Part II Sorting and Order Statistics
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- Chapter 6. Heapsort, the use of priority queue
- Chapter 7. Quicksort, probabilistic analysis, randomized algorithms
- Chapter 8. Sorting in linear time, lower bounds
- Chapter 9. Medians and order statistics
Chapter 6. Heapsort

Chapter 6. Heapsort and the use of priority queue
Chapter 6. Heapsort and the use of priority queue
Chapter 6. Heapsort and the use of priority queue

- key(parent) ≥ key(leftChild), key(rightChild);
Chapter 6. Heapsort

Chapter 6. Heapsort and the use of priority queue

- key(parent) ≥ key(leftChild), key(rightChild);
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Chapter 6. Heapsort

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- key(parent) ≥ key(leftChild), key(rightChild);
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Chapter 6. Heapsort

Chapter 6. Heapsort and the use of priority queue

- key(parent) \geq key(leftChild), key(rightChild);
- relationships are modeled with a complete binary tree
- can be stored in arrays (indexes begin with 0),
  \text{index(leftChild)} = 2 \times \text{index(parent)} + 1
Chapter 6. Heapsort and the use of priority queue

- key(parent) ≥ key(leftChild), key(rightChild);
- relationships are modeled with a complete binary tree
- can be stored in arrays (indexes begin with 0),
  index(leftChild) = 2 × index(parent) + 1
  index(rightChild) = 2 × index(parent) + 2
Chapter 6. Heapsort

The heap sort algorithm consists of subroutines:
Chapter 6. Heapsort

The heap sort algorithm consists of subroutines:

- **Build-Max-Heap** \((A)\)
Chapter 6. Heapsort

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- **Build-Max-Heap**(\( A \))
- **Max-Heapify**(\( A, i \))
Chapter 6. Heapsort

The heap sort algorithm consists of subroutines:

- **Build-Max-Heap**(A)
- **Max-Heapify**(A, i)
- **HeapSort**(A)
Chapter 6. Heapsort

The heap sort algorithm consists of subroutines:

- **Build-Max-Heap** \((A)\)
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- **HeapSort** \((A)\)

Heaps as priority queues
Chapter 6. Heapsort

The heap sort algorithm consists of subroutines:

- **Build-Max-Heap**($A$)
- **Max-Heapify**($A, i$)
- **HeapSort**($A$)

Heaps as priority queues

- **Heap-Maximum**($A$)
- **Heap-Extract-Max**($A$)
- **Heap-Increase-Key**($A, i, key$)
- **Max-Heap-Insert**($A, key$)
Chapter 6. Heapsort

The heap sort algorithm consists of subroutines:

- **Build-Max-Heap**\( (A) \)
- **Max-Heapify**\( (A, i) \)
- **HeapSort**\( (A) \)

Heaps as priority queues

- **Heap-Maximum**\( (A) \)
- **Heap-Extract-Max**\( (A) \)
The heap sort algorithm consists of subroutines:

- Build-Max-Heap(A)
- Max-Heapify(A, i)
- HeapSort(A)

Its algorithms as priority queues:

- Heap-Maximum(A)
- Heap-Extract-Max(A)
- Heap-Increase-Key(A, I, key)
Chapter 6. Heapsort

The heap sort algorithm consists of subroutines:

- **Build-Max-Heap** \((A)\)
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Heaps as priority queues

- **Heap-Maximum** \((A)\)
- **Heap-Extract-Max** \((A)\)
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- **Max-Heap-Insert** \((A, key)\)
Chapter 6. Heapsort

Algorithm `HEAPSORT(A)`
Chapter 6. Heapsort

Algorithm HeapSort\((A)\)

1. Build-Max-Heap\((A)\)
Chapter 6. Heapsort

Algorithm \textsc{HeapSort}(A)

1. \textsc{Build-Max-Heap}(A)
2. \textbf{for} \( i = \text{length}[A] - 1 \) \textbf{downto} 1 \quad \{ \text{indexes begin from 0} \}

Subroutine \textsc{Build-Max-Heap}(A)

1. \textsc{heapsize}[A] = \text{length}[A]
2. \textbf{for} \( i = \lfloor \frac{1}{2} \text{length}[A] \rfloor \) \textbf{downto} 0 \quad \{ \text{indexes begin from 0} \}
3. \textsc{Max-Heapify}(A,i)
Chapter 6. Heapsort

Algorithm `HEAPSORT(A)`

1. `BUILD-MAX-HEAP(A)`
2. `for i = length[A] − 1 downto 1 { indexes begin from 0}`
Chapter 6. Heapsort

Algorithm \textsc{HeapSort}(A)

1. \textbf{Build-Max-Heap}(A)
2. \textbf{for} \( i = \text{length}[A] - 1 \ \textbf{downto} \ 1 \) \{ indexes begin from 0\}
4. \( \text{heapsize}[A] = \text{heapsize}[A] - 1 \)
Chapter 6. Heapsort

Algorithm \texttt{HeapSort}(A)

1. \texttt{Build-Max-Heap}(A)
2. \texttt{for} \ i = \text{length}[A] - 1 \ \texttt{downto} \ 1 \ \{ \ \text{indexes begin from 0}\}
3. \text{exchange} \ A[1] \leftrightarrow A[i]
4. \text{heapsize}[A] = \text{heapsize}[A] - 1
5. \texttt{Max-Heapify}(A, 0)
Algorithm `HeapSort(A)`

1. `Build-Max-Heap(A)`
2. `for i = length[A] - 1` `downto` `1` `{ indexes begin from 0}`
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\[ T_{HS}(n) = T_{BMH}(n) + (n - 1)T_{MH}(n, 1), \text{ where } n = |A| \]
Algorithm \textsc{HeapSort}(A)

1. \textsc{Build-Max-Heap}(A)
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\[
T_{HS}(n) = T_{BMH}(n) + (n - 1)T_{MH}(n, 1), \text{ where } n = |A|
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Subroutine \textsc{Build-Max-Heap}(A)
Chapter 6. Heapsort

Algorithm \textbf{HEAPSORT}(A)

1. \textbf{BUILD-MAX-HEAP}(A)
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\[ T_{HS}(n) = T_{BMH}(n) + (n - 1)T_{MH}(n, 1), \text{ where } n = |A| \]

Subroutine \textbf{BUILD-MAX-HEAP}(A)

1. \text{heapsize}[A] = \text{length}[A]
Chapter 6. Heapsort

Algorithm **HeapSort**($A$)

1. **Build-Max-Heap**($A$)
2. for $i = \text{length}[A] - 1$ downto 1 { indexes begin from 0}
4. $\text{heapsize}[A] = \text{heapsize}[A] - 1$
5. **Max-Heapify**($A, 0$)

$$T_{HS}(n) = T_{BMH}(n) + (n - 1)T_{MH}(n, 1), \text{ where } n = |A|$$

Subroutine **Build-Max-Heap**($A$)

1. $\text{heapsize}[A] = \text{length}[A]$
2. for $i = \lfloor \frac{1}{2}\text{length}[A] \rfloor$ downto 0 { indexes begin from 0}
Chapter 6. Heapsort

Algorithm HeapSort($A$)

1. Build-Max-Heap($A$)
2. for $i = \text{length}[A] - 1$ downto 1 { indexes begin from 0 }
4. heapsize[$A$] = heapsize[$A$] - 1
5. Max-Heapify($A, 0$)

$T_{HS}(n) = T_{BMH}(n) + (n - 1)T_{MH}(n, 1)$, where $n = |A|$

Subroutine Build-Max-Heap($A$)

1. heapsize[$A$] = length[$A$]
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3. Max-Heapify($A, i$)
Chapter 6. Heapsort

Algorithm $\text{HEAPSort}(A)$

1. $\text{BUILD-MAX-HEAP}(A)$
2. for $i = \text{length}[A] - 1$ downto 1 \{ indexes begin from 0\}
4. $\text{heapsize}[A] = \text{heapsize}[A] - 1$
5. $\text{MAX-HEAPIFY}(A, 0)$

$T_{HS}(n) = T_{BMH}(n) + (n - 1)T_{MH}(n, 1)$, where $n = |A|$

Subroutine $\text{BUILD-MAX-HEAP}(A)$

1. $\text{heapsize}[A] = \text{length}[A]$
2. for $i = \lfloor \frac{1}{2} \text{length}[A] \rfloor$ downto 0 \{ indexes begin from 0\}
3. $\text{MAX-HEAPIFY}(A, i)$

$T_{BMH}(n) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} T_{MH}(n, i)$
Chapter 6. Heapsort

Subroutine MAX-HEAPIFY($A, i$)

1. $l = 2 \times i + 1$
2. $r = 2 \times i + 2$
3. If ($l \leq \text{heapsize}[A]$) and ($A[l] > A[i]$) then
   
   4. largest = $l$
5. else
6. largest = $i$
7. If ($r \leq \text{heapsize}[A]$) and ($A[r] > A[\text{largest}]$) then
8. largest = $r$

9. If largest $\neq i$ then exchange $A[i] \leftrightarrow A[\text{largest}]$

10. Max-Heapiify($A, \text{largest}$)

$T_{MH}(n, i) \leq c + T_{MH}(n, 2i)$

Because $T_{MH}(n, i) = c + T_{MH}(n, 2i + 1)$, or $= c + T_{MH}(n, 2i + 2)$

$T_{MH}(n, i) \leq c \log_2 n$, for all $i = 0, 1, \ldots, n - 1$.

$T_{BMH}(n) = \lfloor \frac{n}{2} \rfloor \sum_{i=0}^{n} T_{MH}(n, i)$

$\leq cn^2 \log_2 n$

$T_{HS}(n) = T_{BMH}(n) + (n - 1)T_{MH}(n, 0)$

$\leq cn^2 \log_2 n + (n - 1)c \log_2 n \leq O(n \log n)$
Chapter 6. Heapsort

Subroutine MAX-HEAPIFY(A, i)
1. \( l = 2 \times i + 1 \)
Chapter 6. Heapsort

Subroutine MAX-HEAPIFY(A, i)

1. \( l = 2 \times i + 1 \)
2. \( r = 2 \times i + 2 \)
Chapter 6. Heapsort

Subroutine MAX-HEAPIFY\((A, i)\)

1. \( l = 2 \times i + 1 \)
2. \( r = 2 \times i + 2 \)
3. \textbf{if} \((l \leq \text{heapsize}[A]) \textbf{ and } (A[l] > A[i])\)
Chapter 6. Heapsort

Subroutine MAX-HEAPIFY(A, i)

1. \( l = 2 \times i + 1 \)
2. \( r = 2 \times i + 2 \)
3. if \( l \leq \text{heapsize}[A] \) and \( A[l] > A[i] \)
4. then \( \text{largest} = l \)
Chapter 6. Heapsort

Subroutine MAX-HEAPIFY$(A, i)$

1. $l = 2 \times i + 1$
2. $r = 2 \times i + 2$
3. if $(l \leq \text{heapsize}[A])$ and $(A[l] > A[i])$
4. then $\text{largest} = l$
5. else $\text{largest} = i$
Chapter 6. Heapsort

Subroutine MAX-HEAPIFY($A$, $i$)

1. $l = 2 \times i + 1$
2. $r = 2 \times i + 2$
3. if ($l \leq \text{heapsize}[A]$) and ($A[l] > A[i]$)
   4. then largest = $l$
   5. else largest = $i$
6. if ($r \leq \text{heapsize}[A]$) and ($A[r] > A[\text{largest}]$)
Chapter 6. Heapsort

Subroutine MAX-HEAPIFY(A, i)

1. \( l = 2 \times i + 1 \)
2. \( r = 2 \times i + 2 \)
3. if \( l \leq \text{heapsize}[A] \) and \( A[l] > A[i] \)
   then largest = l
4. else largest = i
5. if \( r \leq \text{heapsize}[A] \) and \( A[r] > A[\text{largest}] \)
   then largest = r

T_{MH}(n,i) \leq c + T_{MH}(n,2i)
Because
T_{MH}(n,i) = c + T_{MH}(n,2i+1)
T_{MH}(n,i) \leq c \log_2 n,
for all i = 0, 1, ..., n - 1.
T_{BMH}(n) = \lfloor \frac{n}{2} \rfloor \sum_{i=0}^{n} T_{MH}(n,i)
\leq cn^2 \log_2 n
T_{HS}(n) = T_{BMH}(n) + (n - 1)T_{MH}(n,0)
\leq cn^2 \log_2 n + (n - 1)c \log_2 n
\leq O(n \log n)
Chapter 6. Heapsort

Subroutine MAX-HEAPIFY($A, i$)

1. $l = 2 \times i + 1$
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3. if ($l \leq \text{heapsize}[A]$) and ($A[l] > A[i]$)
   then largest = $l$
4. else largest = $i$
5. if ($r \leq \text{heapsize}[A]$) and ($A[r] > A[\text{largest}$])
   then largest = $r$
6. if largest $\neq i$

$T_{BMH}(n) = \lfloor \frac{n}{2} \rfloor \sum_{i=0}^{n-1} T_{MH}(n,i)$
$\leq cn^2 \log_2 n$
$T_{HS}(n) = T_{BMH}(n) + (n-1)T_{MH}(n,0)$
$\leq cn^2 \log_2 n + (n-1)c \log_2 n$
$\leq O(n \log n)$
Chapter 6. Heapsort

Subroutine MAX-HEAPIFY($A, i$)

1. $l = 2 \times i + 1$
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3. if ($l \leq \text{heapsize}[A]$) and ($A[l] > A[i]$)
   4. then $\text{largest} = l$
   5. else $\text{largest} = i$
6. if ($r \leq \text{heapsize}[A]$) and ($A[r] > A[\text{largest}]$)
   7. then $\text{largest} = r$
8. if $\text{largest} \neq i$
   9. then exchange $A[i] \leftrightarrow A[\text{largest}]$
Subroutine `MAX-HEAPIFY(A, i)`

1. \( l = 2 \times i + 1 \)
2. \( r = 2 \times i + 2 \)
3. \( \text{if } (l \leq \text{heapsize}[A]) \text{ and } (A[l] > A[i]) \)
4. \( \text{then } largest = l \)
5. \( \text{else } largest = i \)
6. \( \text{if } (r \leq \text{heapsize}[A]) \text{ and } (A[r] > A[largest]) \)
7. \( \text{then } largest = r \)
8. \( \text{if } largest \neq i \)
9. \( \text{then exchange } A[i] \leftrightarrow A[largest] \)
10. \( \text{MAX-HEAPIFY}(A, largest) \)
Chapter 6. Heapsort

Subroutine MAX-HEAPIFY(A, i)

1. $l = 2 \times i + 1$
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3. if $(l \leq \text{heapsize}[A])$ and $(A[l] > A[i])$
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   then $\text{largest} = r$
6. if $\text{largest} \neq i$
   then exchange $A[i] \leftrightarrow A[\text{largest}]$
7. MAX-HEAPIFY(A, largest)

$T_{MH}(n, i) \leq c + T_{MH}(n, 2i)$  Because $T_{MH}(n, i) = c + T_{MH}(n, 2i + 1)$, or
   $= c + T_{MH}(n, 2i + 2)$
Chapter 6. Heapsort

Subroutine MAX-HEAPIFY($A, i$)

1. $l = 2 \times i + 1$
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3. if $(l \leq \text{heapsize}[A])$ and $(A[l] > A[i])$
   then $\text{largest} = l$
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   then $\text{largest} = r$
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   then exchange $A[i] \leftrightarrow A[\text{largest}]$
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$T_{MH}(n, i) \leq c + T_{MH}(n, 2i)$  Because $T_{MH}(n, i) = c + T_{MH}(n, 2i + 1)$, or
$= c + T_{MH}(n, 2i + 2)$

$T_{MH}(n, i) \leq c \log_2 n$, for all $i = 0, 1, \ldots, n - 1$.  

$T_{BMH}(n) = \lfloor n/2 \rfloor \sum_{i=0}^{n/2} T_{MH}(n, i) \leq c n^2 \log_2 n$

$T_{HS}(n) = T_{BMH}(n) + (n - 1)T_{MH}(n, 0) \leq c n^2 \log_2 n + (n - 1)c \log_2 n \leq O(n \log n)$
Chapter 6. Heapsort

Subroutine $\text{MAX-HEAPIFY}(A, i)$

1. $l = 2 \times i + 1$
2. $r = 2 \times i + 2$
3. if $(l \leq \text{heapsize}[A]) \text{ and } (A[l] > A[i])$
4. then largest $= l$
5. else largest $= i$
6. if $(r \leq \text{heapsize}[A]) \text{ and } (A[r] > A[\text{largest}])$
7. then largest $= r$
8. if largest $\neq i$
9. then exchange $A[i] \leftrightarrow A[\text{largest}]$
10. $\text{MAX-HEAPIFY}(A, \text{largest})$

$T_{MH}(n, i) \leq c + T_{MH}(n, 2i) \quad \text{Because } T_{MH}(n, i) = c + T_{MH}(n, 2i + 1), \text{ or }$

$= c + T_{MH}(n, 2i + 2)$

$T_{MH}(n, i) \leq c \log_2 n$, for all $i = 0, 1, \ldots, n - 1$.

$T_{BMH}(n) = \sum_{i=0}^{\lfloor n/2 \rfloor} T_{MH}(n, i)$
Chapter 6. Heapsort

Subroutine \textbf{MAX-HEAPIFY}(A, i)

1. \( l = 2 \times i + 1 \)
2. \( r = 2 \times i + 2 \)
3. \textbf{if} \((l \leq \text{heapsize}[A]) \text{ and } (A[l] > A[i])\)
4. \textbf{then} \( \text{largest} = l \)
5. \textbf{else} \( \text{largest} = i \)
6. \textbf{if} \((r \leq \text{heapsize}[A]) \text{ and } (A[r] > A[\text{largest}])\)
7. \textbf{then} \( \text{largest} = r \)
8. \textbf{if} \( \text{largest} \neq i \)
9. \textbf{then} exchange \( A[i] \leftrightarrow A[\text{largest}] \)
10. \textbf{MAX-HEAPIFY}(A, \text{largest})

\( T_{MH}(n, i) \leq c + T_{MH}(n, 2i) \quad \text{Because } T_{MH}(n, i) = c + T_{MH}(n, 2i + 1), \text{ or } = c + T_{MH}(n, 2i + 2) \)

\( T_{MH}(n, i) \leq c \log_2 n, \text{ for all } i = 0, 1, \ldots, n - 1. \)

\( T_{BMH}(n) = \sum_{i=0}^{\lfloor n/2 \rfloor} T_{MH}(n, i) \leq c \frac{n}{2} \log_2 n \)
Chapter 6. Heapsort

Subroutine MAX-HEAPIFY\((A, i)\)

1. \(l = 2 \times i + 1\)
2. \(r = 2 \times i + 2\)
3. if \((l \leq \text{heapsize}[A]) \text{ and } (A[l] > A[i])\) then
   largest = \(l\)
4. else
   largest = \(i\)
5. if \((r \leq \text{heapsize}[A]) \text{ and } (A[r] > A[\text{largest}])\) then
   largest = \(r\)
6. if largest \(\neq i\) then
   exchange \(A[i] \leftrightarrow A[\text{largest}]\)
7. \(\text{MAX-HEAPIFY}(A, \text{largest})\)

\(T_{MH}(n, i) \leq c + T_{MH}(n, 2i)\) Because \(T_{MH}(n, i) = c + T_{MH}(n, 2i + 1)\), or
\(= c + T_{MH}(n, 2i + 2)\)

\(T_{MH}(n, i) \leq c \log_2 n\), for all \(i = 0, 1, \ldots, n - 1\).

\(T_{BMH}(n) = \sum_{i=0}^{\lfloor n/2 \rfloor} T_{MH}(n, i) \leq c \frac{n}{2} \log_2 n\)

\(T_{HS}(n) = T_{BMH}(n) + (n - 1)T_{MH}(n, 0)\)
Chapter 6. Heapsort

Subroutine MAX-HEAPIFY\((A, i)\)

1. \(l = 2 \times i + 1\)
2. \(r = 2 \times i + 2\)
3. \(\text{if } (l \leq \text{heapsize}[A]) \text{ and } (A[l] > A[i])\)
4. \(\text{then } \text{largest} = l\)
5. \(\text{else } \text{largest} = i\)
6. \(\text{if } (r \leq \text{heapsize}[A]) \text{ and } (A[r] > A[\text{largest}])\)
7. \(\text{then } \text{largest} = r\)
8. \(\text{if } \text{largest} \neq i\)
9. \(\text{then exchange } A[i] \leftrightarrow A[\text{largest}]\)
10. \(\text{MAX-HEAPIFY}(A, \text{largest})\)

\(T_{MH}(n, i) \leq c + T_{MH}(n, 2i) \quad \text{Because } T_{MH}(n, i) = c + T_{MH}(n, 2i + 1), \text{ or}\)

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\(T_{MH}(n, i) \leq c \log_2 n, \text{ for all } i = 0, 1, \ldots, n - 1.\)

\(T_{BMH}(n) = \sum_{i=0}^{\lfloor n/2 \rfloor} T_{MH}(n, i) \leq c \frac{n}{2} \log_2 n\)

\(T_{HS}(n) = T_{BMH}(n) + (n - 1)T_{MH}(n, 0) \leq c \frac{n}{2} \log_2 n + (n - 1)c \log_2 n\)
Chapter 6. Heapsort

Subroutine MAX-HEAPIFY(A, i)

1. \( l = 2 \times i + 1 \)
2. \( r = 2 \times i + 2 \)
3. \( \text{if } (l \leq \text{heapsize}[A]) \text{ and } (A[l] > A[i]) \)
4. \( \text{then } \text{largest} = l \)
5. \( \text{else } \text{largest} = i \)
6. \( \text{if } (r \leq \text{heapsize}[A]) \text{ and } (A[r] > A[\text{largest}]) \)
7. \( \text{then } \text{largest} = r \)
8. \( \text{if } \text{largest} \neq i \)
9. \( \text{then exchange } A[i] \leftrightarrow A[\text{largest}] \)
10. \( \text{MAX-HEAPIFY}(A, \text{largest}) \)

\[ T_{MH}(n, i) \leq c + T_{MH}(n, 2i) \quad \text{Because } T_{MH}(n, i) = c + T_{MH}(n, 2i + 1), \text{ or} \]
\[ = c + T_{MH}(n, 2i + 2) \]

\[ T_{MH}(n, i) \leq c \log_2 n, \text{ for all } i = 0, 1, \ldots, n - 1. \]

\[ T_{BMH}(n) = \sum_{i=0}^{\lfloor n/2 \rfloor} T_{MH}(n, i) \leq c \frac{n}{2} \log_2 n \]

\[ T_{HS}(n) = T_{BMH}(n) + (n - 1)T_{MH}(n, 0) \leq c \frac{n}{2} \log_2 n + (n - 1)c \log_2 n \leq O(n \log n) \]
Chapter 6. Heapsort

Operations on heaps:

Function Heap-Maximum(A) obtain the maximum
1. return (A[1])

Function Heap-Extract-Max(A) obtain and remove the maximum
1. if heapsize[A] < 1 then return ("heap underflow")
2. max = A[1]
4. heapsize[A] = heapsize[A] - 1
5. Max-Heapify(A, 1)
6. return (max)

Function Heap-Increase-Key(A,i,key) replace a key with a larger value
1. if key < A[i] then return ("new key is smaller than current key")
2. A[i] = key
3. while i > 1 and A[PARENT[i]] < A[i]
5. i = PARENT[i]

Function Max-Heap-Insert(A,key) insert a new key to heap
1. heapsize[A] = heapsize[A] + 1
2. A[heapsize[A]] = −∞
3. Heap-Increase-Key(A, heapsize[A], key)
Chapter 6. Heapsort

Operations on heaps:

Function `HEAP-MAXIMUM(A)`

1. return \(A[1]\)

obtain the maximum
Chapter 6. Heapsort

Operations on heaps:

Function **Heap-Maximum**\( (A) \)
1. \textbf{return} \( (A[1]) \)

Function **Heap-Extract-Max**\( (A) \)
1. \textbf{if} heapsize\[A]\ < 1
2. \textbf{then return} ("heap underflow")
3. \( max = A[1] \)
4. \( A[1] = A[\text{heapsize}\[A]] \)
5. \( \text{heapsize}\[A] = \text{heapsize}\[A] - 1 \)
6. \textbf{Max-Heapify}(A, 1)
7. \textbf{return} \( (max) \)
Chapter 6. Heapsort

Operations on heaps:

Function $\text{HEAP-MAXIMUM}(A)$ obtain the maximum
1. $\text{return } (A[1])$

Function $\text{HEAP-EXTRACT-MAX}(A)$ obtain and remove the maximum
1. if $\text{heapsize}[A] < 1$
2. then $\text{return } ("\text{heap underflow}")$
3. $\text{max} = A[1]$
5. $\text{heapsize}[A] = \text{heapsize}[A] - 1$
6. $\text{MAX-HEAPIFY}(A, 1)$
7. $\text{return } (\text{max})$

Function $\text{HEAP-INCREASE-KEY}(A, i, \text{key})$ replace a key with a larger value
1. if $\text{key} < A[i]$
2. then $\text{return } ("\text{new key is smaller than current key}")$
3. $A[i] = \text{key}$
4. while $i > 1$ and $A[\text{PARENT}[i]] < A[i]$
5. exchange $A[i] \leftrightarrow A[\text{PARENT}[i]]$
6. $i = \text{PARENT}[i]$

Function $\text{Max-Heap-Insert}(A, \text{key})$ insert a new key to heap
1. $\text{heapsize}[A] = \text{heapsize}[A] + 1$
2. $A[\text{heapsize}[A]] = -\infty$
3. $\text{HEAP-INCREASE-KEY}(A, \text{heapsize}[A], \text{key})$
Chapter 6. Heapsort

Operations on heaps:

Function **HEAP-MAXIMUM**(A)
1. return (A[1])

Function **HEAP-EXTRACT-MAX**(A)
1. if heapsize[A] < 1
2. then return ("heap underflow")
3. max = A[1]
5. heapsize[A] = heapsize[A] − 1
6. **MAX-HEAPIFY**(A, 1)
7. return (max)

Function **HEAP-INCREASE-KEY**(A, i, key)
1. if key < A[i]
2. then return ("new key is smaller than current key")
3. A[i] = key
4. while i > 1 and A[PARENT[i]] < A[i]
6. i = PARENT[i]

Function **MAX-HEAP-INSERT**(A, key)
1. heapsize[A] = heapsize[A] + 1
2. A[heapsize[A]] = −∞
3. **HEAP-INCREASE-KEY**(A, heapsize[A], key)
Chapter 7. Quicksort and randomized algorithms
Chapter 7. Quicksort

Chapter 7. Quicksort and randomized algorithms

Idea of the Quicksort: divide-and-conquer
Chapter 7. Quicksort and randomized algorithms

Idea of the Quicksort: divide-and-conquer

• divide: re-organize list $A[p, r]$ into two sublists $A[p, q - 1]$ and $A[q + 1, r]$ based on pivot $A[q]$, such that
Chapter 7. Quicksort and randomized algorithms

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Chapter 7. Quicksort and randomized algorithms

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Chapter 7. Quicksort

Chapter 7. Quicksort and randomized algorithms

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Chapter 7. Quicksort

Chapter 7. Quicksort and Randomized algorithms
Algorithm \textsc{QuickSort} $(A, p, r)$

1. if $p < r$
2. then \[ q = \text{Partition}(A, p, r) \]
3. \text{QuickSort}(A, p, q-1)
4. \text{QuickSort}(A, q+1, r)

How the pivot $A[q]$ is identified is crucial to the performance of QuickSort.

- Assume $A[q]$ partitions list $A[p, r]$ evenly, then $T(n) \leq 2T(n/2) + cn = O(n \log_2 n)$
- Assume $A[q]$ partitions the list 20% vs 80%, then $T(n) \leq T(5n) + T(4n) + cn = O(n \log_2 n)$
- Assume $A[q]$ partitions the list 1% vs 99%, then $T(n) \leq T(100n) + T(99n) + cn = O(n \log_2 n)$

How can we identify such a pivot?
Chapter 7. Quicksort

Chapter 7. Quicksort and Randomized algorithms

Algorithm QUICKSORT \((A, p, r)\)
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Chapter 7. Quicksort

Chapter 7. Quicksort and Randomized algorithms

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Chapter 7. Quicksort

Chapter 7. Quicksort and Randomized algorithms

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Chapter 7. Quicksort

Chapter 7. Quicksort and Randomized algorithms

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Chapter 7. Quicksort

Chapter 7. Quicksort and Randomized algorithms

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How can we identify such a pivot?
Chapter 7. Quicksort

\[ \begin{array}{cccccccc}
    i & j & p & q & r \\
    \hline
    2 & 8 & 7 & 1 & 3 & 5 & 6 & 4 \\
    \hline
    2 & 8 & 7 & 1 & 3 & 5 & 6 & 4 \\
    \hline
    2 & 8 & 7 & 1 & 3 & 5 & 6 & 4 \\
    \hline
    2 & 8 & 7 & 1 & 3 & 5 & 6 & 4 \\
    \hline
    2 & 8 & 7 & 1 & 3 & 5 & 6 & 4 \\
    \hline
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    \hline
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    \hline
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    \hline
    2 & 8 & 7 & 1 & 3 & 5 & 6 & 4 \\
    \hline
    \end{array} \]

quicksort(A, p, q-1)  quicksort(A, q+1, r)
PARTITION($A, p, r$)
1 $x \leftarrow A[r]$
2 $i \leftarrow p - 1$
3 for $j \leftarrow p$ to $r - 1$
4 do if $A[j] \leq x$
5 then $i \leftarrow i + 1$
6 exchange $A[i] \leftrightarrow A[j]$
7 exchange $A[i + 1] \leftrightarrow A[r]$
8 return $i + 1$
Partition may not guarantee to partition the list to two fractions of sizes $\epsilon n : (1 - \epsilon)n$, for a constant $\epsilon > 0$. 
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- however, chances for skewed cases like above are very small.
- that is, the cases other than the skewed ones occur much more often.

So the idea of Quicksort may work well on a majority of data.
Chapter 7. Quicksort

Assume that the equal likely chance for every number to be in the last position, what is the chance to partition the list into

\[ x\% \text{ vs } (100 - x)\% \]

fragments, for \( 10 \leq x \leq 90 \)?
Chapter 7. Quicksort

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fragments, for \( 10 \leq x \leq 90 \)?

The chance is \( = 80\% \)
Chapter 7. Quicksort

What running time would it be if 10:90 partition is always guaranteed?
Chapter 7. Quicksort

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\[ T(n) \leq T(n/10) + T(9n/10) + cn \]
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Chapter 7. Quicksort

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Using the recursive-tree method, we have

- \( l_0: \quad cn \)
- \( cn \)
- \( cn \)

\[ T(n) \leq cn \log_{10} 9 \]

\[ T(n) \leq cn \log_{10} 9 = c'n \log_{10} n = O(n \log n) \]
Chapter 7. Quicksort

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\[ l_0: \quad cn \quad cn \]
\[ l_1: \quad cn/10 \quad 9cn/10 \]

\[ l_h: \quad cn/10^h \quad 9cn/10^h \]

\[ l_k: \quad c9n/10^k \quad 9cn/10^k \]

where \( l_0 = 1 \), i.e., \( h = \log_{10} n \),

\[ T(n) \leq cn \log_{10} 9n \]

\[ T(n) \leq cn \log_{10} 9n = c' n \log_2 n = O(n \log_2 n) \]

where \( c' = c/\log_{10} 9 \).
What running time would it be if 10:90 partition is always guaranteed?

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\[
\begin{align*}
    l_0: & \quad cn \\
    l_1: & \quad cn/10 \quad \quad 9cn/10 \\
    l_2: & \quad \quad cn/10^2 \quad 9cn/10^2 \quad 9cn/10^2 \quad 9^2cn/10^2 \\
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Chapter 7. Quicksort

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Using the recursive-tree method, we have

\[ l_0: \quad \text{cn} \]
\[ l_1: \quad \text{cn}/10 \quad \text{cn}/10 \quad 9\text{cn}/10 \]
\[ l_2: \quad \text{cn}/10^2 \quad 9\text{cn}/10^2 \quad 9\text{cn}/10^2 \quad 9^2\text{cn}/10^2 \]
\[ \ldots \]

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Chapter 7. Quicksort

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Using the recursive-tree method, we have

<table>
<thead>
<tr>
<th>Level ( l )</th>
<th>( cn )</th>
<th>( cn/10 )</th>
<th>( 9cn/10 )</th>
<th>( cn/10^2 )</th>
<th>( 9cn/10^2 )</th>
<th>( 9^2cn/10^2 )</th>
<th>( \cdots )</th>
<th>( c9^n n/10^h )</th>
<th>( cn )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( l_0 )</td>
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<td>( cn )</td>
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Chapter 7. Quicksort

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    \cdots & & \cdots & & \cdots & & \cdots \\
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Chapter 7. Quicksort

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  l_2: & \quad cn/10^2, 9cn/10^2, 9^2cn/10^2 \\
  & \quad \ldots \\
  \vdots & \quad \ldots \\
  l_h: & \quad cn/10^h, \ldots, c9^h n/10^h, cn \\
  l_k: & \quad \ldots, c9^k n/10^k \leq cn
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where \((\frac{1}{10})^h n = 1\), i.e., \(h = \log_{10} n\)
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\end{align*}
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where \( \left( \frac{1}{10} \right)^h n = 1 \), i.e., \( h = \log_{10} n \)

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where \( \left(\frac{1}{10}\right)^h n = 1 \), i.e., \( h = \log_{10} n \)

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\[ cn \log_{10} n \leq T(n) \leq cn \log_{\frac{10}{9}} n \]
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\[cn \log_{10} n \leq T(n) \leq cn \log_{\frac{10}{9}} n\]

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\[ cn \log_{10} n \leq T(n) \leq cn \log_{\frac{10}{9}} n \]

\[ T(n) \leq cn \log_{\frac{10}{9}} n = cn \frac{\log_2 n}{\log_2 \frac{10}{9}} \]
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\begin{align*}
  l_0: & \quad cn \\
  l_1: & \quad cn/10 \quad 9cn/10 \\
  l_2: & \quad cn/10^2 \quad 9cn/10^2 \quad 9^2cn/10^2 \\
  \vdots \\
  l_h: & \quad cn/10^h \quad \cdots \quad c9^h n/10^h \\
  l_k: & \quad \cdots \quad c9^k n/10^k \quad \leq cn
\end{align*}
\]

where \((\frac{1}{10})^h n = 1\), i.e., \(h = \log_{10} n\)

\((\frac{9}{10})^k n = 1\), i.e., \(k = \log_{\frac{10}{9}} n\)

\[ cn \log_{10} n \leq T(n) \leq cn \log_{\frac{10}{9}} n \]

\[ T(n) \leq cn \log_{\frac{10}{9}} n = cn \frac{\log_2 n}{\log_2 \frac{10}{9}} = c\' n \log_2 n \]
Chapter 7. Quicksort

What running time would it be if 10:90 partition is always guaranteed?

\[ T(n) \leq T(n/10) + T(9n/10) + cn \]

Using the recursive-tree method, we have

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\begin{align*}
\text{l}_0: & \quad cn \\
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\end{align*}
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where \( \left(\frac{1}{10}\right)^h n = 1 \), i.e., \( h = \log_{10} n \)

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Chapter 7. Quicksort

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where \( c' = c/ \log_2 \frac{10}{9} \)
Instead of analyzing \texttt{QuickSort} (with uniformly distributed for input) we design a randomized version of the algorithm and analyze it.

\begin{algorithm}
\caption{Randomized-Partition}\label{alg:randomized-partition}
\begin{algorithmic}[1]
\State $i \leftarrow \text{random}(p, r)$
\State $\text{exchange } A[r] \leftrightarrow A[i]$
\State return $\text{Partition}(A, p, r)$
\end{algorithmic}
\end{algorithm}

\begin{algorithm}
\caption{Randomized QuickSort}\label{alg:randomized-quick-sort}
\begin{algorithmic}[1]
\If{$p < r$}
\State $q \leftarrow \text{Randomized-Partition}(A, p, r)$
\State $\text{Randomized QuickSort}(A, p, q - 1)$
\State $\text{Randomized QuickSort}(A, q + 1, r)$
\EndIf
\end{algorithmic}
\end{algorithm}
Chapter 7. Quicksort

Instead of analyzing QuickSort (with uniformly distributed for input) we design a randomized version of the algorithm and analyze it.

Algorithm Randomized-Partition\((A,p,r)\)
1. \(i = \text{random}(p,r)\)
2. exchange \(A[r] \leftrightarrow A[i]\)
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Instead of analyzing **QuickSort** (with uniformly distributed for input) we design a randomized version of the algorithm and analyze it.

Algorithm **RANDOMIZED-PARTITION** \((A, p, r)\)

1. \(i = \text{random}(p, r)\)
2. exchange \(A[r] \leftrightarrow A[i]\)
3. return \((\text{PARTITION}(A, p, r))\)

Algorithm **RANDOMIZED QUICKSORT** \((A, p, r)\)
Instead of analyzing \textsc{QuickSort} (with uniformly distributed for input) we design a randomized version of the algorithm and analyze it.

Algorithm \textsc{Randomized-Partition}(A, p, r)
1. \(i = \text{random}(p, r)\)
2. exchange \(A[r] \leftrightarrow A[i]\)
3. \textbf{return} (\textsc{Partition}(A, p, r))

Algorithm \textsc{Randomized QuickSort} (A, p, r)
1. \textbf{if} \(p < r\)
Chapter 7. Quicksort

Instead of analyzing \textsc{QuickSort} (with uniformly distributed for input) we design a randomized version of the algorithm and analyze it.

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2. exchange $A[r] \leftrightarrow A[i]$
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Chapter 7. Quicksort

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Analysis of \textsc{Randomized-QuickSort}
Analysis of **RANDOMIZED-QUICKSORT**

- count the expected number of comparisons between \( x_i \) and \( x_j \);
Chapter 7. Quicksort

Analysis of **RANDOMIZED-QUICKSORT**

- count the expected number of comparisons between \(x_i\) and \(x_j\);

**Observation 1**: \(x_i\) is compared with \(x_j\) only when either is a pivot;
Chapter 7. Quicksort

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Chapter 7. Quicksort

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- let \(X = \sum_{i=1}^{n} \sum_{j=1, i<j}^{n} X_{i,j}\), written as \(X = \sum_{i<j} X_{i,j}\)
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Analysis of Randomized-QuickSort

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$$E(X) = E\left(\sum_{i<j} X_{i,j}\right) = \sum_{i<j} E(X_{i,j}) = \sum_{i<j} \, P(X_{i,j} = 1)$$
Chapter 7. Quicksort

Analysis of Randomized-QuickSort

• count the expected number of comparisons between \(x_i\) and \(x_j\);

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\[
E(X) = E(\sum_{i<j} X_{i,j}) = \sum_{i<j} E(X_{i,j}) = \sum_{i<j} P(X_{i,j} = 1)
\]
Analysis of \textsc{Randomized-QuickSort} (cont.)

\[ E(X) = E\left(\sum_{i<j} X_{i,j}\right) \]
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Analysis of RANDOMIZED-QUICKSORT (cont.)

\[ E(X) = E\left(\sum_{i<j} X_{i,j}\right) = \sum_{i<j} E(X_{i,j}) = \]
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Analysis of **Randomized-QuickSort** (cont.)

\[
E(X) = E\left( \sum_{i<j} X_{i,j} \right) = \sum_{i<j} E(X_{i,j}) = \sum_{i<j} \text{Prob}(X_{i,j} = 1)
\]
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Analysis of RANDOMIZED-QUICKSORT (cont.)

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\( X_{i,j} = 1 \), i.e., comparison between \( x_i \) and \( x_j \) occurs only when
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Analysis of \texttt{RANDOMIZED-QUICKSORT} (cont.)

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Analysis of RANDOMIZED-QUICKSORT (cont.)

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\[
P(X_{i,j} = 1) = 2 \frac{1}{|L|}, \text{ where } |L| \text{ is the size of the sublist. why?}
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Analysis of \textsc{Randomized-QuickSort} (cont.)

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but we do not know the size of the sublist \( L \)!
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Analysis of **RANDOMIZED-QUICKSORT** (cont.)

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E(X) = E\left(\sum_{i<j} X_{i,j}\right) = \sum_{i<j} E(X_{i,j}) = \sum_{i<j} \text{Prob}(X_{i,j} = 1)
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however, if \(x_i, x_j\) are so indexed in the final sorted list,
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Analysis of **Randomized-QuickSort** (cont.)

\[ E(X) = E\left(\sum_{i<j} X_{i,j}\right) = \sum_{i<j} E(X_{i,j}) = \sum_{i<j} \text{Prob}(X_{i,j} = 1) \]

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but we do not know the size of the sublist \(L\)!

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Analysis of \textsc{Randomized-QuickSort} (cont.)

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$X_{i,j} = 1$, i.e., comparison between $x_i$ and $x_j$ occurs only when

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but we do not know the size of the sublist $L$!

however, if $x_i, x_j$ are so indexed in the final sorted list, then

size of the sublist (which $x_i, x_j$ belongs to)

$$|L| \geq (j - i + 1)$$
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Analysis of **RANDOMIZED-QUICKSORT** (cont.)

\[ E(X) = E\left(\sum_{i<j} X_{i,j}\right) = \sum_{i<j} E(X_{i,j}) = \sum_{i<j} \text{Prob}(X_{i,j} = 1) \]

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\[ |L| \geq (j - i + 1) \]

So \( P(X_{i,j} = 1) \leq 2 \frac{1}{|L|} \leq 2 \frac{1}{j-i+1} \)
original unsorted list

sublist $L$ containing elements 5 and 10
10 is a pivot

$L$ has to contain elements between 5 and 10
i.e., $L$ has to contain elements 6, 7, 8, 9
$|L| \geq j - i + 1 = 10 - 5 + 1 = 6$

final sorted list

$x_5$ $x_{10}$
Analysis of \texttt{RANDOMIZED-QUICKSORT} (cont.)

\begin{align*}
E(X) &= E\left(\sum_{i<j} X_{i,j}\right) = \sum_{i<j} E(X_{i,j}) = \sum_{i<j} \text{Prob}(X_{i,j} = 1) \\
&\leq \sum_{i<j} 2 \frac{1}{j - i + 1}
\end{align*}

for some constant $c > 0$.

So $E(X) = O(n \log_2 n)$. 
Chapter 7. Quicksort

Analysis of \textsc{Randomized-QuickSort} (cont.)

\[ E(X) = E(\sum_{i<j} X_{i,j}) = \sum_{i<j} E(X_{i,j}) = \sum_{i<j} \text{Prob}(X_{i,j} = 1) \]

\[ \leq \sum_{i<j} 2 \frac{1}{j - i + 1} = \sum_{i=1}^{n-1} \sum_{j=2}^{n} 2 \frac{1}{j - i + 1} \leq n \sum_{i=1}^{n-1} \sum_{j=2}^{n} \frac{1}{j - i + 1} \leq cn \log_2 n \]

for some constant \( c > 0 \). So \( E(X) = O(n \log_2 n) \).
Chapter 7. Quicksort

Analysis of \texttt{RANDOMIZED-QUICKSORT (cont.)}

\begin{align*}
E(X) &= E\left(\sum_{i<j} X_{i,j}\right) = \sum_{i<j} E(X_{i,j}) = \sum_{i<j} \text{Prob}(X_{i,j} = 1) \\
&\leq \sum_{i<j} 2 \frac{1}{j - i + 1} = \sum_{i=1}^{n-1} \sum_{j=2}^{n} 2 \frac{1}{j - i + 1} \\
&\leq \sum_{i=1}^{n-1} 2 \sum_{k=1}^{n-1} \frac{1}{k + 1} \leq
\end{align*}
Chapter 7. Quicksort

Analysis of \textsc{Randomized-QuickSort} (cont.)

\[ E(X) = E\left(\sum_{i<j} X_{i,j}\right) = \sum_{i<j} E(X_{i,j}) = \sum_{i<j} \text{Prob}(X_{i,j} = 1) \]

\[ \leq \sum_{i<j} 2 \frac{1}{j - i + 1} = \sum_{i=1}^{n-1} \sum_{j=2}^{n} 2 \frac{1}{j - i + 1} \]

\[ \leq \sum_{i=1}^{n-1} 2 \sum_{k=1}^{n-1} \frac{1}{k + 1} \leq \sum_{i=1}^{n-1} c \log_2 n \]
**Chapter 7. Quicksort**

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**Analysis of RANDOMIZED-QUICKSORT (cont.)**

\[
E(X) = E\left(\sum_{i<j} X_{i,j}\right) = \sum_{i<j} E(X_{i,j}) = \sum_{i<j} \text{Prob}(X_{i,j} = 1)
\]

\[
\leq \sum_{i<j} 2 \frac{1}{j - i + 1} = \sum_{i=1}^{n-1} \sum_{j=2}^{n} 2 \frac{1}{j - i + 1}
\]

\[
\leq \sum_{i=1}^{n-1} 2 \sum_{k=1}^{n-1} \frac{1}{k + 1} \leq \sum_{i=1}^{n-1} c \log_2 n \leq cn \log_2 n
\]
Chapter 7. Quicksort

Analysis of \textsc{Randomized-QuickSort} (cont.)

\[ E(X) = E(\sum_{i<j} X_{i,j}) = \sum_{i<j} E(X_{i,j}) = \sum_{i<j} \text{Prob}(X_{i,j} = 1) \]

\[ \leq \sum_{i<j} \frac{2}{j - i + 1} = \sum_{i=1}^{n-1} \sum_{j=2}^{n} \frac{1}{j - i + 1} \]

\[ \leq \sum_{i=1}^{n-1} 2 \sum_{k=1}^{n-1} \frac{1}{k + 1} \leq \sum_{i=1}^{n-1} c \log_2 n = cn \log_2 n \]

for some constant \( c > 0 \).
Analysis of Randomized-QuickSort (cont.)

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So \( E(X) = O(n \log_2 n) \).
Chapter 7. Quicksort

O(n log n) Sorting Algorithms
Chapter 8. Lower Bounds and Sorting in Linear Time

Chapter 8. Lower bounds and sorting in linear time
Chapter 8. Lower Bounds and Sorting in Linear Time

Chapter 8. Lower bounds and sorting in linear time

- We have used Big-\( O \) for upper bounds.
Chapter 8. Lower bounds and sorting in linear time

- We have used Big-$O$ for upper bounds.
- We need another notation for lower bounds.

\[ \Omega(g(n)) = \{ f(n) : \exists c > 0, k > 0 \text{ such that } f(n) \geq cg(n) \text{ for all } n \geq k \} \]

In other words, \( \lim_{n \to \infty} \frac{f(n)}{g(n)} = \text{constant} > 0 \) or \( \infty \).

For example, we have shown \( T(n) = \Omega(n^2) \) for Insertion Sort.

Proof techniques for Big-$\Omega$ are similar to those for Big-$O$. 
Chapter 8. Lower Bounds and Sorting in Linear Time

Chapter 8. Lower bounds and sorting in linear time

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- We need another notation for lower bounds.

Define $\Omega(g(n))$ be the set of functions that have growth rates not slower than $cg(n)$ for any given constant $c > 0$.
Chapter 8. Lower Bounds and Sorting in Linear Time

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Chapter 8. Lower bounds and sorting in linear time

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Chapter 8. Lower bounds and sorting in linear time

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Define $\Omega(g(n))$ be the set of functions that have growth rates not slower than $cg(n)$ for any given constant $c > 0$.

$$\Omega(g(n)) = \{ f(n) : \exists c > 0, k > 0 \text{ such that } f(n) \geq cg(n) \text{ for all } n \geq k \}$$

In other word, $\lim_{n \to \infty} \frac{f(n)}{g(n)} = \text{constant} > 0$ or $\infty$
Chapter 8. Lower bounds and sorting in linear time

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For example, we have shown $T(n) = \Omega(n^2)$ for Insertion Sort.
Proof techniques for Big-$\Omega$ are similar to those for Big-$O$. 
Important notes on lower bound and upper bound

- Insertion Sort runs in time $\Theta(n^2)$.
- MergeSort runs in time $\Theta(n \log_2 n)$.
- The sorting problem has $\Theta(n \log_2 n)$ time complexity.

This means the problem can be solved in time $O(n \log_2 n)$ and $\Omega(n \log_2 n)$ is necessary to solve the problem.
Chapter 8. Lower Bounds and Sorting in Linear Time

Important notes on lower bound and upper bound

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Insertion Sort: $O(n^2) \rightarrow O(n^2)$

Merge Sort: $O(n \log_2 n) \rightarrow O(n \log_2 n)$

Lower bound for Insertion Sort: $\Omega(n^2)$

Lower bound for Merge Sort: $\Omega(n \log_2 n)$

Function $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$ if and only if $f(n) = \Theta(g(n))$.

So we say:

- Insertion Sort runs in time $\Theta(n^2)$,
- Merge Sort runs in time $\Theta(n \log_2 n)$,
- the sorting problem has $\Theta(n \log_2 n)$ time complexity, meaning: the problem can be solved in time $O(n \log_2 n)$ and $\Omega(n \log_2 n)$ is necessary to solve the problem.
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## Chapter 8. Lower Bounds and Sorting in Linear Time

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Chapter 8. Lower Bounds and Sorting in Linear Time

Deriving a lower bound for sorting

with decision tree as algorithm/computation model

Claim 1: total number of leaves is $\geq n!$.

Claim 2: the height of the tree at least $\geq \log n!$.

(The minimum of heights of all such trees!)
Chapter 8. Lower Bounds and Sorting in Linear Time

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**Theorem:** Sorting needs $\Omega(n \log n)$ comparisons on comparison-based computation models.
Chapter 8. Lower Bounds and Sorting in Linear Time

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**Prove**.
The longest path from the root to a leaf is $\Omega(\log n!)$. I.e., the number of comparisons needed in the worst case is $\Omega(\log n!)$. 

$n! = n(n-1)(n-2) \cdots (n-n) = (n^2) \times (n^2-1) \times \cdots \times 2 \times 1 = \Omega((n^2)^n)$ or by Stirling's formula:

$n! = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \left(1 + O\left(\frac{1}{n}\right)\right) = \Omega(n \log n)$.
Theorem: Sorting needs $\Omega(n \log n)$ comparisons on comparison-based computation models.

Prove.
The longest path from the root to a leave is $\Omega(\log n!)$. I.e., the number of comparisons needed in the worst case is $\Omega(\log n!)$.

\[
n! = n(n-1)(n-2)\cdots(n-\frac{n}{2})(n-\frac{n}{2}-1)\cdots2 \times 1
\]

\[
\geq \left(\frac{n}{2}\right)^{\frac{n}{2}} \times 2^{\frac{n}{2} - 1} \geq \frac{1}{2} n^{\frac{n}{2}}
\]

or by Stirling’s formula:

\[
n! = \sqrt{2\pi n}(n/e)^n(1 + O(1/n))
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\[
\Omega(\log(n!)) = \Omega(n \log n)
\]
Chapter 8. Lower Bounds and Sorting in Linear Time

Sorting algorithms with worst case linear time
Chapter 8. Lower Bounds and Sorting in Linear Time

Sorting algorithms with worst case linear time

- count sort
- radix sort
- bucket sort
Chapter 8. Lower Bounds and Sorting in Linear Time

Count sort

Algorithm Counting-Sort (A, B, k)

1. A contains n integers;
2. \( k \) is the max;
3. \( C[i] = 0 \);
4. for \( j = 1 \) to \( \text{length}[A] \)
   5. \( C[A[j]] = C[A[j]] + 1 \);
6. for \( i = 0 \) to \( k \)
   7. \( C[i] = C[i] + C[i-1] \);
8. \( C[i] \) contains the number of elements whose values = \( i \);
9. for \( j = \text{length}[A] \) down to 1
   11. \( C[A[j]] = C[A[j]] - 1 \);

Example: A: 2 5 3 0 2 3 0 3, \( k = 5 \), C: 2 0 2 3 0 1

analysis: \( T(n) = O(k + n) \)
Count sort

Algorithm COUNTING-SORT \((A, B, k)\)  \{\(A\) contains \(n\) integers; \(k\) is the max\}
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Chapter 8. Lower Bounds and Sorting in Linear Time

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8. \{\(C[i]\) contains the number of elements whose values \(\leq i\)\}
9. \textbf{for} \(j = \text{length}[A]\) \textbf{downto} \(1\)
10. \(B[C[A[j]]] = A[j]\)
11. \(C[A[j]] = C[A[j]] - 1\)

Example: \(A: 2\ 5\ 3\ 0\ 2\ 3\ 0\ 3\), \(k = 5\), \(C: 2\ 0\ 2\ 3\ 0\ 1\)

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6. \(\text{for } i = 1 \text{ to } k\)
7. \(C[i] = C[i] + C[i - 1]\)
8. \(\{C[i] \text{ contains the number of elements whose values } \leq i\}\)
Chapter 8. Lower Bounds and Sorting in Linear Time

Count sort

Algorithm COUNTING-SORT \((A, B, k)\) \{\(A\) contains \(n\) integers; \(k\) is the max\}

1. for \(i = 0\) to \(k\)
2. \(\quad C[i] = 0\)
3. for \(j = 1\) to \(\text{length}[A]\)
4. \(\quad C[A[j]] = C[A[j]] + 1\)
5. \(\{C[i]\ \text{contains the number of elements whose values} = i\}\)
6. for \(i = 1\) to \(k\)
7. \(\quad C[i] = C[i] + C[i - 1]\)
8. \(\{C[i]\ \text{contains the number of elements whose values} \leq i\}\)
9. for \(j = \text{length}[A]\) downto 1
Count sort

Algorithm COUNTING-SORT \((A, B, k)\) \(\{A \text{ contains } n \text{ integers; } k \text{ is the max}\}\)

1. \textbf{for} \(i = 0\) \textbf{to} \(k\)
2. \hspace{1em} \(C[i] = 0\)
3. \textbf{for} \(j = 1\) \textbf{to} \(\text{length}[A]\)
4. \hspace{1em} \(C[A[j]] = C[A[j]] + 1\)
5. \(\{C[i] \text{ contains the number of elements whose values } = i\}\)
6. \textbf{for} \(i = 1\) \textbf{to} \(k\)
7. \hspace{1em} \(C[i] = C[i] + C[i - 1]\)
8. \(\{C[i] \text{ contains the number of elements whose values } \leq i\}\)
9. \textbf{for} \(j = \text{length}[A]\) \textbf{downto} \(1\)
10. \(B[C[A[j]]] = A[j]\)

Example: \(A: 2 \ 5 \ 3 \ 0 \ 2 \ 3 \ 0 \ 3\), \(k = 5\), \(C: 2 \ 0 \ 2 \ 3 \ 0 \ 1\)

analysis: \(T(n) = O(k + n)\)
Chapter 8. Lower Bounds and Sorting in Linear Time

Count sort

Algorithm COUNTING-SORT $(A, B, k)$ \{ $A$ contains $n$ integers; $k$ is the max \}

1. \textbf{for} $i = 0$ to $k$
2. \hspace{1em} $C[i] = 0$
3. \textbf{for} $j = 1$ to length[$A$]
4. \hspace{1em} $C[A[j]] = C[A[j]] + 1$
5. \{ $C[i]$ contains the number of elements whose values $= i$ \}
6. \textbf{for} $i = 1$ to $k$
7. \hspace{1em} $C[i] = C[i] + C[i - 1]$
8. \{ $C[i]$ contains the number of elements whose values $\leq i$ \}
9. \textbf{for} $j = \text{length}[A]$ \textbf{downto} 1
10. \hspace{1em} $B[C[A[j]]] = A[j]$
11. \hspace{1em} $C[A[j]] = C[A[j]] - 1$

Example: $A$: 2 5 3 0 2 3 0 3, $k$ = 5, $C$: 2 0 2 3 0 1

Analysis: $T(n) = O(k + n)$
Count sort

Algorithm COUNTING-SORT \((A, B, k)\) \{\(A\) contains \(n\) integers; \(k\) is the max\}
1. \(\text{for } i = 0 \text{ to } k\)
2. \(C[i] = 0\)
3. \(\text{for } j = 1 \text{ to } \text{length}[A]\)
4. \(C[A[j]] = C[A[j]] + 1\)
5. \(\{C[i] \text{ contains the number of elements whose values } = i\}\)
6. \(\text{for } i = 1 \text{ to } k\)
7. \(C[i] = C[i] + C[i - 1]\)
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9. \(\text{for } j = \text{length}[A] \text{ downto } 1\)
10. \(B[C[A[j]]] = A[j]\)
11. \(C[A[j]] = C[A[j]] - 1\)

Example: \(A: 2 \ 5 \ 3 \ 0 \ 2 \ 3 \ 0 \ 3, \ k = 5,\)
Count sort

Algorithm COUNTING-SORT \((A, B, k)\)  \(\{A\) contains \(n\) integers; \(k\) is the max\}

1.  \textbf{for} \(i = 0\) \textbf{to} \(k\)
2.       \(C[i] = 0\)
3.  \textbf{for} \(j = 1\) \textbf{to} \text{length}[A]
4.       \(C[A[j]] = C[A[j]] + 1\)
5.  \{\(C[i]\) contains the number of elements whose values \(= i\)\}
6.  \textbf{for} \(i = 1\) \textbf{to} \(k\)
7.       \(C[i] = C[i] + C[i - 1]\)
8.  \{\(C[i]\) contains the number of elements whose values \(\leq i\)\}
9.  \textbf{for} \(j = \text{length}[A]\) \textbf{downto} 1
10.     \(B[C[A[j]]] = A[j]\)
11.     \(C[A[j]] = C[A[j]] - 1\)

Example: \(A: 2\ 5\ 3\ 0\ 2\ 3\ 0\ 3, \quad k = 5, \quad C: 2\ 0\ 2\ 3\ 0\ 1\)
Chapter 8. Lower Bounds and Sorting in Linear Time

Count sort

Algorithm Counting-Sort \((A, B, k)\) \(\{A\ contains\ n\ integers;\ k\ is\ the\ max\}\)

1. \(\text{for } i = 0 \text{ to } k\)
2. \(C[i] = 0\)
3. \(\text{for } j = 1 \text{ to } \text{length}[A]\)
4. \(C[A[j]] = C[A[j]] + 1\)
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11. \(C[A[j]] = C[A[j]] - 1\)

Example: \(A: 2\ 5\ 3\ 0\ 2\ 3\ 0\ 3,\ \ k = 5,\ \ C: 2\ 0\ 2\ 3\ 0\ 1\)

analysis:
Chapter 8. Lower Bounds and Sorting in Linear Time

Count sort

Algorithm COUNTING-SORT \(A, B, k\) \(\{A\ contains\ n\ integers;\ k\ is\ the\ max\}\)

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10. \hspace{1em} \(B[C[A[j]]] = A[j]\)
11. \hspace{1em} \(C[A[j]] = C[A[j]] - 1\)

Example: \(A: 2\ 5\ 3\ 0\ 2\ 3\ 0\ 3,\quad k = 5,\quad C: 2\ 0\ 2\ 3\ 0\ 1\)

analysis: \(T(n) = O(k + n)\)
Radix Sort:

Algorithm Radix-Sort \((A, d)\)

1. for \(i = 1\) to \(d\)
2. sort \(A\) on the \(i\)th digit

Lemma. Given \(n\) \(b\)-bit binary numbers and any positive \(r \leq b\).

Radix-Sort uses \(\Theta\left(\lceil \frac{b}{r} \rceil (n + 2r)\right)\) time.
Chapter 8. Lower Bounds and Sorting in Linear Time

Radix Sort:

<table>
<thead>
<tr>
<th>329</th>
<th>720</th>
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<tbody>
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<td>657</td>
<td>839</td>
</tr>
</tbody>
</table>
Radix Sort:

329     720     720     329
457     355     329     355
657     436     436     436
839     457     839     457
436     657     355     657
720     329     457     720
355     839     657     839

Algorithm $\text{Radix-Sort}(A, d)$
Radix Sort:

329    720    720   329
457    355    329   355
657    436    436   436
839    457    839   457
436    657    355   657
720    329    457   720
355    839    657   839

Algorithm RADIUS-SORT(A, d)
1. for i = 1 to d
### Chapter 8. Lower Bounds and Sorting in Linear Time

**Radix Sort:**

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**Algorithm** Radix-Sort($A, d$)

1. `for i = 1 to d`
2. `sort A on the i-th digit`
Chapter 8. Lower Bounds and Sorting in Linear Time

Radix Sort:

329  720  720  329
457  355  329  355
657  436  436  436
839  457  839  457
436  657  355  657
720  329  457  720
355  839  657  839

Algorithm Radix-Sort\((A, d)\)
1. \textbf{for} \(i = 1\) \textbf{to} \(d\)
2. \textbf{sort} \(A\) on the \(i\)th digit

Lemma. Given \(n\) \(b\)-bit binary numbers and any positive \(r \leq b\). Radix-Sort uses \(\Theta([b/r](n + 2^r))\) time.
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Radix-Sort uses $\Theta(\lceil b/r \rceil (n + 2^r))$ time.
Lemma. Given $n$ $b$-bit binary numbers and any positive $r \leq b$. Radix-Sort uses $\Theta([b/r](n + 2^r))$ time.

Proof. Each $b$-digit binary number can be regarded as $[b/r]$ $r$-digit binary numbers. These $r$-digit binary numbers are of integer values in the range of $\{0, 1, \ldots, 2^r - 1\}$.
Lemma. Given \( n \) \( b \)-bit binary numbers and any positive \( r \leq b \), 
\textsc{Radix-Sort} uses \( \Theta([b/r](n + 2^r)) \) time.

**Proof.** Each \( b \)-digit binary number can be regarded as \([b/r]\) \( r \)-digit binary numbers. These \( r \)-digit binary numbers are of integer values in the range of \( \{0, 1, \ldots, 2^r - 1\} \).

Run \textsc{Radix-Sort} on the original binary numbers assumed to be \([b/r]\) columns.
Lemma. Given \( n \) \( b \)-bit binary numbers and any positive \( r \leq b \). \textsc{Radix-Sort} uses \( \Theta(\lceil b/r \rceil (n + 2^r)) \) time.

Proof. Each \( b \)-digit binary number can be regarded as \( \lceil b/r \rceil \) \( r \)-digit binary numbers. These \( r \)-digit binary numbers are of integer values in the range of \( \{0, 1, \ldots, 2^r - 1\} \).

Run \textsc{Radix-Sort} on the original binary numbers assumed to be \( \lceil b/r \rceil \) columns.

For every column, sorting by \textsc{Counting-Sort} with \( 2^r - 1 \) being the maximum.
Lemma. Given \( n \) \( b \)-bit binary numbers and any positive \( r \leq b \). Radix-Sort uses \( \Theta([b/r](n + 2^r)) \) time.

Proof. Each \( b \)-digit binary number can be regarded as \( [b/r] \) \( r \)-digit binary numbers. These \( r \)-digit binary numbers are of integer values in the range of \( \{0, 1, \ldots, 2^r - 1\} \).

Run Radix-Sort on the original binary numbers assumed to be \( [b/r] \) columns.

For every column, sorting by Counting-Sort with \( 2^r - 1 \) being the maximum.

The total time is \( O([b/r](n + 2^r)) \), where \( (n + 2^r) \) is time for Counting-Sort.
Lemma. Given $n$ $b$-bit binary numbers and any positive $r \leq b$. Radix-Sort uses $\Theta([b/r](n + 2^r))$ time.

Proof. Each $b$-digit binary number can be regarded as $[b/r]$ $r$-digit binary numbers. These $r$-digit binary numbers are of integer values in the range of $\{0, 1, \ldots, 2^r - 1\}$.

- Run Radix-Sort on the original binary numbers assumed to be $[b/r]$ columns.
- For every column, sorting by Counting-Sort with $2^r - 1$ being the maximum.
- The total time is $O([b/r](n + 2^r))$, where $(n + 2^r)$ is time for Counting-Sort.

Since all steps in the two algorithms are mandatory, the total time is also $\Omega([b/r](n + 2^r))$, thus $\Theta([b/r](n + 2^r))$. 


**Lemma.** Given $n$ $b$-bit binary numbers and any positive $r \leq b$. Radix-Sort uses $\Theta(\lceil b/r \rceil (n + 2^r))$ time.

**Proof.** Each $b$-digit binary number can be regarded as $\lceil b/r \rceil$ $r$-digit binary numbers. These $r$-digit binary numbers are of integer values in the range of $\{0, 1, \ldots, 2^r - 1\}$.

Run Radix-Sort on the original binary numbers assumed to be $\lceil b/r \rceil$ columns.

For every column, sorting by Counting-Sort with $2^r - 1$ being the maximum.

The total time is $O(\lceil b/r \rceil (n + 2^r))$, where $(n + 2^r)$ is time for Counting-Sort.

Since all steps in the two algorithms are mandatory, the total time is also $\Omega(\lceil b/r \rceil (n + 2^r))$, thus $\Theta(\lceil b/r \rceil (n + 2^r))$.

Once $b$ and $n$ are given, we can choose $r$ to minimize the quantity $\lceil b/r \rceil (n + 2^r)$. 
Bucket Sort (assuming uniform distribution of inputs)
Bucket Sort (assuming uniform distribution of inputs)

Algorithm Bucket-Sort(A)
Chapter 8. Lower Bounds and Sorting in Linear Time

**Bucket Sort** (assuming uniform distribution of inputs)

Algorithm **BUCKET-SORT**($A$)
1. $n = length[A]$
Bucket Sort (assuming uniform distribution of inputs)

Algorithm Bucket-Sort($A$)
1. $n = length[A]$
2. for $i = 1$ to $n$
3. insert $A[i]$ into list $B[\lfloor n A[i] \rfloor]$
4. for $i = 0$ to $n - 1$
5. sort list $B[i]$ with Insertion Sort
6. concatenate the lists $B[0], B[1], ..., B[n - 1]$
Bucket Sort (assuming uniform distribution of inputs)

Algorithm BUCKET-SORT(A)
1. \( n = \text{length}[A] \)
2. \textbf{for} \( i = 1 \) \textbf{to} \( n \)
3. \hspace{1em} insert \( A[i] \) into list \( B[\lfloor nA[i] \rfloor] \)
Chapter 8. Lower Bounds and Sorting in Linear Time

Bucket Sort (assuming uniform distribution of inputs)

Algorithm \textsc{Bucket-Sort}(A)
1. \( n = \text{length}[A] \)
2. \textbf{for} \( i = 1 \) \textbf{to} \( n \)
3. \quad \text{insert} \( A[i] \) \text{ into list } \text{B}[[nA[i]]]
4. \textbf{for} \( i = 0 \) \textbf{to} \( n - 1 \)
Bucket Sort (assuming uniform distribution of inputs)

Algorithm $\text{Bucket-Sort}(A)$
1. $n = \text{length}[A]$
2. for $i = 1$ to $n$
3. insert $A[i]$ into list $B[\lfloor nA[i] \rfloor]$  
4. for $i = 0$ to $n - 1$
5. sort list $B[i]$ with $\text{Insertion Sort}$

A: .78 .17 .39 .26 .72 .94 .21 .12 .23 .68
B: 0 / 1 → .12 / 2 → .21 / 3 → .23 / 4 / 5 / 6 → .68 / 7 → .72 / 8 / 9 → .94
Bucket Sort (assuming uniform distribution of inputs)

Algorithm Bucket-Sort($A$)
1. $n = length[A]$
2. for $i = 1$ to $n$
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Chapter 8. Lower Bounds and Sorting in Linear Time

**Bucket Sort** (assuming uniform distribution of inputs)

Algorithm Bucket-Sort(A)
1. \( n = length[A] \)
2. **for** \( i = 1 \) **to** \( n \)
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4. **for** \( i = 0 \) **to** \( n - 1 \)
5. \( \text{sort list} \ B[i] \text{ with} \ \text{Insertion Sort} \)
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A: .78 .17 .39 .26 .72 .94 .21 .12 .23 .68
Bucket Sort (assuming uniform distribution of inputs)

Algorithm Bucket-Sort(A)
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A: 0.78 0.17 0.39 0.26 0.72 0.94 0.21 0.12 0.23 0.68

B: 0 /
Bucket Sort (assuming uniform distribution of inputs)

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6. concatenate the lists \( B[0], B[1], ..., B[n - 1] \)

A: .78 .17 .39 .26 .72 .94 .21 .12 .23 .68

B: 0 /
1 → .12 → .17
Chapter 8. Lower Bounds and Sorting in Linear Time

**Bucket Sort** (assuming uniform distribution of inputs)

Algorithm **Bucket-Sort**($A$)

1. $n = \text{length}[A]$
2. for $i = 1$ to $n$
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4. for $i = 0$ to $n - 1$
5. sort list $B[i]$ with **Insertion Sort**
6. concatenate the lists $B[0], B[1], ..., B[n - 1]$

A:  .78 .17 .39 .26 .72 .94 .21 .12 .23 .68

B:  0 /
     1 $\rightarrow$ .12 $\rightarrow$ .17
     2 $\rightarrow$ .21 $\rightarrow$ .23 $\rightarrow$ .26
**Bucket Sort** (assuming uniform distribution of inputs)

Algorithm **Bucket-Sort**\((A)\)

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\[A: \quad .78 \quad .17 \quad .39 \quad .26 \quad .72 \quad .94 \quad .21 \quad .12 \quad .23 \quad .68\]

\[B: \quad 0 \quad /\]
\[1 \rightarrow .12 \rightarrow .17\]
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Bucket Sort (assuming uniform distribution of inputs)

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A: \( .78 \ .17 \ .39 \ .26 \ .72 \ .94 \ .21 \ .12 \ .23 \ .68 \)

B: \( 0 / \)
1 → .12 → .17
2 → .21 → .23 → .26
3 → .39
4 /
Bucket Sort (assuming uniform distribution of inputs)

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6. concatenate the lists $B[0], B[1], ..., B[n - 1]$

A: 0.78 0.17 0.39 0.26 0.72 0.94 0.21 0.12 0.23 0.68

B: 0 /
    1 $\rightarrow$ 0.12 $\rightarrow$ 0.17
    2 $\rightarrow$ 0.21 $\rightarrow$ 0.23 $\rightarrow$ 0.26
    3 $\rightarrow$ 0.39
    4 /
    5 /

---
Bucket Sort (assuming uniform distribution of inputs)

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A: .78 .17 .39 .26 .72 .94 .21 .12 .23 .68

B: 0 /
  1 → .12 → .17
  2 → .21 → .23 → .26
  3 → .39
  4 /
  5 /
  6 → .68
Bucket Sort (assuming uniform distribution of inputs)

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A:  .78 .17 .39 .26 .72 .94 .21 .12 .23 .68

B:  0 /
     1 $\rightarrow$ .12 $\rightarrow$ .17
     2 $\rightarrow$ .21 $\rightarrow$ .23 $\rightarrow$ .26
     3 $\rightarrow$ .39
     4 /
     5 /
     6 $\rightarrow$ .68
     7 $\rightarrow$ .72 $\rightarrow$ .78
Bucket Sort (assuming uniform distribution of inputs)

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A:  .78 .17 .39 .26 .72 .94 .21 .12 .23 .68

B:  0 /
   1 → .12 → .17
   2 → .21 → .23 → .26
   3 → .39
   4 /
   5 /
   6 → .68
   7 → .72 → .78
   8 /
Chapter 8. Lower Bounds and Sorting in Linear Time

**Bucket Sort** (assuming uniform distribution of inputs)

Algorithm `Bucket-Sort(A)`

1. \( n = \text{length}[A] \)
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4. \( \text{for } i = 0 \text{ to } n - 1 \)
5. \( \text{sort list } B[i] \text{ with Insertion Sort} \)
6. \( \text{concatenate the lists } B[0], B[1], \ldots, B[n - 1] \)

\( A: \) .78 .17 .39 .26 .72 .94 .21 .12 .23 .68

\( B: \) 0 /
   1 → .12 → .17
   2 → .21 → .23 → .26
   3 → .39
   4 /
   5 /
   6 → .68
   7 → .72 → .78
   8 /
   9 → .94
Bucket Sort (assuming uniform distribution of inputs)

Algorithm Bucket-Sort(A)
1. \( n = \text{length}[A] \)
2. for \( i = 1 \) to \( n \)
3. insert \( A[i] \) into list \( B[\lfloor nA[i] \rfloor] \)
4. for \( i = 0 \) to \( n - 1 \)
5. sort list \( B[i] \) with Insertion Sort
6. concatenate the lists \( B[0], B[1], ..., B[n - 1] \)

A: .78 .17 .39 .26 .72 .94 .21 .12 .23 .68

B: 0 /
  1 → .12 → .17
  2 → .21 → .23 → .26
  3 → .39
  4 /
  5 /
  6 → .68
  7 → .72 → .78
  8 /
  9 → .94
Chapter 9. Medians and Order Statistics

Chapter 9. Medians and order statistics

• find the maximum: linear time
• find the minimum: linear time
• find the median (i.e., the $n/2$th smallest element)?

The problem has upper bound $O(n \log_2 n)$.

Why? Can we do better?
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Can we do better?
Chapter 9. Medians and Order Statistics

Selection problem

Input: a list \( A \) of elements, an integer \( i \);
Output: the \( i \)th smallest element in \( A \);
There are algorithms solving it in linear time.

Two types of algorithms:

• Selection in expected linear time (but worst case \( \Theta(n^2) \))
• Selection in worst case linear time
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Selection in *expected* linear time

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Selection in *expected* linear time

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Idea of the algorithm:

- randomly identify a pivot $x$ and partition the list $A$ into two sublists $A_l$ and $A_u$; assume the rank of $x$ is $k$;
Selection in \textit{expected} linear time

\textbf{Input:} a list $A$ of elements, an integer $i$;  
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Idea of the algorithm:

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  \item randomly identify a pivot $x$ and partition the list $A$ into two sublists $A_l$ and $A_u$;  
    assume the \textbf{rank} of $x$ is $k$;
  \item if $i = k$, done, return ($x$);
\end{itemize}
Selection in *expected* linear time

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  - else if $k > i$, recursively do for $A_l$ with $i$;
    - else recursively do for $A_u$ with $i - k$;
Algorithm RANDOMIZED-SELECT \((A, p, r, i)\)
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Algorithm \textsc{Randomized-Select} \((A, p, r, i)\)

1. \textbf{if} \(p = r\)
Algorithm \textsc{Randomized-Select} (A, p, r, i)
1. \textbf{if} $p = r$

If pivots always partition lists into $n^r$: $r \leq n$, for some $r > 1$,
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assuming $r \geq 2$,

$$T(n) \leq cn(r-1)^r + cn(r-1)^{r-1} + \ldots + cn(r-1)^1 + cn \leq O(n)$$
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8. return ($\text{RANDOMIZED-SELECT} \ (A, p, q - 1, i)$)
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If pivots always partition lists into \(\frac{n}{r} : \frac{r-1}{r}n\), for some \(r > 1\),
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Algorithm `Randomized-Select (A, p, r, i)`

1. `if p = r`
2. `return (A[p])`
3. `q = Randomized Partition (A, p, r)`
4. `k = q - p + 1`
5. `if i = k`
6. `return (A[q])`
7. `else if i < k`
8. `return (Randomized-Select (A, p, q - 1, i))`
9. `else return (Randomized-Select (A, q + 1, r, i - k))`

If pivots always partition lists into \( \frac{n}{r} : \frac{r-1}{r}n \), for some \( r > 1 \),
time \( T(n) \) would have the recurrence

\[
T(n) \leq \max\{T\left(\frac{n}{r}\right), T\left(\frac{(r - 1)n}{r}\right)\} + nc
\]
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Algorithm \textsc{Randomized-Select} \( (A, p, r, i) \)
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assuming \(r \geq 2\),

\[
T(n) \leq cn\left(\frac{r-1}{r}\right) + cn\left(\frac{r-1}{r}\right)^2 + cn\left(\frac{r-1}{r}\right)^3 + \ldots cn\left(\frac{r-1}{r}\right)^m = O(n)
\]

where \(\left(\frac{r-1}{r}\right)^m n = 1\), \(m = \log_{\frac{r-1}{r}} n\).
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Performance analysis
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Performance analysis

The worst case: running time $\Theta(n^2)$. 
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Average case: $E[T(n)]$
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The worst case: running time $\Theta(n^2)$.

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• on sublist $A[p..r]$, assume $n = r - p + 1$;
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Performance analysis

The worst case: running time $\Theta(n^2)$.

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- on sublist $A[p..r]$, assume $n = r - p + 1$;
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Performance analysis

The worst case: running time $\Theta(n^2)$.

Average case: $E[T(n)]$

- on sublist $A[p..r]$, assume $n = r - p + 1$;
- the algorithm identifies a pivot and recursively computes on sublist $A[p..q]$ (or $A[q + 1..r]$);
- the pivot is chosen with probability $\frac{1}{n}$;
Average case: $E[T(n)]$ (cont’)

- so the expected time $E[T(n)]$ needs to include the average time of recursion on the case when sublist $A[p..q]$ possibly has lengths $k = 0, 1, 2, \ldots, n - 1$
Average case: $E[T(n)]$ (cont’)

- so the expected time $E[T(n)]$ needs to include the average time of recursion on the case when sublist $A[p..q]$ possibly has lengths $k = 0, 1, 2, \ldots, n - 1$

- thus the expected time $E[T(n)]$ is computed as

$$E[T(n)] \leq \sum_{k=1}^{n} \frac{1}{n} \cdot E[T(\max\{k-1, n-k\})] + an, \text{ for some constant } a > 0$$
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Average case: $E[T(n)]$ (cont’)

- so the expected time $E[T(n)]$ needs to include the average time of recursion on the case when sublist $A[p..q]$ possibly has lengths $k = 0, 1, 2, \ldots, n - 1$

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$$E[T(n)] \leq \sum_{k=1}^{n} \frac{1}{n} \cdot E[T(\max\{k-1, n-k\})] + an,$$

for some constant $a > 0$

because $\max\{k - 1, n - k\} = k - 1$ if $k > n/2$ and $\max\{k - 1, n - k\} = n - k$ if $k \leq n/2$

$$E[T(n)] \leq 2/n \sum_{k=n/2}^{n-1} E[T(k)] + an$$
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We conclude that

\[ E[T(n)] \leq \frac{2}{n} \sum_{k=1}^{n-1} k = \frac{n}{2} \]

Theorem.

\[ E[T(n)] = O(n). \]

Proof (by substitution method).

We will prove that \( E[T(n)] \leq cn \) for some \( c > 0 \).

• Base case: \( n = ? \), we will decide later;
• Assumption: for all \( k \leq n - 1 \), \( E[T(k)] \leq ck \);
• Induction: \( E[T(k)] \leq \frac{2}{n} \sum_{k=1}^{n-1} k = \frac{n}{2} E[T(k)] + an \leq \frac{2}{n} \sum_{k=1}^{n-1} k = \frac{n}{2} ck + an \leq 2c n \left[ \sum_{k=1}^{n-1} k - \frac{n}{2} \right] + an = \cdots = 3cn/4 + c/2 + an \leq cn \) when \( (cn/4 - c/2 - an) \geq 0 \).

• Base case: \( T(n) \leq cn \), for \( n < 2c/(c - 4a) \),

How to prove?
We conclude that $E[T(n)] \leq \frac{2}{n} \sum_{k=n/2}^{n-1} E[T(k)] + an$
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We conclude that $E[T(n)] \leq \frac{2}{n} \sum_{k=\frac{n}{2}}^{n-1} E[T(k)] + an$

**Theorem.** $E[T(n)] = O(n)$. 
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**Theorem.** $E[T(n)] = O(n)$.

**Proof** (by substitution method). We will prove that $E[T(n)] \leq cn$ for some $c > 0$. 

[Base case: $T(n) \leq cn$, for $n < 2c/4$]

How to prove?
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We conclude that $E[T(n)] \leq \frac{2}{n} \sum_{k=n/2}^{n-1} E[T(k)] + an$

**Theorem.** $E[T(n)] = O(n)$.

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- Base case: $n =$
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We conclude that $E[T(n)] \leq 2/n\sum_{k=n/2}^{n-1} E[T(k)] + an$

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We conclude that \( E[T(n)] \leq 2/n \sum_{k=n/2}^{n-1} E[T(k)] + an \)

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- Induction:

\[
E[T(k)] \leq 2/n \sum_{k=n/2}^{n-1} E[T(k)] + an
\]

\[= \cdots \leq 3cn/4 + c/2 + an \]

That is when \( n \geq 2c/(c - 4a) \).
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We conclude that $E[T(n)] \leq 2/n\sum_{k=n/2}^{n-1} E[T(k)] + an$

**Theorem.** $E[T(n)] = O(n)$.

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  $$= \frac{2c}{n} \left[ \sum_{k=1}^{n-1} k - \sum_{k=1}^{n/2-1} k \right] + an$$
We conclude that $E[T(n)] \leq 2/n \sum_{k=n/2}^{n-1} E[T(k)] + an$

**Theorem.** $E[T(n)] = O(n)$.

**Proof** (by substitution method). We will prove that $E[T(n)] \leq cn$ for some $c > 0$.

- Base case: $n = 2$, we will decide later;
- Assumption: for all $k \leq n - 1$, $E[T(k)] \leq ck$;
- Induction:

$$E[T(k)] \leq 2/n \sum_{k=n/2}^{n-1} E[T(k)] + an \leq 2/n \sum_{k=n/2}^{n-1} ck + an$$

$$= \frac{2c}{n} \left[ \sum_{k=1}^{n-1} k - \sum_{k=1}^{n/2-1} k \right] + an = \frac{2c}{n} \left[ \frac{n-1}{2} (n) - \frac{n/2-1}{2} (n/2) \right] + an$$
We conclude that $E[T(n)] \leq 2/n \sum_{k=n/2}^{n-1} E[T(k)] + an$

**Theorem.** $E[T(n)] = O(n)$.

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- **Base case:** $n = ?$, we will decide later;
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- **Induction:**

$$E[T(k)] \leq 2/n \sum_{k=n/2}^{n-1} E[T(k)] + an \leq 2/n \sum_{k=n/2}^{n-1} ck + an$$

$$= \frac{2c}{n} \left[ \sum_{k=1}^{n-1} k - \sum_{k=1}^{n/2 - 1} k \right] + an = \frac{2c}{n} \left[ \frac{n - 1}{2} (n) - \frac{n/2 - 1}{2} (n/2) \right] + an$$

$$= \cdots =$$
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We conclude that $E[T(n)] \leq 2/n \sum_{k=n/2}^{n-1} E[T(k)] + an$

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  $$= \frac{2c}{n} \left[ \sum_{k=1}^{n-1} k - \sum_{k=1}^{n/2-1} k \right] + an = \frac{2c}{n} \left[ \frac{n-1}{2} (n) - \frac{n/2 - 1}{2} (n/2) \right] + an$$

  $$= \cdots = \frac{3cn}{4} + c/2 + an = cn - (cn/4 - c/2 - an) \leq cn$$
We conclude that $E[T(n)] \leq 2/n\sum_{k=n/2}^{n-1} E[T(k)] + an$

**Theorem.** $E[T(n)] = O(n)$.

**Proof** (by substitution method). We will prove that $E[T(n)] \leq cn$ for some $c > 0$.

- Base case: $n = ?$, we will decide later;
- Assumption: for all $k \leq n - 1$, $E[T(k)] \leq ck$;
- Induction:

\[
E[T(k)] \leq 2/n\sum_{k=n/2}^{n-1} E[T(k)] + an \leq 2/n\sum_{k=n/2}^{n-1} ck + an
\]

\[
= \frac{2c}{n} \left[ \sum_{k=1}^{n-1} k - \sum_{k=1}^{n/2-1} k \right] + an = \frac{2c}{n} \left[ \frac{n-1}{2} (n) - \frac{n/2 - 1}{2} (n/2) \right] + an
\]

\[
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Chapter 9. Medians and Order Statistics

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- **Base case:** $T(n) \leq cn$, for $n < 2c/(c - 4a)$, How to prove?
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Selection in worst case linear time

Input: set $S$ of $n$ elements and $i$

Output: the $i$th smallest element in $S$

Main idea:
• find a pivot $x$ to partition the list $S$ into two sublists $S_1$ and $S_2$, such that $\forall y \in S_1 \ y < x$ and $\forall z \in S_2 \ z > x$
• both $S_1$ and $S_2$ are guaranteed only a fraction of $S$;
• the $i$th smallest element is either $x$, or in $S_1$ or in $S_2$ (but not both);
• in either of the latter two cases, the algorithm is applied recursively.

$T(n) \leq T(\beta n) + cn$ where $0 < \beta < 1$, such that
$T(n) \leq cn + c\beta n + c\beta^2 n + \ldots + c\beta^m n \leq cn + c\beta n + c\beta^2 n + \ldots \leq c \frac{1}{1-\beta} n = O(n)$
Chapter 9. Medians and Order Statistics

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Chapter 9. Medians and Order Statistics

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  \[
  \forall y \in S_1 \quad y < x
  \]  
  and
  \[
  \forall z \in S_2 \quad z > x
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Chapter 9. Medians and Order Statistics

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Chapter 9. Medians and Order Statistics

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Chapter 9. Medians and Order Statistics

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\[ T(n) \leq T(\beta n) + cn \]

such that

\[ T(n) \leq cn + c\beta n + c\beta^2 n + \ldots + c\beta^m n \leq c \frac{1}{1-\beta} n = O(n) \]
Chapter 9. Medians and Order Statistics

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Chapter 9. Medians and Order Statistics

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Chapter 9. Medians and Order Statistics

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Chapter 9. Medians and Order Statistics

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Chapter 9. Medians and Order Statistics

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$$\leq cn + c\beta n + c\beta^2 n + \cdots \leq c\frac{1}{1-\beta} n = O(n)$$
How to find such a pivot?

- The very selection algorithm is recursively called for finding the pivot.
- The size of the sublist to find the pivot is also a fraction $\alpha n$ of the original list $S$, $|S| = n$.
- The total time actually is $T(n) \leq T(\alpha n) + T(\beta n) + cn$ where $\alpha + \beta < 1$. 
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- the very selection algorithm is recursively called for finding the pivot
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$$T(n) \leq T(\alpha n) + T(\beta n) + cn$$

where $\alpha + \beta < 1$
Algorithm \texttt{Select} \((S, i)\); \{ where \(S\) contains \(n\) distinct elements\}
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Algorithm `SELECT (S, i);` { where $S$ contains $n$ distinct elements }
(1) divide $S$ into $\lceil n/5 \rceil$ groups of 5 elements
Algorithm \textsc{Select} \((S, i)\); \{ where \(S\) contains \(n\) distinct elements\}
(1) divide \(S\) into \(\lceil n/5 \rceil\) groups of 5 elements
(2) sort each group (of 5) and find the median of each group;
Chapter 9. Medians and Order Statistics

Algorithm \texttt{Select} \((S, i)\); \{ where \(S\) contains \(n\) distinct elements\}

1. divide \(S\) into \(\lceil n/5 \rceil\) groups of 5 elements
2. sort each group (of 5) and find the median of each group;
   let \(M\) contain all these medians; where \(|M| = \lceil n/5 \rceil\)

(3) recursively call \texttt{Select} \((M, \lceil n/10 \rceil)\);
let the result be \(x\) and let the rank of \(x\) be \(k\) in \(S\)

(4) if \(i = k\) return \((x)\)

(5) else use \(x\) as the pivot to partition \(S\) resulting in \(S_1\) and \(S_2\),
    such that \(\forall y \in S_1\) \(y < x\) and \(\forall z \in S_2\) \(z > x\)

(6) if \(i < k\) recursively call \texttt{Select} \((S_1, i)\)
    else recursively call \texttt{Select} \((S_2, i - k)\)
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Algorithm \texttt{Select} \((S, i)\); \{ where \(S\) contains \(n\) distinct elements\}

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Algorithm $\text{Select} \ (S, i); \ \{ \text{where } S \text{ contains } n \text{ distinct elements} \}$

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2. sort each group (of 5) and find the median of each group;
   let $M$ contain all these medians; where $|M| = \lceil n/5 \rceil$
3. recursively call $\text{Select}(M, \lceil n/10 \rceil)$;
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Chapter 9. Medians and Order Statistics

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Algorithm \textbf{Select} \((S, i); \{\text{ where } S \text{ contains } n \text{ distinct elements}\}

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3. recursively call \textbf{Select} \((M, \lceil n/10 \rceil)\);
   let the result be \(x\) and let the rank of \(x\) be \(k\) in \(S\)
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Algorithm Select \((S, i)\); \{ where \(S\) contains \(n\) distinct elements\}
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(3) recursively call Select\((M, \lceil n/10 \rceil)\);
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Algorithm $\text{Selective}(S, i);$ \{ where $S$ contains $n$ distinct elements\}

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2. sort each group (of 5) and find the median of each group;
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3. recursively call $\text{Selective}(M, \lceil n/10 \rceil)$;
   let the result be $x$ and let the rank of $x$ be $k$ in $S$
4. if $i = k$ return $(x)$
5. else use $x$ as the pivot to partition $S$ resulting in $S_1$ and $S_2$,
   such that $\forall y \in S_1 \ y < x$ and $\forall z \in S_2 \ z > x$
6. if $i < k$ recursively call $\text{Selective}(S_1, i)$
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Algorithm Select $(S, i)$; { where $S$ contains $n$ distinct elements }

1. divide $S$ into $\lceil n/5 \rceil$ groups of 5 elements
2. sort each group (of 5) and find the median of each group;  
   let $M$ contain all these medians; where $|M| = \lceil n/5 \rceil$
3. **recursively call** Select($M, \lceil n/10 \rceil$);  
   let the result be $x$ and let the rank of $x$ be $k$ in $S$
4. **if** $i = k$ **return** $(x)$
5. **else** use $x$ as the pivot to partition $S$ resulting in $S_1$ and $S_2$,  
   such that $\forall y \in S_1 \ y < x$ and $\forall z \in S_2 \ z > x$
6. **if** $i < k$ **recursively call** Select($S_1, i$)  
   **else recursively call** Select($S_2, i - k$)
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Note: the number of elements \( \leq x \) is at least:

\[ |S_1| \geq 3(\lceil n/5 \rceil^2) \geq 3 \frac{n}{10} \]

\[ \Rightarrow |S_2| < n - 3 \frac{n}{10} = 7 \frac{n}{10} \]

Similarly, the number of elements \( \geq x \) is at least:

\[ |S_2| \geq 3(\lceil n/5 \rceil^2 - 2) \geq 3 \frac{n}{10} - 6 \geq 3 \frac{n}{10} \]

\[ \Rightarrow |S_1| < n - 3 \frac{n}{10} + 6 = 7 \frac{n}{10} + 6 \]

So a time upper bound for \( \text{Select} \) is

\[ T(n) = T_{\text{mom}} + T_{\text{sub}} + O(n) \]

\[ T(n) \leq T(\lceil n/5 \rceil) + T(\lceil 7n/10 + 6 \rceil) + O(n) \]

when \( n \geq 140 \).
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Note: the number of elements $\leq x$ is at least:

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So a time upper bound for \texttt{SELECT} is \( T(n) = T_{mom} + T_{sub} + O(n) \)
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Summary of Algorithm Analysis Scenarios
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Given an algorithm, carry out the following in order:
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Given an algorithm, carry out the following in order:

- analyzing time $T(n)$ of the algorithm
- obtain an expression $T(n) = \ldots$
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For example, given Insertion Sort:

- we first analyzed the algorithm and obtained
  
  $T(n) = c_1 n + c_2 (n - 1) + c_4 (n - 1) + c_5 n \sum_{j=2}^{n} t_j + c_6 n \sum_{j=2}^{n} (t_j - 1) + c_7 n \sum_{j=2}^{n} (t_j - 1) + c_8 (n - 1)$

- we guessed upper bound $T(n) = O(n^2)$, i.e., $T(n) \leq c n^2$;
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Summary of Algorithm Analysis Scenarios
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**Summary of Algorithm Analysis Scenarios**

For recursive algorithms
   For example, given **Binary Search** algorithm,

• we first analyze the time $T(n)$ of the algorithm and obtained a recurrence for $T(n)$:
   
   $T(n) \leq T(\lfloor n/2 \rfloor) + c$

• we guess upper bound $T(n) = O(\log_2 n)$, i.e.,
   
   $T(n) \leq c \log_2 n$;

• we prove the guessed bound.
   (1) we can use the recursive tree method by unfolding the time function;
   or
   (2) we can use the substitution method by the principle of induction.
   But we need the recurrence to apply induction.
   using the recurrence:
   
   $T(n) \leq T(\lfloor n/2 \rfloor) + c$

   to prove
   
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see previous lecture notes
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