{for } i = 0 \text{ to } m \)
3. \( T[i, 0] = 0 \)
4. \( \text{for } j = 0 \text{ to } n \)
5. \( T[0, j] = 0 \)
6. \( \text{for } i = 1 \text{ to } m \)
7. \( \text{for } j = 1 \text{ to } n \)
8. \( \text{if } x_i = y_j \)
9. \( T[i, j] = T[i - 1, j - 1] + 1; \ P[i, j] = \text{'\downarrow'} \)
10. \( \text{else if } T[i, j - 1] > T[i - 1, j] \)
11. \( T[i, j] = T[i, j - 1]; \ P[i, j] = \text{'\leftarrow'} \)
12. \( \text{else } T[i, j] = T[i - 1, j]; \ P[i, j] = \text{'\uparrow'} \)

Time complexity: \( O(mn) \)
Algorithm LCS($x, y$)
1. $m = \text{length}(x), n = \text{length}(y)$;
2. for $i = 0$ to $m$
3.   $T[i, 0] = 0$
4. for $j = 0$ to $n$
5.   $T[0, j] = 0$
6. for $i = 1$ to $m$
7.   for $j = 1$ to $n$
8.     if $x_i = y_j$
9.       $T[i, j] = T[i - 1, j - 1] + 1; P[i, j] = \text{↖}$
10.    else if $T[i, j - 1] > T[i - 1, j]$
11.       $T[i, j] = T[i, j - 1]; P[i, j] = \text{←}$
12.    else $T[i, j] = T[i - 1, j]; P[i, j] = \text{↑}$
13. return $(T, P)$
Chapter 15. Dynamic Programming

Algorithm LCS($x, y$)
1. $m = \text{length}(x)$, $n = \text{length}(y)$;
2. for $i = 0$ to $m$
3. \hspace{1em} $T[i, 0] = 0$
4. for $j = 0$ to $n$
5. \hspace{1em} $T[0, j] = 0$
6. for $i = 1$ to $m$
7. \hspace{1em} for $j = 1$ to $n$
8. \hspace{2em} if $x_i = y_j$
9. \hspace{3em} $T[i, j] = T[i - 1, j - 1] + 1$; $P[i, j] = \leftarrow$
10. \hspace{2em} else if $T[i, j - 1] > T[i - 1, j]$
11. \hspace{3em} $T[i, j] = T[i, j - 1]$; $P[i, j] = \downarrow$
12. \hspace{2em} else $T[i, j] = T[i - 1, j]$; $P[i, j] = \uparrow$
13. return $(T, P)$

Time complexity: $O(mn)$
Chapter 15. Dynamic Programming

Traceback the longest common subsequence
either recursive or iterative algorithm
Chapter 15. Dynamic Programming

Traceback the longest common subsequence either recursive or iterative algorithm

Algorithm PrintLCS($P, x, i, j$)
Chapter 15. Dynamic Programming

Traceback the longest common subsequence

either recursive or iterative algorithm

Algorithm `PRINTLCS(P, x, i, j)`
1. `if (i > 0) ∧ (j > 0)`
Chapter 15. Dynamic Programming

Traceback the longest common subsequence

either recursive or iterative algorithm

Algorithm \textsc{PrintLCS}(P, x, i, j)
1. \hspace{1em} \textbf{if} \ (i > 0) \land (j > 0)
2. \hspace{1em} \textbf{if} \ P[i, j] = \text{"\\"}

Chapter 15. Dynamic Programming

Traceback the longest common subsequence
either recursive or iterative algorithm

Algorithm PRINTLCS($P, x, i, j$)
1. if $(i > 0) \land (j > 0)$
2. if $P[i, j] = \prime\prime\prime$
3. PRINTLCS($P, x, i - 1, j - 1$)
Chapter 15. Dynamic Programming

Traceback the longest common subsequence

either recursive or iterative algorithm

Algorithm PrintLCS($P, x, i, j$)
1. if $(i > 0) \land (j > 0)$
2. if $P[i, j] = ^\text{↖}$
3. PrintLCS($P, x, i - 1, j - 1$)
4. Print ($x_i$)
Chapter 15. Dynamic Programming

Traceback the longest common subsequence

either recursive or iterative algorithm

Algorithm \texttt{PrintLCS}(P, x, i, j)

1. if \((i > 0) \land (j > 0)\)
2. \hspace{1em} if \(P[i, j] = '\leftarrow'\)
3. \hspace{2em} \texttt{PrintLCS}(P, x, i - 1, j - 1)
4. \hspace{1em} Print \((x_i)\)
5. \hspace{1em} else if \(P[i, j] = '\leftarrow'\)
Chapter 15. Dynamic Programming

Traceback the longest common subsequence

either recursive or iterative algorithm

Algorithm PrintLCS(P, x, i, j)
1. \textbf{if} \ (i > 0) \land (j > 0)
2. \textbf{if} \ P[i, j] = '\n'
3. \quad PrintLCS(P, x, i - 1, j - 1)
4. \quad Print (x_i)
5. \textbf{else if} \ P[i, j] = '\leftarrow'
6. \quad PrintLCS(P, x, i, j - 1)
Traceback the longest common subsequence

either recursive or iterative algorithm

Algorithm \(\text{PRINTLCS}(P, x, i, j)\)
1. \(\text{if } (i > 0) \land (j > 0)\)
2. \(\text{if } P[i, j] = \backslash\) ← \(\text{PRINTLCS}(P, x, i - 1, j - 1)\)
3. \(\text{Print } (x_i)\)
4. \(\text{else if } P[i, j] = \leftarrow\) ← \(\text{PRINTLCS}(P, x, i, j - 1)\)
5. \(\text{else } \text{PRINTLCS}(P, x, i - 1, j)\)
Traceback the longest common subsequence

either recursive or iterative algorithm

Algorithm \textsc{PrintLCS}(P, x, i, j)

1. \textbf{if} \ (i > 0) \land (j > 0)
2. \hspace{1em} \textbf{if} \ P[i, j] = '↖'
3. \hspace{1em} \textsc{PrintLCS}(P, x, i - 1, j - 1)
4. \hspace{1em} \textbf{Print} (x_i)
5. \hspace{1em} \textbf{else if} \ P[i, j] = '←'
6. \hspace{1em} \textsc{PrintLCS}(P, x, i, j - 1)
7. \hspace{1em} \textbf{else} \ \textsc{PrintLCS}(P, x, i - 1, j)
8. \hspace{1em} \textbf{else return} ()
Chapter 15. Dynamic Programming

Pairwise Alignment
Chapter 15. Dynamic Programming

Pairwise Alignment

ACCGGTCGAGTGCG  
   CGTCGATGC ← common sequence  
GTCGTTCGGATGCCC
Chapter 15. Dynamic Programming

Pairwise Alignment

ACCGGTCGAGTGCG
  CGTCGATGC ← common sequence
GTCGTTCGGATGCC

biologically more meaningful view:

ACCGGTC  GAGTGC
  ||  ||  ||  ||
  CG  TC  GA  TGC ← common sequence
  ||  ||  ||  ||
GTCGTTCGGA  TGCCC
Chapter 15. Dynamic Programming

Pairwise Alignment

ACCGGTCGAGTGCG
   CGTCGATGC ← common sequence
GTCGTTCGGATGCCC

biologically more meaningful view:

ACCGGTC  GAGTGC
   || || || ||||
CG  TC  GA  TGC ← common sequence
   || || || ||||
GTCGTTCGGA  TGCCC

blue letters are conserved;
red letters are mutated;
purple letters are inserted or deleted;
Chapter 15. Dynamic Programming

Alignment between two sequences:
Chapter 15. Dynamic Programming

Alignment between two sequences:

```
ACCGGTC_GAGTGCG
  || || || |||
 CG TC GA TGC ← common sequence
  || || || |||
GTCGTTCGGA_TGCC
```
Chapter 15. Dynamic Programming

Alignment between two sequences:

```
ACCGGTC_GAGTGCG_  
|| || || ||||
CG TC GA TGC ← common sequence
|| || || ||||
GTCGTTCGGA_TGCC
```

- two sequences padded with ‘_’s called **gaps**;
Alignment between two sequences:

```
ACCGGTC_GAGTGCG_
    || || || |||
    CG TC GA TGC ← common sequence
    || || || |||
GTCGTTCGGA_TGCC
```

- two sequences padded with ‘_’s called **gaps**;
- two sequences are of the same length, aligned;
Alignment between two sequences:

\[
\begin{align*}
\text{ACCGGTC} & \quad \text{GAGTGC}G \\
\text{CG} & \quad \text{TC} & \quad \text{GA} & \quad \text{TGC} \\
\text{GTCGTTCGGA} & \quad \text{TGCCC}
\end{align*}
\]

- two sequences padded with \_'s called \text{gaps};
- two sequences are of the same length, aligned;
- sum of column scores is used to identify the best alignment
Alignment between two sequences:

```
ACCGGTC_GAGTGCG_
|| || || ||||
CG TC GA TGC ← common sequence
|| || || ||||
GTCGTTCGGA_TGCC
```

- two sequences padded with ‘_’s called gaps;
- two sequences are of the same length, aligned;
- sum of column scores is used to identify the best alignment

For example, a score scheme
+5 for conservation columns, (reward)
-2 for substitution columns, (penalty)
-6 for delete/insert columns, (penalty)
the above alignment has the total score

\[
(−2) + (−2) + 5 + 5 + (−2) + 5 + 5 + (−6) + 5 + 5 + (−6) + 5 + 5 + 5 + (−2) + (−6)
\]

LCS is a special case

\[
0 + 0 + 1 + 1 + 0 + 1 + 1 + 0 + 1 + 1 + 0 + 1 + 1 + 1 + 0 + 0
\]
Alignment between two sequences:

```
ACCGGTC_GAGTGCG_
  || || || |||
CG  TC  GA  TGC  ← common sequence
  || || || |||
GTCGTTCGGA_TGCC
```

- two sequences padded with ‘_’s called *gaps*;
- two sequences are of the same length, aligned;
- sum of column scores is used to identify the best alignment

For example, a score scheme

- +5 for conservation columns, (reward)
- -2 for substitution columns, (penalty)
- -6 for delete/insert columns, (penalty)
Chapter 15. Dynamic Programming

Alignment between two sequences:

```
ACCGGTC_GAGTGCG_
  || || || |||
CG TC GA TGC ← common sequence
  || || || |||
GTCGTTCGGA_TGCC
```

- two sequences padded with ‘_’s called gaps;
- two sequences are of the same length, aligned;
- sum of column scores is used to identify the best alignment

For example, a score scheme

- +5 for conservation columns, (reward)
- -2 for substitution columns, (penalty)
- -6 for delete/insert columns, (penalty)

the above alignment has the total score

\[-2 + (-2) + 5 + 5 + (-2) + 5 + 5 + (-6) + 5 + 5 + (-6) + 5 + 5 + 5 + (-2) + (-6)\]
Alignment between two sequences:

ACCGGTC\_GAGTGC\_G
  || || || ||
CG TC GA TGC ← common sequence
  || || || ||
GTCGTTCGGA\_TGCCC

- two sequences padded with ‘_’s called gaps;
- two sequences are of the same length, aligned;
- sum of column scores is used to identify the best alignment

For example, a score scheme
- +5 for conservation columns, (reward)
- -2 for substitution columns, (penalty)
- -6 for delete/insert columns, (penalty)

the above alignment has the total score
\((-2) + (-2) + 5 + 5 + (-2) + 5 + 5 + (-6) + 5 + 5 + (-6) + 5 + 5 + 5 + (-2) + (-6)\)

LCS is a special case
Chapter 15. Dynamic Programming

Alignment between two sequences:

ACCGGTC_GAGTGCG_
\[
\begin{array}{ccccccc}
& & & & & & \\
C & G & T & C & G & A & T & G & C & G & T & C & G & A & T & G & C & \leftarrow & \text{common sequence} & \\
& & & & & & \\
\end{array}
\]

GTCGTTCGGA_TGCC

- two sequences padded with ‘_’s called gaps;
- two sequences are of the same length, aligned;
- sum of column scores is used to identify the best alignment

For example, a score scheme
- +5 for conservation columns, (reward)
- -2 for substitution columns, (penalty)
- -6 for delete/insert columns, (penalty)

the above alignment has the total score
\[
( -2 ) + ( -2 ) + 5 + 5 + ( -2 ) + 5 + 5 + ( -6 ) + 5 + 5 + ( -2 ) + 5 + 5 + ( -2 ) + ( -6 )
\]

LCS is a special case
\[
0 + 0 + 1 + 1 + 0 + 1 + 1 + 0 + 1 + 1 + 0 + 1 + 1 + 0 + 0
\]
Chapter 15. Dynamic Programming

Significance of sequence alignments:

Sequence Homology Reveals Functions

Homology reveals evolution of structure/function

<table>
<thead>
<tr>
<th>Protein</th>
<th>Sequence</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>FOS_RAT</td>
<td>MMFSGFNADYEASSSRCSASSPAGDSLSSYH</td>
<td>60</td>
</tr>
<tr>
<td>FOS_MOUSE</td>
<td>MMFSGFNADYEASSSRCSASSPAGDSLSSYH</td>
<td>60</td>
</tr>
<tr>
<td>FOS_CHICK</td>
<td>MMYQGFAGEYVEAPRSSRCSASSPAGDSLSSYH</td>
<td>60</td>
</tr>
<tr>
<td>FOSB_MOUSE</td>
<td>MFQAFPGYDSGSRCSSPSAESQ---YSVDSFSPPTAASSEQ-CAGLGEEMPFSF</td>
<td>54</td>
</tr>
<tr>
<td>FOSB_HUMAN</td>
<td>MFQAFPGYDSGSRCSSPSAESQ---YSVDSFSPPTAASSEQ-CAGLGEEMPFSF</td>
<td>54</td>
</tr>
<tr>
<td>Consensus</td>
<td><em>......</em>......<em>......</em>......<em>......</em>......<em>......</em>......<em>......</em>......<em>......</em>......<em>......</em>......<em>......</em></td>
<td></td>
</tr>
</tbody>
</table>

Homology reveals regulatory structure (E. Coli promoters)

<table>
<thead>
<tr>
<th>Promoter</th>
<th>Sequence</th>
</tr>
</thead>
<tbody>
<tr>
<td>T3</td>
<td>TCTCAACGTAACTTTTACATGATGTACGCCGCTTCCCGATAAGGG</td>
</tr>
<tr>
<td>T7</td>
<td>GTACAAAAATACATTTGATCTTACTACACCCGTGACGCAAGG</td>
</tr>
<tr>
<td>E1</td>
<td>ATGCAATTATTTTTTTGCCGTCGTCGAGGCTGACGCAAGG</td>
</tr>
<tr>
<td>E2</td>
<td>AAACGAAAATATGATCTTACTACACCCGTGACGCAAGG</td>
</tr>
<tr>
<td>PR</td>
<td>GTCGACAACATTTTTGCCGTCGTCGAGGCTGACGCAAGG</td>
</tr>
<tr>
<td>P1</td>
<td>TTAACGCGTGGATTTTTTGATCTTACTACACCCGTGACGCAAGG</td>
</tr>
<tr>
<td>T7_A3</td>
<td>TTACGCGTGGATTTTTTGATCTTACTACACCCGTGACGCAAGG</td>
</tr>
<tr>
<td>T7_A1</td>
<td>TTACGCGTGGATTTTTTGATCTTACTACACCCGTGACGCAAGG</td>
</tr>
<tr>
<td>T7_A2</td>
<td>TTACGCGTGGATTTTTTGATCTTACTACACCCGTGACGCAAGG</td>
</tr>
<tr>
<td>fdVIII</td>
<td>TTACGCGTGGATTTTTTGATCTTACTACACCCGTGACGCAAGG</td>
</tr>
</tbody>
</table>

-35
-10
+1
Pairwise alignment problem:

**INPUT**: sequences $x = x_1x_2 \ldots x_m$, $y = y_1y_2 \ldots y_n$, and scoring scheme $\text{score}$

**OUTPUT**: $x'$ and $y'$, which are $x$ and $y$ padded with '_'s, respectively such that total score $\sum_{i=1}^{l} \text{score}(x'_i, y'_i)$ is the maximum,

where $l = |x'| = |y'|$.

How to solve it?
Chapter 15. Dynamic Programming

Pairwise alignment problem:

**Input:** sequences $x = x_1x_2 \ldots x_m$, $y = y_1y_2 \ldots y_n$, and scoring scheme $score$

**Output:** $x'$ and $y'$, which are $x$ and $y$ padded with '-'s, respectively such that total score $\sum_{i=1}^{l} score(x'_i, y'_i)$ is the maximum, where $l = |x'| = |y'|$.

How to solve it?

Similar to LCS problem, we consider prefixes $x_1 \ldots x_i$ of $x$, $y_1 \ldots y_j$ of $y$. 
Chapter 15. Dynamic Programming

Pairwise alignment problem:

**INPUT:** sequences $x = x_1 x_2 \ldots x_m$, $y = y_1 y_2 \ldots y_n$, and scoring scheme $score$

**OUTPUT:** $x'$ and $y'$, which are $x$ and $y$ padded with '-'s, respectively such that total score $\sum_{i=1}^{l} score(x'_i, y'_i)$ is the maximum, where $l = |x'| = |y'|$.

How to solve it?

Similar to LCS problem, we consider prefixes $x_1 \ldots x_i$ of $x$, $y_1 \ldots y_j$ of $y$.

(1) \[ x_1 \ldots x_{i-1} x_i \\
    y_1 \ldots y_{j-1} y_j \]

(2) \[ x_1 \ldots x_{i-1} x_i \\
    y_1 \ldots y_{j-1} y_j \]

(3) \[ x_1 \ldots x_{i-1} x_i \\
    y_1 \ldots y_{j-1} y_j \]
Chapter 15. Dynamic Programming

Pairwise alignment problem:

**INPUT**: sequences $x = x_1 x_2 \ldots x_m$, $y = y_1 y_2 \ldots y_n$, and scoring scheme $\text{score}$

**OUTPUT**: $x'$ and $y'$, which are $x$ and $y$ padded with '-'s, respectively such that total score $\sum_{i=1}^{l} \text{score}(x'_i, y'_i)$ is the maximum, where $l = |x'| = |y'|$.

How to solve it?

Similar to LCS problem, we consider prefixes $x_1 \ldots x_i$ of $x$, $y_1 \ldots y_j$ of $y$.

1. $x_1 \ldots x_{i-1} x_i$
2. $x_1 \ldots x_{i-1} x_i$
3. $x_1 \ldots x_{i-1} x_i$

(1) (2) (3)

case (1) contributes to a match/mismatch score
Chapter 15. Dynamic Programming

Pairwise alignment problem:

**Input**: sequences \( x = x_1x_2 \ldots x_m \), \( y = y_1y_2 \ldots y_n \), and scoring scheme \( \text{score} \)

**Output**: \( x' \) and \( y' \), which are \( x \) and \( y \) padded with ' 's, respectively such that total score \( \sum_{i=1}^{l} \text{score}(x'_i, y'_i) \) is the maximum, where \( l = |x'| = |y'|. \)

How to solve it?

Similar to LCS problem, we consider prefixes \( x_1 \ldots x_i \) of \( x \), \( y_1 \ldots y_j \) of \( y \).

\begin{align*}
\text{(1)} & \quad x_1 \ldots x_{i-1} x_i \quad y_1 \ldots y_{j-1} y_j \\
\text{(2)} & \quad x_1 \ldots x_{i-1} x_i \quad y_1 \ldots y_{j-1} y_j \\
\text{(3)} & \quad x_1 \ldots x_{i-1} x_i \quad y_1 \ldots y_{j-1} y_j \\
\end{align*}

case (1) contributes to a match/mismatch score
cases (2) and (3) contribute to an insertion/deletion score.
Scoring function for pairwise alignment
Chapter 15. Dynamic Programming

Scoring function for pairwise alignment

Example (1)
- +5 for conservation columns, (reward)
- -2 for substitution columns, (penalty)
- -6 for delete/insert columns, (penalty)
Scoring function for pairwise alignment

Example (1)
- +5 for conservation columns, (reward)
- -2 for substitution columns, (penalty)
- -6 for delete/insert columns, (penalty)

Example (2)

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>C</th>
<th>G</th>
<th>T</th>
<th>-</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>5</td>
<td>-1</td>
<td>-2</td>
<td>-1</td>
<td>-4</td>
</tr>
<tr>
<td>C</td>
<td>-1</td>
<td>5</td>
<td>-3</td>
<td>-2</td>
<td>-4</td>
</tr>
<tr>
<td>G</td>
<td>-2</td>
<td>-3</td>
<td>5</td>
<td>-2</td>
<td>-4</td>
</tr>
<tr>
<td>T</td>
<td>-1</td>
<td>-2</td>
<td>-2</td>
<td>5</td>
<td>-4</td>
</tr>
<tr>
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<td>-4</td>
<td>-4</td>
<td>-4</td>
<td>-4</td>
<td>*</td>
</tr>
</tbody>
</table>
Chapter 15. Dynamic Programming

Scoring function for pairwise alignment

Example (1)
- +5 for conservation columns, (reward)
- -2 for substitution columns, (penalty)
- -6 for delete/insert columns, (penalty)

Example (2)

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>C</th>
<th>G</th>
<th>T</th>
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</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>5</td>
<td>-1</td>
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<td>-1</td>
<td>-4</td>
</tr>
<tr>
<td>C</td>
<td>-1</td>
<td>5</td>
<td>-3</td>
<td>-2</td>
<td>-4</td>
</tr>
<tr>
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<td>-4</td>
</tr>
<tr>
<td>T</td>
<td>-1</td>
<td>-2</td>
<td>-2</td>
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<td>-4</td>
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Why sum of column scores?
Chapter 15. Dynamic Programming

Scoring function for pairwise alignment

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Why sum of column scores?

product of independent column probabilities
Chapter 15. Dynamic Programming

As for the LCS problem, we can define

\[ A(i, j) \] is the maximum score alignment between \( x_1 \ldots x_i \) and \( y_1 \ldots y_j \), then

\[
A(i, j) = \begin{cases} 
  \text{concat}(A(i - 1, j - 1), x_i) & \text{if this has the highest score;}
\end{cases}
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Instead, we define objective function \( S(i, j) \) to be the maximum alignment score between \( x_1 \ldots x_i \) and \( y_1 \ldots y_j \):

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S(i, j) = \max \begin{cases} 
  S(i - 1, j - 1) + \text{score}(x_i, y_j) & i \geq 1, j \geq 1;
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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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\]

scoring function \text{score}

\[
\begin{array}{cccccc}
A & C & G & T & - \\
A & 5 & -1 & -2 & -1 & -4 \\
C & -1 & 5 & -3 & -2 & -4 \\
G & -2 & -3 & 5 & -2 & -4 \\
T & -1 & -2 & -2 & 5 & -4 \\
- & -4 & -4 & -4 & -4 & * \\
\end{array}
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Chapter 15. Dynamic Programming

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\[
\begin{array}{|c|c|c|c|c|}
\hline
& A & C & G & T & - \\
\hline
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- & -4 & -4 & -4 & -4 & * \\
\hline
\end{array}
\]

exercise: Table filling and traceback
Chapter 15. Dynamic Programming

Affine gap penalty: a challenge for DP

Consider: new gap penalty score:

\[ \text{penalty} = -\text{op} - \text{ext} (l-1) \]

for a gap covering \( l \) positions, where \( \text{op} > \text{ext} \).

how to modify recurrences:

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Chapter 15. Dynamic Programming

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- $op$: penalty to open a gap;
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Affine gap penalty: a challenge for DP

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

Issue: we do not know how long a gap is.

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S(i, j) = \max \begin{cases} 
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\]

**Solution 1:** we can try all different length \(l\)

\[
S(i, j) = \max \begin{cases} 
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\max_{1 \leq l \leq i} S(i - l, j) - \text{op} \times \text{ext} 
\end{cases}
\]

But the complexity increased to: \(O(nm \times \max\{n, m\})\)

Why does the complexity increase? a lot of recomputations, but where?
Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

So we want to do recursion "one step at a time":

[Further text not visible in the image]
Chapter 15. Dynamic Programming

So we want to do recursion "one step at a time":

**but at the same time, differentiate different "states"**
where the two tails are

**Substitute**

![Dynamic Programming Diagram](image)
Chapter 15. Dynamic Programming

So we want to do recursion "one step at a time":

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**Insertion**
Chapter 15. Dynamic Programming

So we want to do recursion "one step at a time":

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**Substitute**

**Insertion**

**Deletion**
Chapter 15. Dynamic Programming

Define $S(i, j)$ to be the maximum score of alignment between $x_1 \ldots x_i$ and $y_1 \ldots y_j$ with $x_i$ being aligned to $y_j$. 

\[
S(i, j) = \max \begin{cases} 
S(i-1, j-1) + \text{score}(x_i, y_j) \\
I(i-1, j-1) - \text{ext} \\
D(i-1, j-1) - \text{op} 
\end{cases}
\]

Define $I(i, j)$ to be the maximum score of alignment between $x_1 \ldots x_i$ and $y_1 \ldots y_j$ with $y_j$ being insertion.
Define $S(i, j)$ to be the maximum score of alignment between $x_1 \ldots x_i$ and $y_1 \ldots y_j$ with $x_i$ being aligned to $y_j$.

\[
S(i, j) = \max \left\{ S(i-1, j-1) + \text{score}(x_i, y_j), I(i-1, j-1) + \text{score}(x_i, y_j), D(i-1, j-1) + \text{score}(x_i, y_j) \right\}
\]
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Chapter 15. Dynamic Programming

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    D(i - 1, j - 1) + \text{score}(x_i, y_j) 
\right\}$$

Define $I(i, j)$ to be the maximum score of alignment between $x_1 \ldots x_i$ and $y_1 \ldots y_j$ with $y_j$ being insertion.

$$I(i, j) = \max \left\{ 
    I(i, j - 1) - \text{ext}, 
    S(i, j - 1) - \text{op} 
\right\}$$
Chapter 15. Dynamic Programming

Define $D(i,j)$ to be the maximum score of alignment between $x_1 \ldots x_i$ and $y_1 \ldots y_j$ with $x_i$ being deletion. $D(i,j) = \max\{D(i-1,j) - \text{ext} S(i-1,j), D(i,j-1) - \text{op} \}$

- Note that there is no deletion right after insertion, neither insertion right after deletion. Why?
- What is the desired value? $\max\{S(m,n), I(m,n), D(m,n)\}$
- How to fill the table(s) and to traceback the optimal alignment?
Define $D(i, j)$ to be the maximum score of alignment between $x_1 \ldots x_i$ and $y_1 \ldots y_j$ with $x_i$ being deletion.
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D(i-1, j) - ext \\
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```
A C A _ G T A
A C C T _ _ A  →  A C A G T A
A C C T _ _ A
```
Chapter 15. Dynamic Programming

Define $D(i, j)$ to be the maximum score of alignment between $x_1 \ldots x_i$ and $y_1 \ldots y_j$ with $x_i$ being deletion.

$$D(i, j) = \max \begin{cases} 
D(i - 1, j) - ext \\
S(i - 1, j) - op 
\end{cases}$$

- Note that there is no deletion right after insertion, neither insertion right after deletion. *why?*

- What is the desired value?
Chapter 15. Dynamic Programming

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- Note that there is no deletion right after insertion, neither insertion right after deletion. why?

![Diagram](image)

- What is the desired value? $\max \left\{ S(m, n), I(m, n), D(m, n) \right\}$
Chapter 15. Dynamic Programming

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$$\begin{array}{c}
\text{ACACT__A} \\
\text{A C A G T A}
\end{array} \quad \rightarrow \quad \begin{array}{c}
\text{ACACT__A} \\
\text{A C A G T A}
\end{array}$$

- What is the desired value? $\max \left\{ S(m, n), I(m, n), D(m, n) \right\}$

- How to fill the table(s) and to traceback the optimal alignment?
Chapter 15. Dynamic Programming

DP applied to summation problems

Unmatched Socks Problem

There are $n$ pairs of socks, each a different color; each day, remove arbitrarily two individual socks; compute the probability that, on none of the $k$ days, a pair of color-matched socks is removed.

When $n = 5$, $k = 1$

$$P_1 = \frac{\text{number of color-not-matched pairs}}{\text{number of total pairs}} = \frac{10 \times 8}{\binom{10}{2}} = \frac{80}{45} = \frac{8}{9}$$

When $n = 5$, $k = 2$

$$P_2 = P_1 \times \left( P_u + P_m + P_{u,m} \right) = \frac{8}{9} \times \left( \frac{1}{\binom{8}{2}} + \frac{6 \times 4}{\binom{8}{2}} + \frac{2 \times 6}{\binom{8}{2}} \right) = \frac{8}{9} \times \frac{25}{28}$$

$$P_3 = P_2 \times \ldots$$
DP applied to summation problems

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When $n = 5$, $k = 1$

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$$P_3 = P_2 \times ??$$
DP applied to summation problems

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\]

When \( n = 5 \), \( k = 2 \)

\[
P_2 = P_1 \times \left( \frac{P_u + P_m + P_{u,m}}{2} \right) = \frac{8}{9} \times \left( \frac{1}{\binom{8}{2}} + \frac{6 \times 4}{2 \binom{8}{2}} + \frac{2 \times 6}{\binom{8}{2}} \right) = \frac{8}{9} \times \frac{25}{28}
\]

\[
P_3 = P_2 \times \text{??}
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Chapter 15. Dynamic Programming

DP applied to summation problems

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Chapter 15. Dynamic Programming

**DP applied to summation problems**

**Unmatched Socks Problem**

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When \( n = 5, \ k = 1 \)

\[
P_1 = \]

\[
\text{number of color-not-matched pairs} \over \text{number of total pairs} = {10 \times 8 \over 2} \over {10 \choose 2} = {40 \over 45} = {8 \over 9}
\]

When \( n = 5, \ k = 2 \)

\[
P_2 = P_1 \times (P_u + P_m + P_{u,m}) = {8 \over 9} \times \left( 1 \over {8 \choose 2} + (6 \times 4) \over 2 \times {8 \choose 2} + 2 \times 6 \times {8 \choose 2} \right) = {8 \over 9} \times {25 \over 28}
\]

\[
P_3 = P_2 
\]
DP applied to summation problems

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P_1 = \frac{\text{number of color-not-matched pairs}}{\text{number of total pairs}}
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Chapter 15. Dynamic Programming

DP applied to summation problems

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When $n = 5$, $k = 1$

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Chapter 15. Dynamic Programming

DP applied to summation problems

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$$P_1 = \frac{\text{number of color-not-matched pairs}}{\text{number of total pairs}} = \frac{(10 \times 8)/2}{\binom{10}{2}} = \frac{80}{45} = \frac{16}{9}$$
Chapter 15. Dynamic Programming

DP applied to summation problems

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Chapter 15. Dynamic Programming

**DP applied to summation problems**

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

DP applied to summation problems

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P_2 = P_1
\]
Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

**DP applied to summation problems**

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DP applied to summation problems

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Chapter 15. Dynamic Programming

Unfortunately, the probability of individual days cannot be independently calculated,
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Chapter 15. Dynamic Programming

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Let $m_k$ be the number of pairs of color-matched socks after selection on the $k$th day.
Chapter 15. Dynamic Programming

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Let $m_k$ be the number of pairs of color-matched socks after selection on the $k$th day.

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Note that

$$2m_k + u_k = 2n - 2k, \text{ for every } k \leq n$$
Chapter 15. Dynamic Programming

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We define $P_k(m_k, u_k)$ to be the total probability that not a pair of matched socks is selected on any day of the first $k$ days, resulting in $m_k$ and $u_k$. Then
Chapter 15. Dynamic Programming

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$$P_k(m_k, u_k) = \sum \begin{cases} 
\frac{P_{k-1}(m_k + 2, u_k - 2) \times (m_k + 2)(2m_k + 2)}{(2m_k + 2 + u_k)^2} \\
\frac{P_{k-1}(m_k + 1, u_k) \times 2(m_k + 1)u_k}{(2m_k + 2 + u_k)^2} \\
\frac{P_{k-1}(m_k, u_k + 2) \times (u_k + 2)}{(2m_k + 2 + u_k)^2}
\end{cases}$$
Chapter 15. Dynamic Programming

We have seen a number of examples with ”forward DP” solutions.
Chapter 15. Dynamic Programming

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(Most of these problems are solvable with "backward DP" as well).

Matrix Chain Multiplication problem

Input: matrices $A_1, \ldots, A_n$, where $A_i$ has dimension $p_{i-1} \times p_i$;

Output: a parenthesization by which the product $A_1 \times A_2 \times \cdots \times A_n$ uses the minimum number of scalar multiplications.
We have seen a number of examples with "forward DP" solutions. (Most of these problems are solvable with "backward DP" as well). We now consider an example with an "inside-out DP" solution.
Chapter 15. Dynamic Programming

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Matrix Chain Multiplication problem
Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

5x5 \times 5x4 \times 4x8 \times 8x2 = 5x2
Chapter 15. Dynamic Programming

scalable multiplications

= 5 × 4 × 8
Chapter 15. Dynamic Programming

Many possible parenthesizations

\[
P(n) = \sum_{k=1}^{n-1} P(k) P(n-k),
\]
Catalan number
Chapter 15. Dynamic Programming

Many possible parenthesizations

1. \((A (B (C D)))\)
   \[5(5)^2 + 5(4)^2 + 4(8)^2 = 154\]
2. \((A ((B C) D))\)
   \[5(5)^2 + 5(4)8 + 5(8)^2 = 290\]
3. \(((A B) (C D))\)
   \[5(5)^2 + 4(8)^2 + 5(4)^2 = 204\]
4. \(((A (B C)) D)\)
   \[5(4)8 + 5(5)8 + 5(8)^2 = 440\]
5. \(((A B) C) D\)
   \[5(5)^2 + 5(4)^2 + 5(8)^2 = 340\]
Chapter 15. Dynamic Programming

Many possible parenthesizations

Note: The number of all possible parenthesizations is

\[ P(n) = \sum_{k=1}^{n-1} P(k)P(n-k), \text{ Catalan number} \]
Chapter 15. Dynamic Programming

**Step 1**: identify *optimal substructure*:
Optimal solution in terms of optimal solutions to subproblems:

1. The best parenthesization of $A_i A_{i+1} \cdots A_j$ must be $(A_i \cdots A_k)(A_k+1 \cdots A_j)$ for some $k, i \leq k < j$.

2. The optimal cost of $A_i A_{i+1} \cdots A_j$ must be the smallest among optimal costs for $(A_i \cdots A_k)(A_k+1 \cdots A_j)$ $k = i, i+1, \ldots, j-1$.

3. For each $k$, the optimal cost for the above is the optimal cost for $(A_i \cdots A_k) +$ the optimal cost for $(A_k+1 \cdots A_j) +$ the number of scalar multiplications to multiply the two terms.
Chapter 15. Dynamic Programming

**Step 1:** identify *optimal substructure*:

Optimal solution in terms of optimal solutions to subproblems:

1. The best parenthesization of $A_i A_{i+1} \cdots A_j$ must be

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Step 1: identify optimal substructure:

Optimal solution in terms of optimal solutions to subproblems:

(1) the best parentheses of $A_i A_{i+1} \cdots A_j$ must be

$$(A_i \cdots A_k)(A_{k+1} \cdots A_j)$$

for some $k, i \leq k < j$.

(2) The optimal cost of $A_i A_{i+1} \cdots A_j$ must be the smallest among optimal costs for

$$(A_i \cdots A_k)(A_{k+1} \cdots A_j)$$

$k = i, i + 1, \cdots, j - 1$. 


Step 1: identify *optimal substructure*:
Optimal solution in terms of optimal solutions to subproblems:

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(2) The optimal cost of $A_i A_{i+1} \cdots A_j$ must be the smallest among optimal costs for

$$k = i, i + 1, \cdots, j - 1.$$
Chapter 15. Dynamic Programming

Step 2: objective function and recurrence

Define $f(i,j)$ to be the minimum number of scalar multiplications needed for $A_i A_{i+1} \cdots A_j$.

Then $f(i,j) = \min_{i \leq k < j} \{ f(i,k) + f(k+1,j) + p_{i-1} p_k p_{j-1} \}$ where base case is $f(i,i) = 0$ when $i = j$, for single matrix $A_i$. 
Step 2: objective function and recurrence

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Chapter 15. Dynamic Programming

Step 2: objective function and recurrence

Define $f(i, j)$ to be the minimum number of scalar multiplications needed for $A_i A_{i+1} \cdots A_j$. Then

$$f(i, j) = \min_{i \leq k < j} \{ f(i, k) + f(k+1, j) + p_i p_{k+1} p_j \}$$

where base case is

$$f(i, j) = 0 \text{ when } i = j, \text{ for single matrix } A_i.$$
Step 3. Fill out the DP table for function $f$. 
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- Table size $n \times n$;
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- Table size $n \times n$;
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- diagonal cells have the same $j-1$ values;
Chapter 15. Dynamic Programming

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<th>3</th>
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<td>0</td>
<td></td>
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<td></td>
<td></td>
</tr>
</tbody>
</table>

Side lengths: 1, 4, 5, 3, 6, 7, 1

\[
A_1^{(1\times4)} \times A_2^{(4\times5)} \times A_3^{(5\times3)} \times A_4^{(3\times6)} \times A_5^{(6\times7)} \times A_6^{(7\times1)}
\]
Algorithm `MatrixChainOrder(p)`
1. \( n = \text{length}[p] - 1; \)
Algorithm \textsc{MatrixChainOrder}(p)
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Algorithm \textsc{MatrixChainOrder}(p)

1. \hspace{1em} n = \text{length}[p] - 1;
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3. \hspace{1em} \hspace{1em} M[i, i] = 0;
Chapter 15. Dynamic Programming

Algorithm MatrixChainOrder$(p)$
1. $n = \text{length}[p] - 1$;
2. for $i = 1$ to $n$
3. \hspace{1em} $M[i, i] = 0$;
4. for $l = 2$ to $n$
5. \hspace{1em} for $i = 1$ to $n - l + 1$
6. \hspace{2em} $j = i + l - 1$;
7. \hspace{1em} $M[i, j] = \infty$;
8. \hspace{1em} for $k = i$ to $j - 1$
9. \hspace{2em} $q = M[i, k] + M[k + 1, j] + p[i - 1]p[k]p[j]$
10. \hspace{1em} if $q < M[i, j]$
11. \hspace{2em} $M[i, j] = q$
12. \hspace{1em} $S[i, j] = k$
13. return $(M, S)$

Time complexity: $O(n^2 \times n)$
Algorithm `MatrixChainOrder(p)`
1. \( n = \text{length}[p] - 1; \)
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Algorithm MatrixChainOrder\( (p) \)
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3. \( M[i, i] = 0; \)
4. \( \text{for } l = 2 \text{ to } n \)
5. \( \text{for } i = 1 \text{ to } n - l + 1 \)
6. \( j = i + l - 1; \)
Algorithm **MatrixChainOrder**\( (p) \)

1. \( n = length[p] - 1; \)
2. \( \textbf{for} \ i = 1 \ \textbf{to} \ n \)
3. \( M[i, i] = 0; \)
4. \( \textbf{for} \ l = 2 \ \textbf{to} \ n \)
5. \( \textbf{for} \ i = 1 \ \textbf{to} \ n - l + 1 \)
6. \( j = i + l - 1; \)
7. \( M[i, j] = \infty; \)

Time complexity: \( O(n^2 \times n) \)
Algorithm `MATRIXCHAINORDER(p)`
1. \( n = \text{length}[p] - 1; \)
2. \( \text{for } i = 1 \text{ to } n \)
3. \( M[i, i] = 0; \)
4. \( \text{for } l = 2 \text{ to } n \)
5. \( \text{for } i = 1 \text{ to } n - l + 1 \)
6. \( j = i + l - 1; \)
7. \( M[i, j] = \infty; \)
8. \( \text{for } k = i \text{ to } j - 1 \)

Time complexity: \( O(n^2 \times n) \)
Algorithm **MatrixChainOrder**($p$)

1. $n = length[p] - 1$;
2. for $i = 1$ to $n$
3. $M[i, i] = 0$;
4. for $l = 2$ to $n$
5. for $i = 1$ to $n - l + 1$
6. $j = i + l - 1$;
7. $M[i, j] = \infty$;
8. for $k = i$ to $j - 1$
Chapter 15. Dynamic Programming

Algorithm \textsc{MatrixChainOrder}(p)

1. \( n = \text{length}[p] - 1; \)
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6. \( j = i + l - 1; \)
7. \( M[i, j] = \infty; \)
8. \textbf{for} \( k = i \) \textbf{to} \( j - 1 \)
10. \textbf{if} \( q < M[i, j] \)
Algorithm $\textsc{MatrixChainOrder}(p)$

1. $n = \text{length}[p] - 1$;
2. $\textbf{for } i = 1 \textbf{ to } n$
3. $M[i, i] = 0$;
4. $\textbf{for } l = 2 \textbf{ to } n$
5. $\textbf{for } i = 1 \textbf{ to } n - l + 1$
6. $j = i + l - 1$;
7. $M[i, j] = \infty$;
8. $\textbf{for } k = i \textbf{ to } j - 1$
10. $\textbf{if } q < M[i, j]$ 
11. $M[i, j] = q$
Algorithm \textsc{MatrixChainOrder}(p)

1. \hspace{1em} \textit{n} = \text{length}[p] - 1;
2. \hspace{1em} \textbf{for} \ i = 1 \ \textbf{to} \ n \ \textbf{do}
3. \hspace{2.5em} M[i, i] = 0;
4. \hspace{1em} \textbf{for} \ l = 2 \ \textbf{to} \ n \ \textbf{do}
5. \hspace{2.5em} \textbf{for} \ i = 1 \ \textbf{to} \ n - l + 1 \ \textbf{do}
6. \hspace{7em} j = i + l - 1;
7. \hspace{2.5em} M[i, j] = \infty;
8. \hspace{2.5em} \textbf{for} \ k = i \ \textbf{to} \ j - 1 \ \textbf{do}
9. \hspace{7em} q = M[i, k] + M[k + 1, j] + p[i - 1]p[k]p[j]
10. \hspace{2.5em} \textbf{if} \ q < M[i, j] \ \textbf{then}
11. \hspace{7em} M[i, j] = q
12. \hspace{2.5em} S[i, j] = k
13. \hspace{1em} \textbf{return} (M, S)

Time complexity: \(O(n^2 \times n)\)
Algorithm `MatrixChainOrder(p)`

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Time complexity: \( O(n^2 \times n) \)
Chapter 15. Dynamic Programming

**Step 4.** Obtain the optimization parenthesization
Chapter 15. Dynamic Programming

**Step 4.** obtain the optimization parenthesization

Algorithm `MatrixChainParenthesization(i, j, S)`

1. if `i < j`
2. \( k = s[i, j] \)
3. print \((i, j, " : ", k)\)
4. `MatrixChainParenthesization(i, k, S)`
5. `MatrixChainParenthesization(k + 1, j, S)`
6. return
Step 4. obtain the optimization parenthesization

Algorithm \texttt{MatrixChainParenthesization}(i, j, S)

1. \textbf{if} $i < j$
2. \hspace{1em} $k = s[i, j]$
3. \hspace{1em} \textbf{print} (i, j, ” : ”, k)
4. \hspace{1em} \texttt{MatrixChainParenthesization}(i, k, S)
5. \hspace{1em} \texttt{MatrixChainParenthesization}(k + 1, j, S)
6. \textbf{return}

Usage: \textbf{call} \texttt{MatrixChainParenthesization}(1, n, S)
Step 4. obtain the optimization parenthesization

Algorithm MatrixChainParenthesization\((i, j, S)\)

1. \textbf{if} \(i < j\)
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5.  \textbf{MatrixChainParenthesization}(k + 1, j, S)
6. \textbf{return}

Usage: \textbf{call} MatrixChainParenthesization\((1, n, S)\)

Time complexity for MatrixChainParenthesization:
Chapter 15. Dynamic Programming

**Step 4.** obtain the optimization parenthesization

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Usage: **call** `MatrixChainParenthesization(1, n, S)`

Time complexity for `MatrixChainParenthesization`: $O(n)$. 
Chapter 16. Greedy Algorithms

Chapter 16 Greedy Algorithms
Chapter 16. Greedy Algorithms

- Dynamic programming is to consider all possible choices and select the best.
Chapter 16. Greedy Algorithms

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  a DP approach always leads to the optimal solution
Chapter 16. Greedy Algorithms

Chapter 16 Greedy Algorithms

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Chapter 16. Greedy Algorithms

- Dynamic programming is to consider all possible choices and select the best.  
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- A greedy algorithm may ignore some choices and select the one that is locally the best.  
  a greedy strategy may or may NOT lead to the optimal solution
Chapter 16. Greedy Algorithms

Activity-selection problem

Input:
\[ n \text{ activities } S = \{ a_1, \ldots, a_n \}, \text{ each with a start time } s_i \text{ and finish time } f_i, \quad 1 \leq i \leq n \]

Output: subset \( A \subseteq S \) of mutually compatible activities such that \( |A| \) is the maximum.

\[ A = \{ a_1, a_3, a_6, a_8 \}, \text{ or } A = \{ a_2, a_5, a_7, a_8 \}, \text{ or } A = \{ a_2, a_5, a_7, a_9 \}, \ldots. \]
Chapter 16. Greedy Algorithms

Activity-selection problem

**Input:** $n$ activities $S = \{a_1, \ldots, a_n\}$, each with a start time $s_i$ and finish time $f_i$, $1 \leq i \leq n$

**Output:** subset $A \subseteq S$ of mutually compatible activities such that $|A|$ is the maximum.
Chapter 16. Greedy Algorithms

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Chapter 16. Greedy Algorithms

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Chapter 16. Greedy Algorithms

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Chapter 16. Greedy Algorithms

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\[ \ldots \]
Chapter 16. Greedy Algorithms

A dynamic programming solution
Chapter 16. Greedy Algorithms

A dynamic programming solution

**Step 1** analysis of problem
Chapter 16. Greedy Algorithms

A dynamic programming solution

**Step 1** analysis of problem
Chapter 16. Greedy Algorithms

A dynamic programming solution

Step 1 analysis of problem

How to solve it recursively?
Chapter 16. Greedy Algorithms

For example, selecting \( a_3 \) would result in two subsets of activities to further consider: \( \{a_1\} \) and \( \{a_6, a_7, a_8, a_9\} \); selecting \( a_5 \) would result in two subsets of activities to further consider: \( \{a_1, a_2\} \) and \( \{a_7, a_8\} \).
Chapter 16. Greedy Algorithms

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selecting $a_5$ would result in two subsets of activities to further consider: $\{a_1, a_2\}$ and $\{a_7, a_8, a_9\}$;
Chapter 16. Greedy Algorithms

selecting \(a_5\) would result in two subsets of activities to further consider: \(\{a_1, a_2\}\) and \(\{a_7, a_8, a_9\}\); but if selecting again \(a_8\) would result in two smaller subsets: \(\{a_7\}\) and \(\{\}\); these subproblems are not “prefixes” or “suffixes”, they could be “middle segments”; we approach such problems with inside-out instead of forward or backward methods.
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Chapter 16. Greedy Algorithms

So we consider solving the activity selection problem on subsets of tasks:

Define subset $S_{i,j} = \{ a_k \in S : f_i \leq s_k < f_k \leq s_j \}$

*Example:* $S_{2,9} = \{ a_5, a_6, a_7 \}$, $S_{3,8} = \{ a_6, a_7 \}$.

Introducing dummy activities: $a_0$ with $s_0 = f_0 = \text{the earliest start time of all}$, and $a_{n+1}$ with $s_{n+1} = f_{n+1} = \text{the latest finish time of all}$. 
Chapter 16. Greedy Algorithms

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Chapter 16. Greedy Algorithms

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Chapter 16. Greedy Algorithms

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Chapter 16. Greedy Algorithms

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Then

\[
|A_{i,j}| = \max_{a_k \in S_{i,j}} \{|A_{i,k} \cup \{a_k\} \cup A_{k,j}|\}
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Chapter 16. Greedy Algorithms

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Then

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\]

\[
|A_{0,10}| = \max \left\{ \ldots, |A_{0,6} \cup \{a_6\} \cup A_{6,10}|, \quad \text{where} \quad S_{0,6} = \{a_1, \ldots, a_4\}, \quad S_{6,10} = \{a_8, a_9\} \right\}
\]
Chapter 16. Greedy Algorithms

Instead of solving this with DP, we wonder if all but one of the subproblems could be made empty.

\[ A_{i,j} = \max_{a_k \in S_{i,j}} \{ |A_{i,k} \cup \{a_k\} \cup A_{k,j}| \} \]

i.e., a greedy choice of \( a_k \in S_{i,j} \) makes \( A_{i,l} \cup \{a_l\} \cup A_{l,i} \) not needed, for all \( l \neq k \)

But the chosen \( a_k \) has to guarantee to maintain \( |A_{i,j}| \) is the maximum.

called greedy-choice property (need to be proved)
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a greedy choice

Solution Structure

DP
Chapter 16. Greedy Algorithms

Activity Selection problem has the greedy-choice property.
Activity Selection problem has the greedy-choice property.

**Theorem 16.1**: Let $S_{i,j}$ be a subproblem and $a_k \in S_{i,j}$ be the activity of the earliest finish time, then $a_k$ is in some optimal solution for $S_{i,j}$.
Activity Selection problem has the greedy-choice property.

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Chapter 16. Greedy Algorithms

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**Proof**: (Using the “swapping method”, or “exchange method”)
Activity Selection problem has the greedy-choice property.

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Let $A_{i,j}$ be an optimal solution for $S_{i,j}$.

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Let \( A_{i,j} \) be an optimal solution for \( S_{i,j} \).

(1) If \( a_k \in A_{i,j} \), then the theorem is true.

(2) If \( a_k \notin A_{i,j} \), assume \( a_p \) is the one with earliest finish time in \( A_{i,j} \).

\[ B_{i,j} = A_{i,j} \cup \{a_k\} - \{a_p\} \]

- because \( f_p \geq f_k \), \( a_k \) is compatible with other activities in \( B_{i,j} \).
- \( |B_{i,j}| = |A_{i,j}| \), so \( B_{i,j} \) is also an optimal solution for \( S_{i,j} \).

Since \( a_k \in B_{i,j} \), we prove the theorem.
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We define \( B_{i,j} = A_{i,j} - \{a_p\} \cup \{a_k\} \). Then

- because \( f_p \geq f_k \), \( a_k \) is compatible with other activities in \( B_{i,j} \).
  So \( B_{i,j} \) is a solution for \( S_{i,j} \).
Chapter 16. Greedy Algorithms

Activity Selection problem has the greedy-choice property.

**Theorem 16.1**: Let $S_{i,j}$ be a subproblem and $a_k \in S_{i,j}$ be the activity of the earliest finish time, then $a_k$ is in some optimal solution for $S_{i,j}$.

The theorem is the same as that selecting the activity with the earliest finish time

- makes all other subproblems disappearing, and
- maintains the optimality of solution

**Proof**: (Using the “swapping method”, or “exchange method”)

Let $A_{i,j}$ be an optimal solution for $S_{i,j}$.

(1) If $a_k \in A_{i,j}$, then the theorem is true.

(2) If $a_k \not\in A_{i,j}$, assume $a_p$ is the one with earliest finish time in $A_{i,j}$.

We define $B_{i,j} = A_{i,j} - \{a_p\} \cup \{a_k\}$. Then

- because $f_p \geq f_k$, $a_k$ is compatible with other activities in $B_{i,j}$.
  So $B_{i,j}$ is a solution for $S_{i,j}$.
- $|B_{i,j}| = |A_{i,j}|$, so $B_{i,j}$ is also an optimal solution for $S_{i,j}$.
Activity Selection problem has the greedy-choice property.

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Let $A_{i,j}$ be an optimal solution for $S_{i,j}$.

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We define $B_{i,j} = A_{i,j} \setminus \{a_p\} \cup \{a_k\}$. Then

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  So $B_{i,j}$ is a solution for $S_{i,j}$.
- $|B_{i,j}| = |A_{i,j}|$, so $B_{i,j}$ is also an optimal solution for $S_{i,j}$.

Since $a_k \in B_{i,j}$, we prove the theorem.
Chapter 16. Greedy Algorithms

Using the Theorem 16.1 and the recurrence

\[ |A_{i,j}| = \max_{a_k \in S_{ij}} \{|A_{i,k} \cup \{a_k\} \cup A_{k,j}|\} \]

we can derive greedy algorithms for the Activity Selection problem.
Chapter 16. Greedy Algorithms

Assume that the activities are sorted according to their finish times.
\[ f_1 \leq f_2 \leq \cdots \leq f_n. \]
Chapter 16. Greedy Algorithms

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Adding hypothetical activity $a_0$ (with $s_0 = f_0 = s_1$)
Chapter 16. Greedy Algorithms

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Adding hypothetical activity \( a_0 \) (with \( s_0 = f_0 = s_1 \)) and activity \( a_{n+1} \) (with \( s_{n+1} = f_{n+1} = f_n \)).
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Algorithm \textsc{Activity-Selection}(s, f, k, n)
Chapter 16. Greedy Algorithms

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Algorithm \texttt{Activity-Selection}(s, f, k, n)
1. \( m = k + 1 \)
Chapter 16. Greedy Algorithms

Assume that the activities are sorted according to their finish times. 

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Algorithm \textsc{Activity-Selection}(s, f, k, n)
1. \( m = k + 1 \)
2. \textbf{while} \( m \leq n \) and \( s_m < f_k \)
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Chapter 16. Greedy Algorithms

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1. \( m = k + 1 \)
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3. \( m = m + 1 \)
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Chapter 16. Greedy Algorithms

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1. \( m = k + 1 \)
2. \textbf{while} \( m \leq n \) and \( s_m < f_k \)
3. \( m = m + 1 \)
4. \textbf{if} \( m \leq n \)
5. \textbf{return} \( \{a_m\} \cup \texttt{Activity-Selection}(s, f, m, n) \)
Chapter 16. Greedy Algorithms

Assume that the activities are sorted according to their finish times.
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Algorithm \textsc{Activity-Selection}(\( s, f, k, n \))

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4. if \( m \leq n \)
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First called with \textsc{Activity-Selection}(s, f_0, n)

Time complexity: \( O(n) \).
Fractional Knapsack Problem:

Fractional Knapsack Problem:

Input:
n items of sizes $s_1, ..., s_n$ and values $v_1, ..., v_n$; and a knapsack size $B$.

Output: a subset $A \subseteq \{1, 2, ..., n\}$, each with a fraction $0 < f_i \leq 1$, which maximizes $\sum_{i \in A} f_i v_i$ under the constraint $\sum_{i \in A} f_i s_i \leq B$.

Define:

$K(S, X) = \text{the maximum value of packing items from set } S \text{ into space } X$.

Then

$K(S, X) = \max_{0 < f_i \leq 1, i \in S} \{ f_i v_i + K(S - \{i\}, X - f_i s_i) \}$.

• We hope to select some item $i$ that makes other subproblems disappear.
• We select item $i$ such that it has the highest density $d_i = \frac{v_i}{s_i}$.
• For convenience, assume the items are sorted according to the non-decreasing order of density.

Theorem: Fractional Knapsack problem has the greedy-choice property.
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Chapter 16. Greedy Algorithms

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Chapter 16. Greedy Algorithms

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Chapter 16. Greedy Algorithms

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Chapter 16. Greedy Algorithms

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Chapter 16. Greedy Algorithms

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**Theorem:** Fractional Knapsack problem has the greedy-choice property.
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We need to prove the following:
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Given a subset of items $S \subseteq \{1, 2, \ldots, n\}$ and space $X$. Assume item $i$ has the highest density in $S$. Then there is an optimal solution for $S$ which contains the fraction $f_i$ of the item $i$, for some maximum $f_i$, $0 \leq f_i \leq 1$ such that $f_is_i \leq X$. (This is left as an exercise)
Chapter 16. Greedy Algorithms

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Given a subset of items \( S \subseteq \{1, 2, \ldots, n\} \) and space \( X \). Assume item \( i \) has the highest density in \( S \). Then there is an optimal solution for \( S \) which contains the fraction \( f_i \) of the item \( i \), for some maximum \( f_i \), \( 0 \leq f_i \leq 1 \) such that \( f_i s_i \leq X \).

(This is left as an exercise)
Solve the following problem of **Coin Change** problem with **DP**:

**Input**: $X$ cents of money and coin denominations \{d_1, d_2, d_3\} where $1 \leq d_1 < d_2 < d_3$;

**Output**: the minimum number of coins with total amount $= X$. 

1. What does your recursive solution look like?
2. What should the objective function be?
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e.g., \( X = 12, d_1 = 1, d_2 = 3, d_3 = 5 \),
then an optimal solution is: \( 5, 5, 1, 1 \),
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Review for Midterm I

- Big-$O$, Big-$\Omega$, lower bound, upper bound
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Huffman Code
Chapter 16. Greedy Algorithms

Huffman Code

compressing data using binary bits
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Huffman Code

compressing data using binary bits
code: a compressing scheme,
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**Huffman Code**

compressing data using binary bits
code: a compressing scheme, fixed length vs variable length code
Chapter 16. Greedy Algorithms

Huffman Code

compressing data using binary bits
code: a compressing scheme, fixed length vs variable length code

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<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
<th>f</th>
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<tr>
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<td>.12</td>
<td>.16</td>
<td>.09</td>
<td>.05</td>
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<tr>
<td>fixed length code</td>
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<td>001</td>
<td>010</td>
<td>011</td>
<td>100</td>
<td>101</td>
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<tr>
<td>variable length code</td>
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<td>111</td>
<td>1101</td>
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Huffman Code

compressing data using binary bits
code: a compressing scheme, fixed length vs variable length code

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(a)  

(b)
prefix code: a prefix of a codeword cannot be another codeword.
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The Optimal Prefix Code Problem:
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**Input**: character set $C$ and frequencies $f$;
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Chapter 16. Greedy Algorithms

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**Input**: character set $C$ and frequencies $f$;
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achieves the minimum

where $d_T(c)$ is the depth of character $c$ in tree $T$. 

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Note that

• $d_T(c)$ is the length of codeword for $c$ under the code scheme $T$. 

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\]

achieves the minimum

where \( d_T(c) \) is the depth of character \( c \) in tree \( T \).

Note that

- \( d_T(c) \) is the length of codeword for \( c \) under the code scheme \( T \).
- \( B(T) \times n \) is the number of bits required to code a file (of \( n \) characters).
Algorithm \textsc{Huffman}(C, f)

1. \( n = |C| \)
2. \( Q = C \)
3. for \( i = 1 \) to \( n - 1 \)
4. \( \text{newnode}(z) \)
5. \( x = \text{Extract-Min}(Q) \)
6. \( z.\text{leftchild} = x \)
7. \( y = \text{Extract-Min}(Q) \)
8. \( z.\text{rightchild} = y \)
9. \( f(z) = f(x) + f(y) \)
9. \( \text{Insert}(Q, z) \)
9. return \( \text{Extract-Min}(Q) \)
Algorithm \textsc{Huffman}(C, f)

1. \quad n = |C|
Chapter 16. Greedy Algorithms

Algorithm Huffman($C, f$)

1. $n = |C|$
2. $Q = C$

3. for $i = 1$ to $n - 1$
   4. newnode($z$)
   5. $x = \text{Extract-Min}(Q)$
      extract two least frequent characters
   6. $z.\text{leftchild} = x$
   7. $y = \text{Extract-Min}(Q)$
   8. $z.\text{rightchild} = y$
   9. $f(z) = f(x) + f(y)$
   10. $\text{Insert}(Q, z)$
5. return $\text{Extract-Min}(Q)$
Chapter 16. Greedy Algorithms

Algorithm \textsc{Huffman}(C, f)

1. $n = |C|$  
2. $Q = C$  
3. \textbf{for} $i = 1 \text{ to } n - 1$
Algorithm \textsc{Huffman}(C, f)

1. \hspace{1em} n = |C|
2. \hspace{1em} Q = C
3. \hspace{1em} \textbf{for } i = 1 \textbf{ to } n - 1
4. \hspace{1em} \textbf{newnode}(z)
Algorithm \textsc{Huffman}(C, f)

1. \( n = |C| \)
2. \( Q = C \)
3. \textbf{for} \( i = 1 \) \textbf{to} \( n - 1 \)
4. \textbf{newnode}(z)
5. \( x = \textsc{Extract-Min}(Q) \) \quad \text{extract two least frequent characters}
Algorithm $\text{HUFFMAN}(C, f)$

1. $n = |C|$
2. $Q = C$
3. for $i = 1$ to $n - 1$
4. \hspace{1em} newnode($z$)
5. \hspace{1em} $x = \text{EXTRACT-MIN}(Q)$ \hspace{1em} \text{extract two least frequent characters}
6. \hspace{1em} $z.leftchild = x$
Chapter 16. Greedy Algorithms

Algorithm \textsc{Huffman}(C, f)

1. \( n = |C| \)
2. \( Q = C \)
3. \hspace{0.5cm} \textbf{for} \; i = 1 \; \textbf{to} \; n - 1 \\
4. \hspace{1cm} \textbf{newnode}(z) \\
5. \hspace{1cm} x = \textsc{Extract-Min}(Q) \; \; \text{extract two least frequent characters} \\
6. \hspace{1cm} z.\text{leftchild} = x \\
7. \hspace{1cm} y = \textsc{Extract-Min}(Q)
Chapter 16. Greedy Algorithms

Algorithm \textsc{Huffman}(C, f)

1. \hspace{1em} n = |C|
2. \hspace{1em} Q = C
3. \hspace{1em} \textbf{for} i = 1 \textbf{to} n - 1
4. \hspace{2em} \textsc{newnode}(z)
5. \hspace{1em} x = \textsc{Extract-Min}(Q) \hspace{1em} \textit{extract two least frequent characters}
6. \hspace{2em} z.leftchild = x
7. \hspace{1em} y = \textsc{Extract-Min}(Q)
8. \hspace{2em} z.rightchild = y
Algorithm **HUFFMAN**(\(C, f\))

1. \(n = |C|\)
2. \(Q = C\)
3. for \(i = 1\) to \(n - 1\)
4. \[\text{newnode}(z)\]
5. \(x = \text{EXTRACT-MIN}(Q)\) \hspace{1cm} \text{extract two least frequent characters}
6. \(z\).leftchild = \(x\)
7. \(y = \text{EXTRACT-MIN}(Q)\)
8. \(z\).rightchild = \(y\)
9. \(f(z) = f(x) + f(y)\)
Algorithm \textsc{Huffman}(\(C, f\))

1. \( n = |C| \)
2. \( Q = C \)
3. \textbf{for} \( i = 1 \) \textbf{to} \( n - 1 \)
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Algorithm \textsc{Huffman}(C, f)

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9. \hspace{1em} \( f(z) = f(x) + f(y) \)
9. \hspace{1em} \textsc{Insert}(Q, z)
9. \hspace{1em} \textbf{return} \ (\textsc{Extract-Min}(Q))
Chapter 16. Greedy Algorithms

An example of Huffman algorithm

- Symbols: A, B, C, D, E, F, G
  freq.: 2, 3, 5, 8, 13, 15, 18

- Huffman codes:
  A: 10100  B: 10101  C: 1011
  D: 100     E: 00    F: 01
  G: 11

A Huffman code Tree
Correctness of Huffman’s algorithm

Lemma 16.2
(Greedy choice property)
Let $C$ be an alphabet and $f$ be the frequency function for characters in $C$. Let $x$ and $y$ be two characters in $C$ having the lowest frequencies. Then there exists an optimal prefix code for $C$ in which the codewords for $x$ and $y$ have the same length and differ in the last bit.

Proof: Assume $C$, $f$, $x$ and $y$ are as described in the lemma condition. Let $T$ be any optimal prefix code for $C$.

(a) if $T$ contains the situation about $x$ and $y$ stated in the lemma. Done for the proof!

(b) let $a,b \in C$ be two characters such that their codewords have the same length and differ in the last bit. Also assume that their codeword length $d_T(a) = d_T(b) \geq d_T(c)$, for every $c \in C$. We note that such $a$ and $b$ can be found from $T$ without difficulty.
Chapter 16. Greedy Algorithms

Correctness of Huffman’s algorithm

Lemma 16.2 (Greedy choice property)
Correctness of Huffman’s algorithm

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Let $C$ be an alphabet and $f$ be the frequency function for characters in $C$. Let $x$ and $y$ be two characters in $C$ having the lowest frequencies. Then there exists an optimal prefix code for $C$ in which the codewords for $x$ and $y$ have the same length and differ in the last bit.
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Chapter 16. Greedy Algorithms

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Chapter 16. Greedy Algorithms

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Chapter 16. Greedy Algorithms

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Correctness of Huffman’s algorithm

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(a) if $T$ contains the situation about $x$ and $y$ stated in the lemma. Done for the proof!

(b) let $a, b \in C$ be two characters such that their codewords have the same length and differ in the last bit. Also assume that their codeword length

$$d_T(a) = d_T(b) \geq d_T(c), \text{ for every } c \in C$$

We note that such $a$ and $b$ can be found from $T$ without difficulty.
We use the following swapping strategy to obtain another prefix code $T'$ for $C$.

Without loss of generality, assume $\{a, b\} \cap \{x, y\} = \emptyset$. The created code $T'$ is the same as $T$ except the positions $a$ and $x$ are swapped in $T'$; so are the positions of $b$ and $y$ swapped in $T'$. Therefore,

$$B(T) - B(T') = \sum_{c \in C} f(c) d_T(c) - \sum_{c \in C} f(c) d_{T'}(c) = f(a) d_T(a) + f(b) d_T(b) + f(x) d_T(x) + f(y) d_T(y) - f(a) d_{T'}(a) - f(b) d_{T'}(b) - f(x) d_{T'}(x) - f(y) d_{T'}(y) = (f(a) - f(x)) d_T(a) + (f(x) - f(a)) d_T(x) + (f(b) - f(y)) d_T(b) + (f(y) - f(b)) d_T(y) \geq 0$$

which implies $B(T) \geq B(T')$. So $T'$ is also an optimal code for $C$. Therefore, $T'$ satisfies the lemma.
We use the following swapping strategy to obtain another prefix code $T'$ for $C$. Without loss of generality, assume $\{a, b\} \cap \{x, y\} = \emptyset$. The created code $T'$ is the same as $T$ except the positions $a$ and $x$ are swapped in $T'$; so are the positions of $b$ and $y$ swapped in $T'$. 

\[
B(T) - B(T') = \sum_{c \in C} f(c) d_T(c) - \sum_{c \in C} f(c) d_{T'}(c) = f(a) d_T(a) + f(b) d_T(b) + f(x) d_T(x) + f(y) d_T(y) - f(a) d_{T'}(a) - f(b) d_{T'}(b) - f(x) d_{T'}(x) - f(y) d_{T'}(y) \geq 0
\]

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We use the following swapping strategy to obtain another prefix code $T'$ for $C$.

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\[ B(T) - B(T') = \sum_{c \in C} f(c) d_T(c) - \sum_{c \in C} f(c) d_{T'}(c) = f(a) d_T(a) + f(b) d_T(b) + f(x) d_T(x) + f(y) d_T(y) - f(a) d_{T'}(a) - f(b) d_{T'}(b) - f(x) d_{T'}(x) - f(y) d_{T'}(y) = \]

\[ = (f(a) - f(x)) d_T(a) + (f(x) - f(a)) d_T(x) + (f(b) - f(y)) d_T(b) + (f(y) - f(b)) d_T(y) \geq 0 \]

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Without loss of generality, assume $\{a, b\} \cap \{x, y\} = \emptyset$.

The created code $T'$ is the same as $T$ except the positions $a$ and $x$ are swapped in $T'$; so are the positions of $b$ and $y$ swapped in $T'$.

$$B(T) - B(T') = \sum_{c \in C} f(c)d_T(c) - \sum_{c \in C} f(c)d_{T'}(c)$$
We use the following swapping strategy to obtain another prefix code \( T' \) for \( C \).

Without loss of generality, assume \( \{a, b\} \cap \{x, y\} = \emptyset \).

The created code \( T' \) is the same as \( T \) except the positions \( a \) and \( x \) are swapped in \( T' \); so are the positions of \( b \) and \( y \) swapped in \( T' \).

\[
B(T) - B(T') = \sum_{c \in C} f(c)d_T(c) - \sum_{c \in C} f(c)d_{T'}(c)
= f(a)d_T(a) + f(b)d_T(b) + f(x)d_T(x) + f(y)d_T(y)
\]

which implies \( B(T) \geq B(T') \). So \( T' \) is also an optimal code for \( C \).

So \( T' \) satisfies the lemma.
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$$B(T) - B(T') = \sum_{c \in C} f(c)d_T(c) - \sum_{c \in C} f(c)d_{T'}(c)$$

$$= f(a)d_T(a) + f(b)d_T(b) + f(x)d_T(x) + f(y)d_T(y)$$

$$- f(a)d_{T'}(a) - f(b)d_{T'}(b) - f(x)d_{T'}(x) - f(y)d_{T'}(y)$$

which implies $B(T) \geq B(T')$. So $T'$ is also an optimal code for $C$.
We use the following swapping strategy to obtain another prefix code $T'$ for $C$.

Without loss of generality, assume $\{a, b\} \cap \{x, y\} = \phi$.

The created code $T'$ is the same as $T$ except the positions $a$ and $x$ are swapped in $T'$; so are the positions of $b$ and $y$ swapped in $T'$.

$$B(T) - B(T') = \sum_{c \in C} f(c)d_T(c) - \sum_{c \in C} f(c)d_{T'}(c)$$

$$= f(a)d_T(a) + f(b)d_T(b) + f(x)d_T(x) + f(y)d_T(y)$$

$$- f(a)d_{T'}(a) - f(b)d_{T'}(b) - f(x)d_{T'}(x) - f(y)d_{T'}(y)$$

$$= (f(a) - f(x))d_T(a) + (f(x) - f(a))d_T(x) + (f(b) - f(y))d_T(b)$$

$$+ (f(y) - f(b))d_T(y)$$

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\[
B(T) - B(T') = \sum_{c \in C} f(c)d_T(c) - \sum_{c \in C} f(c)d_{T'}(c)
\]

\[
= f(a)d_T(a) + f(b)d_T(b) + f(x)d_T(x) + f(y)d_T(y) \\
- f(a)d_{T'}(a) - f(b)d_{T'}(b) - f(x)d_{T'}(x) - f(y)d_{T'}(y)
\]

\[
= (f(a) - f(x))d_T(a) + (f(x) - f(a))d_T(x) + (f(b) - f(y))d_T(b) + (f(y) - f(b))d_T(y)
\]

\[
= (f(a) - f(x))(d_T(a) - d_T(x)) + (f(b) - f(y))(d_T(b) - d_T(y)) \geq 0
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We use the following swapping strategy to obtain another prefix code $T'$ for $C$.

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B(T) - B(T') = \sum_{c \in C} f(c)d_T(c) - \sum_{c \in C} f(c)d_{T'}(c)
\]

\[
= f(a)d_T(a) + f(b)d_T(b) + f(x)d_T(x) + f(y)d_T(y)
\]

\[
- f(a)d_{T'}(a) - f(b)d_{T'}(b) - f(x)d_{T'}(x) - f(y)d_{T'}(y)
\]

\[
= (f(a) - f(x))d_T(a) + (f(x) - f(a))d_T(x) + (f(b) - f(y))d_T(b)
\]

\[
+ (f(y) - f(b))d_T(y)
\]

\[
= (f(a) - f(x))(d_T(a) - d_T(x)) + (f(b) - f(y))(d_T(b) - d_T(y)) \geq 0
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which implies $B(T) \geq B(T')$. So $T'$ is also an optimal code for $C$. 
Chapter 16. Greedy Algorithms

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B(T) - B(T') = \sum_{c \in C} f(c)d_T(c) - \sum_{c \in C} f(c)d_{T'}(c)
\]

\[
= f(a)d_T(a) + f(b)d_T(b) + f(x)d_T(x) + f(y)d_T(y) - f(a)d_{T'}(a) - f(b)d_{T'}(b) - f(x)d_{T'}(x) - f(y)d_{T'}(y)
\]

\[
= (f(a) - f(x))d_T(a) + (f(x) - f(a))d_T(x) + (f(b) - f(y))d_T(b) + (f(y) - f(b))d_T(y)
\]

\[
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Chapter 16. Greedy Algorithms

Lemma 16.3 (Optimal substructure)
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Let $C$ be an alphabet and $f$ be the frequency function for characters in $C$. Let $x$ and $y$ be two characters in $C$ having the lowest frequencies.
Lemma 16.3 (Optimal substructure)

Let $C$ be an alphabet and $f$ be the frequency function for characters in $C$. Let $x$ and $y$ be two characters in $C$ having the lowest frequencies. Let

$$C' = C - \{x, y\} \cup \{z\}$$

and the frequency function for $C'$ is the same as $f$ except that $f(z) = f(x) + f(y)$.
Lemma 16.3 (Optimal substructure)

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and the frequency function for $C'$ is the same as $f$ except that $f(z) = f(x) + f(y)$.

Let $T'$ be any an optimal prefix code tree for $C'$. Then the tree $T$ obtained from $T'$ by replacing the leaf node $z$ with an internal node having $x$ and $y$ as children, is an optimal prefix code tree for $C$. 

Proof:

The code $T$ for $C$ constructed from $T'$, a code for $C'$, has the following objective function value

$$B(T) = \sum_{c \in C} f(c) \cdot d_T(c) = \sum_{c \in C - \{x, y\}} f(c) \cdot d_{T'}(c) + [f(x) + f(y)] \cdot d_{T'}(z) + 1$$

$$= \sum_{c \in C - \{x, y\}} f(c) \cdot d_{T'}(c) + [f(x) + f(y)] \cdot d_{T'}(z) + [f(x) + f(y)]$$

$$= \sum_{c \in C - \{x, y\} \cup \{z\}} f(c) \cdot d_{T'}(c) + [f(x) + f(y)]$$

$$= B(T') + f(x) + f(y)$$
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$$= \sum_{c \in C - \{x, y\}} f(c)d_{T'}(c) + [f(x) + f(y)](d_{T'}(z) + 1)$$
Lemma 16.3 (Optimal substructure)

Let $C$ be an alphabet and $f$ be the frequency function for characters in $C$. Let $x$ and $y$ be two characters in $C$ having the lowest frequencies. Let

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and the frequency function for $C'$ is the same as $f$ except that $f(z) = f(x) + f(y)$.

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The code $T$ for $C$ constructed from $T'$, a code for $C'$, has the following objective function value

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Chapter 16. Greedy Algorithms

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Now if $T$ is not optimal for $C$, assume optimal prefix code $S$ for $C$ such that $B(S) < B(T)$.
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By Lemma 16.2, we assume $S$ to be such that codewords for $x$ and $y$ are of the same depth and differ in the last bit.
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We construct another code $S'$ for $C'$ by removing the branches for $x$ and $y$ and assign $z$ to its parent node. We have $B(S') = B(S) - f(x) - f(y)$. But

$$B(S') < B(T) - f(x) - f(y) = B(T') + f(x) + f(y) - f(x) - f(y) = B(T')$$

This implies $T'$ is not optimal code for $C'$. Contradict. So $T$ should be optimal for $C$.

Theorem 16.4
Huffman’s algorithm produces an optimal prefix code.
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**Theorem 16.4** Huffman’s algorithm produces an optimal prefix code.
Chapter 16. Greedy Algorithms

Coin change problem
Chapter 16. Greedy Algorithms

Coin change problem

**Input:** money of value $n$, currency coin denominations $D = \{v_1, \ldots, v_m\}$
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**INPUT:** money of value \( n \), currency coin denominations \( D = \{ v_1, \ldots, v_m \} \)

**OUTPUT:** the least number of coins with denominations in \( D \) whose values sum up exactly to \( n \).
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A mathematical formulation of the problem

**Input:** positive integer \( n \), and set of positive integers \( D = \{v_1, \ldots, v_m\} \)
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\[
\sum_{i=1}^{m} x_i v_i = n \text{ to minimize } \left( \sum_{i=1}^{m} x_i \right)
\]
Chapter 16. Greedy Algorithms

A dynamic programming solution

• for any value $k$, one coin of the $m$ denominations has to be used.
• then for the left value, one coin of the $m$ denominations has to be used.

Define objective function:

$c(k)$ to be the least number of coins to change value $k$. Then

$$c(k) = \min \begin{cases} c(k - v_1) + 1 & k \geq v_1 \\ c(k - v_2) + 1 & k \geq v_2 \\ \vdots \\ c(k - v_m) + 1 & k \geq v_m \end{cases}$$

• base case:
  
  $c(v_1) = 1$, \ldots, $c(v_m) = 1$. 

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A dynamic programming solution

**analysis:**
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Chapter 16. Greedy Algorithms

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A greedy algorithm

Theorem: For the US coin denominations $D_{us} = \{1, 5, 10, 25\}$, the Coin Change problem has the greedy-choice property.

- always use the coin of the largest possible value.

but not all coin denominations have such property, e.g., $\{1, 3, 4, 10\}$, for $n = 6$.
A greedy algorithm

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Chapter 16. Greedy Algorithms

**Proof:** Assume $A$ is the optimal solution for $n$. 

• I. For $5 \leq n < 10$, assume $5 \notin A$, but all coins in $A$ have to be one-cent coins. Let $B = A - \{1, 1, 1, 1, 1\} \cup \{5\}$. $|B| < |A|$, which is impossible. So $5 \in A$.

• II. For $10 \leq n < 25$, assume $10 \notin A$, but all coins in $A$ are either five-cent or one-cent coins.

   (1) all coins in $A$ are one-cent coins, let $B = A - \{1, 1, 1, 1, 1, 1, 1, 1, 1, 1\} \cup \{10\}$, then $|B| < |A|$, which is impossible. So $10 \in A$;

   (2) there is one five-cent coin and some one-cent coins in $A$, let $B = A - \{1, 1, 1, 1, 1\} \cup \{10\}$, $|B| < |A|$, which is impossible. So $10 \in A$;

   (3) there is at least two five-cent coins in $A$, so $B = A - \{5, 5\} \cup \{10\}$, $|B| < |A|$, which is impossible. So $10 \in A$;
Proof: Assume $A$ is the optimal solution for $n$. Consider various cases

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Chapter 16. Greedy Algorithms

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  - (2) there is one five-cent coin and some one-cent coins in $A$, let $B = A - \{1, 1, 1, 1, 1\} \cup \{10\}$, $|B| < |A|$, which is impossible. So $10 \in A$;
  - (3) there is at least two five-cent coins in $A$, so $B = A - \{5, 5\} \cup \{10\}$, $|B| < |A|$, which is impossible. So $10 \in A$;
Proof: Assume $A$ is the optimal solution for $n$. Consider various cases

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**Chapter 16. Greedy Algorithms**

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So we conclude that there are at least two ten-cent coins are in $A$. 
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    • If $10 \leq m < 25$, according to II, $10$ is in the solution for $m$.
      So we conclude that there are at least two ten-cent coins are in $A$. Now we assume $k = n - 2 \times 10$. 
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If $10 \leq k < 25$, according to II, 10 is in the solution for $k$, then $\{10, 10, 10\} \subseteq A$, \end{quote}
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- **III.** For $25 \leq n$, assume $25 \not\in A$, but all coins in $A$ are either ten-cents, five-cents or one-cent.

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Review for Parts I, II, and III

Summaries for Parts I, II, and III
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Part I: Fundamentals of Analysis of algorithms
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Part I: Fundamentals of Analysis of algorithms

- time/space complexity functions
Review for Parts I, II, and III

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Part I: Fundamentals of Analysis of algorithms

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Summaries for Parts I, II, and III

Part I: Fundamentals of Analysis of algorithms

- time/space complexity functions
  - input size
  - upper bound

- asymptotic notations
  - \( \bigO \)
  - \( \bigOmega \)

- recurrences for complexities of recursive algorithms
- recurrence proof methods:
  - recursive tree (unfolding)
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  - recursive tree (unfolding)
  - substitution (induction)
Review for Parts I, II, and III

Part II: Sorting and order statistics
Review for Parts I, II, and III

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- sorting algorithms: heap, insertion, merge, quick
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- sorting algorithms: heap, insertion, merge, quick
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- linear time to find $k$th smallest element
Part III. Exhaustive search, DP and greedy algorithms
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  - simple enumeration of all possible solutions
  - search tree method
  - resulting in $T(n) = T(n-1) + T(n-1-m), m \geq 1$.
- dynamic programming
  - recursive solution (with optimal substructure)
  - recurrence for numerical objective function
  - iterative, bottom-up table-filling
  - traceback of solution.
- greedy algorithm
  - DP solution
  - proof of greedy-choice property (via swapping method)
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