Part VI. Graph Algorithms
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- Chapter 22. Elementary graph algorithms
- Chapter 23. Minimum spanning trees
- Chapter 24. Single-source shortest paths
- Chapter 25. All-pairs shortest paths
Chapter 22. Elementary graph algorithms

Chapter 22. Elementary graph algorithms
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- Representations of graphs
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- Traverse graphs:
  1. breadth-first-search (BFS)
  2. depth-first-search (DFS)
- Applications:
  1. topological sort
  2. strongly connected components
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• traverse graphs:
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Chapter 22. Elementary graph algorithms

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Chapter 22 Elementary graph algorithms

Graph: $G = (V, E)$
Chapter 22. Elementary graph algorithms

Terminologies and notations:

• graph $G = (V, E)$, where $V = \{v_1, ..., v_n\}$ and $E \subseteq V \times V$

  $V = \{1, 2, 3, 4, 5, 6, 7\}$

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• weight $w$: $E \to \mathbb{R}$, e.g., $w(1, 2) = 4$, $w(5, 6) = 3$, etc.

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\[\text{Diagram of graph } G = (V, E)\]
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Chapter 22. Elementary graph algorithms

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![Graph Diagram]
Chapter 22. Elementary graph algorithms

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- **complete graphs**: $K_n$

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- **planar graphs**: embedded in the plane without crossing edges: However, $K_5$ is not planar, neither is $K_{3,3}$
Chapter 22. Elementary graph algorithms

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Chapter 22. Elementary graph algorithms

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  1-tree is tree;

  2-tree is a graph but with **tree width** = 2
Chapter 22. Elementary graph algorithms

Representations of graphs

adjacency-matrix
adjacency-list
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Representations of graphs

adjacency-matrix
adjacency-list
Chapter 22. Elementary graph algorithms

adjacency-matrix for a weighted graph
Chapter 22. Elementary graph algorithms

Traverse graphs

basic ideas of depth-first-search (DFS) and breadth-first-search (BFS)

Both methods yield "search trees"
or "search forest" (if the graph is not connected)
Chapter 22. Elementary graph algorithms

DFS on directed graphs, search tree
Chapter 22. Elementary graph algorithms

DFS on directed graphs, search tree
Chapter 22. Elementary graph algorithms

DFS on directed graphs, search tree

DFS on non-directed graphs, search tree
Chapter 22. Elementary graph algorithms

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Chapter 22. Elementary graph algorithms

DFS on directed graphs, search tree

![Directed Graph Example](image1.png)

DFS on non-directed graphs, search tree

![Non-directed Graph Example](image2.png)
Traversing graphs is an important task:

- Navigating the whole graph
- For connectivity check
- For circle check
- etc.

DFS and BFS are two fundamental algorithms for graph traversal!
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Chapter 22. Elementary graph algorithms

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Chapter 22. Elementary graph algorithms

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DFS and BFS are two fundamental algorithms for graph traversal!
Chapter 22. Elementary graph algorithms

First recursive DFS algorithm, assuming $G$ is connected.

How does the algorithm start?

- Initially set $u.\text{visit} = \text{false}$ for every vertex $u \in G.V$;
- $s.\pi = \text{NULL}$ for some specific $s \in G.V$;
- Call $\text{Recursive-DFS}(G, s)$.

But if $G$ is not connected, what should we do?
Chapter 22. Elementary graph algorithms

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RECURSIVE-DFS(G, u);
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Chapter 22. Elementary graph algorithms

First recursive DFS algorithm, assuming $G$ is connected.

\textsc{Recursive-DFS}(G, u);
1. \textbf{if} not $u.visit$
2. \hspace{1em} $u.visit = \text{true};$ \hspace{2em} \{ mark $u$ "visited" \}
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Chapter 22. Elementary graph algorithms

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Chapter 22. Elementary graph algorithms

To-Start-DFS(G)
Chapter 22. Elementary graph algorithms

**To-Start-DFS**($G$)
1. for each $s \in G.V$ \hspace{1cm} \{ initialize visit values \}
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Chapter 22. Elementary graph algorithms

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Chapter 22. Elementary graph algorithms

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Chapter 22. Elementary graph algorithms

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Chapter 22. Elementary graph algorithms

DFS (from the textbook) computes discover and finish time stamps \( (u.d \text{ and } u.f) \) for every visited vertex \( u \).

```plaintext
DFS(G)
1   for each vertex \( u \in G.V \)
2       \( u.color = WHITE \)
3       \( u.\pi = NIL \)
4   \( time = 0 \)
5   for each vertex \( u \in G.V \)
6       if \( u.color == WHITE \)
7           DFS-VISIT(G, u)
```

```plaintext
DFS-VISIT(G, u)
1   \( time = time + 1 \)  // white vertex \( u \) has just been discovered
2   \( u.d = time \)
3   \( u.color = GRAY \)
4   for each \( v \in G.Adj[u] \)  // explore edge \((u, v)\)
5       if \( v.color == WHITE \)
6           \( v.\pi = u \)
7           DFS-VISIT(G, v)
8   \( u.color = BLACK \)  // blacken \( u \); it is finished
9   \( time = time + 1 \)
10  \( u.f = time \)
```
Chapter 22. Elementary graph algorithms

DFS-Visit \((G, v)\)
1. \(\text{time} = \text{time} + 1\)  // white vertex \(v\) has just been discovered
2. \(v.d = \text{time}\)
3. \(v.color = \text{GRAY}\)
4. for each \(v \in G.Adj[u]\)  // explore edge \((u, v)\)
5. \(\text{if } v.color == \text{WHITE}\)
6. \(v.\pi = u\)
7. \(\text{DFS-Visit}\((G, v)\)\)
8. \(v.color = \text{BLACK}\)  // blacken \(v\); it is finished
9. \(\text{time} = \text{time} + 1\)
10. \(v.f = \text{time}\)

→: edge being explored;
→: edge path taken by DFS
Chapter 22. Elementary graph algorithms

DFS-Visit(G, u)
1. time = time + 1 \quad \text{// white vertex } u \text{ has just been discovered}
2. u.d = time
3. u.color = GRAY
4. for each v ∈ G.Adj[u] \quad \text{// explore edge } (u, v)
5. \quad \text{if } v.color == \text{WHITE}
6. \quad \quad v.π = u
7. \quad \quad DFS-Visit(G, v)
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Chapter 22. Elementary graph algorithms

DFS-VISIT(G, u)
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→: edge being explored;
→: edge path taken by DFS
Chapter 22. Elementary graph algorithms

**DFS-VISIT(G, u)**

1. \( time = time + 1 \) // white vertex \( u \) has just been discovered
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5. \( \text{if } v.color == \text{WHITE} \)
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→: edge being explored;
→: edge path taken by DFS
Chapter 22. Elementary graph algorithms
Chapter 22. Elementary graph algorithms

Another example of DFS execution (page 605)
Time complexity of DFS algorithm

\[ \Theta(|E| + |V|) \]

where \(|E|\) is the number of edges in \(G\).
Chapter 22. Elementary graph algorithms

Time complexity of DFS algorithm

DFS(G)
1 for each vertex \( u \in G.V \)
2 \( u.color = \text{WHITE} \)
3 \( u.\pi = \text{NIL} \)
4 \( \text{time} = 0 \)
5 for each vertex \( u \in G.V \)
6 \( \text{if} \ u.color == \text{WHITE} \)
7 \( \text{DFS-Visit}(G, u) \)

DFS-Visit(G, u)
1 \( \text{time} = \text{time} + 1 \) // white vertex \( u \) has just been discovered
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Chapter 22. Elementary graph algorithms

Time complexity of DFS algorithm

\[
\text{DFS}(G)
\]

1. \textbf{for} each vertex \( u \in G.V \)
2. \hspace{1em} \( u.color = \text{WHITE} \)
3. \hspace{1em} \( u.\pi = \text{NIL} \)
4. \hspace{1em} \text{time} = 0
5. \hspace{1em} \textbf{for} each vertex \( u \in G.V \)
6. \hspace{2em} \textbf{if} \( u.color == \text{WHITE} \)
7. \hspace{3em} \text{DFS-VISIT}(G, u)

\[
\text{DFS-VISIT}(G, u)
\]

1. \text{time} = \text{time} + 1 \hspace{1em} // \text{white vertex } u \text{ has just been discovered}
2. \hspace{1em} u.d = \text{time}
3. \hspace{1em} u.color = \text{GRAY}
4. \hspace{1em} \textbf{for} each \( v \in G.Adj[u] \) \hspace{1em} // \text{explore edge } (u, v)
5. \hspace{2em} \textbf{if} \ v.color == \text{WHITE}
6. \hspace{3em} \hspace{1em} v.\pi = u
7. \hspace{3em} \text{DFS-VISIT}(G, v)
8. \hspace{1em} u.color = \text{BLACK} \hspace{1em} // \text{blacken } u; \text{ it is finished}
9. \hspace{1em} \text{time} = \text{time} + 1
10. \hspace{1em} u.f = \text{time}

\( \Theta(|E| + |V|) \), where \(|E|\) is the number of edges in \( G \).
Chapter 22. Elementary graph algorithms

Properties of depth-first-search:
Chapter 22. Elementary graph algorithms

Properties of depth-first-search:

(1) \( u = v.\pi \)
Properties of depth-first-search:

(1) $u = v.\pi$ iff DFS-Visit$(G, v)$ is called.
Chapter 22. Elementary graph algorithms

Properties of depth-first-search:

(1) \( u = v.\pi \) iff \( \text{DFS-Visit}(G, v) \) is called.

(2) Theorem 22.7 (Parenthesis Theorem): for any \( u, v \), exactly one of the following three conditions holds:
Properties of depth-first-search:

(1) \( u = v.\pi \) iff \( \text{DFS-Visit}(G, v) \) is called.

(2) **Theorem 22.7 (Parenthesis Theorem):** for any \( u, v \), exactly one of the following three conditions holds:

- \([u.d, u.f]\) and \([v.d, v.f]\) are entirely disjoint, and neither \( u \) nor \( v \) is a descendant of the other in the search tree.

**Corollary 22.8 (Nesting of descendants' intervals)** Vertex \( v \) is a proper descendant of \( u \) in the depth-first search forest if and only if \( u.d < v.d < v.f < u.f \).
Chapter 22. Elementary graph algorithms

Properties of depth-first-search:

(1) $u = v.\pi$ iff DFS-\textsc{Visit}$(G, v)$ is called.

(2) Theorem 22.7 (Parenthesis Theorem): for any $u, v$, exactly one of the following three conditions holds:

- $[u.d, u.f]$ and $[v.d, v.f]$ are entirely disjoint, and neither $u$ nor $v$ is a descendant of the other in the search tree.

- $[u.d, u.f]$ is contained entirely within $[v.d, v.f]$ and $u$ is a descendant of $v$, or
Chapter 22. Elementary graph algorithms

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(1) \( u = v.\pi \) iff DFS-\textsc{Visit}(G, v) is called.

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- \([v.d, v.f]\) is contained entirely within \([u.d, u.f]\) and \( v \) is a descendant of \( u \).
Chapter 22. Elementary graph algorithms

Properties of depth-first-search:

(1) $u = v.\pi$ iff $\text{DFS-Visit}(G, v)$ is called.

(2) Theorem 22.7 (Parenthesis Theorem): for any $u, v$, exactly one of the following three conditions holds:

- $[u.d, u.f]$ and $[v.d, v.f]$ are entirely disjoint, and neither $u$ nor $v$ is a descendant of the other in the search tree.
- $[u.d, u.f]$ is contained entirely within $[v.d, v.f]$ and $u$ is a descendant of $v$, or
- $[v.d, v.f]$ is contained entirely within $[u.d, u.f]$ and $v$ is a descendant of $u$.

Corollary 22.8 (Nesting of descendants' intervals) Vertex $v$ is a proper descendant of $u$ in the depth-first search forest if and only if $u.d < v.d < v.f < u.f$. 
Chapter 22. Elementary graph algorithms
Chapter 22. Elementary graph algorithms

**Theorem 22.9 (White-path Theorem)** In a depth-first search forest of graph $G$, vertex $v$ is a descendant of $u$ if and only if at the time $u.d$ that the search discovers $u$, there is a path from $u$ to $v$ consisting of entirely of white vertices.

**Proof**:

$\Rightarrow$

- **Case 1**:
  - $u = v$, apparently the claim is true;

- **Case 2**:
  - $v$ is a proper descendant of $u$, use Corollary 22.8 on every vertex on the path from $u$ to $v$; the claim is true;

$\Leftarrow$

Assume that at the time $u.d$, there is a path from $u$ to $v$ as stated in the theorem but $v$ is not descendant of $u$.

Assume $(w, x)$ be an edge on the path and $x$ is the first vertex on the path which is not descendant of $u$.

Note that $w$ is a descendant of $u$ (or just $u$) because at time $u.d$ $x$ is of WHITE color, $u.d < x.d$.

Because $(w, x)$ is an edge, there are two possible scenarios:

1. When $(w, x)$ is being explored, $x$ has already been discovered; we thus have $x.d < w.f$;
2. When $(w, x)$ is being explored, $x$ has WHITE color but will then be discovered we thus also have $x.d < w.f$;

According to Corollary 22.8, $u.d < x.d < w.f < u.f$.

Thus, $u.d < x.d < u.f$.

By Theorem 22.7, the interval $[x.d, x.f]$ is entirely contained within interval $[u.d, u.f]$.

Therefore, $x$ is a descendant of $u$. Contradicts to the earlier assumption.

$v$ should be a descendant of $u$. 
Chapter 22. Elementary graph algorithms

**Theorem 22.9 (White-path Theorem)** In a depth-first search forest of graph $G$, vertex $v$ is a descendant of $u$ if and only if at the time $u.d$ that the search discovers $u$, there is a path from $u$ to $v$ consisting of entirely of white vertices.

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Chapter 22. Elementary graph algorithms

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**Proof:** $\Rightarrow$
- case 1: $u = v$, apparently the claim is true;
- case 2: $v$ is a proper descendant of $u$, use **Corollary 22.8** on every vertex on the path from $u$ to $v$; the claim is true;
Chapter 22. Elementary graph algorithms

**Theorem 22.9 (White-path Theorem)** In a depth-first search forest of graph $G$, vertex $v$ is a descendant of $u$ if and only if at the time $u.d$ that the search discovers $u$, there is a path from $u$ to $v$ consisting of entirely of white vertices.

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case 1: $u = v$, apparently the claim is true;

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$\Leftarrow$

Assume that at the time $u.d$, there is a path from $u$ to $v$ as stated in the theorem but $v$ is not descendant of $u$. 
Chapter 22. Elementary graph algorithms

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Assume that at the time $u.d$, there is a path from $u$ to $v$ as stated in the theorem but $v$ is not descendant of $u$.

Assume $(w, x)$ be an edge on the path and $x$ is the first vertex on the path which is not descendant of $u$. 


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Assume $(w, x)$ be an edge on the path and $x$ is the first vertex on the path which is not descendant of $u$. Note that $w$ is a descendant of $u$ (or just $u$)
**Chapter 22. Elementary graph algorithms**

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Assume $(w, x)$ be an edge on the path and $x$ is the first vertex on the path which is not descendant of $u$. Note that $w$ is a descendant of $u$ (or just $u$). Because at time $u.d$ $x$ is of WHITE color, $u.d < x.d$. 
Chapter 22. Elementary graph algorithms

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Because $(w, x)$ is an edge, there are two possible scenarios:
- (1) when $(w, x)$ is being explored, $x$ has already been discovered;
Chapter 22. Elementary graph algorithms

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case 1: $u = v$, apparently the claim is true;

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Assume that at the time $u.d$, there is a path from $u$ to $v$ as stated in the theorem but $v$ is not descendant of $u$.

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Because at time $u.d$ $x$ is of WHITE color, $u.d < x.d$.

Because $(w, x)$ is an edge, there are two possible scenarios:

(1) when $(w, x)$ is being explored, $x$ has already been discovered; we thus have $x.d < w.f$;
Chapter 22. Elementary graph algorithms

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Because $(w, x)$ is an edge, there are two possible scenarios:
- (1) when $(w, x)$ is being explored, $x$ has already been discovered; we thus have $x.d < w.f$;
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Chapter 22. Elementary graph algorithms

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Chapter 22. Elementary graph algorithms

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Chapter 22. Elementary graph algorithms

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Chapter 22. Elementary graph algorithms

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According to Corollary 22.8, $u.d < x.d < w.f < u.f$. Thus, $u.d < x.d < u.f$

By Theorem 22.7, the interval $[x.d, x.f]$ is entirely contained within interval $[u.d, u.f]$. 

Chapter 22. Elementary graph algorithms

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**Proof:** $\Rightarrow$

- case 1: $u = v$, apparently the claim is true;
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According to Corollary 22.8, $u.d < x.d < w.f < u.f$. Thus, $u.d < x.d < u.f$

By Theorem 22.7, the interval $[x.d, x.f]$ is entirely contained within interval $[u.d, u.f]$. Therefore, $x$ is a descendant of $u$. Contradicts to the earlier assumption.
Chapter 22. Elementary graph algorithms

**Theorem 22.9 (White-path Theorem)** In a depth-first search forest of graph $G$, vertex $v$ is a descendant of $u$ if and only if at the time $u.d$ that the search discovers $u$, there is a path from $u$ to $v$ consisting of entirely of white vertices.

**Proof:** ⇒
- case 1: $u = v$, apparently the claim is true;
- case 2: $v$ is a proper descendant of $u$, use **Corollary 22.8** on every vertex on the path from $u$ to $v$; the claim is true;

⇐
Assume that at the time $u.d$, there is a path from $u$ to $v$ as stated in the theorem but $v$ is not descendant of $u$.

Assume $(w, x)$ be an edge on the path and $x$ is the first vertex on the path which is not descendant of $u$. Note that $w$ is a descendant of $u$ (or just $u$)

Because at time $u.d$ $x$ is of WHITE color, $u.d < x.d$.

Because $(w, x)$ is an edge, there are two possible scenarios:
- (1) when $(w, x)$ is being explored, $x$ has already been discovered; we thus have $x.d < w.f$;
- (2) when $(w, x)$ is being explored, $x$ has WHITE color but will then be discovered we thus also have $x.d < w.f$;

According to **Corollary 22.8**, $u.d < x.d < w.f < u.f$. Thus, $u.d < x.d < u.f$.

By **Theorem 22.7**, the interval $[x.d, x.f]$ is entirely contained within interval $[u.d, u.f]$. Therefore, $x$ is a descendant of $u$. Contradicts to the earlier assumption. $v$ should be a descendant of $u$. 
Chapter 22. Elementary graph algorithms

Classification of edges (for directed graphs)
Chapter 22. Elementary graph algorithms

Classification of edges (for directed graphs)

- Tree edges: those in the search tree (forest);
- Back edges: those connecting a vertex to an ancestor; a selfloop, in a directed graph, can be a back edge;
- Forward edges: those connecting a vertex to a descendant;
- Cross edges: all other edges;

DFS forest with back, forward, & cross edges.
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![DFS forest with back, forward, & cross edges.](image)
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Chapter 22. Elementary graph algorithms

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Chapter 22. Elementary graph algorithms

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Chapter 22. Elementary graph algorithms

To identify the type of edge \((u,v)\) with the color of \(v\):
- **WHITE**: tree edge;
- **GRAY**: back edge;
- **BLACK**: forward or cross edge;

First stages of a Directed DFS, showing **Edges**, the **DFS TREE**, a **Tree Edge**, a **Back Edge**, a **Forward Edge**, and a **Cross Edge**.
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Chapter 22. Elementary graph algorithms

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First stages of a Directed DFS, showing *Edges*, the *DFS TREE*, a *Tree Edge*, a *Back Edge*, a *Forward Edge*, and a *Cross Edge*.
Chapter 22. Elementary graph algorithms

**Theorem 22.10** In a depth-first search of undirected graph $G$, every edge of $G$ is either tree edge or back edge.
Chapter 22. Elementary graph algorithms

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Proof. Let $(u, v)$ be an edge in $G$. Assume in the depth-first search of $G$, $u.d < v.d$. Then there are two scenarios:
Chapter 22. Elementary graph algorithms

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1. $v$ is discovered by exploring edge $(u, v)$, then $(u, v)$ is a tree edge;
2. $v$ is discovered not through exploring edge $(u, v)$.

   Because $(u, v)$ is an edge, $v$ is discovered when $u$ is in gray color.
Theorem 22.10 In a depth-first search of undirected graph $G$, every edge of $G$ is either tree edge or back edge.

Proof. Let $(u,v)$ be an edge in $G$. Assume in the depth-first search of $G$, $u.d < v.d$. Then there are two scenarios:

1. $v$ is discovered by exploring edge $(u,v)$, then $(u,v)$ is a tree edge;
2. $v$ is discovered not through exploring edge $(u,v)$.

Because $(u,v)$ is an edge, $v$ is discovered when $u$ is in gray color. Since $u$ is in the adjacency list of $v$, $(v,u)$ will eventually be explored and thus a back edge.
Chapter 22. Elementary graph algorithms

Breadth First Search (BFS)
Chapter 22. Elementary graph algorithms

Breadth First Search (BFS)
Chapter 22. Elementary graph algorithms

Breadth First Search Algorithm (with a queue)

Time complexity of BFS: $O(|V| + |E|)$

Note: BFS can find a shortest path from $s$ to all other nodes (non-weighted). (Why?)
Chapter 22. Elementary graph algorithms

Breadth First Search Algorithm (with a queue)
Chapter 22. Elementary graph algorithms

Breadth First Search Algorithm (with a queue)

BFS(G, s)
1 for each vertex u ∈ G.V − {s}
2 u.color = WHITE
3 u.d = ∞
4 u.π = NIL
5 s.color = GRAY
6 s.d = 0
7 s.π = NIL
8 Q = ∅
9 ENQUEUE(Q, s)
10 while Q ≠ ∅
11 u = DEQUEUE(Q)
12 for each v ∈ G.Adj[u]
13 if v.color == WHITE
14 v.color = GRAY
15 v.d = u.d + 1
16 v.π = u
17 ENQUEUE(Q, v)
18 u.color = BLACK
Chapter 22. Elementary graph algorithms

Breadth First Search Algorithm (with a queue)

BFS\((G, s)\)

1. for each vertex \(u \in G.V - \{s\}\)
2. \(u.color = \text{WHITE}\)
3. \(u.d = \infty\)
4. \(u.\pi = \text{NIL}\)
5. \(s.color = \text{GRAY}\)
6. \(s.d = 0\)
7. \(s.\pi = \text{NIL}\)
8. \(Q = \emptyset\)
9. \(\text{ENQUEUE}(Q, s)\)
10. while \(Q \neq \emptyset\)
11. \(u = \text{DEQUEUE}(Q)\)
12. for each \(v \in G.\text{Adj}[u]\)
13. \(\text{if } v.color == \text{WHITE}\)
14. \(v.color = \text{GRAY}\)
15. \(v.d = u.d + 1\)
16. \(v.\pi = u\)
17. \(\text{ENQUEUE}(Q, v)\)
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Chapter 22. Elementary graph algorithms

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Chapter 22. Elementary graph algorithms

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12 for each $v \in G.Adj[u]$
13 \hspace{1em} if $v.color == \text{WHITE}$
14 \hspace{2em} $v.color = \text{GRAY}$
15 \hspace{2em} $v.d = u.d + 1$
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Time complexity of BFS: $O(|V| + |E|)$
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Chapter 22. Elementary graph algorithms

Topological sorting

• On directed acyclic graphs (DAGs)
  • A sorted order:
    socks, shorts, pants, shoes, shirt, tie, belt, jacket, watch.
Chapter 22. Elementary graph algorithms

Topological sorting
Chapter 22. Elementary graph algorithms

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Chapter 22. Elementary graph algorithms

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A sorted order: socks, shorts, pants, shoes, shirt, tie, belt, jacket, watch.
Chapter 22. Elementary graph algorithms

• apply DFS algorithm.
• reversed order of finish times: p,n,o,s,m,r,y,v,x,w,z,u,q,t
• Correctness proof?
Chapter 22. Elementary graph algorithms

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- apply DFS algorithm.

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\[ p, n, o, s, m, r, y, v, x, w, z, u, q, t \]
Chapter 22. Elementary graph algorithms

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Chapter 22. Elementary graph algorithms

Let $G = (V, E)$ be a digraph. A strongly connected component (SCC) is a maximal subgraph $H = (V_H, E_H)$ of $G$ such that for every two nodes $v, u \in V_H$,

1. there is a directed path $v \Rightarrow u$ consisting of edges in $E_H$; and
2. there is a directed path $u \Rightarrow v$ consisting of edges in $E_H$. 
Chapter 22. Elementary graph algorithms

Strongly connected components (SCC)
Chapter 22. Elementary graph algorithms

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Chapter 22. Elementary graph algorithms

**Strongly connected components (SCC)**

Let $G = (V, E)$ be a digraph. A *strongly connected component* is a maximal subgraph $H = (V_H, E_H)$ of $G$ such that for every two nodes $v, u \in V_H$,
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Chapter 22. Elementary graph algorithms

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Chapter 22. Elementary graph algorithms

Idea of an algorithm to use DFS to solve SCC problem.

• Use DFS to generate DFS forest; each search tree $T_u$ (rooted at $u$) consists of vertices $v$ such that $u \Rightarrow v$.

• Use DFS again on $T_u$; hope to search from every one $v$ within $T_u$ to make sure $v \Rightarrow u$ as well.

• However, this may be difficult (proof is left as an exercise).
Chapter 22. Elementary graph algorithms

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Chapter 22. Elementary graph algorithms

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Chapter 22. Elementary graph algorithms

Algorithm Strongly Connected Components \((G)\)

1. call \(\text{DFS}(G)\) to compute \(u.f\) for each \(u \in G.V\)
2. compute \(G^T\) the transpose of \(G\) {reverse all edges in \(G\)}
3. call \(\text{DFS}(G^T)\) (vertices are considered in the decreasing order of finish times computed in step 1)
4. output each tree in the depth-first forest produced by step 3.
Algorithm \textsc{Strongly Connected Components}(G)

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Chapter 22. Elementary graph algorithms

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Chapter 22. Elementary graph algorithms

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Chapter 22. Elementary graph algorithms

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Chapter 22. Elementary graph algorithms

Ideas behind the algorithm:
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- the first pass DFS results in a DFS forest, let $T$ be a tree with root $r$;
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Chapter 22. Elementary graph algorithms

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- the first pass DFS results in a DFS forest, let $T$ be a tree with root $r$;
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• the first pass DFS results in a DFS forest, let $T$ be a tree with root $r$; for every vertex $u \in T$, $r \leadsto u$,
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• let vertex $v \in T$ but $v \neq r$, we are not sure $v \leadsto u$;

• that is the same as to use second-DFS (starting from $r$) to check if $r \leadsto v$ after edge directions are reversed;
Chapter 22. Elementary graph algorithms

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• instead, we would like to check if $v \leadsto r$ for $v \in T$. 
Chapter 22. Elementary graph algorithms

Ideas behind the algorithm:

- the first pass DFS results in a DFS forest, let $T$ be a tree with root $r$; for every vertex $u \in T$, $r \rightsquigarrow u$,
  so an SCC can only be produced from some tree in the forest;
- let vertex $v \in T$ but $v \neq r$, we are not sure $v \rightsquigarrow u$;
- instead, we would like to check if $v \rightsquigarrow r$ for $v \in T$.
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Chapter 22. Elementary graph algorithms

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• let vertex $v \in T$ but $v \neq r$, we are not sure $v \rightsquigarrow u$;

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• that is the same as to use second-DFS (starting from $r$)
  to check if $r \rightsquigarrow v$ after edge directions are reversed;

• only those in the same second-DSF tree belongs to the same SCC.
Chapter 22. Elementary graph algorithms

Chapter 22. Elementary graph algorithms

Properties from algorithm

Strongly Connected Components

\((G)\)

1. Component graph:

\(G_{SCC} = (V_{SCC}, E_{SCC})\)

Let

\(C_1, C_2, ..., C_k\) be \(k\) distinct SCCs for \(G\). Then

\(V_{SCC} = \{v_1, v_2, ..., v_k\}\);

\(E_{SCC} = \{(v_i, v_j) : \exists u \in C_i, v \in C_j, (u, v) \in E\}\).

Then \(G_{SCC}\) is a DAG (directed acyclic graph).

Proof. Assume the opposite to the claim that, for some \(v_i, v_j \in V_{SCC}\), there is a path \(v_i \Rightarrow v_j\) and another path \(v_j \Rightarrow v_i\), forming a cycle in \(V_{SCC}\).

By the definition of \(G_{SCC}\), there must be a path in \(G\), from some vertex in \(C_i\) to some vertex in \(C_j\); at the same time, there is a path in \(G\), from some vertex in \(C_j\) to some vertex in \(C_i\). Then \(C_i\) and \(C_j\) should form a single SCC, not two distinct SCCs. Contradicts.
Properties from algorithm $\textsc{Strongly Connected Components}(G)$
Chapter 22. Elementary graph algorithms

Properties from algorithm **STRONGLY CONNECTED COMPONENTS**(G)

(1) Component graph: \( G^{SCC} = (V^{SCC}, E^{SCC}) \) is defined as follow:
Properties from algorithm \textbf{Strongly Connected Components}(G)

(1) Component graph: \(G^{SCC} = (V^{SCC}, E^{SCC})\) is defined as follow:

Let \(C_1, C_2, \ldots, C_k\) be \(k\) distinct SCCs for \(G\). Then
Properties from algorithm **Strongly Connected Components**($G$)

(1) Component graph: $G^{SCC} = (V^{SCC}, E^{SCC})$ is defined as follow:

let $C_1, C_2, \ldots, C_k$ be $k$ distinct SCCs for $G$. Then

$V^{SCC} = \{v_1, v_2, v_k\}$;

$E^{SCC} = \{(v_i, v_j) : \exists u \in C_i, v \in C_j, (u, v) \in E\}$. 
Properties from algorithm `Strongly Connected Components(G)`

(1) Component graph: \( G^{SCC} = (V^{SCC}, E^{SCC}) \) is defined as follow:

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\[
\begin{align*}
V^{SCC} &= \{v_1, v_2, v_k\}; \\
E^{SCC} &= \{(v_i, v_j) : \exists u \in C_i, v \in C_j, (u, v) \in E\}.
\end{align*}
\]

Then \( G^{SCC} \) is a DAG (directed acyclic graph).
Chapter 22. Elementary graph algorithms

Properties from algorithm **Strongly Connected Components** $(G)$

(1) Component graph: $G^{SCC} = (V^{SCC}, E^{SCC})$ is defined as follow:

   let $C_1, C_2, \ldots, C_k$ be $k$ distinct SCCs for $G$. Then

   $V^{SCC} = \{v_1, v_2, v_k\};$

   $E^{SCC} = \{(v_i, v_j) : \exists u \in C_i, v \in C_j, (u, v) \in E\}.$

   Then $G^{SCC}$ is a DAG (directed acyclic graph).

**Proof.** Assume the opposite to the claim that, for some $v_i, v_j \in V^{SCC}$, there is a path $v_i \leadsto v_j$ and another path $v_j \leadsto v_i$, forming a cycle in $V^{SCC}$. 
Properties from algorithm **Strongly Connected Components**($G$)

(1) Component graph: $G^{SCC} = (V^{SCC}, E^{SCC})$ is defined as follow:

let $C_1, C_2, \ldots, C_k$ be $k$ distinct SCCs for $G$. Then

$V^{SCC} = \{v_1, v_2, \ldots, v_k\}$;
$E^{SCC} = \{(v_i, v_j) : \exists u \in C_i, v \in C_j, (u, v) \in E\}$.

Then $G^{SCC}$ is a DAG (directed acyclic graph).

**Proof.** Assume the opposite to the claim that, for some $v_i, v_j \in V^{SCC}$, there is a path $v_i \leadsto v_j$ and another path $v_j \leadsto v_i$, forming a cycle in $V^{SCC}$.

By the definition of $G^{SCC}$, there must be a path in $G$, from some vertex in $C_i$ to some vertex in $C_j$; at the same time, there is a path in $G$, from some vertex in $C_j$ to some vertex in $C_i$. Then $C_i$ and $C_j$ should form a single SCC, not two distinct SCCs. **Contradicts.**
Chapter 22. Elementary graph algorithms

Let $C$ be a SCC, define $f(C) = \max_{u \in C} \{ u.f \}$, (with the finish times from the first DFS call).

Lemma 22.14: Let $C$ and $C'$ be distinct strongly connected components for directed graph $G$. If $(u, v) \in E$, where $u, \in C$ and $v \in C'$, then $f(C) > f(C')$.

Proof: Assume the opposite, i.e., $f(C) < f(C')$. Then there must be vertices $x \in C$ and $y \in C'$ such that $x.f < y.f$.

Now consider the first DFS call, there are two situations:

1. $y$ was searched before $x$: by property (1) there is no path from $y$ to $x$, $x.f > y.f$.
2. $y$ was searched after $x$: since there is a path from $x$ to $y$ because of $(u, v)$, $x.f > y.f$.

Both cases contradict the assumption. So $f(C) > f(C')$. 
Chapter 22. Elementary graph algorithms

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Chapter 22. Elementary graph algorithms

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Chapter 22. Elementary graph algorithms

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Chapter 22. Elementary graph algorithms

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Chapter 22. Elementary graph algorithms

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Chapter 22. Elementary graph algorithms

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Chapter 22. Elementary graph algorithms

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Both cases contradicts the assumption.
Chapter 22. Elementary graph algorithms

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Both cases contradicts the assumption. So $f(C) > f(C')$. 
Chapter 22. Elementary graph algorithms

The algorithm

Strongly Connected Components

(G)

correctly computes the strongly connected components for a directed graph G.

We need to prove two statements:

1. If v⇝u and u⇝v in G, then u and v belong to the same component C produced by the algorithm.

2. If u,v ∈ C, then we have v⇝u and u⇝v in G.
(3) The algorithm \texttt{Strongly Connected Components}(G) correctly computes the strongly connected components for a directed graph \( G \).
Chapter 22. Elementary graph algorithms

(3) The algorithm \textsc{Strongly Connected Components}(G) correctly computes the strongly connected components for a directed graph \(G\).

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(3) The algorithm Strongly Connected Components($G$) correctly computes the strongly connected components for a directed graph $G$.

We need to prove two statements:

(1) If $v \leadsto u$ and $u \leadsto v$ in $G$, then $u$ and $v$ belong to the same component $C$ produced by the algorithm.
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We need to prove two statements:

(1) If \( v \leadsto u \) and \( u \leadsto v \) in G, then \( u \) and \( v \) belong to the same component \( C \) produced by the algorithm.

(2) If \( u, v \in C \), then we have \( v \leadsto u \) and \( u \leadsto v \) in G.
Proof:

(1) If \( v \rightarrow u \) and \( u \rightarrow v \) in \( G \), then \( u \) and \( v \) belong to the same component \( C \) produced by the algorithm.

Sketch of proof:

• Assume in 1st DFS, \( v \) was discovered before \( u \) (or opposite);

• As \( v \rightarrow u \) in \( G \), \( u \) and \( v \) belong to the same search tree rooted at \( r \) with \( r.f \geq v.f > u.f \) (note: \( r \) could be just \( v \))

• As \( u \rightarrow v \) in \( G \), \( v \rightarrow u \) in \( G_T \);

• Now consider the 2nd DFS; there are 2 situations:

  (1) Searching from some \( w \) with \( w.f \geq v.f \) (note: \( w \) could be \( v \)) finds \( v \) first; then it finds \( u \);

  (2) The search finds \( u \) first; because \( v \rightarrow u \) in \( G \), \( u \rightarrow v \) is in \( G_T \), it finds also \( v \).

In both situations, \( u \) and \( v \) belong to the same search tree in the 2nd DFS search. Therefore, \( u \) and \( u \) belong to the same component.
Proof:

• Assume in the 1st DFS, \( v \) was discovered before \( u \) (or opposite);
• As \( v \to u \) in \( G \), \( u \) and \( v \) belong to the same search tree rooted at \( r \) with \( r.f \geq v.f > u.f \); (note: \( r \) could be just \( v \));
• As \( u \to v \) in \( G \), \( v \to u \) in \( G_{T} \);
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  1) Searching from some \( w \) with \( w.f \geq v.f \); (note: \( w \) could be \( v \)) finds \( v \) first; then it finds \( u \);
  2) The search finds \( u \) first; because \( v \to u \) in \( G \), \( u \to v \) is in \( G_{T} \), it finds also \( v \). In both situations, \( u \) and \( v \) belong to the same search tree in the 2nd DFS search. Therefore, \( u \) and \( v \) belong to the same component.
Proof:

(1) If \( v \sim u \) and \( u \sim v \) in \( G \) in \( G \), then \( u \) and \( v \) belong to the same component \( C \) produced by the algorithm.
Chapter 22. Elementary graph algorithms

Proof:

(1) If \( v \leadsto u \) and \( u \leadsto v \) in \( G \) in \( G \), then \( u \) and \( v \) belong to the same component \( C \) produced by the algorithm.

Sketch of proof:
Chapter 22. Elementary graph algorithms

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Chapter 22. Elementary graph algorithms

Proof:

(1) If $v \sim u$ and $u \sim v$ in $G$ in $G$, then $u$ and $v$ belong to
the same component $C$ produced by the algorithm.

Sketch of proof:

• assume in 1st DFS, $v$ was discovered before $u$ (or opposite);
• as $v \sim u$ in $G$, $u$ and $v$ belong to the same search tree rooted at $r$
  with $r.f \geq v.f > u.f$ (note: $r$ could be just $v$)
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\begin{itemize}
  \item assume in 1st DFS, \( v \) was discovered before \( u \) (or opposite);
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  \item as \( u \leadsto v \) in \( G \), \( v \leadsto u \) in \( G^T \);
\end{itemize}
Chapter 22. Elementary graph algorithms

Proof:

(1) If \( v \leadsto u \) and \( u \leadsto v \) in \( G \), then \( u \) and \( v \) belong to the same component \( C \) produced by the algorithm.

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Chapter 22. Elementary graph algorithms

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Chapter 22. Elementary graph algorithms

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Chapter 22. Elementary graph algorithms

Proof:

(1) If \( v \leadsto u \) and \( u \leadsto v \) in \( G \) in \( G \), then \( u \) and \( v \) belong to the same component \( C \) produced by the algorithm.

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- as \( u \leadsto v \) in \( G \), \( v \leadsto u \) in \( G^T \);
- now consider the 2nd DFS; there are 2 situations:
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  - (2) the search finds \( u \) first; because \( v \leadsto u \) in \( G \), \( u \leadsto v \) is in \( G^T \),
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In both situations, \( u \) and \( v \) belongs to the same search tree in the 2nd DFS search. Therefore, \( u \) and \( u \) belong to the same component.
Chapter 22. Elementary graph algorithms

If $u,v \in C$, then we have $v \Rightarrow u$ and $u \Rightarrow v$ in $G$.

Sketch of proof:

1. Assume $u, v$ belong to the same tree of root $r$ in 2nd DFS;
2. Then $r.f > u.f$; and $r.f > v.f$ in 1st DFS;
3. The assumption in (1) also implies:
   - $r \Rightarrow u$ and $r \Rightarrow v$ in $G_T$;
   - That is, $u \Rightarrow r$ and $v \Rightarrow r$ in $G$;
   - Then $u.f > r.f$ and $v.f > r.f$ in 1st DFS, which conflict with conclusions in (2), UNLESS $r \Rightarrow u$ and $r \Rightarrow v$ in $G$ also.

4. This means: through $r$, $v \Rightarrow u$ and $u \Rightarrow v$ in $G$. 
(2) If \( u, v \in C \), then we have \( v \leadsto u \) and \( u \leadsto v \) in \( G \).
(2) If $u, v \in C$, then we have $v \leadsto u$ and $u \leadsto v$ in $G$.

Sketch of proof:
Chapter 22. Elementary graph algorithms

(2) If $u, v \in C$, then we have $v \Rightarrow u$ and $u \Rightarrow v$ in $G$.

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(1) assume $u, v$ belong to the same tree of root $r$ in 2nd DFS;

(2) then $r.f > u.f$; and $r.f > v.f$ in 1st DFS;

(3) the assumption in (1) also implies:

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• that is, $u \Rightarrow r$ and $v \Rightarrow r$ in $G$;

• then $u.f > r.f$ and $v.f > r.f$ in 1st DFS, which conflict with conclusions in (2), UNLESS $r \Rightarrow u$ and $r \Rightarrow v$ in $G$ also.

(4) This means: through $r$, $v \Rightarrow u$ and $u \Rightarrow v$ in $G$. 
(2) If \( u, v \in C \), then we have \( v \prec u \) and \( u \prec v \) in \( G \).

Sketch of proof:

(1) assume \( u, v \) belong to the same tree of root \( r \) in 2nd DFS;
(2) then \( r.f > u.f \); and \( r.f > v.f \) in 1st DFS;
Chapter 22. Elementary graph algorithms

(2) If $u, v \in C$, then we have $v \leadsto u$ and $u \leadsto v$ in $G$.

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Chapter 22. Elementary graph algorithms

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   • that is, \( u \leadsto r \) and \( v \leadsto r \) in \( G^T \);
   • then \( u.f > r.f \) and \( v.f > r.f \) in 1st DFS,
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(2) If \( u, v \in C \), then we have \( v \leadsto u \) and \( u \leadsto v \) in \( G \).

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   - \( r \leadsto u \) and \( r \leadsto v \) in \( G^T \);
   - that is, \( u \leadsto r \) and \( v \leadsto r \) in \( G \);
(2) If \( u, v \in C \), then we have \( v \leadsto u \) and \( u \leadsto v \) in \( G \).

**Sketch of proof:**

1. Assume \( u, v \) belong to the same tree of root \( r \) in 2nd DFS;
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   - That is, \( u \leadsto r \) and \( v \leadsto r \) in \( G \);
   - Then \( u.f > r.f \) and
Chapter 22. Elementary graph algorithms

(2) If $u, v \in C$, then we have $v \rightarrow u$ and $u \rightarrow v$ in $G$.

Sketch of proof:

(1) assume $u, v$ belong to the same tree of root $r$ in 2nd DFS;
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(3) the assumption in (1) also implies:
   - $r \rightarrow u$ and $r \rightarrow v$ in $G^T$;
   - that is, $u \rightarrow r$ and $v \rightarrow r$ in $G$;
   - then $u.f > r.f$ and $v.f > r.f$ in 1st DFS,
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Chapter 22. Elementary graph algorithms

(2) If \( u, v \in C \), then we have \( v \leadsto u \) and \( u \leadsto v \) in \( G \).

Sketch of proof:

(1) assume \( u, v \) belong to the same tree of root \( r \) in 2nd DFS;
(2) then \( r.f > u.f \); and \( r.f > v.f \) in 1st DFS;
(3) the assumption in (1) also implies:
   • \( r \leadsto u \) and \( r \leadsto v \) in \( G^T \);
   • that is, \( u \leadsto r \) and \( v \leadsto r \) in \( G \);
   • then \( u.f > r.f \) and \( v.f > r.f \) in 1st DFS,

which conflict with conclusions in (2), \textbf{UNLESS}
(2) If $u, v \in C$, then we have $v \leadsto u$ and $u \leadsto v$ in $G$.

Sketch of proof:

(1) assume $u, v$ belong to the same tree of root $r$ in 2nd DFS;
(2) then $r.f > u.f$; and $r.f > v.f$ in 1st DFS;
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  • $r \leadsto u$ and $r \leadsto v$ in $G_T$;
  • that is, $u \leadsto r$ and $v \leadsto r$ in $G$;
  • then $u.f > r.f$ and $v.f > r.f$ in 1st DFS,
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Chapter 22. Elementary graph algorithms

(2) If \( u, v \in C \), then we have \( v \leadsto u \) and \( u \leadsto v \) in \( G \).

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     which conflict with conclusions in (2), UNLESS
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(4) This means: through \( r, v \leadsto u \) and \( u \leadsto v \) in \( G \).
Reachability Problem

Input: $G = (V,E)$, and $s,t \in V$;
Output: YES if and only there is a path $s \leadsto t$ in $G$.

• The problem can be solved with DFS and BFS by search on the graph from $s$ until $t$ shows up.
  Linear time $O(|E| + |V|)$. Can we do better?

• But first answer the following question: Can you write an SQL program to solve Reachability?

It appears that a loop is needed to solve Reachability. Why?
Inherent difficulty in parallel computation. P-complete, it cannot be solved in time $O(\log n)$ even if $\Theta(n)$ CPUs are used.
Reachability Problem

Input:

- \( G = (V, E) \)
- \( s, t \in V \)

Output:

- YES if and only if there is a path \( s \Rightarrow t \) in \( G \).

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Chapter 22. Elementary graph algorithms

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Chapter 22. Elementary graph algorithms

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Chapter 22. Elementary graph algorithms

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Chapter 22. Elementary graph algorithms

Reachability Problem

**Input:** \( G = (V, E) \), and \( s, t \in V \);

**Output:** YES if and only there is a path \( s \rightarrow t \) in \( G \).

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Chapter 23. Minimum Spanning Trees

A spanning tree of a graph \( G = (V,E) \) is a tree as a subgraph in \( G \) which contains all vertices in \( V \).

A minimum spanning tree (MST) of an edge-weighted graph \( G \) is a spanning tree with the least edge weight sum.
A spanning tree of a graph $G = (V, E)$ is a tree as a subgraph in $G$ which contains all vertices in $V$. A minimum spanning tree (MST) of an edge-weighted graph $G$ is a spanning tree with the least edge weight sum.
Chapter 23. Minimum Spanning Trees

- A spanning tree of a graph $G = (V, E)$ is a tree as a subgraph in $G$ which contains all vertices in $V$. 

![Diagram of a graph with a spanning tree highlighted in blue]

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Chapter 23. Minimum Spanning Trees

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Chapter 23. Minimum Spanning Trees

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Chapter 23. Minimum Spanning Trees

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Chapter 23. Minimum Spanning Trees

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Chapter 23. Minimum Spanning Trees

The MST problem
Chapter 23. Minimum Spanning Trees

The MST problem

**INPUT:** connected, undirected graph $G = (V, E)$ with weight $w : E \rightarrow R$, ...
Chapter 23. Minimum Spanning Trees

The MST problem

**Input:** connected, undirected graph $G = (V, E)$ with weight $w : E \to R$,  
**Output:** a spanning tree $T = (V, E')$ such that

$$W(T) = \sum_{(u,v) \in E'} w(u,v)$$

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Chapter 23. Minimum Spanning Trees

The MST problem

**Input:** connected, undirected graph \( G = (V, E) \) with weight \( w : E \rightarrow \mathbb{R} \),

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We will introduce two **greedy algorithms**: (1) Kruskal’s and (2) Prim’s
Chapter 23. Minimum Spanning Trees

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• They have the same generic process to grow a spanning tree;
Chapter 23. Minimum Spanning Trees

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We will introduce two **greedy algorithms**: (1) Kruskal’s and (2) Prim’s

- They **have the same generic process** to grow a spanning tree;
- but **differ** in which edge to add the partially grown tree.
Chapter 23. Minimum Spanning Trees

Growing an MST

A generic process to grow an MST.

Generic MST \((G,w)\) {

given graph \(G\) and weight function \(w\)

1. \(A = \emptyset\);

2. while \(A\) does not form a spanning tree

3. find an edge \((u,v)\) that is safe for \(A\)

4. \(A = A \cup \{(u,v)\}\)

5. return \((A)\)

Loop invariant: \(A\) is always a subset of some MST; Note: when the loop terminates, \(A\) is a MST.

Safe edge: edge \((u,v)\) is safe for \(A\) if it does not violate the loop invariant, i.e, \(A \cup \{(u,v)\}\) is a subset of some MST.
Chapter 23. Minimum Spanning Trees

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Chapter 23. Minimum Spanning Trees

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Chapter 23. Minimum Spanning Trees

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Chapter 23. Minimum Spanning Trees

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Chapter 23. Minimum Spanning Trees

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Chapter 23. Minimum Spanning Trees

Growing an MST
A generic process to grow an MST.

\textsc{Generic} MST($G, w$) \{ given graph $G$ and weight function $w$ \}

1. $A = \emptyset$;
2. \textbf{while} $A$ does not form a spanning tree
3. \hspace{1em} find an edge $(u, v)$ that is \textbf{safe} for $A$
4. \hspace{1em} $A = A \cup \{(u, v)\}$
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Chapter 23. Minimum Spanning Trees

We first need some terminologies
Chapter 23. Minimum Spanning Trees

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- **cut** : \((S, V - S)\), a partition of \(V\)
Chapter 23. Minimum Spanning Trees

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- **cut**: \((S, V - S)\), a partition of \(V\)
Chapter 23. Minimum Spanning Trees

We first need some terminologies

- **cut**: \((S, V-S)\), a partition of \(V\)

- **crossing**: \((u, v)\) crosses cut \((S, V - S)\)
  if \(u\) and \(v\) are in \(S\) and \(V - S\), respectively
Chapter 23. Minimum Spanning Trees

Some more terminologies
Some more terminologies

- **respect**: a cut respects a set $A$ of edges if no edge in $A$ crosses the cut.
Some more terminologies

- **respect**: a cut respects a set $A$ of edges if no edge in $A$ crosses the cut.

- **light edge**: an edge is a light edge crossing a cut if its weight is the minimum of any edge that crosses the cut.
Chapter 23. Minimum Spanning Trees

**Theorem 23.1** Let $G = (V, E)$. 
Chapter 23. Minimum Spanning Trees

**Theorem 23.1** Let $G = (V, E)$.
Let $A \subseteq E$, contained in some MST for $G$. 

**Sketch of proof**: 
(1) If $A \cup \{(u, v)\}$ forms a cycle there must have been another edge in $A$ that crosses cut $(S, V - S)$ (WHY?), implying the cut did not respect $A$. Contradicts.
(2) Assume that some MST $T$, $A \subset T$. First, $T \cup \{(u, v)\}$ forms a cycle! (WHY?) There must be another edge $(x, y)$ crossing the cut $(S, V - S)$. Since $(u, v)$ is light edge, $T' = T - \{(x, y)\} \cup \{(u, v)\}$ is an MST. Now $A \cup \{(u, v)\} \subseteq T'$ because $(x, y) \not\in A$ (otherwise, the cut would not respect $A$).
Chapter 23. Minimum Spanning Trees

**Theorem 23.1** Let $G = (V, E)$.
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Chapter 23. Minimum Spanning Trees

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Chapter 23. Minimum Spanning Trees

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For the theorem, we need to prove:
Chapter 23. Minimum Spanning Trees

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Chapter 23. Minimum Spanning Trees

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Chapter 23. Minimum Spanning Trees

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(1) If $A \cup \{(u, v)\}$ forms a cycle there must have been another edge in $A$ that crosses cut $(S, V - S)$ (WHY?),
Chapter 23. Minimum Spanning Trees

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Let \((u, v)\) be a light edge crossing the cut.
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For the theorem, we need to prove:
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(1) If \( A \cup \{(u, v)\} \) forms a cycle there must have been another edge in \( A \)
that crosses cut \((S, V - S)\) (**WHY?**), implying the cut did not respect \( A \).
**Contradicts.**
Chapter 23. Minimum Spanning Trees

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   First, $T \cup \{(u, v)\}$ forms a circle!
Chapter 23. Minimum Spanning Trees

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Chapter 23. Minimum Spanning Trees

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(2) $A$, after including $(u, v)$, is still a subset of some MST.

**Sketch of proof:**

(1) If $A \cup \{(u, v)\}$ forms a cycle there must have been another edge in $A$
    that crosses cut $(S, V - S)$ (WHY?), implying the cut did not respect $A$.
    **Contradicts.**

(2) Assume that some MST $T$, $A \subset T$.
    First, $T \cup \{(u, v)\}$ forms a circle! **Why?**
    There must be another edge $(x, y)$ cross the cut $(S, V - S)$.
Chapter 23. Minimum Spanning Trees

**Theorem 23.1** Let $G = (V, E)$.
Let $A \subseteq E$, contained in some MST for $G$.
Let $(S, V - S)$ be any cut of $G$ that respects $A$.
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Since $(u, v)$ is light edge, $T' = T - \{(x, y)\} \cup \{(u, v)\}$ is an MST.
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Chapter 23. Minimum Spanning Trees

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For the theorem, we need to prove:
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(1) If \( A \cup \{(u, v)\} \) forms a cycle there must have been another edge in \( A \) that crosses cut \((S, V - S)\) \( (\text{WHY?}) \), implying the cut did not respect \( A \).
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(2) Assume that some MST \( T, A \subset T \).
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There must be another edge \((x, y)\) cross the cut \((S, V - S)\).
Since \((u, v)\) is light edge, \( T' = T - \{(x, y)\} \cup \{(u, v)\} \) is an MST.

Now \( A \cup \{(u, v)\} \subseteq T' \) because \((x, y) \notin A \)
(otherwise, the cut would not respect \( A \).
Chapter 23. Minimum Spanning Trees

Theorem 23.1 gives a sufficient information for how to identify a safe edge to make the GENERIC MST algorithm work.
Chapter 23. Minimum Spanning Trees

Theorem 23.1 gives a sufficient information for how to identify a safe edge to make the Generic MST algorithm work.

- specific algorithms can be produced from Generic MST based on how the set $A$ is grown.

\textbf{MST-Kruskal} $(G,w)$

1. $A = \emptyset$
2. for each vertex $v \in G.V$
3. Make-Set ($v$)
4. sort edges in $E$ into non-decreasing order by their weight $w$
5. for each edge $(u,v) \in E$, taken in the order
6. if $\text{Find Set} (u) \neq \text{Find Set} (v)$
7. $A = A \cup \{(u,v)\}$
8. $\text{Union} (u,v)$
9. return $(A)$
Chapter 23. Minimum Spanning Trees

Theorem 23.1 gives a sufficient information for how to identify a safe edge to make the Generic MST algorithm work.

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Chapter 23. Minimum Spanning Trees

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Chapter 23. Minimum Spanning Trees

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Chapter 23. Minimum Spanning Trees

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Chapter 23. Minimum Spanning Trees

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**MST-Kruskal** ($G, w$)
1. $A = \emptyset$;
2. **for** each vertex $v \in G.V$
3. **Make-Set**($v$)
4. **sort** edges in $E$ into non-decreasing order by their weight $w$
5. **for** each edge $(u, v) \in E$, taken in the order
6. **if** **Find Set** ($u$) $\neq$ **Find Set**($v$)
7. $A = A \cup \{(u, v)\}$
8. **Union**($u, v$)
Chapter 23. Minimum Spanning Trees

Theorem 23.1 gives a sufficient information for how to identify a safe edge to make the Generic MST algorithm work.

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5. for each edge $(u, v) \in E$, taken in the order
6.  if Find Set ($u$) $\neq$ Find Set($v$)
   7.  $A = A \cup \{(u, v)\}$
   8.  Union($u, v$)
9. return ($A$)
Chapter 23. Minimum Spanning Trees

Execution of Kruskal's algorithm for MST

Disjoint sets:
- [A] [B] [C] [D] [E] [F] [G] [H]
- [A] [B] [C] [D] [E] [F] [G] [H]
- [A] [B] [C] [D] [E] [F, G] [H]
- [A] [B] [D] [E] [C, F, G] [H]
- [A] [D] [E] [B, C, F, G] [H]
- [D] [E] [A, B, C, F, G] [H]
- [D, E] [A, B, C, F, G] [H]
- [D, E, H] [A, B, C, F, G]
- [A, B, C, D, E, F, G, H]
Chapter 23. Minimum Spanning Trees

At each iteration of the for loop, e.g., identify

\[ [D, E] [A, B, C, F, G] [H] \]

\[ [D, E] [A, B, C, F, G] [H] \]
Chapter 23. Minimum Spanning Trees

At each iteration of the for loop, e.g., identify

\[ A = \{ (A, F), (B, F), (C, G), (F, G) \} \]

Cut that respects \( A \):

\[ S = \{ A, B, C, D, F, G \} \]

\[ V - S = \{ E, H \} \]

Light edge \((D, E)\) crosses the cut;

\[ A = \{ (A, F), (B, F), (C, G), (F, G), (D, E) \} \]

Cut that respects \( A \):

\[ S = \{ A, B, C, F, G, H \} \]

\[ V - S = \{ E, H \} \]

Light edge \((E, H)\) crosses the cut;

\[ A = \]

\[ [D, E] [A, B, C, F, G] [H] \]

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Chapter 23. Minimum Spanning Trees

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Chapter 23. Minimum Spanning Trees

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\[ A = \{(A, F), (B, F), (C, G), (F, G)\} \]

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At each iteration of the for loop, e.g., identify

- $A = \{(A,F), (B,F), (C,G), (F,G)\}$,
- cut that respects $A$: $S = \{A, B, C, D, F, G\}$, $V - S = \{E, H\}$,
- light edge $(D, E)$ crosses the cut;
Chapter 23. Minimum Spanning Trees

At each iteration of the for loop, e.g., identify

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- cut that respects \( A \): \( S = \{A, B, C, D, F, G\} \), \( V - S = \{E, H\} \),
  
  light edge \((D, E)\) crosses the cut;

\[ A = \]

\[ \text{[D]} \quad \text{[E]} \quad [A, B, C, F, G] \quad [H] \]

\[ \text{[D, E]} \quad [A, B, C, F, G] \quad [H] \]
Chapter 23. Minimum Spanning Trees

At each iteration of the for loop, e.g., identify

- $\mathcal{A} = \{(A, F), (B, F), (C, G), (F, G)\}$,
  cut that respects $\mathcal{A}$: $S = \{A, B, C, D, F, G\}$, $V - S = \{E, H\}$,
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Chapter 23. Minimum Spanning Trees

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  - cut that respects \( A \): \( S = \{ A, B, C, D, F, G \} \), \( V - S = \{ E, H \} \),
  - light edge \( (D, E) \) crosses the cut;

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Chapter 23. Minimum Spanning Trees

The Kruskal's algorithm uses disjoint-set data structures
(where elements are partitioned into disjoint sets)
Chapter 23. Minimum Spanning Trees

The Kruskal’s algorithm uses disjoint-set data structures (where elements are partitioned into disjoint sets)

- **Make Set**($x$): create a set of single element $x$;
Chapter 23. Minimum Spanning Trees

The Kruskal’s algorithm uses disjoint-set data structures (where elements are partitioned into disjoint sets)

- **Make Set** \(x\): create a set of single element \(x\);
- **Find Set** \(x\): identify the set that contains element \(x\);
Chapter 23. Minimum Spanning Trees

The Kruskal’s algorithm uses disjoint-set data structures (where elements are partitioned into disjoint sets)

- **Make Set**\((x)\): create a set of single element \(x\);
- **Find Set**\((x)\): identify the set that contains element \(x\);
- **Union**\((x,y)\): union the two sets containing \(x\) and \(y\) into one;

Implementations (left: linked lists, Right: disjoint-set forest)

Time complexity: \(O(\log n)\) for *Make Set*\((x)\), *Find Set*\((x)\), *Union*\((x,y)\) with disjoint-set forest implementation.

Time complexity of Kruskal’s algorithm: \(O(|E| \log |V| + |V|)\).
Chapter 23. Minimum Spanning Trees

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Chapter 23. Minimum Spanning Trees

The Kruskal’s algorithm uses disjoint-set data structures (where elements are partitioned into disjoint sets)

- **MAKE_SET(x)**: create a set of single element x;
- **FIND_SET(x)**: identify the set that contains element x;
- **UNION(x, y)**: union the two sets containing x and y into one;

Implementations (left: linked lists, Right: disjoint-set forest)

Time complexity:
Chapter 23. Minimum Spanning Trees

The Kruskal’s algorithm uses disjoint-set data structures (where elements are partitioned into disjoint sets)

- **Make Set**\( (x) \): create a set of single element \( x \);
- **Find Set**\( (x) \): identify the set that contains element \( x \);
- **Union**\( (x, y) \): union the two sets containing \( x \) and \( y \) into one;

Implementations (left: linked lists, Right: disjoint-set forest)

Time complexity: \( O(\log n) \) for **Make Set**\( (x) \), **Find Set**\( (x) \), **Union**\( (x, y) \) with disjoint-set forest implementation.
Chapter 23. Minimum Spanning Trees

The Kruskal’s algorithm uses disjoint-set data structures (where elements are partitioned into disjoint sets)

- **Make Set**(x): create a set of single element x;
- **Find Set**(x): identify the set that contains element x;
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Chapter 23. Minimum Spanning Trees

MST-Prim(\(G, w, r\))
Chapter 23. Minimum Spanning Trees

MST-Prim($G, w, r$)
1. for each $u \in G.V$

usage of Priority queue: $Q$, Extract Min takes $O(\log n)$ time.

running time $O(|E| + |V| \log |V|)$. 
Chapter 23. Minimum Spanning Trees

MST-Prim($G, w, r$)
1. for each $u \in G.V$
2. $u.key = \infty$  \{ $u.key$ is the $u$'s shortest distance to set $A = V-Q$\}

runnig time $O(|E| + |V| \log |V|)$. 

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MST-Prim($G, w, r$)
1. for each $u \in G.V$
2. $u.key = \infty$ \quad \{ u.key is the u’s shortest distance to set $A = V-Q$ \}
3. $u.\pi = NULL$
MST-Prim$(G, w, r)$

1. for each $u \in G.V$
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4. $r.key = 0$ \{ start from vertex $r$ \}
Chapter 23. Minimum Spanning Trees

MST-Prim\((G, w, r)\)
1. \textbf{for} each \(u \in G.V\)
2. \(u.\text{key} = \infty\) \hspace{1em} \{ \textit{u.key} is the \(u\)'s shortest distance to set } \(A = V-Q\} \)
3. \(u.\pi = \text{NULL}\)
4. \(r.\text{key} = 0\) \hspace{1em} \{ start from vertex } \(r\}\)
5. \(Q = G.V\) \hspace{1em} \{ establish priority queue } \(Q\) with \textit{key} values\)

running time \(O(|E| + |V| \log |V|)\).
Chapter 23. Minimum Spanning Trees

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4. \(r.key = 0\) \{ start from vertex \(r\) \}
5. \(Q = G.V\) \{ establish priority queue \(Q\) wit \(key\) values\}
6. \textbf{while} \(Q \neq \emptyset\)
Chapter 23. Minimum Spanning Trees

MST-Prim\((G, w, r)\)
1. \textbf{for each} \(u \in G.V\) \{ \(u\.key\) is the \(u\)'s shortest distance to set \(A = V - Q\) \}
2. \(u\.key = \infty\)
3. \(u\.\pi = NULL\)
4. \(r\.key = 0\) \{ start from vertex \(r\) \}
5. \(Q = G.V\) \{ establish priority queue \(Q\) wit key values\}
6. \textbf{while} \(Q \neq \emptyset\)
7. \(u = \text{Extract Min}(Q)\)
Chapter 23. Minimum Spanning Trees

MST-PRI(M, w, r)
1. for each u ∈ G.V
2. u.key = ∞ \{ u.key is the u’s shortest distance to set A = V - Q} 
3. u.π = NULL
4. r.key = 0 \{ start from vertex r \}
5. Q = G.V \{ establish priority queue Q wit key values \}
6. while Q ≠ ∅
7. u = EXTRACT-MIN(Q)
8. for each v ∈ Adj[u]

running time \( O(|E| + |V| \log |V|) \).
Chapter 23. Minimum Spanning Trees

MST-PRIM(G, w, r)
1. for each \( u \in G.V \)  
   \( u.key = \infty \) \{ \( u.key \) is the \( u \)'s shortest distance to set \( A = V-Q \) \}
2. \( u.\pi = NULL \)
3. \( r.key = 0 \) \{ start from vertex \( r \) \}
4. \( Q = G.V \) \{ establish priority queue \( Q \) wit key values \}
5. while \( Q \neq \emptyset \)
   6. \( u = \text{Extract Min}(Q) \)
   7. for each \( v \in Adj[u] \)
   8. if \( v \in Q \) and \( w(u, v) < v.key \) \{ for those not in \( A \), update distances \}
Chapter 23. Minimum Spanning Trees

MST-Prim\((G, w, r)\)

1. \textbf{for} each \(u \in G.V\) \hfill \{ \textit{u.key} is the \(u\)'s shortest distance to set \(A = V-Q\) \}
2. \(u.\text{key} = \infty\)
3. \(u.\pi = \text{NULL}\)
4. \(r.\text{key} = 0\) \hfill \{ \text{start from vertex } r \} \hfill \{ \text{establish priority queue } Q \text{ wit } \text{key} \text{ values} \}
5. \(Q = G.V\)
6. \textbf{while} \(Q \neq \emptyset\) \hfill \{ \text{running time } O( |E| + |V| \log |V| ) \}
7. \(u = \text{Extract Min}(Q)\)
8. \textbf{for} each \(v \in \text{Adj}[u]\)
9. \hfill \textbf{if} \(v \in Q \text{ and } w(u,v) < v.\text{key} \) \hfill \{ \text{for those not in } A, \text{ update distances} \}
10. \hfill \textbf{then} \(v.\pi = u\)
MST-Prim\((G, w, r)\)
1. \textbf{for} each \(u \in G.V\) 
2. \(u.\text{key} = \infty\) \{ \text{\textit{u.key} is the \textit{u}'s shortest distance to set } A = V-Q \}\n3. \(u.\pi = \text{NULL}\)
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6. \textbf{while} \(Q \neq \emptyset\)
7. \(u = \text{Extract Min}(Q)\)
8. \textbf{for} each \(v \in Adj[u]\)
9. \quad \textbf{if} \(v \in Q \) \text{ and } \(w(u, v) < v.\text{key}\) \{ \text{for those not in } A, \text{ update distances} \}
10. \quad \text{then } v.\pi = u
11. \quad v.\text{key} = w(u, v)
Chapter 23. Minimum Spanning Trees

MST-Prim($G, w, r$)
1. for each $u \in G.V$
2. $u.key = \infty$ \{ $u.key$ is the $u$’s shortest distance to set $A = V-Q$ \}
3. $u.\pi = NULL$
4. $r.key = 0$ \{ start from vertex $r$ \}
5. $Q = G.V$ \{ establish priority queue $Q$ wit key values \}
6. while $Q \neq \emptyset$
7. $u = \text{Extract Min}(Q)$
8. for each $v \in \text{Adj}[u]$
9. if $v \in Q$ and $w(u, v) < v.key$ \{ for those not in $A$, update distances \}
10. then $v.\pi = u$
11. $v.key = w(u, v)$
12. return $\pi$

Usage of Priority queue: $Q$, Extract Min takes $O(\log n)$ time.
Running time $O(|E| + |V| \log |V|)$. 


Chapter 23. Minimum Spanning Trees

MST-Prim\((G, w, r)\)

1. \textbf{for} each \(u \in G.V\) \{ \(u.key\) is the \(u\)'s shortest distance to set \(A = V - Q\) \}
2. \(u.key = \infty\)
3. \(u.\pi = NULL\)
4. \(r.key = 0\) \{ start from vertex \(r\) \}
5. \(Q = G.V\) \{ establish priority queue \(Q\) wit key values \}
6. \textbf{while} \(Q \neq \emptyset\)
7. \(u = \text{Extract Min}(Q)\)
8. \textbf{for} each \(v \in Adj[u]\)
9. \quad \text{if} \(v \in Q\) and \(w(u, v) < v.key\) \{ for those not in \(A\), update distances \}
10. \quad \text{then} \(v.\pi = u\)
11. \quad \(v.key = w(u, v)\)
12. \textbf{return} \(\pi\)

usage of Priority queue: \(Q\), \textbf{Extract Min} takes \(O(\log n)\) time.
Chapter 23. Minimum Spanning Trees

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running time \( O(|E| + |V| \log |V|) \).
Chapter 23. Minimum Spanning Trees
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Some questions about MST
Chapter 23. Minimum Spanning Trees

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  what are subproblems, or what do subsolutions look like?
Chapter 24. Single Source Shortest Paths

Chapter 24. Single-source shortest paths
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Chapter 24. Single-source shortest paths

Given a graph \( G = (V, E) \), with weight \( w : E \to \mathbb{R} \); and single vertex \( s \in V \);
Chapter 24. Single Source Shortest Paths

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Chapter 24. Single Source Shortest Paths

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- Shortest distance between $u$ and $v$ is

\[
\delta(u, v) = \min_{u \rightsquigarrow v} \{w(p)\}
\]
Chapter 24. Single Source Shortest Paths

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Chapter 24. Single Source Shortest Paths

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Chapter 24. Single Source Shortest Paths

- Single-source shortest paths: from $s$ to each vertex $v \in V$
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- All-pairs shortest paths: from $s$ to $t$ for all pairs $s, t \in V$. 
Lemma 24.1 (a subpath of a shortest path is a shortest path)

Proof idea: (proof by contradiction)
Assume that $p_{i,j}$ is not the shortest path from $v_i$ to $v_j$. Then there is a shorter path $q_{i,j}$ from $v_i$ to $v_j$.

Define path $q = (v_0, \ldots, v_i, q_{i,j}, v_j, \ldots, v_k)$ has weight $w(q) = i \sum_{t=1}^{i} w(v_t-1, v_t) + w(q_{i,j}) + k - 1 \sum_{t=j}^{k} w(v_t, v_t+1) < i \sum_{t=1}^{i} w(v_t-1, v_t) + w(p_{i,j}) + k - 1 \sum_{t=j}^{k} w(v_t, v_t+1) = w(p)$
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Chapter 24. Single Source Shortest Paths

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Chapter 24. Single Source Shortest Paths

Some terminologies:

• negative weights are allowed;
• cycles on a path: not a simple path;
• negative weight cycles, 0 weight cycles representing shortest paths: predecessor
• shortest path tree:

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\text{http://graphserver.sourceforge.net/gallery.html (width } \propto \frac{1}{\text{distance}})\]
Chapter 24. Single Source Shortest Paths

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Chapter 24. Single Source Shortest Paths

Technique: relaxation

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Chapter 24. Single Source Shortest Paths

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Chapter 24. Single Source Shortest Paths

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- **In other words:**
  
  Let $v.d$ be an weight upper bound of a shortest path from $s$ to $v$,
  
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  The process of relaxing edge $(u, v)$: improves $v.d$ by taking the path
  
  through $u$, and update $v.d$ and $v.\pi$. 
Bellman-Ford algorithm

1. for each vertex \( v \in G.V \) initialization
2. \( v.d = \infty \)
3. \( v.\pi = \text{NULL} \)
4. \( s.d = 0 \)
5. for \( i = 1 \) to \(|V| - 1\) relaxation
6. for each edge \( (u,v) \in G.E \)
7. if \( v.d > u.d + w(u,v) \)
8. \( v.d = u.d + w(u,v) \)
9. \( v.\pi = u \)
10. for each edge \( (u,v) \in G.E \) checking negative weight cycle
11. if \( v.d > u.d + w(u,v) \)
12. return (FALSE)
13. return (TRUE)

Running time: \( O(|V||E|) \)
Chapter 24. Single Source Shortest Paths

Bellman-Ford algorithm

\textsc{Bellman-Ford}(G, w, s)
Chapter 24. Single Source Shortest Paths

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**Bellman-Ford**(\(G, w, s\))

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Running time: \(O(|V||E|)\)
Bellman-Ford algorithm

**Bellman-Ford**($G, w, s$)

1. for each vertex $v \in G.V$ initialization
2. $v.d = \infty$

Running time: $O(|V||E|)$
Chapter 24. Single Source Shortest Paths

Bellman-Ford algorithm

\textsc{Bellman-Ford}(G, w, s)
1. \textbf{for} each vertex \( v \in G.V \) \hspace{1cm} \text{initialization}
2. \hspace{1cm} \( v.d = \infty \)
3. \hspace{1cm} \( v.\pi = NULL \)
4. \hspace{1cm} \( s.d = 0 \)
5. \hspace{1cm} \textbf{for} \( i = 1 \) to \( |V| - 1 \) \hspace{1cm} \text{relaxation}
6. \hspace{2cm} \textbf{for} each edge \((u,v) \in G.E \)
7. \hspace{3cm} \textbf{if} \( v.d > u.d + w(u,v) \)
8. \hspace{3cm} \hspace{1cm} \( v.d = u.d + w(u,v) \)
9. \hspace{3cm} \hspace{1cm} \( v.\pi = u \)
10. \hspace{2cm} \textbf{for} each edge \((u,v) \in G.E \) \hspace{1cm} \text{checking negative weight cycle}
11. \hspace{3cm} \textbf{if} \( v.d > u.d + w(u,v) \)
12. \hspace{3cm} \hspace{1cm} \text{return} (FALSE)
13. \hspace{3cm} \hspace{1cm} \text{return} (TRUE)

Running time: \( O(|V||E|) \)
Bellman-Ford algorithm

\textsc{Bellman-Ford}(G, w, s)
1. for each vertex \( v \in G.V \) initialization
2. \( v.d = \infty \)
3. \( v.\pi = NULL \)
4. \( s.d = 0 \)
5. for \( i = 1 \) to \(|V| - 1\) relaxation
6. for each edge \((u, v) \in G.E\)
7. if \( v.d > u.d + w(u,v) \)
8. \( v.d = u.d + w(u,v) \)
9. \( v.\pi = u \)
10. for each edge \((u, v) \in G.E\) checking negative weight cycle
11. if \( v.d > u.d + w(u,v) \)
12. return (FALSE)
13. return (TRUE)

Running time: \( O(|V||E|) \)
Chapter 24. Single Source Shortest Paths

Bellman-Ford algorithm

**Bellman-Ford** \((G, w, s)\)

1. **for** each vertex \(v \in G.V\) initialization
2. \(v.d = \infty\)
3. \(v.\pi = NULL\)
4. \(s.d = 0\)
5. **for** \(i = 1\) to \(|V| - 1\) relaxation

Running time: \(O(|V| |E|)\)
Bellman-Ford algorithm

\textsc{Bellman-Ford}(G, w, s)

1. \textbf{for} each vertex \(v \in G.V\) \hspace{1cm} \text{initialization}
2. \(v.d = \infty\)
3. \(v.\pi = NULL\)
4. \(s.d = 0\)
5. \textbf{for} \(i = 1\) to \(|V| - 1\) \hspace{1cm} \text{relaxation}
6. \textbf{for} each edge \((u, v) \in G.E\)
Bellman-Ford algorithm

**Bellman-Ford** (*G*, *w*, *s*)
1. for each vertex *v* ∈ *G.V* initialization
2. *v.d* = $\infty$
3. *v.π* = NULL
4. *s.d* = 0
5. for *i* = 1 to |*V*| − 1 relaxation
6. for each edge (*u*, *v*) ∈ *G.E*
7. if *v.d* > *u.d* + *w*(*u*, *v*)

Running time: $O(|V| \cdot |E|)$
Bellman-Ford algorithm

**Bellman-Ford**($G, w, s$)

1. for each vertex $v \in G.V$ initialization
2. $v.d = \infty$
3. $v.\pi = NULL$
4. $s.d = 0$
5. for $i = 1$ to $|V| - 1$ relaxation
6. for each edge $(u, v) \in G.E$
7. if $v.d > u.d + w(u, v)$
8. $v.d = u.d + w(u, v)$

Running time : $O(|V||E|)$
Bellman-Ford algorithm

**Bellman-Ford** \((G, w, s)\)

1. for each vertex \(v \in G.V\) initialization
2. \(v.d = \infty\)
3. \(v.\pi = NULL\)
4. \(s.d = 0\)
5. for \(i = 1\) to \(|V| - 1\) relaxation
6. for each edge \((u, v) \in G.E\)
7. if \(v.d > u.d + w(u, v)\)
8. \(v.d = u.d + w(u, v)\)
9. \(v.\pi = u\)

**Running time:** \(O(|V||E|)\)
Bellman-Ford algorithm

\textsc{Bellman-Ford}(G, w, s)

1. \textbf{for} each vertex \( v \in G.V \) \textbf{ initialization}
2. \( v.d = \infty \)
3. \( v.\pi = NULL \)
4. \( s.d = 0 \)
5. \textbf{for} \( i = 1 \) to \(|V| - 1\) \textbf{ relaxation}
6. \textbf{ for} each edge \((u, v) \in G.E\)
7. \textbf{ if} \( v.d > u.d + w(u, v) \)
8. \( v.d = u.d + w(u, v) \)
9. \( v.\pi = u \)
10. \textbf{ for} each edge \((u, v) \in G.E\) \textbf{ checking negative weight cycle}

Running time: \( O(|V||E|) \)
Chapter 24. Single Source Shortest Paths

Bellman-Ford algorithm

\textit{Bellman-Ford}(G, w, s)

1. \textbf{for} each vertex \( v \in G.V \) \hspace{1cm} \text{initialization}
2. \hspace{1cm} \( v.d = \infty \)
3. \hspace{1cm} \( v.\pi = NULL \)
4. \hspace{1cm} \( s.d = 0 \)
5. \hspace{1cm} \textbf{for} \( i = 1 \) \textbf{to} \( |V| - 1 \) \hspace{1cm} \text{relaxation}
6. \hspace{2cm} \textbf{for} each edge \((u, v) \in G.E\)
7. \hspace{3cm} \textbf{if} \( v.d > u.d + w(u, v) \)
8. \hspace{4cm} \( v.d = u.d + w(u, v) \)
9. \hspace{4cm} \( v.\pi = u \)
10. \hspace{1cm} \textbf{for} each edge \((u, v) \in G.E\) \hspace{1cm} \text{checking negative weight cycle}
11. \hspace{2cm} \textbf{if} \( v.d > u.d + w(u, v) \)

Running time: \( O(|V||E|) \)
Bellman-Ford algorithm

Bellman-Ford \((G, w, s)\)

1. for each vertex \(v \in G.V\) initialization
2. \(v.d = \infty\)
3. \(v.\pi = Null\)
4. \(s.d = 0\)
5. for \(i = 1\) to \(|V| - 1\) relaxation
6. for each edge \((u, v) \in G.E\)
7. if \(v.d > u.d + w(u, v)\)
8. \(v.d = u.d + w(u, v)\)
9. \(v.\pi = u\)
10. for each edge \((u, v) \in G.E\) checking negative weight cycle
11. if \(v.d > u.d + w(u, v)\)
12. return (FALSE)

Running time: \(O(|V||E|)\)
Bellman-Ford algorithm

\text{Bellman-Ford}(G, w, s)

1. \textbf{for} each vertex \( v \in G.V \) \textit{ initialization}
2. \( v.d = \infty \)
3. \( v.\pi = NULL \)
4. \( s.d = 0 \)
5. \textbf{for} \( i = 1 \text{ to } |V| - 1 \) \textit{ relaxation}
6. \textbf{for} each edge \((u, v) \in G.E\)
7. \hspace{1em} \textbf{if} \( v.d > u.d + w(u, v) \)
8. \hspace{1em} \( v.d = u.d + w(u, v) \)
9. \hspace{1em} \( v.\pi = u \)
10. \textbf{for} each edge \((u, v) \in G.E\) \textit{ checking negative weight cycle}
11. \hspace{1em} \textbf{if} \( v.d > u.d + w(u, v) \)
12. \hspace{1em} \textbf{return} (FALSE)
13. \textbf{return} (TRUE)

**Running time:** \( O(|V||E|) \)
Chapter 24. Single Source Shortest Paths

Bellman-Ford algorithm

\textbf{Bellman-Ford}(G, w, s)

1. \textbf{for} each vertex \( v \in G.V \) \hspace{1cm} \text{initialization}
2. \( v.d = \infty \)
3. \( v.\pi = \text{NULL} \)
4. \( s.d = 0 \)
5. \textbf{for} \( i = 1 \) \textbf{to} \( |V| - 1 \) \hspace{1cm} \text{relaxation}
6. \textbf{for} each edge \( (u, v) \in G.E \)
7. \hspace{1cm} \textbf{if} \( v.d > u.d + w(u, v) \)
8. \hspace{1cm} \( v.d = u.d + w(u, v) \)
9. \hspace{1cm} \( v.\pi = u \)
10. \textbf{for} each edge \( (u, v) \in G.E \) \hspace{1cm} \text{checking negative weight cycle}
11. \hspace{1cm} \textbf{if} \( v.d > u.d + w(u, v) \)
12. \hspace{1cm} \textbf{return} (FALSE)
13. \hspace{1cm} \textbf{return} (TRUE)

Running time: \( O(|V||E|) \)
Chapter 24. Single Source Shortest Paths
Chapter 24. Single Source Shortest Paths

1. 

2. 

3. 

4. 

5.
Properties of shortest paths and relaxation
Properties of shortest paths and relaxation

\texttt{RELAX}(u, v, w)
Chapter 24. Single Source Shortest Paths

Properties of shortest paths and relaxation

**RELAX**\((u, v, w)\)

1. **if** \(v.d > u.d + w(u, v)\)
Chapter 24. Single Source Shortest Paths

Properties of shortest paths and relaxation

**Relax** \((u, v, w)\)

1. if \(v.d > u.d + w(u, v)\)
2. \(v.d = u.d + w(u, v)\)
Properties of shortest paths and relaxation

\textsc{Relax}(u, v, w)

1. if \( v.d > u.d + w(u, v) \)
2. \( v.d = u.d + w(u, v) \)
3. \( v.\pi = u \)
Chapter 24. Single Source Shortest Paths

Properties of shortest paths and relaxation

\texttt{RELAX}(u, v, w)

1. \textbf{if} \( v.d > u.d + w(u,v) \)
2. \( v.d = u.d + w(u,v) \)
3. \( v.\pi = u \)

\textbf{Lemma 24.14, Convergence property}: Let \( s \rightsquigarrow u \rightarrow v \) is a shortest path. If \( u.d = \delta(s,u) \) holds before \texttt{RELAX}(u, v, w) is called, then \( v.d = \delta(s,v) \) after the call.
Chapter 24. Single Source Shortest Paths

Properties of shortest paths and relaxation

\textbf{RELAX}(u, v, w)

1. \textbf{if} \ v.d > u.d + w(u, v)
2. \ v.d = u.d + w(u, v)
3. \ v.\pi = u

\textbf{Lemma 24.14, Convergence property}: Let \( s \leadsto u \rightarrow v \) is a shortest path. If \( u.d = \delta(s, u) \) holds before \textbf{RELAX}(u, v, w) is called, then \( v.d = \delta(s, v) \) after the call.

\textbf{Proof}: 

\ldots
Chapter 24. Single Source Shortest Paths

Properties of shortest paths and relaxation

\texttt{RELAX}(u, v, w)

1. if $v.d > u.d + w(u, v)$
2. $v.d = u.d + w(u, v)$
3. $v.\pi = u$

\textbf{Lemma 24.14, Convergence property}: Let $s \sim u \rightarrow v$ is a shortest path. If $u.d = \delta(s, u)$ holds before \texttt{RELAX}(u, v, w) is called, then $v.d = \delta(s, v)$ after the call.

\textbf{Proof}: $v.d \leq u.d + w(u, v)$
Properties of shortest paths and relaxation

\text{RELAX}(u, v, w)

1. if \( v.d > u.d + w(u, v) \)
2. \( v.d = u.d + w(u, v) \)
3. \( v.\pi = u \)

\textbf{Lemma 24.14, Convergence property}: Let \( s \leadsto u \rightarrow v \) is a shortest path. If \( u.d = \delta(s, u) \) holds before \text{RELAX}(u, v, w) \) is called, then \( v.d = \delta(s, v) \) after the call.

\textbf{Proof}: \( v.d \leq u.d + w(u, v) = \delta(s, u) + w(u, v) \)
Properties of shortest paths and relaxation

\textbf{RELAX}(u, v, w)
1. if \( v.d > u.d + w(u, v) \)
2. \( v.d = u.d + w(u, v) \)
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\textbf{Proof}: \( v.d \leq u.d + w(u, v) = \delta(s, u) + w(u, v) = \delta(s, v) \).
Chapter 24. Single Source Shortest Paths

Properties of shortest paths and relaxation

\texttt{RELAX}(u, v, w)
1. if \( v.d > u.d + w(u, v) \)
2. \( v.d = u.d + w(u, v) \)
3. \( v.\pi = u \)

\textbf{Lemma 24.14, Convergence property}: Let \( s \leadsto u \rightarrow v \) is a shortest path. If \( u.d = \delta(s, u) \) holds before \texttt{RELAX}(u, v, w) is called, then \( v.d = \delta(s, v) \) after the call.

\textbf{Proof}: \( v.d \leq u.d + w(u, v) = \delta(s, u) + w(u, v) = \delta(s, v) \). So \( v.d = \delta(s, v) \).
We want to prove that, if a shortest path $s \leadsto v$ consists of $k$ edges, Bellman-Fold obtains value $v.d = \delta(s, v)$ after the $k$th round of relaxation (assuming there is no negative cycle).
Chapter 24. Single Source Shortest Paths

We want to prove that, if a shortest path \( s \leadsto v \) consists of \( k \) edges, Bellman-Fold obtains value \( v.d = \delta(s, v) \) after the \( k \)th round of relaxation (assuming there is no negative cycle).

Proof idea: Induction on \( k \).
Chapter 24. Single Source Shortest Paths

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Proof idea: Induction on \( k \).

- \( k = 0 \), \( v \) can only be \( s \). Proved!
We want to prove that, if a shortest path $s \rightarrow v$ consists of $k$ edges, Bellman-Fold obtains value $v.d = \delta(s, v)$ after the $k$th round of relaxation (assuming there is no negative cycle).

**Proof idea:** Induction on $k$.

- $k = 0$, $v$ can only be $s$. Proved!
- Assume the **claim** is proved for all vertices $v$ that have a shortest path of length $k$. 
We want to prove that, if a shortest path \( s \leadsto v \) consists of \( k \) edges, Bellman-Fold obtains value \( v.d = \delta(s, v) \) after the \( k \)th round of relaxation (assuming there is no negative cycle).

**Proof idea:** Induction on \( k \).

- \( k = 0 \), \( v \) can only be \( s \). Proved!
- Assume the **claim** is proved for all vertices \( v \) that have a shortest path of length \( k \). What claim again??
Chapter 24. Single Source Shortest Paths

We want to prove that, if a shortest path $s \leadsto v$ consists of $k$ edges, Bellman-Fold obtains value $v.d = \delta(s,v)$ after the $k$th round of relaxation (assuming there is no negative cycle).

Proof idea: Induction on $k$.

- $k = 0$, $v$ can only be $s$. Proved!

- Assume the claim is proved for all vertices $v$ that have a shortest path of length $k$. What claim again??

- Let $v$ be any vertex that has a shortest path $s \leadsto u \rightarrow v$, consisting of $k + 1$ edges;
We want to prove that, if a shortest path \( s \leadsto v \) consists of \( k \) edges, Bellman-Fold obtains value \( v.d = \delta(s,v) \) after the \( k \)th round of relaxation (assuming there is no negative cycle).

Proof idea: Induction on \( k \).

- \( k = 0 \), \( v \) can only be \( s \). Proved!

- Assume the claim is proved for all vertices \( v \) that have a shortest path of length \( k \). What claim again??

- Let \( v \) be any vertex that has a shortest path \( s \leadsto u \rightarrow v \), consisting of \( k + 1 \) edges;

  Then \( s \leadsto u \) is a shortest path for \( u \) consisting of \( k \) edges;
We want to prove that, if a shortest path \( s \leadsto v \) consists of \( k \) edges, Bellman-Fold obtains value \( v.d = \delta(s, v) \) after the \( k \)th round of relaxation (assuming there is no negative cycle).

Proof idea: Induction on \( k \).

- \( k = 0 \), \( v \) can only be \( s \). Proved!

- Assume the claim is proved for all vertices \( v \) that have a shortest path of length \( k \). What claim again??

- Let \( v \) be any vertex that has a shortest path \( s \leadsto u \rightarrow v \), consisting of \( k + 1 \) edges;

Then \( s \leadsto u \) is a shortest path for \( u \) consisting of \( k \) edges;

Now by assumption, \( u.d = \delta(s, u) \) after \( k \) round of relaxation.
Chapter 24. Single Source Shortest Paths

We want to prove that, if a shortest path \( s \leadsto v \) consists of \( k \) edges, Bellman-Fold obtains value \( v.d = \delta(s, v) \) after the \( k \)th round of relaxation (assuming there is no negative cycle).

Proof idea: Induction on \( k \).

- \( k = 0 \), \( v \) can only be \( s \). Proved!

- Assume the claim is proved for all vertices \( v \) that have a shortest path of length \( k \). What claim again??

- Let \( v \) be any vertex that has a shortest path \( s \leadsto u \rightarrow v \), consisting of \( k + 1 \) edges;

Then \( s \leadsto u \) is a shortest path for \( u \) consisting of \( k \) edges;

Now by assumption, \( u.d = \delta(s, u) \) after \( k \) round of relaxation.

By Convergence property Lemma, \( v.d = \delta(s, v) \)
after another round of relaxation.
Lemma 24.15, Path-relaxation property: Let $p = (v_0, v_1, \ldots, v_k)$ be a shortest path from $s = v_0$ to $v_k$. If a sequence relaxation steps occur that includes, in order, relaxing the edges $(v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k)$, then $v_k.d = \delta(s, v_k)$ after these relaxations and at all times afterward. This property holds no matter what other edge relaxations occur, including relaxations that are intermixed with relaxations of these edges.
Lemma 24.15, Path-relaxation property: Let $p = (v_0, v_1, \ldots, v_k)$ be a shortest path from $s = v_0$ to $v_k$. If a sequence relaxation steps occur that includes, in order, relaxing the edges $(v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k)$, then $v_k.d = \delta(s, v_k)$ after these relaxations and at all times afterward. This property holds no matter what other edge relaxations occur, including relaxations that are intermixed with relaxations of these edges.

Proof:
Lemma 24.15, Path-relaxation property: Let $p = (v_0, v_1, \ldots, v_k)$ be a shortest path from $s = v_0$ to $v_k$. If a sequence relaxation steps occur that includes, in order, relaxing the edges $(v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k)$, then $v_k.d = \delta(s, v_k)$ after these relaxations and at all times afterward. This property holds no matter what other edge relaxations occur, including relaxations that are intermixed with relaxations of these edges.

Proof: We prove by induction on $i$ that after the $i$th edge $(v_{i-1}, v_i)$ on path $p$ is relaxed, $v_i.d = \delta(s, v_i)$.
Lemma 24.15, Path-relaxation property: Let \( p = (v_0, v_1, \ldots, v_k) \) be a shortest path from \( s = v_0 \) to \( v_k \). If a sequence relaxation steps occur that includes, in order, relaxing the edges \((v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k)\), then \( v_k.d = \delta(s, v_k) \) after these relaxations and at all times afterward. This property holds no matter what other edge relaxations occur, including relaxations that are intermixed with relaxations of these edges.

Proof: We prove by induction on \( i \) that after the \( i \)th edge \((v_{i-1}, v_i)\) on path \( p \) is relaxed, \( v_i.d = \delta(s, v_i) \).

basis: \( i = 0 \). \( v_0 = s \), \( s.d = 0 = \delta(s, s) \).
Lemma 24.15, Path-relaxation property: Let \( p = (v_0, v_1, \ldots, v_k) \) be a shortest path from \( s = v_0 \) to \( v_k \). If a sequence relaxation steps occur that includes, in order, relaxing the edges \((v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k)\), then \( v_k.d = \delta(s, v_k) \) after these relaxations and at all times afterward. This property holds no matter what other edge relaxations occur, including relaxations that are intermixed with relaxations of these edges.

Proof: We prove by induction on \( i \) that after the \( i \)th edge \((v_{i-1}, v_i)\) on path \( p \) is relaxed, \( v_i.d = \delta(s, v_i) \).

basis: \( i = 0. \ v_0 = s, \ s.d = 0 = \delta(s, s) \)!
Assume: \( v_{i-1}.d = \delta(s, v_{i-1}) \).
**Chapter 24. Single Source Shortest Paths**

**Lemma 24.15, Path-relaxation property:** Let $p = (v_0, v_1, \ldots, v_k)$ be a shortest path from $s = v_0$ to $v_k$. If a sequence relaxation steps occur that includes, in order, relaxing the edges $(v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k)$, then $v_k.d = \delta(s, v_k)$ after these relaxations and at all times afterward. This property holds no matter what other edge relaxations occur, including relaxations that are intermixed with relaxations of these edges.

**Proof:** We prove by induction on $i$ that after the $i$th edge $(v_{i-1}, v_i)$ on path $p$ is relaxed, $v_i.d = \delta(s, v_i)$.

**basis:** $i = 0$. $v_0 = s$, $s.d = 0 = \delta(s, s)$

**Assume:** $v_{i-1}.d = \delta(s, v_{i-1})$.

**Induction:** After we relax edge $(v_{i-1}, v_i)$, by convergence property, we have $v_i.d = \delta(s, v_i)$. 


Lemma 24.15, Path-relaxation property: Let $p = (v_0, v_1, \ldots, v_k)$ be a shortest path from $s = v_0$ to $v_k$. If a sequence relaxation steps occur that includes, in order, relaxing the edges $(v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k)$, then $v_k.d = \delta(s, v_k)$ after these relaxations and at all times afterward. This property holds no matter what other edge relaxations occur, including relaxations that are intermixed with relaxations of these edges.

Proof: We prove by induction on $i$ that after the $i$th edge $(v_{i-1}, v_i)$ on path $p$ is relaxed, $v_i.d = \delta(s, v_i)$.

basis: $i = 0$. $v_0 = s$, $s.d = 0 = \delta(s, s)$ !
Assume: $v_{i-1}.d = \delta(s, v_{i-1})$.
Induction: After we relax edge $(v_{i-1}, v_i)$, by convergence property, we have $v_i.d = \delta(s, v_i)$. And this holds for all times afterward.
Chapter 24. Single Source Shortest Paths

Correctness of **Bellman-Ford** algorithm

1. On graphs without negative cycles.

**Lemma 24.2**

Let $G = (V,E)$ be a weighted, directed graph with source $s$ and weight function $w: E \rightarrow \mathbb{R}$ and assume that $G$ contains no negative weight cycles that can be reached from $s$. Then after $|V| - 1$ iterations of line 5 in the algorithm, $v.d = \delta(s,v)$ for all vertices $v$ that are reachable from $s$.

**Proof**: (By induction on $k$, the number of edges on the computed path $p$: $s \xrightarrow{p} v$, to prove the claim to be true).

**Base**: $k = 0$. $v = s$. It is true.

**Assume**: the claim is true for $k - 1$.

**Induction**: computed path $p$: $s \xrightarrow{p} v$ has $k$ edges and $p$ arrives at $x$ before reaching $v$ via $(x,v)$. So $v.d = x.d + w(x,v)$.

By Lemma 24.1, $\delta(s,v) = \delta(s,y) + w(y,v)$ for some $y$.

Since after $k$ iterations, $v.d$ has been updated with the statement

$$\text{if } v.d > u.d + w(u,v) \text{ then } v.d = u.d + w(u,v),$$

for all $u$, including $x$ and $y$.

By the assumption, for every $u$, including $x$ and $y$, $u.d = \delta(s,u)$ because the computed path $s \xrightarrow{} u$ contains $k - 1$ edges. So we have

$$v.d = x.d + w(x,v) \leq y.d + w(y,v) = \delta(s,y) + w(y,v) = \delta(s,v)$$
Chapter 24. Single Source Shortest Paths

Correctness of \textbf{Bellman-Ford} algorithm

1. On graphs without negative cycles)
Chapter 24. Single Source Shortest Paths

Correctness of Bellman-Ford algorithm

1. On graphs without negative cycles

**Lemma 24.2** Let \( G = (V, E) \) be a weighted, directed graph with source \( s \) and weight function \( w : E \rightarrow \mathbb{R} \) and assume that \( G \) contains no negative weight cycles that can be reached from \( s \). Then after \( |V| - 1 \) iterations of line 5 in the algorithm, \( v.d = \delta(s, v) \) for all vertices \( v \) that are reachable from \( s \).
Chapter 24. Single Source Shortest Paths

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\textbf{Proof:} (By induction on $k$,}
Chapter 24. Single Source Shortest Paths

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Chapter 24. Single Source Shortest Paths

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\textbf{Proof}: (By induction on $k$, the number of edges on the computed path $p$: $s \xrightarrow{p} v$, to prove the claim to be true).

\textbf{Base}: $k = 0$. $v = s$. It is true.
Chapter 24. Single Source Shortest Paths

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Chapter 24. Single Source Shortest Paths

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Chapter 24. Single Source Shortest Paths

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By Lemma 24.1, $\delta(s, v) = \delta(s, y) + w(y, v)$ for some $y$. 
Chapter 24. Single Source Shortest Paths

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Chapter 24. Single Source Shortest Paths

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Chapter 24. Single Source Shortest Paths

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Chapter 24. Single Source Shortest Paths

**Theorem 24.4** Bellman-Ford algorithm is correct on weighted, directed graphs.
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Chapter 24. Single Source Shortest Paths

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Let the cycle to be $c = (v_0, v_1, \cdots, v_k)$, where $v_0 = v_k$ and $\sum_{i=1}^{k} w(v_{i-1}, v_i) < 0$.

Assume for all $i$, $v_i.d \leq v_{i-1}.d + w(v_{i-1}, v_i)$.

Then $\sum_{i=1}^{k} v_i.d \leq \sum_{i=1}^{k} v_{i-1}.d + \sum_{i=1}^{k} w(v_{i-1}, v_i)$, implying $\sum_{i=1}^{k} w(v_{i-1}, v_i) \geq 0$ contradicting $c$ being a negative cycle.
Chapter 24. Single Source Shortest Paths

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Chapter 24. Single Source Shortest Paths

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$$
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contradicting to $c$ being a negative cycle where $\sum_{i=1}^{k} w(v_{i-1}, v_i) < 0$
Finding shortest paths on DAGs (directed acyclic graphs)
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- Algorithms can take the advantage of the non-cyclicity.
Finding shortest paths on DAGs (directed acyclic graphs)

- Algorithms can take the advantage of the non-cyclicity.
- How would your algorithm be?
Finding shortest paths on DAGs (directed acyclic graphs)

- Algorithms can take the advantage of the non-cyclicity.
- How would your algorithm be?

Topological order of vertices!
Chapter 24. Single Source Shortest Paths

\textbf{Dag-Shortest Paths}(G, w, s)

1. topologically sort the vertices of $G.V$
2. for each vertex $v \in G.V$
3. $v.d = \infty$
4. $v.\pi = \text{NULL}$
5. $s.d = 0$
6. for each $u \in G.V$, in the topologically sorted order
7. for each vertex $v \in \text{Adj}[u]$
8. if $v.d > u.d + w(u,v)$
9. $v.d = u.d + w(u,v)$
10. $v.\pi = u$
11. return $(d, \pi)$

• Should we improve lines 6-7?
• Running time: ?
Chapter 24. Single Source Shortest Paths

**Dag-Shortest Paths** ($G, w, s$)

1. topologically sort the vertices of $G.V$
Chapter 24. Single Source Shortest Paths

**DAG-Shortest Paths** \((G, w, s)\)

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Chapter 24. Single Source Shortest Paths

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Chapter 24. Single Source Shortest Paths

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Chapter 24. Single Source Shortest Paths

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7. \hspace{1em} for each vertex $v \in Adj[u]$
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Chapter 24. Single Source Shortest Paths

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Chapter 24. Single Source Shortest Paths

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Chapter 24. Single Source Shortest Paths

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● Should we improve lines 6-7?
● Running time: ?
Chapter 24. Single Source Shortest Paths

note: the root is $s$. 
Dijkstra’s algorithm

On weighted, directed graphs in which each edge has non-negative weight.
Dijkstra’s algorithm

On weighted, directed graphs in which each edge has non-negative weight.

\textsc{Dijkstra}(G, w, s)
Dijkstra’s algorithm

On weighted, directed graphs in which each edge has non-negative weight.

\textsc{Dijkstra}(G, w, s)
1. for each vertex \( v \in G.V \)
Chapter 24. Single Source Shortest Paths

**Dijkstra's algorithm**

On weighted, directed graphs in which each edge has non-negative weight.

\[
\text{Dijkstra}(G, w, s)
\]

1. for each vertex \( v \in G.V \)
2. \( v.d = \infty \)

Running time:?
Chapter 24. Single Source Shortest Paths

**Dijkstra’s algorithm**

On weighted, directed graphs in which each edge has non-negative weight.

\[ \text{DIJKSTRA}(G, w, s) \]

1. for each vertex \( v \in G.V \)
2. \( v.d = \infty \)
3. \( v.\pi = \text{NULL} \)
4. \( s.d = 0 \)
5. \( S = \emptyset \)
6. \( Q = G.V \)
7. while \( Q \) is not empty
8. \( u = \text{Extract Min}(Q) \)
9. \( S = S \cup \{u\} \)
10. for each vertex \( v \in \text{Adj}[u] \)
11. if \( v.d > u.d + w(u, v) \)
12. \( v.d = u.d + w(u, v) \)
13. \( v.\pi = u \)
14. return \((d, \pi)\)

Running time:?
Dijkstra’s algorithm

On weighted, directed graphs in which each edge has non-negative weight.

\text{Dijkstra}(G, w, s)
1. \text{for each vertex } v \in G.V
2. \hspace{0.5cm} v.d = \infty
3. \hspace{0.5cm} v.\pi = NULL
4. \hspace{0.5cm} s.d = 0

Running time:?
Dijkstra’s algorithm

On weighted, directed graphs in which each edge has non-negative weight.

\textsc{Dijkstra}(G, w, s)
1. for each vertex \( v \in G.V \)
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On weighted, directed graphs in which each edge has non-negative weight.

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4. \(s.d = 0\)
5. \(S = \emptyset\)
6. \(Q = G.V\)
7. while \(Q\) is not empty
8. \(u = \text{Extract Min} (Q)\)
Chapter 24. Single Source Shortest Paths

Dijkstra’s algorithm

On weighted, directed graphs in which each edge has non-negative weight.

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6. \( Q = G.V \)
7. while \( Q \) is not empty
8. \( u = \text{EXTRACT MIN}(Q) \)
9. \( S = S \cup \{u\} \)
Chapter 24. Single Source Shortest Paths

Dijkstra’s algorithm

On weighted, directed graphs in which each edge has non-negative weight.

$\text{DIJKSTRA}(G, w, s)$
1. for each vertex $v \in G.V$
2. \hspace{1em} $v.d = \infty$
3. \hspace{1em} $v.\pi = NULL$
4. \hspace{1em} $s.d = 0$
5. \hspace{1em} $S = \emptyset$
6. \hspace{1em} $Q = G.V$
7. while $Q$ is not empty
8. \hspace{1em} $u = \text{EXTRACT MIN} (Q)$
9. \hspace{1em} $S = S \cup \{u\}$
10. for each vertex $v \in \text{Adj}[u]$
Dijkstra’s algorithm

On weighted, directed graphs in which each edge has non-negative weight.

Dijkstra\((G, w, s)\)
1. for each vertex \(v \in G.V\)
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7. while \(Q\) is not empty
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9. \(S = S \cup \{u\}\)
10. for each vertex \(v \in Adj[u]\)
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On weighted, directed graphs in which each edge has non-negative weight.

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12. \( v.d = u.d + w(u, v) \)
Chapter 24. Single Source Shortest Paths

Dijkstra’s algorithm

On weighted, directed graphs in which each edge has non-negative weight.

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10. for each vertex \( v \in Adj[u] \)
11. \( \text{if } v.d > u.d + w(u, v) \)
12. \( v.d = u.d + w(u, v) \)
13. \( v.\pi = u \)

Running time:?
Chapter 24. Single Source Shortest Paths

Dijkstra’s algorithm

On weighted, directed graphs in which each edge has non-negative weight.

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Dijkstra’s algorithm

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12. \( v.d = u.d + w(u, v) \)
13. \( v.\pi = u \)
14. return \((d, \pi)\)

Running time:?
Note: the black-colored vertices are in set $S$. 
Chapter 24. Single Source Shortest Paths

**Correctness of algorithm** Dijkstra

Theorem 24.6

Dijkstra's algorithm, run on a weighted, directed graph \( G = (V,E) \) with non-negative weight function \( w \) and source \( s \), terminates with \( u.d = \delta(s,u) \) for all vertices \( u \in V \).

Proof: We need to show the while loop has loop invariant:

\[ u.d = \delta(s,u) \]

for each \( u \in S \).

Assume \( u \) to be the first such vertex that \( u.d > \delta(s,u) \) when it is being added to \( S \) then there must be a shortest path \( p : s \Rightarrow x \Rightarrow y \Rightarrow u \), for some \( x \in S \) and some \( y \not\in S \).

\( y.d = \delta(s,y) \) when \( u \) is being added to \( S \). This is because \( x \in S \), \( x.d = \delta(s,x) \) when \( x \) was added to \( S \). Edge \((x,y)\) was related at that time, and \( y.d = \delta(s,y) \) by Convergence-property.

So when \( u \) was chosen, \( u.d \leq y.d = \delta(s,y) \leq \delta(s,u) \). Contradicts the choice of \( u \).

So \( u.d = \delta(s,u) \) when it is being included to \( S \).
Chapter 24. Single Source Shortest Paths

Correctness of algorithm **Dijkstra**

**Theorem 24.6** Dijkstra’s algorithm, run on a weighted, directed graph $G = (V, E)$ with non-negative weight function $w$ and source $s$, terminates with $u.d = \delta(s, u)$ for all vertices $u \in V$. 

Proof: We need to show the while loop has loop invariant:

$u.d = \delta(s, u)$ for each $u \in S$

Assume $u$ to be the first such vertex that $u.d > \delta(s, u)$ when it is being added to $S$ then there must be a shortest path $p: s \rightarrow x \rightarrow y \rightarrow u$, for some $x \in S$ and some $y \not\in S$.

$y.d = \delta(s, y)$ when $u$ is being added to $S$. This is because $x \in S$, $x.d = \delta(s, x)$ when $x$ was added to $S$.

Edge $(x, y)$ was related at that time, and $y.d = \delta(s, y)$ by Convergence-property.

So when $u$ was chosen, $u.d \leq y.d = \delta(s, y) \leq \delta(s, u)$. Contradicts the choice of $u$.

So $u.d = \delta(s, u)$ when it is being included to $S$. 

Chapter 24. Single Source Shortest Paths

**Correctness of algorithm Dijkstra**

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**Proof**: We need to show the while loop has loop invariant:
$u.d = \delta(s, u)$ for each $u \in S$
**Chapter 24. Single Source Shortest Paths**

**Correctness of algorithm Dijkstra**

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**Proof:** We need to show the **while** loop has loop invariant:

$u.d = \delta(s, u)$ for each $u \in S$

Assume $u$ to be the first such vertex that $u.d > \delta(s, u)$ when it is being added to $S$. 

Edge $(x, y)$ was related at that time, and $y.d = \delta(s, y)$ by Convergence-property. So when $u$ was chosen, $u.d \leq y.d = \delta(s, y) \leq \delta(s, u)$. Contradicts the choice of $u$. So $u.d = \delta(s, u)$ when it is being included to $S$. 

Correctness of algorithm **Dijkstra**

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Assume $u$ to be the first such vertex that $u.d > \delta(s, u)$ when it is being added to $S$ then there must be a shortest path $p: s \rightsquigarrow x \rightarrow y \rightsquigarrow u$, for some $x \in S$ and some $y \notin S$. 
Chapter 24. Single Source Shortest Paths

Correctness of algorithm DIJKSTRA

Theorem 24.6 Dijkstra’s algorithm, run on a weighted, directed graph \( G = (V, E) \) with non-negative weight function \( w \) and source \( s \), terminates with \( u.d = \delta(s, u) \) for all vertices \( u \in V \).

Proof: We need to show the while loop has loop invariant:
\( u.d = \delta(s, u) \) for each \( u \in S \)

Assume \( u \) to be the first such vertex that \( u.d > \delta(s, u) \) when it is being added to \( S \) then there must be a shortest path \( p: s \leadsto x \rightarrow y \leadsto u \), for some \( x \in S \) and some \( y \not\in S \).

\( y.d = \delta(s, y) \) when \( u \) is being added to \( S \). This is because \( x \in S \), \( x.d = \delta(s, x) \) when \( x \) was added to \( S \).
Chapter 24. Single Source Shortest Paths

Correctness of algorithm **Dijkstra**

**Theorem 24.6** Dijkstra’s algorithm, run on a weighted, directed graph \( G = (V, E) \) with non-negative weight function \( w \) and source \( s \), terminates with \( u.d = \delta(s, u) \) for all vertices \( u \in V \).

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\( u.d = \delta(s, u) \) for each \( u \in S \)

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Chapter 24. Single Source Shortest Paths

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Chapter 24. Single Source Shortest Paths

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Assume \( u \) to be the first such vertex that \( u.d > \delta(s, u) \) when it is being added to \( S \) then there must be a shortest path \( p: s \rightsquigarrow x \rightarrow y \rightsquigarrow u \), for some \( x \in S \) and some \( y \notin S \).

\( y.d = \delta(s, y) \) when \( u \) is being added to \( S \). This is because \( x \in S \), \( x.d = \delta(s, x) \) when \( x \) was added to \( S \). Edge \((x, y)\) was related at that time, and \( y.d = \delta(s, y) \) by Convergence-property. So

When \( u \) was chosen, \( u.d \leq y.d = \delta(s, y) \leq \delta(s, u) \). **Contradicts** the choice of \( u \).
Chapter 24. Single Source Shortest Paths

Correctness of algorithm Dijkstra

Theorem 24.6 Dijkstra’s algorithm, run on a weighted, directed graph $G = (V, E)$ with non-negative weight function $w$ and source $s$, terminates with $u.d = \delta(s, u)$ for all vertices $u \in V$.

Proof: We need to show the while loop has loop invariant:

$u.d = \delta(s, u)$ for each $u \in S$

Assume $u$ to be the first such vertex that $u.d > \delta(s, u)$ when it is being added to $S$ then there must be a shortest path $p$: $s \leadsto x \rightarrow y \leadsto u$, for some $x \in S$ and some $y \notin S$.

$y.d = \delta(s, y)$ when $u$ is being added to $S$. This is because $x \in S$, $x.d = \delta(s, x)$ when $x$ was added to $S$. Edge $(x, y)$ was related at that time, and $y.d = \delta(s, y)$ by Convergence-property. So

When $u$ was chosen, $u.d \leq y.d = \delta(s, y) \leq \delta(s, u)$. Contradicts the choice of $u$. So $u.d = \delta(s, u)$ when it is being included to $S$. 
Chapter 24. Single Source Shortest Paths

- Running time of Dijkstra?
Chapter 24. Single Source Shortest Paths

- Running time of \textsc{Dijkstra}?
- Can \textsc{Dijkstra} deals with negative edges or cycles?
Chapter 24. Single Source Shortest Paths

- Running time of Dijkstra?
- Can Dijkstra deal with negative edges or cycles?
- Fundamental differences between Bellman-Ford and Dijkstra?
Chapter 24. Single Source Shortest Paths

- Running time of Dijkstra?
- Can Dijkstra deals with negative edges or cycles?
- Fundamental differences between Bellman-Ford and Dijkstra?

\[ \text{Dijkstra}(G, w, s) \]
1. for each vertex \( v \in G.V \)
2. \( v.d = \infty \)
3. \( v.\pi = \text{NULL} \)
4. \( s.d = 0 \)
5. \( S = \emptyset \)
6. \( Q = G.V \)
7. while \( Q \) is not empty
8. \( u = \text{EXTRACT MIN } (Q) \)
9. \( S = S \cup \{u\} \)
10. for each vertex \( v \in \text{Adj}[u] \)
11. if \( v.d > u.d + w(u, v) \)
12. \( v.d = u.d + w(u, v) \)
13. \( v.\pi = u \)
14. return \( (d, \pi) \)

\[ \text{Bellman-Ford}(G, w, s) \]
1. for each vertex \( v \in G.V \)
2. \( v.d = \infty \)
3. \( v.\pi = \text{NULL} \)
4. \( s.d = 0 \)
5. for \( i = 1 \) to \( |V| - 1 \)
6. for each edge \( (u, v) \in G.E \)
7. if \( v.d > u.d + w(u, v) \)
8. \( v.d = u.d + w(u, v) \)
9. \( v.\pi = u \)
10. for each edge \( (u, v) \in G.E \)
11. if \( v.d > u.d + w(u, v) \)
12. return (FALSE)
13. return (TRUE)
Chapter 24. Single Source Shortest Paths

• Fundamental differences between Dijkstra and MST-Prim?
Chapter 24. Single Source Shortest Paths

» Fundamental differences between \texttt{Dijkstra} and \texttt{MST-Prim}?
Chapter 24. Single Source Shortest Paths

- Fundamental differences between Dijkstra and MST-Prim?

Dijkstra\((G, w, s)\)
1. for each vertex \(v \in G.V\)
2. \(v.d = \infty\)
3. \(v.\pi = NULL\)
4. \(s.d = 0\)
5. \(S = \emptyset\)
6. \(Q = G.V\)
7. while \(Q\) is not empty
8. \(u = \text{EXTRACT MIN}(Q)\)
9. \(S = S \cup \{u\}\)
10. for each vertex \(v \in Adj[u]\)
11. if \(v.d > u.d + w(u, v)\)
12. \(v.d = u.d + w(u, v)\)
13. \(v.\pi = u\)
14. return \((d, \pi)\)

MST-Prim\((G, w, r)\)
1. for each \(u \in G.V\)
2. \(u.key = \infty\) \{ \(u.key\) is too large \}
3. \(u.\pi = NULL\)
4. \(r.key = 0\)
5. \(Q = G.V\)
6. while \(Q \neq \emptyset\)
7. \(u = \text{EXTRACT MIN}(Q)\)
8. for each \(v \in Adj[u]\)
9. if \(v \in Q\) and \(w(u, v) < v.key\)
10. then \(v.\pi = u\)
11. \(v.key = w(u, v)\)
12. return \(\pi\)
Chapter 25. All-pairs shortest paths

Chapter 25. All-pairs shortest paths
Chapter 25. All-pairs shortest paths

All Pair Shortest Paths Problem
Chapter 25. All-pairs shortest paths

All Pair Shortest Paths Problem

**Input:** A weighted graph $G = (V, E)$ with edge weight function $w$;
Chapter 25. All-pairs shortest paths

All Pair Shortest Paths Problem

**Input:** A weighted graph $G = (V, E)$ with edge weight function $w$;
**Output:** Shortest paths between every pair of vertices in $G$. 
All Pair Shortest Paths Problem

**Input:** A weighted graph \( G = (V, E) \) with edge weight function \( w \);

**Output:** Shortest paths between every pair of vertices in \( G \).

- **Dijkstra** would run in time \( O(|V|^2 \log |V| + |V||E|) \) on non-negative edges.

**Floyd-Warshall algorithm:** \( O(|V|^3) \).
Chapter 25. All-pairs shortest paths

All Pair Shortest Paths Problem

**Input:** A weighted graph $G = (V, E)$ with edge weight function $w$;  
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- **Dijkstra** would run in time $O(|V|^2 \log |V| + |V||E|)$ on non-negative edges.
- **Bellman-Ford** would run in time $O(|V|^2 |E|)$ for general graphs, but $O(|V|^4)$ on “dense” graphs.
Chapter 25. All-pairs shortest paths

All Pair Shortest Paths Problem

**Input**: A weighted graph \( G = (V, E) \) with edge weight function \( w \);

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New algorithms
Chapter 25. All-pairs shortest paths

All Pair Shortest Paths Problem

**Input**: A weighted graph $G = (V, E)$ with edge weight function $w$;

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New algorithms

- A dynamic programming algorithm $O(|V|^4)$, improved to $O(|V|^3 \log |V|)$
Chapter 25. All-pairs shortest paths

All Pair Shortest Paths Problem

INPUT: A weighted graph $G = (V, E)$ with edge weight function $w$;
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- Floyd-Warshall algorithm: $O(|V|^3)$.
Chapter 25. All-pairs shortest paths

All Pair Shortest Paths Problem

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New algorithms

- A dynamic programming algorithm $O(|V|^4)$, improved to $O(|V|^3 \log |V|)$

- **Floyd-Warshall** algorithm: $O(|V|^3)$.

**Graph representation:** adjacency matrix $W = (w_{ij})$
A dynamic programming approach
Chapter 25. All-pairs shortest paths

A dynamic programming approach

- Optimal substructure
- Objective function
Chapter 25. All-pairs shortest paths

A dynamic programming approach

- Optimal substructure
- Objective function

Define $l_{ij}$ be the minimum weight of any path from $v_i$ to $v_j$. 
Chapter 25. All-pairs shortest paths

A dynamic programming approach

- Optimal substructure
- Objective function

Define $l_{ij}$ be the minimum weight of any path from $v_i$ to $v_j$ does not work! having a data dependency issue.
Chapter 25. All-pairs shortest paths

A dynamic programming approach

- Optimal substructure
- Objective function

Define $l_{ij}$ be the minimum weight of any path from $v_i$ to $v_j$ 
does not work! having a data dependency issue.

Define $l_{ij}^m$ be the minimum weight of any path from $v_i$ to $v_j$ that contains at most $m$ edges.
Chapter 25. All-pairs shortest paths

A dynamic programming approach

- Optimal substructure
- Objective function

Define $l_{ij}$ be the minimum weight of any path from $v_i$ to $v_j$.

Define $l^m_{ij}$ be the minimum weight of any path from $v_i$ to $v_j$ that contains at most $m$ edges.

or alternatively,
Chapter 25. All-pairs shortest paths

A dynamic programming approach

• Optimal substructure
• Objective function

Define $l_{ij}$ be the minimum weight of any path from $v_i$ to $v_j$ does not work! having a data dependency issue.

Define $l^m_{ij}$ be the minimum weight of any path from $v_i$ to $v_j$ that contains at most $m$ edges.

or alternatively,

Define $l^k_{ij}$ be the minimum weight of any path from $v_i$ to $v_j$ in which vertices are of indexes $\leq k$. 
Chapter 25. All-pairs shortest paths

Define $l^m_{ij}$ be the minimum weight of any path from $v_i$ to $v_j$ that contains at most $m$ edges.
Define $l^m_{ij}$ be the minimum weight of any path from $v_i$ to $v_j$ that contains at most $m$ edges.

$$l^m_{ij} = \min(l^{m-1}_{ij}, \min_{1 \leq k \leq n} \{l^{m-1}_{ik} + w_{kj}\})$$
Chapter 25. All-pairs shortest paths

Define $l^m_{ij}$ be the minimum weight of any path from $v_i$ to $v_j$ that contains at most $m$ edges.

$$l^m_{ij} = \min (l^{m-1}_{ij}, \min_{1 \leq k \leq n} \{l^{m-1}_{ik} + w_{kj}\}) = \min_{1 \leq k \leq n} \{l^{m-1}_{ik} + w_{kj}\}$$

using $w_{jj} = 0$. 
Chapter 25. All-pairs shortest paths

Define $l^m_{ij}$ be the minimum weight of any path from $v_i$ to $v_j$ that contains at most $m$ edges.

$$
l^m_{ij} = \min(l^{m-1}_{ij}, \min_{1 \leq k \leq n} \{l^{m-1}_{ik} + w_{kj}\}) = \min_{1 \leq k \leq n} \{l^{m-1}_{ik} + w_{kj}\}
$$

using $w_{jj} = 0$.

and base cases:

$$
l^1_{ij} = w_{ij}
$$
Define $l_{ij}^m$ be the minimum weight of any path from $v_i$ to $v_j$ that contains at most $m$ edges.

$$l_{ij}^m = \min(l_{ij}^{m-1}, \min_{1 \leq k \leq n} \{l_{ik}^{m-1} + w_{kj}\}) = \min_{1 \leq k \leq n} \{l_{ik}^{m-1} + w_{kj}\}$$

using $w_{jj} = 0$.

and base cases:

$$l_{ij}^1 = w_{ij}$$

Adjacency matrix $W = (w_{ij})$ is the default.
Chapter 25. All-pairs shortest paths

DP table filling algorithm:
Chapter 25. All-pairs shortest paths

DP table filling algorithm:
For $L^1 = W$ and $m = 2, \ldots, n - 1$, compute table $L^m$ from table $L^{m-1}$;
Chapter 25. All-pairs shortest paths

DP table filling algorithm:

For $L^1 = W$ and $m = 2, \ldots, n - 1$, compute table $L^m$ from table $L^{m-1}$; technically two tables are enough.
Chapter 25. All-pairs shortest paths

DP table filling algorithm:

For $L^1 = W$ and $m = 2, \ldots, n - 1$, compute table $L^m$ from table $L^{m-1}$; technically two tables are enough.

Extended Shortest Paths($L, W$)
Chapter 25. All-pairs shortest paths

DP table filling algorithm:

For \( L^1 = W \) and \( m = 2, \ldots , n - 1 \), compute table \( L^m \) from table \( L^{m-1} \); technically two tables are enough.

**Extended Shortest Paths** \((L, W)\)

1. \( n = \text{rows}[L] \);
Chapter 25. All-pairs shortest paths

DP table filling algorithm:

For $L^1 = W$ and $m = 2, \ldots, n - 1$, compute table $L^m$ from table $L^{m-1}$;
  technically two tables are enough.

**Extended Shortest Paths**($L, W$)
1. $n = \text{rows}[L]$;
2. let $L'$ be an $n \times n$ table;
Chapter 25. All-pairs shortest paths

DP table filling algorithm:

For $L^1 = W$ and $m = 2, \ldots, n - 1$, compute table $L^m$ from table $L^{m-1}$; technically two tables are enough.

**Extended Shortest Paths** $(L,W)$

1. $n = \text{rows}[L]$;
2. let $L'$ be an $n \times n$ table;
3. for $i = 1$ to $n$
Chapter 25. All-pairs shortest paths

DP table filling algorithm:

For \( L^1 = W \) and \( m = 2, \ldots, n - 1 \), compute table \( L^m \) from table \( L^{m-1} \); technically two tables are enough.

**Extended Shortest Paths** \((L, W)\)

1. \( n = \text{rows}[L] \);
2. let \( L' \) be an \( n \times n \) table;
3. for \( i = 1 \) to \( n \)
4. for \( j = 1 \) to \( n \)
Chapter 25. All-pairs shortest paths

DP table filling algorithm:

For $L^1 = W$ and $m = 2, \ldots, n - 1$, compute table $L^m$ from table $L^{m-1}$; technically two tables are enough.

**Extended Shortest Paths** $(L, W)$
1. $n = \text{rows}[L]$;
2. let $L'$ be an $n \times n$ table;
3. for $i = 1$ to $n$
4. \hspace{1em} for $j = 1$ to $n$
5. \hspace{2em} $L'[i, j] = \infty$
Chapter 25. All-pairs shortest paths

DP table filling algorithm:

For $L^1 = W$ and $m = 2, \ldots, n - 1$, compute table $L^m$ from table $L^{m-1}$; technically two tables are enough.

**Extended Shortest Paths** $(L, W)$

1. $n = \text{rows}[L]$;
2. let $L'$ be an $n \times n$ table;
3. for $i = 1$ to $n$
   4. for $j = 1$ to $n$
   5. $L'[i, j] = \infty$
   6. for $k = 1$ to $n$

Call **Extended Shortest Paths**$(L, W)$ for $m = 2, 3, \ldots, n - 1$

$L^m \leftarrow \text{Extended Shortest Paths}(L^{m-1}, W)$
Chapter 25. All-pairs shortest paths

DP table filling algorithm:

For $L^1 = W$ and $m = 2, \ldots, n - 1$, compute table $L^m$ from table $L^{m-1}$; technically two tables are enough.

**Extended Shortest Paths**($L, W$)
1. $n = rows[L]$;
2. let $L'$ be an $n \times n$ table;
3. for $i = 1$ to $n$
4. for $j = 1$ to $n$
5. $L'[i, j] = \infty$
6. for $k = 1$ to $n$
7. $L'[i, j] = \min\{L'[i, j], L[i, k] + w[k, j]\}$
Chapter 25. All-pairs shortest paths

DP table filling algorithm:

For $L^1 = W$ and $m = 2, \ldots, n - 1$, compute table $L^m$ from table $L^{m-1}$; technically two tables are enough.

**Extended Shortest Paths** $(L,W)$
1. $n = \text{rows}[L]$;
2. let $L'$ be an $n \times n$ table;
3. for $i = 1$ to $n$
4.     for $j = 1$ to $n$
5.         $L'[i,j] = \infty$
6.     for $k = 1$ to $n$
7.         $L'[i,j] = \min\{L'[i,j], L[i,k] + w[k,j]\}$
8. return $(L')$
Chapter 25. All-pairs shortest paths

DP table filling algorithm:

For $L^1 = W$ and $m = 2, \ldots, n - 1$, compute table $L^m$ from table $L^{m-1}$; technically two tables are enough.

Extended Shortest Paths ($L, W$)
1. $n = \text{rows}[L]$;
2. let $L'$ be an $n \times n$ table;
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4. for $j = 1$ to $n$
5. $L'[i, j] = \infty$
6. for $k = 1$ to $n$
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8. return ($L'$)

Call Extended Shortest Paths for $m = 2, 3, \ldots, n - 1$
Chapter 25. All-pairs shortest paths

DP table filling algorithm:

For $L^1 = W$ and $m = 2, \ldots, n - 1$, compute table $L^m$ from table $L^{m-1}$; technically two tables are enough.

**Extended Shortest Paths** $(L, W)$

1. $n = \text{rows}[L]$;
2. let $L'$ be an $n \times n$ table;
3. for $i = 1$ to $n$
   4. for $j = 1$ to $n$
   5. $L'[i, j] = \infty$
   6. for $k = 1$ to $n$
   7. $L'[i, j] = \min\{L'[i, j], L[i, k] + w[k, j]\}$
8. return ( $L'$)

Call **Extended Shortest Paths** for $m = 2, 3, \ldots, n - 1$

$L^m \leftarrow \text{Extended Shortest Paths}(L^{m-1}, W)$
Chapter 25. All-pairs shortest paths

Running on an example:
Chapter 25. All-pairs shortest paths

• running time: $\Theta(n^4)$.
Chapter 25. All-pairs shortest paths

- running time: $\Theta(n^4)$.
- improving the running time by repeatedly squaring:

Let $2^k = n - 1$. Then $k = \lceil \log_2(n - 1) \rceil$. Faster All Pair Shortest Paths

1. $n = \text{rows}[W]$;
2. $L = W$;
3. $m = 1$;
4. while $m < n - 1$;
5. $L = \text{Extended Shortest Paths}(L, L)$;
6. $m = 2 \times m$;
7. return $(L)$.
• running time: $\Theta(n^4)$.

• improving the running time by repeatedly squaring:
  compute: $L^1, L^2, L^4, \cdots, L^{2^k}$.
Chapter 25. All-pairs shortest paths

- running time: $\Theta(n^4)$.  
- improving the running time by repeatedly squaring:  
  compute: $L^1, L^2, L^4, \cdots, L^{2^k}$.  
  what is $k$ here?
Chapter 25. All-pairs shortest paths

- running time: $\Theta(n^4)$.
- improving the running time by repeatedly squaring:
  
  compute: $L^1, L^2, L^4, \cdots, L^{2^k}$.

  what is $k$ here? Let $2^k = n - 1$. Then $k = \lceil \log_2(n - 1) \rceil$. 
Chapter 25. All-pairs shortest paths

- running time: $\Theta(n^4)$.
- improving the running time by repeatedly squaring:
  
  compute: $L^1, L^2, L^4, \ldots, L^{2^k}$.

  what is $k$ here? Let $2^k = n - 1$. Then $k = \lceil \log_2(n - 1) \rceil$.

Faster All Pair Shortest Paths($W$)
Chapter 25. All-pairs shortest paths

- running time: $\Theta(n^4)$. 
- improving the running time by repeatedly squaring:
  - compute: $L^1, L^2, L^4, \cdots, L^{2^k}$.
  - what is $k$ here? Let $2^k = n - 1$. Then $k = \lceil \log_2(n - 1) \rceil$.

**Faster All Pair Shortest Paths**($W$)

1. $n = \text{rows}[W]$;
Chapter 25. All-pairs shortest paths

- running time: $\Theta(n^4)$.
- improving the running time by repeatedly squaring:
  compute: $L^1, L^2, L^4, \ldots, L^{2^k}$.

  what is $k$ here? Let $2^k = n - 1$. Then $k = \lceil \log_2(n - 1) \rceil$.

Faster All Pair Shortest Paths($W$)

1. $n = \text{rows}[W]$;
2. $L = W$;
Chapter 25. All-pairs shortest paths

- running time: $\Theta(n^4)$.
- improving the running time by repeatedly squaring:
  
  compute: $L^1, L^2, L^4, \ldots, L^{2^k}$.

  what is $k$ here? Let $2^k = n - 1$. Then $k = \lceil \log_2(n - 1) \rceil$.

**Faster All Pair Shortest Paths($W$)**

1. $n = \text{rows}[W]$;
2. $L = W$;
3. $m = 1$;
Chapter 25. All-pairs shortest paths

- running time: $\Theta(n^4)$.
- improving the running time by repeatedly squaring:
  
  compute: $L^1, L^2, L^4, \ldots, L^{2^k}$.

  what is $k$ here? Let $2^k = n - 1$. Then $k = \lceil \log_2(n - 1) \rceil$.

Faster All Pair Shortest Paths($W$)

1. $n = \text{rows}[W]$;
2. $L = W$;
3. $m = 1$;
4. while $m < n - 1$
Chapter 25. All-pairs shortest paths

- running time: $\Theta(n^4)$.
- improving the running time by repeatedly squaring:
  
  compute: $L^1, L^2, L^4, \ldots, L^{2^k}$.
  
  what is $k$ here? Let $2^k = n - 1$. Then $k = \lceil \log_2(n - 1) \rceil$.

Faster All Pair Shortest Paths($W$)

1. $n = \text{rows}[W]$;
2. $L = W$;
3. $m = 1$;
4. while $m < n - 1$
5. $L = \text{Extended Shortest Paths}(L, L)$
Chapter 25. All-pairs shortest paths

• running time: $\Theta(n^4)$.

• improving the running time by repeatedly squaring:

  compute: $L^1, L^2, L^4, \ldots, L^{2^k}$.

  what is $k$ here? Let $2^k = n - 1$. Then $k = \lceil \log_2(n - 1) \rceil$.

Faster All Pair Shortest Paths($W$)
1. $n = \text{rows}[W]$;
2. $L = W$;
3. $m = 1$;
4. while $m < n - 1$
5. $L = \text{Extended Shortest Paths}(L, L)$
6. $m = 2 \times m$
Chapter 25. All-pairs shortest paths

- running time: $\Theta(n^4)$.
- improving the running time by repeatedly squaring:
  
  compute: $L^1, L^2, L^4, \ldots, L^{2^k}$.

  what is $k$ here? Let $2^k = n - 1$. Then $k = \lceil \log_2(n - 1) \rceil$.

Faster All Pair Shortest Paths($W$)
1. $n = \text{rows}[W]$;
2. $L = W$;
3. $m = 1$;
4. while $m < n - 1$
5. \hspace{1em} $L = \text{Extended Shortest Paths}(L, L)$
6. \hspace{1em} $m = 2 \times m$
7. return $(L)$
Chapter 25. All-pairs shortest paths

Floyd-Warshall algorithm

Define: $d(k)_{ij}$ to be the shortest path distance from $v_i$ to $v_j$ with no intermediate vertices of indexes higher than $k$.

Thus $d(k)_{ij} = \min\{d(k-1)_{ij}, d(k-1)_{ik} + d(k-1)_{kj}\}$ with base case: $d(0)_{ij} = w_{ij}$.

Floyd-Warshall (W)

1. $n = \text{rows}[W]$
2. $D(0) = W$
3. for $k = 1$ to $n$
4. for $i = 1$ to $n$
5. for $j = 1$ to $n$
6. $D(k)_{ij} = \min\{D(k-1)_{ij}, D(k-1)_{ik} + D(k-1)_{kj}\}$
7. return ($D(n)$)
Chapter 25. All-pairs shortest paths

Floyd-Warshall algorithm

intermediate vertices on a path $v_i \leadsto v_j$: those other than $v_i$ and $v_j$. 

Define: $d(k)_{ij}$ to be the shortest path distance from $v_i$ to $v_j$ with no intermediate vertices of indexes higher than $k$.

Thus $d(k)_{ij} = \min\{d(k-1)_{ij}, d(k-1)_{ik} + d(k-1)_{kj}\}$ with base case: $d(0)_{ij} = w_{ij}$.

Floyd-Warshall ($W$)

1. $n = \text{rows}[W]$
2. $D(0) = W$
3. for $k = 1$ to $n$
4. for $i = 1$ to $n$
5. for $j = 1$ to $n$
6. $D(k)_{ij} = \min\{D(k-1)_{ij}, D(k-1)_{ik} + D(k-1)_{kj}\}$
7. return $(D(n))$
Chapter 25. All-pairs shortest paths

Floyd-Warshall algorithm

intermediate vertices on a path $v_i \rightsquigarrow v_j$: those other than $v_i$ and $v_j$.

Define: $d_{ij}^{(k)}$ to be the shortest path distance from $v_i$ to $v_j$
Chapter 25. All-pairs shortest paths

Floyd-Warshall algorithm

intermediate vertices on a path $v_i \leadsto v_j$: those other than $v_i$ and $v_j$.

Define: $d_i^k(j)$ to be the shortest path distance from $v_i$ to $v_j$ with no intermediate vertices of indexes higher than $k$. 

Floyd-Warshall ($W$)

1. $n = \text{rows} [W]$
2. $D(0) = W$
3. for $k = 1$ to $n$
4. for $i = 1$ to $n$
5. for $j = 1$ to $n$
6. $D(k)[i,j] = \min \{ D(k-1)[i,j], D(k-1)[i,k] + D(k-1)[k,j] \}$
7. return ($D(n)$)
Chapter 25. All-pairs shortest paths

Floyd-Warshall algorithm

intermediate vertices on a path \( v_i \leadsto v_j \): those other than \( v_i \) and \( v_j \).

Define: \( d^{(k)}_{i,j} \) to be the shortest path distance from \( v_i \) to \( v_j \) with no intermediate vertices of indexes higher than \( k \). Thus

\[
d^{(k)}_{i,j} = \min(d^{(k-1)}_{i,j}, d^{(k-1)}_{i,k} + d^{(k-1)}_{k,j})
\]

with base case: \( d^{(0)}_{i,j} = w_{i,j} \).
Chapter 25. All-pairs shortest paths

**Floyd-Warshall algorithm**

intermediate vertices on a path \( v_i \leadsto v_j \): those other than \( v_i \) and \( v_j \).

Define: \( d^{(k)}_{ij} \) to be the shortest path distance from \( v_i \) to \( v_j \) with no intermediate vertices of indexes higher than \( k \). Thus

\[
d^{(k)}_{ij} = \min(d^{(k-1)}_{ij}, d^{(k-1)}_{ik} + d^{(k-1)}_{kj})
\]

with base case: \( d^{(0)}_{ij} = w_{ij} \).

**Floyd-Warshall** \( (W) \)
Chapter 25. All-pairs shortest paths

Floyd-Warshall algorithm

intermediate vertices on a path \( v_i \sim v_j \): those other than \( v_i \) and \( v_j \).

Define: \( d_{ij}^{(k)} \) to be the shortest path distance from \( v_i \) to \( v_j \) with no intermediate vertices of indexes higher than \( k \). Thus

\[
d_{ij}^{(k)} = \min(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)})
\]

with base case: \( d_{ij}^{(0)} = w_{ij} \).

Floyd-Warshall(\( W \))

1. \( n = \text{rows}[W] \)

\[
\]
Chapter 25. All-pairs shortest paths

Floyd-Warshall algorithm

intermediate vertices on a path $v_i \rightsquigarrow v_j$: those other than $v_i$ and $v_j$.

Define: $d_{i,j}^{(k)}$ to be the shortest path distance from $v_i$ to $v_j$ with no intermediate vertices of indexes higher than $k$. Thus

$$d_{i,j}^{(k)} = \min(d_{i,j}^{(k-1)}, d_{i,k}^{(k-1)} + d_{k,j}^{(k-1)})$$

with base case: $d_{i,j}^{(0)} = w_{ij}$.

Floyd-Warshall($W$)
1. $n = \text{rows}[W]$
2. $D^{(0)} = W$
Chapter 25. All-pairs shortest paths

Floyd-Warshall algorithm

intermediate vertices on a path $v_i \leadsto v_j$: those other than $v_i$ and $v_j$.

Define: $d_{ij}^{(k)}$ to be the shortest path distance from $v_i$ to $v_j$ with no intermediate vertices of indexes higher than $k$. Thus

$$d_{ij}^{(k)} = \min(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)})$$

with base case: $d_{ij}^{(0)} = w_{ij}$.

Floyd-Warshall $(W)$

1. $n = \text{rows}[W]$
2. $D^{(0)} = W$
3. for $k = 1$ to $n$


Chapter 25. All-pairs shortest paths

Floyd-Warshall algorithm

intermediate vertices on a path \( v_i \leadsto v_j \): those other than \( v_i \) and \( v_j \).

Define: \( d^{(k)}_{ij} \) to be the shortest path distance from \( v_i \) to \( v_j \) with no intermediate vertices of indexes higher than \( k \). Thus

\[
d^{(k)}_{ij} = \min(d^{(k-1)}_{ij}, d^{(k-1)}_{ik} + d^{(k-1)}_{kj})
\]

with base case: \( d^{(0)}_{ij} = w_{ij} \).

FLOYD-WARSHALL(\( W \))
1. \( n = \text{rows}[W] \)
2. \( D^{(0)} = W \)
3. for \( k = 1 \) to \( n \)
4. for \( i = 1 \) to \( n \)
Chapter 25. All-pairs shortest paths

Floyd-Warshall algorithm

intermediate vertices on a path \( v_i \leadsto v_j \): those other than \( v_i \) and \( v_j \).

Define: \( d_{ij}^{(k)} \) to be the shortest path distance from \( v_i \) to \( v_j \) with no intermediate vertices of indexes higher than \( k \). Thus

\[
d_{ij}^{(k)} = \min(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)})
\]

with base case: \( d_{ij}^{(0)} = w_{ij} \).

\[\text{Floyd-Warshall}(W)\]
1. \( n = \text{rows}[W] \)
2. \( D^{(0)} = W \)
3. \textbf{for} \( k = 1 \) \textbf{to} \( n \)
4. \textbf{for} \( i = 1 \) \textbf{to} \( n \)
5. \textbf{for} \( j = 1 \) \textbf{to} \( n \)
Chapter 25. All-pairs shortest paths

Floyd-Warshall algorithm

Intermediate vertices on a path \( v_i \leadsto v_j \): those other than \( v_i \) and \( v_j \).

Define: \( d^{(k)}_{i,j} \) to be the shortest path distance from \( v_i \) to \( v_j \) with no intermediate vertices of indexes higher than \( k \). Thus

\[
d^{(k)}_{i,j} = \min(d^{(k-1)}_{i,j}, d^{(k-1)}_{i,k} + d^{(k-1)}_{k,j})
\]

with base case: \( d^{(0)}_{i,j} = w_{i,j} \).

Floyd-Warshall(\(W\))
1. \( n = \text{rows}[W] \)
2. \( D^{(0)} = W \)
3. for \( k = 1 \) to \( n \)
4. for \( i = 1 \) to \( n \)
5. for \( j = 1 \) to \( n \)
6. \( D^{(k)}[i, j] = \min\{D^{(k-1)}[i, j], D^{(k-1)}[i, k] + D^{(k-1)}[k, j]\} \)
Chapter 25. All-pairs shortest paths

Floyd-Warshall algorithm

intermediate vertices on a path $v_i \rightsquigarrow v_j$: those other than $v_i$ and $v_j$.

Define: $d_{ij}^{(k)}$ to be the shortest path distance from $v_i$ to $v_j$ with no intermediate vertices of indexes higher than $k$. Thus

$$d_{ij}^{(k)} = \min(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)})$$

with base case: $d_{ij}^{(0)} = w_{ij}$.

Floyd-Warshall($W$)
1. $n = \text{rows}[W]$
2. $D^{(0)} = W$
3. for $k = 1$ to $n$
   4. for $i = 1$ to $n$
      5. for $j = 1$ to $n$
         6. $D^{(k)}[i, j] = \min\{D^{(k-1)}[i, j], D^{(k-1)}[i, k] + D^{(k-1)}[k, j]\}$
7. return ($D^{(n)}$)
Chapter 25. All-pairs shortest paths

\[
D^{(0)} = \begin{pmatrix}
0 & 3 & 8 & \infty & -4 \\
\infty & 0 & \infty & 1 & 7 \\
\infty & 4 & 0 & \infty & \infty \\
2 & \infty & -5 & 0 & \infty \\
\infty & \infty & \infty & 6 & 0 \\
\end{pmatrix}
\]

\[
\Pi^{(0)} = \begin{pmatrix}
\text{NIL} & 1 & 1 & \text{NIL} & 1 \\
\text{NIL} & \text{NIL} & \text{NIL} & 2 & 2 \\
\text{NIL} & 3 & \text{NIL} & \text{NIL} & \text{NIL} \\
4 & \text{NIL} & 4 & \text{NIL} & \text{NIL} \\
\text{NIL} & \text{NIL} & \text{NIL} & 5 & \text{NIL} \\
\end{pmatrix}
\]

\[
D^{(1)} = \begin{pmatrix}
0 & 3 & 8 & \infty & -4 \\
\infty & 0 & \infty & 1 & 7 \\
\infty & 4 & 0 & \infty & \infty \\
2 & 5 & -5 & 0 & -2 \\
\infty & \infty & \infty & 6 & 0 \\
\end{pmatrix}
\]

\[
\Pi^{(1)} = \begin{pmatrix}
\text{NIL} & 1 & 1 & \text{NIL} & 1 \\
\text{NIL} & \text{NIL} & \text{NIL} & 2 & 2 \\
\text{NIL} & 3 & \text{NIL} & \text{NIL} & \text{NIL} \\
4 & 1 & 4 & \text{NIL} & 1 \\
\text{NIL} & \text{NIL} & \text{NIL} & 5 & \text{NIL} \\
\end{pmatrix}
\]

\[
D^{(2)} = \begin{pmatrix}
0 & 3 & 8 & 4 & -4 \\
\infty & 0 & \infty & 1 & 7 \\
\infty & 4 & 0 & 5 & 11 \\
2 & 5 & -5 & 0 & -2 \\
\infty & \infty & \infty & 6 & 0 \\
\end{pmatrix}
\]

\[
\Pi^{(2)} = \begin{pmatrix}
\text{NIL} & 1 & 1 & 2 & 1 \\
\text{NIL} & \text{NIL} & \text{NIL} & 2 & 2 \\
\text{NIL} & 3 & \text{NIL} & 2 & 2 \\
4 & 1 & 4 & \text{NIL} & 1 \\
\text{NIL} & \text{NIL} & \text{NIL} & 5 & \text{NIL} \\
\end{pmatrix}
\]

\[
D^{(3)} = \begin{pmatrix}
0 & 3 & 8 & 4 & -4 \\
\infty & 0 & \infty & 1 & 7 \\
\infty & 4 & 0 & 5 & 11 \\
2 & -1 & -5 & 0 & -2 \\
\infty & \infty & \infty & 6 & 0 \\
\end{pmatrix}
\]

\[
\Pi^{(3)} = \begin{pmatrix}
\text{NIL} & 1 & 1 & 2 & 1 \\
\text{NIL} & \text{NIL} & \text{NIL} & 2 & 2 \\
\text{NIL} & 3 & \text{NIL} & 2 & 2 \\
4 & 3 & 4 & \text{NIL} & 1 \\
\text{NIL} & \text{NIL} & \text{NIL} & 5 & \text{NIL} \\
\end{pmatrix}
\]

\[
D^{(4)} = \begin{pmatrix}
0 & 3 & 1 & 4 & -4 \\
2 & 0 & -4 & 1 & -1 \\
7 & 4 & 0 & 5 & 3 \\
2 & -1 & -5 & 0 & -2 \\
8 & 5 & 1 & 6 & 0 \\
\end{pmatrix}
\]

\[
\Pi^{(4)} = \begin{pmatrix}
\text{NIL} & 1 & 4 & 2 & 1 \\
4 & \text{NIL} & 4 & 2 & 1 \\
4 & 3 & \text{NIL} & 2 & 1 \\
4 & 3 & 4 & \text{NIL} & 1 \\
4 & 3 & 4 & 5 & \text{NIL} \\
\end{pmatrix}
\]

\[
D^{(5)} = \begin{pmatrix}
0 & 1 & -3 & 2 & -4 \\
3 & 0 & -4 & 1 & -1 \\
7 & 4 & 0 & 5 & 3 \\
2 & -1 & -5 & 0 & -2 \\
8 & 5 & 1 & 6 & 0 \\
\end{pmatrix}
\]

\[
\Pi^{(5)} = \begin{pmatrix}
\text{NIL} & 3 & 4 & 5 & 1 \\
4 & \text{NIL} & 4 & 2 & 1 \\
4 & 3 & \text{NIL} & 2 & 1 \\
4 & 3 & 4 & \text{NIL} & 1 \\
4 & 3 & 4 & 5 & \text{NIL} \\
\end{pmatrix}
\]
Chapter 25. All-pairs shortest paths

- Constructing a shortest path
Chapter 25. All-pairs shortest paths

• Constructing a shortest path
• for each $v_i$ and each $v_j$, to remember the last step to reach $j$. 

\[
\pi(0)_{ij} = \begin{cases} 
\text{NULL} & \text{if } i = j \text{ or } w_{ij} = \infty, \\
 i & \text{if } i \neq j \text{ and } w_{ij} < \infty.
\end{cases}
\]

\[
\pi(k)_{ij} = \begin{cases} 
\pi(k-1)_{ij} & \text{if } d(k-1)_{ij} \leq d(k-1)_{ik} + d(k-1)_{kj}, \\
\pi(k-1)_{kj} & \text{if } d(k-1)_{ij} > d(k-1)_{ik} + d(k-1)_{kj}.
\end{cases}
\]
Chapter 25. All-pairs shortest paths

• Constructing a shortest path

• for each $v_i$ and each $v_j$, to remember the last step to reach $j$. predecessor matrix $\pi$, recursively defined as
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\]

\[
\pi^{(k)}_{ij} = \pi^{(k-1)}_{ij} \quad \text{if} \quad d^{(k-1)}_{ij} \leq d^{(k-1)}_{ik} + d^{(k-1)}_{kj}, \quad \text{or}
\]

\[
\pi^{(k)}_{ij} = \pi^{(k-1)}_{kj} \quad \text{if} \quad d^{(k-1)}_{ij} > d^{(k-1)}_{ik} + d^{(k-1)}_{kj}
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    \]
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    \]

Chapter 25. All-pairs shortest paths

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\pi^{(k)}_{ij} = \pi^{(k-1)}_{kj} \text{ if } d^{(k-1)}_{ij} > d^{(k-1)}_{ik} + d^{(k-1)}_{kj}
\]
Chapter 25. All-pairs shortest paths

Summary of shortest path algorithms
Chapter 25. All-pairs shortest paths

Summary of shortest path algorithms

1. Bellman-Ford’s algorithm (able to detect negative weight cycles)
Chapter 25. All-pairs shortest paths

Summary of shortest path algorithms

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2. DAG Shortest Paths (use topological sorting) [Lawler]
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Chapter 25. All-pairs shortest paths

Summary of shortest path algorithms

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Chapter 25. All-pairs shortest paths

Summary of shortest path algorithms

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5. Floyd-Warshall algorithm (DP)