Part VI. Graph Algorithms
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- Chapter 22. Elementary graph algorithms
- Chapter 23. Minimum spanning trees
- Chapter 24. Single-source shortest paths
- Chapter 25. All-pairs shortest paths
Chapter 22. Elementary graph algorithms

• Representations of graphs
• traverse graphs:
  (1) breadth-first-search (BFS);
  (2) depth-first-search (DFS)
• applications:
  (1) topological sort
  (2) strongly connected components
Chapter 22. Elementary graph algorithms

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Chapter 22 Elementary graph algorithms

Graph: $G = (V, E)$
Chapter 22. Elementary graph algorithms

Terminologies and notations:

- **Graph** $G = (V, E)$, where $V = \{v_1, \ldots, v_n\}$ and $E \subseteq V \times V$
  
  - **Weight** $w: E \rightarrow \mathbb{R}$, e.g., $w(1, 2) = 4$, $w(5, 6) = 3$, etc.
  
  - **Degree** $\deg(v) = \text{the number of edges incident on } v$, e.g., $\deg(3) = 4$, $\deg(7) = 4$
  
  - **Path** $a \Rightarrow b$, if $(v_1, v_2), \ldots, (v_{k-1}, v_k) \in E$ and $v_1 = a$ and $v_k = b$. The path is a simple path if $v_1, \ldots, v_k$ are all different.
  
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Chapter 22. Elementary graph algorithms

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![Graph Diagram](image)
Chapter 22. Elementary graph algorithms

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Chapter 22. Elementary graph algorithms

- **digraphs**: directed graphs

- **complete graphs**: $K_n$, e.g., $K_6$

- **bipartite graphs**: $G = (V_1 \cup V_2, E)$, $V_1 \cap V_2 = \emptyset$, $K_3, 3$

- **planar graphs**: embedded in the plane without crossing edges. However, $K_5$ is not planar, neither is $K_3, 3$. 
Chapter 22. Elementary graph algorithms

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  ![Diagram of a digraph](image)

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[Diagram of a complete graph $K_6$ and a bipartite graph]

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![Diagram](image-url)
Chapter 22. Elementary graph algorithms

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Chapter 22. Elementary graph algorithms

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  2-tree is a graph but with **tree width** = 2
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Representations of graphs

adjacency-matrix
adjacency-list
Chapter 22. Elementary graph algorithms

Representations of graphs

- adjacency-matrix
- adjacency-list

(a)

(b)

(c)
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adjacency-matrix for a weighted graph
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Traverse graphs

basic ideas of depth-first-search (DFS) and breadth-first-search (BFS)

Both methods yield "search trees"
or "search forest" (if the graph is not connected)
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DFS on directed graphs, search tree
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DFS on directed graphs, search tree
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DFS on directed graphs, search tree

DFS on non-directed graphs, search tree
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Traversal on graphs is an important task:
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- navigating the whole graph;
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- for connectivity check;
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Traversal on graphs is an important task:

• navigating the whole graph;
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• for circle check;
• etc
Traversing on graphs is an important task:

- navigating the whole graph;
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DFS and BFS are two fundamental algorithms for graph traversal!
Chapter 22. Elementary graph algorithms

First recursive DFS algorithm, assuming $G$ is connected.
Chapter 22. Elementary graph algorithms

First recursive DFS algorithm, assuming $G$ is connected.

\textsc{Recursive-DFS}(G, u);

How does the algorithm start?

- Initially set $u.\text{visit} = \text{false}$ for every vertex $u \in G.V$;
- $s.\pi = \text{NULL}$ for some specific $s \in G.V$;
- call \textsc{Recursive-DFS}(G, s).

But if $G$ is not connected, what should we do?
Chapter 22. Elementary graph algorithms

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**Recursive-DFS($G, u$);**
1. **if not** $u.visit$

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RECURSIVE-DFS(G, u);
1. if not $u.visit$
2. $u.visit = true$;                \{ mark u "visited" \}
3. for each $v \in Adj[u]$ and not $v.visit$; \{ u’s unvisited neighbors \}
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Chapter 22. Elementary graph algorithms

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Chapter 22. Elementary graph algorithms

\textbf{To-Start-DFS}(G)
Chapter 22. Elementary graph algorithms

To-START-DFS($G$)
1. \textbf{for} each $s \in G.V$ \hspace{1cm} \{ initialize visit values \}
2. $s.visit = \textbf{false};$

Recursive-DFS($G, s$)
1. if not $u.visit$
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Chapter 22. Elementary graph algorithms

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Chapter 22. Elementary graph algorithms

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1. $\textbf{if not } u.\text{visit}$
2. $u.\text{visit} = \text{true}$; \hspace{1cm} \{ mark u "visited" \}
3. $\textbf{for each } v \in \text{Adj}[u] \text{ and not } v.\text{visit}$; \hspace{1cm} \{ u’s unvisited neighbors \}
4. $v.\pi = u$; \hspace{1cm} \{ set v’s parent to be u \}
5. $\text{RECURSIVE-DFS}(G, v)$;
6. $\text{return }()$;
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DFS (from the textbook) computes discover and finish time stamps ($u.d$ and $u.f$) for every visited vertex $u$.

DFS($G$)
1   for each vertex $u \in G.V$
2       $u.color = \text{WHITE}$
3       $u.\pi = \text{NIL}$
4   $time = 0$
5   for each vertex $u \in G.V$
6       if $u.color == \text{WHITE}$
7       DFS-VISIT($G, u$)

DFS-VISIT($G, u$)
1   $time = time + 1$ \hfill // white vertex $u$ has just been discovered
2   $u.d = time$
3   $u.color = \text{GRAY}$
4   for each $v \in G.Adj[u]$ \hfill // explore edge $(u, v)$
5       if $v.color == \text{WHITE}$
6           $v.\pi = u$
7           DFS-VISIT($G, v$)
8   $u.color = \text{BLACK}$ \hfill // blacken $u$; it is finished
9   $time = time + 1$
10  $u.f = time$
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DFS-Visit\(G, u\)

1. \(\text{time} = \text{time} + 1\)  \hspace{1cm} // white vertex \(u\) has just been discovered
2. \(u.d = \text{time}\)
3. \(u.color = \text{GRAY}\)
4. \(\text{for each } v \in G.\text{Adj}[u] \) \hspace{1cm} // explore edge \((u, v)\)
5. \(\text{if } v.color == \text{WHITE}\)
6. \hspace{1cm} \(v.\pi = u\)
7. \hspace{1cm} DFS-Visit\((G, v)\)
8. \(u.color = \text{BLACK}\)  \hspace{1cm} // blacken \(u\); it is finished
9. \(\text{time} = \text{time} + 1\)
10. \(u.f = \text{time}\)

\(\rightarrow\): edge being explored;
\(\rightarrow\): edge path taken by DFS
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DFS-Visit(G, u)
1. time = time + 1 // white vertex u has just been discovered
2. u.d = time
3. u.color = GRAY
4. for each v ∈ G.Adj[u] // explore edge (u, v)
5. if v.color == WHITE
6. v.π = u
7. DFS-Visit(G, v)
8. u.color = BLACK // blacken u; it is finished
9. time = time + 1
10. u.f = time

→: edge being explored;
→: edge path taken by DFS
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DFS-VISIT(G, u)

1. time = time + 1  // white vertex u has just been discovered
2. u.d = time
3. u.color = GRAY
4. for each v ∈ G.Adj[u]  // explore edge (u, v)
   if v.color == WHITE
      v.π = u
      DFS-VISIT(G, v)
6. u.color = BLACK  // blacken u; it is finished
7. time = time + 1
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→: edge being explored;
→: edge path taken by DFS
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DFS-VISIT(G, u)
1 \[\text{time} = \text{time} + 1\] \hspace{1cm} // white vertex \( u \) has just been discovered
2 \( u.d = \text{time} \)
3 \( u.color = \text{GRAY} \)
4 for each \( v \in G.\text{Adj}[u] \)
5 \hspace{1cm} if \( v.color == \text{WHITE} \)
6 \hspace{2cm} \( v.\pi = u \)
7 \hspace{2cm} DFS-VISIT(G, v)
8 \( u.color = \text{BLACK} \) \hspace{1cm} // blacken \( u \); it is finished
9 \( \text{time} = \text{time} + 1 \)
10 \( u.f = \text{time} \)

→: edge being explored;
→: edge path taken by DFS
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Another example of DFS execution (page 605)
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Time complexity of DFS algorithm

$\Theta(|E| + |V|)$, where $|E|$ is the number of edges in $G$. 

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Time complexity of DFS algorithm

```
DFS(G)
1   for each vertex u ∈ G.V
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5   for each vertex u ∈ G.V
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DFS-VISIT(G, u)
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2   u.d = time
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4   for each v ∈ G.Adj[u]  // explore edge (u, v)
5       if v.color == WHITE
6           v.π = u
7           DFS-VISIT(G, v)
8   u.color = BLACK  // blacken u; it is finished
9   time = time + 1
10  u.f = time
```
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Time complexity of DFS algorithm

\[
\text{DFS}(G)
\]

1. \textbf{for} each vertex \( u \in G. V \)
2. \hspace{1em} \( u.\text{color} = \text{WHITE} \)
3. \hspace{1em} \( u.\pi = \text{NIL} \)
4. \hspace{1em} \text{time} = 0
5. \textbf{for} each vertex \( u \in G. V \)
6. \hspace{1em} \text{if} \( u.\text{color} == \text{WHITE} \)
7. \hspace{2em} \text{DFS-Visit}(G, u)

\text{DFS-Visit}(G, u)

1. \text{time} = \text{time} + 1 \quad \text{\hspace{1em} // white vertex } u \text{ has just been discovered}
2. \hspace{1em} \( u.d = \text{time} \)
3. \hspace{1em} \( u.\text{color} = \text{GRAY} \)
4. \textbf{for} each \( v \in G.\text{Adj}[u] \) \quad \text{\hspace{1em} // explore edge } (u, v)
5. \hspace{2em} \text{if} \( v.\text{color} == \text{WHITE} \)
6. \hspace{3em} \( v.\pi = u \)
7. \hspace{3em} \text{DFS-Visit}(G, v)
8. \hspace{1em} \( u.\text{color} = \text{BLACK} \) \quad \text{\hspace{1em} // blacken } u; \text{ it is finished}
9. \hspace{1em} \text{time} = \text{time} + 1
10. \hspace{1em} \( u.f = \text{time} \)

\( \Theta(|E| + |V|) \), where \( |E| \) is the number of edges in \( G \).
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Properties of depth-first-search:

(1) $u = v.\pi$ iff DFS-Visit($G,v$) is called.

(2) Theorem 22.7 (Parenthesis Theorem): for any $u,v$, exactly one of the following three conditions holds:
- $[u.d,u.f]$ and $[v.d,v.f]$ are entirely disjoint, and neither $u$ nor $v$ is a descendant of the other in the search tree.
- $[u.d,u.f]$ is contained entirely within $[v.d,v.f]$ and $u$ is a descendant of $v$, or
- $[v.d,v.f]$ is contained entirely within $[u.d,u.f]$ and $v$ is a descendant of $u$.

Corollary 22.8 (Nesting of descendants' intervals) Vertex $v$ is a proper descendant of $u$ in the depth-first search forest if and only if $u.d < v.d < v.f < u.f$. 
Properties of depth-first-search:

(1) \( u = v.\pi \)
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Properties of depth-first-search:

(1) \( u = v.\pi \) iff DFS-\textsc{Visit}(G, v) is called.
Properties of depth-first-search:

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- \([u.d, u.f]\) and \([v.d, v.f]\) are entirely disjoint, and neither \( u \) nor \( v \) is a descendant of the other in the search tree.

Corollary 22.8 (Nesting of descendants' intervals) Vertex \( v \) is a proper descendant of \( u \) in the depth-first search forest if and only if \( u.d < v.d < v.f < u.f \).
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Properties of depth-first-search:

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Properties of depth-first-search:

(1) \( u = v.\pi \iff \text{DFS-Visit}(G, v) \) is called.

(2) **Theorem 22.7 (Parenthesis Theorem):** for any \( u, v \), exactly one of the following three conditions holds:

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- \([v.d, v.f]\) is contained entirely within \([u.d, u.f]\) and \( v \) is a descendant of \( u \).

**Corollary 22.8 (Nesting of descendants' intervals)** Vertex \( v \) is a proper descendant of \( u \) in the depth-first search forest if and only if \( u.d < v.d < v.f < u.f \).
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Theorem 22.9 (White-path Theorem) In a depth-first search forest of graph $G$, vertex $v$ is a descendant of $u$ if and only if at the time $u.d$ that the search discovers $u$, there is a path from $u$ to $v$ consisting of entirely of white vertices.

Proof:

$\Rightarrow$

case 1: $u=v$, apparently the claim is true;

case 2: $v$ is a proper descendant of $u$, use Corollary 22.8 on every vertex on the path from $u$ to $v$; the claim is true;

$\Leftarrow$

Assume that at the time $u.d$, there is a path from $u$ to $v$ as stated in the theorem but $v$ is not descendant of $u$. Assume $(w,x)$ be an edge on the path and $x$ is the first vertex on the path which is not descendant of $u$. Note that $w$ is a descendant of $u$ (or just $u$) because at time $u.d$, $x$ is of WHITE color, $u.d < x.d$. Because $(w,x)$ is an edge, there are two possible scenarios:

(1) when $(w,x)$ is being explored, $x$ has already been discovered; we thus have $x.d < x.f$;

(2) when $(w,x)$ is being explored, $x$ has WHITE color but will then be discovered; we thus also have $x.d < x.f$;

According to Corollary 22.8, $u.d < x.d < x.f < u.f$. Thus, $u.d < x.d < u.f$.

By Theorem 22.7, the interval $[x.d, x.f]$ is entirely contained within interval $[u.d, u.f]$. Therefore, $x$ is a descendant of $u$. Contradicts to the earlier assumption.

$v$ should be a descendant of $u$. 
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**Theorem 22.9 (White-path Theorem)** In a depth-first search forest of graph $G$, vertex $v$ is a descendant of $u$ if and only if at the time $u.d$ that the search discovers $u$, there is a path from $u$ to $v$ consisting of entirely of white vertices.

**Proof:** $\Rightarrow$
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**Proof:** $\Rightarrow$

- case 1: $u = v$, apparently the claim is true;
- case 2: $v$ is a proper descendant of $u$, use Corollary 22.8 on every vertex on the path from $u$ to $v$; the claim is true;

$\Leftarrow$ Assume that at the time $u.d$, there is a path from $u$ to $v$ as stated in the theorem but $v$ is not descendant of $u$. Assume $(w,x)$ be an edge on the path and $x$ is the first vertex on the path which is not descendant of $u$.

Note that $w$ is a descendant of $u$ (or just $u$).

Because at time $u.d$ $x$ is of WHITE color, $u.d < x.d$.

Because $(w,x)$ is an edge, there are two possible scenarios:

1. when $(w,x)$ is being explored, $x$ has already been discovered; we thus have $x.d < w.f$;
2. when $(w,x)$ is being explored, $x$ has WHITE color but will then be discovered we thus also have $x.d < w.f$;

According to Corollary 22.8, $u.d < x.d < w.f < u.f$.

Thus, $u.d < x.d < u.f$.

By Theorem 22.7, the interval $[x.d, x.f]$ is entirely contained within interval $[u.d, u.f]$.

Therefore, $x$ is a descendant of $u$. Contradicts to the earlier assumption.

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Theorem 22.9 (White-path Theorem) In a depth-first search forest of graph $G$, vertex $v$ is a descendant of $u$ if and only if at the time $u.d$ that the search discovers $u$, there is a path from $u$ to $v$ consisting of entirely of white vertices.

Proof: $\Rightarrow$

case 1: $u = v$, apparently the claim is true;

case 2: $v$ is a proper descendant of $u$, use Corollary 22.8 on every vertex on the path from $u$ to $v$; the claim is true;

$\Leftarrow$
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Proof: $\Rightarrow$

- case 1: $u = v$, apparently the claim is true;
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$\Leftarrow$

Assume that at the time $u.d$, there is a path from $u$ to $v$ as stated in the theorem but $v$ is not descendant of $u$. 
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**Theorem 22.9 (White-path Theorem)** In a depth-first search forest of graph $G$, vertex $v$ is a descendant of $u$ if and only if at the time $u.d$ that the search discovers $u$, there is a path from $u$ to $v$ consisting of entirely of white vertices.

**Proof:** $\Rightarrow$
- case 1: $u = v$, apparently the claim is true;
- case 2: $v$ is a proper descendant of $u$, use **Corollary 22.8** on every vertex on the path from $u$ to $v$; the claim is true;

$\Leftarrow$
- Assume that at the time $u.d$, there is a path from $u$ to $v$ as stated in the theorem but $v$ is not descendant of $u$.

Assume $(w, x)$ be an edge on the path and $x$ is the first vertex on the path which is not descendant of $u$. 
Theorem 22.9 (White-path Theorem) In a depth-first search forest of graph $G$, vertex $v$ is a descendant of $u$ if and only if at the time $u.d$ that the search discovers $u$, there is a path from $u$ to $v$ consisting of entirely of white vertices.

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- case 1: $u = v$, apparently the claim is true;
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$\Leftarrow$

Assume that at the time $u.d$, there is a path from $u$ to $v$ as stated in the theorem but $v$ is not descendant of $u$.

Assume $(w, x)$ be an edge on the path and $x$ is the first vertex on the path which is not descendant of $u$. Note that $w$ is a descendant of $u$ (or just $u$)
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**Proof:** \( \Rightarrow \)

- case 1: \( u = v \), apparently the claim is true;
- case 2: \( v \) is a proper descendant of \( u \), use **Corollary 22.8** on every vertex on the path from \( u \) to \( v \); the claim is true;

\( \Leftarrow \)

Assume that at the time \( u.d \), there is a path from \( u \) to \( v \) as stated in the theorem but \( v \) is not descendant of \( u \).

Assume \((w, x)\) be an edge on the path and \( x \) is the first vertex on the path which is not descendant of \( u \). Note that \( w \) is a descendant of \( u \) (or just \( u \))

Because at time \( u.d \) \( x \) is of WHITE color, \( u.d < x.d \).
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Theorem 22.9 (White-path Theorem) In a depth-first search forest of graph $G$, vertex $v$ is a descendant of $u$ if and only if at the time $u.d$ that the search discovers $u$, there is a path from $u$ to $v$ consisting of entirely of white vertices.

Proof: $\Rightarrow$
- case 1: $u = v$, apparently the claim is true;
- case 2: $v$ is a proper descendant of $u$, use Corollary 22.8 on every vertex on the path from $u$ to $v$; the claim is true;

$\Leftarrow$
Assume that at the time $u.d$, there is a path from $u$ to $v$ as stated in the theorem but $v$ is not descendant of $u$.

Assume $(w, x)$ be an edge on the path and $x$ is the first vertex on the path which is not descendant of $u$. Note that $w$ is a descendant of $u$ (or just $u$). Because at time $u.d$ $x$ is of WHITE color, $u.d < x.d$.
Because $(w, x)$ is an edge, there are two possible scenarios:
- (1) when $(w, x)$ is being explored, $x$ has already been discovered;
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- case 1: $u = v$, apparently the claim is true;
- case 2: $v$ is a proper descendant of $u$, use **Corollary 22.8** on every vertex on the path from $u$ to $v$; the claim is true;

$\Leftarrow$

Assume that at the time $u.d$, there is a path from $u$ to $v$ as stated in the theorem but $v$ is not descendant of $u$.

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Chapter 22. Elementary graph algorithms

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1. when $(w, x)$ is being explored, $x$ has already been discovered;
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2. when $(w, x)$ is being explored, $x$ has WHITE color but will then be discovered
Chapter 22. Elementary graph algorithms

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- case 1: $u = v$, apparently the claim is true;
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**Proof:** $\Rightarrow$
- case 1: $u = v$, apparently the claim is true;
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Assume that at the time $u.d$, there is a path from $u$ to $v$ as stated in the theorem but $v$ is not descendant of $u$.

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Because at time $u.d$ $x$ is of WHITE color, $u.d < x.d$.
Because $(w, x)$ is an edge, there are two possible scenarios:
- (1) when $(w, x)$ is being explored, $x$ has already been discovered; we thus have $x.d < w.f$;
- (2) when $(w, x)$ is being explored, $x$ has WHITE color but will then be discovered we thus also have $x.d < w.f$;

According to **Corollary 22.8**, $u.d < x.d < w.f < u.f$. 
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**Proof:** $\Rightarrow$

1. **case 1:** $u = v$, apparently the claim is true;
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Because at time $u.d$ $x$ is of WHITE color, $u.d < x.d$.

Because $(w, x)$ is an edge, there are two possible scenarios:
- (1) when $(w, x)$ is being explored, $x$ has already been discovered; we thus have $x.d < w.f$;
- (2) when $(w, x)$ is being explored, $x$ has WHITE color but will then be discovered we thus also have $x.d < w.f$;

According to Corollary 22.8, $u.d < x.d < w.f < u.f$. Thus, $u.d < x.d < u.f$
By Theorem 22.7, the interval $[x.d, x.f]$ is entirely contained within interval $[u.d, u.f]$.
Chapter 22. Elementary graph algorithms

**Theorem 22.9 (White-path Theorem)** In a depth-first search forest of graph \( G \), vertex \( v \) is a descendant of \( u \) if and only if at the time \( u.d \) that the search discovers \( u \), there is a path from \( u \) to \( v \) consisting of entirely of white vertices.

**Proof:** \( \Rightarrow \)
- case 1: \( u = v \), apparently the claim is true;
- case 2: \( v \) is a proper descendant of \( u \), use **Corollary 22.8** on every vertex on the path from \( u \) to \( v \); the claim is true;

\( \Leftarrow \)

Assume that at the time \( u.d \), there is a path from \( u \) to \( v \) as stated in the theorem but \( v \) is not descendant of \( u \).

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According to **Corollary 22.8**, \( u.d < x.d < w.f < u.f \). Thus, \( u.d < x.d < u.f \)
By **Theorem 22.7**, the interval \([x.d, x.f]\) is entirely contained within interval \([u.d, u.f]\). Therefore, \( x \) is a descendant of \( u \). Contradicts to the earlier assumption.
Chapter 22. Elementary graph algorithms

**Theorem 22.9 (White-path Theorem)** In a depth-first search forest of graph $G$, vertex $v$ is a descendant of $u$ if and only if at the time $u.d$ that the search discovers $u$, there is a path from $u$ to $v$ consisting of entirely of white vertices.

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- case 1: $u = v$, apparently the claim is true;
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$v$ should be a descendant of $u$. 
Classification of edges (for directed graphs)
Chapter 22. Elementary graph algorithms

Classification of edges (for directed graphs)

- Tree edges: those in the search tree (forest);
- Back edges: those connecting a vertex to an ancestor;
- Forward edges: those connecting a vertex to a descendant;
- Cross edges: all other edges.

DFS forest with back, forward, and cross edges.
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Chapter 22. Elementary graph algorithms

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\(\text{DFS forest with back, forward, & cross edges.}\)
Chapter 22. Elementary graph algorithms

To identify the type of edge \((u,v)\) with the color of \(v\):

- **WHITE**: tree edge;
- **GRAY**: back edge;
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First stages of a Directed DFS, showing *Edges*, the *DFS TREE*, a *Tree Edge*, a *Back Edge*, a *Forward Edge*, and a *Cross Edge*.
Chapter 22. Elementary graph algorithms

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Chapter 22. Elementary graph algorithms

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Chapter 22. Elementary graph algorithms

**Theorem 22.10** In a depth-first search of undirected graph $G$, every edge of $G$ is either tree edge or back edge.
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Chapter 22. Elementary graph algorithms

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Proof. Let $(u, v)$ be an edge in $G$. Assume in the depth-first search of $G$, $u.d < v.d$. Then there are two scenarios:
Theorem 22.10 In a depth-first search of undirected graph \( G \), every edge of \( G \) is either tree edge or back edge.

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Chapter 22. Elementary graph algorithms

Breadth First Search (BFS)
Chapter 22. Elementary graph algorithms

Breadth First Search (BFS)
Chapter 22. Elementary graph algorithms

Breadth First Search Algorithm
(with a queue)

Time complexity of BFS: $O(|V| + |E|)$

Note: BFS can find a shortest path from $s$ to all other nodes (non-weighted). (Why?)
Chapter 22. Elementary graph algorithms

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Breadth First Search Algorithm (with a queue)

BFS\( (G, s) \)
1. for each vertex \( u \in G.V - \{s\} \)
   2. \( u.color = \text{WHITE} \)
   3. \( u.d = \infty \)
   4. \( u.\pi = \text{NIL} \)
5. \( s.color = \text{GRAY} \)
6. \( s.d = 0 \)
7. \( s.\pi = \text{NIL} \)
8. \( Q = \emptyset \)
9. ENQUEUE\( (Q, s) \)
10. while \( Q \neq \emptyset \)
11. \( u = \text{DEQUEUE}(Q) \)
12. for each \( v \in G.Adj[u] \)
13. \( \text{if } v.color == \text{WHITE} \)
14. \( v.color = \text{GRAY} \)
15. \( v.d = u.d + 1 \)
16. \( v.\pi = u \)
17. ENQUEUE\( (Q, v) \)
18. \( u.color = \text{BLACK} \)
Chapter 22. Elementary graph algorithms

Breadth First Search Algorithm (with a queue)

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Chapter 22. Elementary graph algorithms

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Chapter 22. Elementary graph algorithms

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Note: BFS can find a shortest path from $s$ to all other nodes (non-weighted). (Why?)
Applications

Reachability Problem

Input: \( G = (V, E) \), and \( s, t \in V \);

Output: YES if and only if there is a path \( s \xrightarrow{} t \) in \( G \).

• The problem can be solved with DFS and BFS by search on the graph from \( s \) until \( t \) shows up.
Chapter 22. Elementary graph algorithms

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Reachability Problem

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Chapter 22. Elementary graph algorithms

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Reachability Problem

**Input:** \( G = (V, E) \), and \( s, t \in V \);

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Reachability Problem

Input: $G = (V, E)$, and $s, t \in V$;
Output: YES if and only there is a path $s \sim t$ in $G$.

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Chapter 22. Elementary graph algorithms

Reachability Problem

Reachability $(G, u, t)$;

1. if not $u$.visit
2. $u$.visit = true
3. for each $v \in \text{Adj}[u]$ and not $v$.visit
4. if $v = t$ then reachable = Yes; exit
5. else $v$.π = $u$
6. Reachability $(G, v, t)$;
7. return ()

Main()
reachable = No;
Reachability $(G, s, t)$;
print (reachable)
Chapter 22. Elementary graph algorithms

Reachability Problem

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Chapter 22. Elementary graph algorithms

Reachability Problem

\text{REACHABILITY}(G, u, t);
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4. \hspace{2em} \textbf{if } v = t \text{ then } \text{reachable } = \text{Yes}; \text{exit};
5. \hspace{2em} \textbf{else } v.\pi = u;
6. \hspace{2em} \hspace{1em} \text{REACHABILITY}(G, v, t);
7. \hspace{1em} \textbf{return } ( );
Reachability Problem

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5. \textbf{else} \textit{v.π} = \textit{u}; \\
6. \textbf{Reachability}(G, v, t); \\
7. \textbf{return} ( );

\textbf{Main( )} \\

reachable = \textbf{No}; \\
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Chapter 22. Elementary graph algorithms

Path Counting Problem

Input:
\( G = (V, E) \), and \( s, t \in V \);

Output: the number of paths from \( s \to t \) in \( G \).

• Note: Reachability algorithm finds only one path

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how to modify to suit our need for counting paths?
Path Counting Problem

Input: $G = (V,E)$, and $s,t \in V$;
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Reachability($G,u,t$);
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Chapter 22. Elementary graph algorithms

**Path Counting Problem**

**Input:** $G = (V, E)$, and $s, t \in V$.

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Path Counting Problem

**Input:** $G = (V, E)$, and $s, t \in V$;

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How to modify to suit our need for counting paths?
Chapter 22. Elementary graph algorithms

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Chapter 22. Elementary graph algorithms

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Chapter 22. Elementary graph algorithms

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how to modify to suit our need for counting paths?
Chapter 22. Elementary graph algorithms

Topological sorting

- On directed acyclic graphs (DAGs)

A sorted order:
socks, shorts, pants, shoes, shirt, tie, belt, jacket, watch.
Topological sorting
Chapter 22. Elementary graph algorithms

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- apply DFS algorithm.
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- apply DFS algorithm.

- reversed order of finish times:

\[ p, n, o, s, m, r, y, v, x, w, z, u, q, t \]
Chapter 22. Elementary graph algorithms

- apply DFS algorithm.

- reversed order of finish times:

  \( p, n, o, s, m, r, y, v, x, w, z, u, q, t \)

- Correctness proof?
Chapter 22. Elementary graph algorithms

Strongly connected components (SCC)

Let $G = (V, E)$ be a digraph. A strongly connected component is a maximal subgraph $H = (V_H, E_H)$ of $G$ such that for every two nodes $v, u \in V_H$,

1. there is a directed path $v \to u$ consisting of edges in $E_H$;
2. there is a directed path $u \to v$ consisting of edges in $E_H$.
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Strongly connected components (SCC)
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Strongly connected components (SCC)

Let $G = (V, E)$ be a digraph. A strongly connected component is a maximal subgraph $H = (V_H, E_H)$ of $G$ such that for every two nodes $v, u \in V_H$, there is a directed path $v \Rightarrow u$ consisting of edges in $E_H$ and there is a directed path $u \Rightarrow v$ consisting of edges in $E_H$. 
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Let $G = (V, E)$ be a digraph. A strongly connected component is a maximal subgraph $H = (V_H, E_H)$ of $G$ such that for every two nodes $v, u \in V_H$,

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Chapter 22. Elementary graph algorithms

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Chapter 22. Elementary graph algorithms

Idea of an algorithm to use DFS to solve SCC problem.

• use DFS to generate DFS forest; each search tree $T_u$ (rooted at $u$) consists of vertices $v$ such that $u \xrightarrow{} v$;

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Algorithm

Strongly Connected Components $(G)$

1. call DFS$(G)$ to compute $u.f$ for each $u \in G.V$
2. compute $G^T$ the transpose of $G$ (reverse all edges in $G$)
3. call DFS$(G^T)$ (vertices are considered in the decreasing order of finish times computed in step 1)
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Properties from algorithm

Strongly Connected Components

\( G \)

Component graph:

\( G_{SCC} = (V_{SCC}, E_{SCC}) \)

let

\[ C_1, C_2, \ldots, C_k \]

be \( k \) distinct SCCs for \( G \). Then

\[ V_{SCC} = \{ v_1, v_2, \ldots, v_k \} \]

\[ E_{SCC} = \{ (v_i, v_j) : \exists u \in C_i, v \in C_j, (u, v) \in E \} \]

Then \( G_{SCC} \) is a DAG (directed acyclic graph).

Proof. Assume the opposite to the claim that, for some \( v_i, v_j \in V_{SCC} \), there is a path \( v_i \xrightarrow{} v_j \) and another path \( v_j \xrightarrow{} v_i \), forming a cycle in \( V_{SCC} \). By the definition of \( G_{SCC} \), there must be a path in \( G \), from some vertex in \( C_i \) to some vertex in \( C_j \); at the same time, there is a path in \( G \), from some vertex in \( C_j \) to some vertex in \( C_i \). Then \( C_i \) and \( C_j \) should form a single SCC, not two distinct SCCs. Contradicts.
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By the definition of \(G^{SCC}\), there must be a path in \(G\), from some vertex in \(C_i\) to some vertex in \(C_j\); at the same time, there is a path in \(G\), from some vertex in \(C_j\) to some vertex in \(C_i\). Then \(C_i\) and \(C_j\) should form a single SCC, not two distinct SCCs. **Contradicts.**
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Let $C$ be a SCC, define $f(C) = \max_{u \in C} \{u.f\}$, (with the finish times from the first DFS call).

Lemma 22.14: Let $C$ and $C'$ be distinct strongly connected components for directed graph $G$. If $(u,v) \in E$, where $u \in C$ and $v \in C'$, then $f(C) > f(C')$.

Proof: Assume the opposite, i.e., $f(C) < f(C')$. Then there must be vertices $x \in C$ and $y \in C'$ such that $x.f < y.f$.

Now consider the first DFS call, there are two situations:

1. $y$ was searched before $x$: by property (1) there is no path from $y$ to $x$, $x.f > y.f$.

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The algorithm

Strongly Connected Components

(G) correctly computes the strongly connected components for a directed graph G.

We need to prove two statements:

1. If \( v \rightarrow u \) and \( u \rightarrow v \) in G, then u and v belong to the same component C produced by the algorithm.

2. If \( u, v \in C \), then we have \( v \rightarrow u \) and \( u \rightarrow v \) in G.
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(3) The algorithm **Strongly Connected Components**($G$) correctly computes the strongly connected components for a directed graph $G$. 

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1. If $v \xrightarrow{} u$ and $u \xrightarrow{} v$ in $G$, then $u$ and $v$ belong to the same component $C$ produced by the algorithm.

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(3) The algorithm **Strongly Connected Components**($G$) correctly computes the strongly connected components for a directed graph $G$.

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(3) The algorithm **Strongly Connected Components**\((G)\) correctly computes the strongly connected components for a directed graph \(G\).

We need to prove two statements:

(1) If \(v \sim u\) and \(u \sim v\) in \(G\), then \(u\) and \(v\) belong to the same component \(C\) produced by the algorithm.

(2) If \(u, v \in C\), then we have \(v \sim u\) and \(u \sim v\) in \(G\).
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Proof:
(1) If \( v \rightarrow u \) and \( u \rightarrow v \) in \( G \), then \( u \) and \( v \) belong to the same component \( C \) produced by the algorithm.

Sketch of proof:
• assume in 1st DFS, \( v \) was discovered before \( u \) (or opposite);
• as \( v \rightarrow u \) in \( G \), \( u \) and \( v \) belong to the same search tree rooted at \( r \) with \( r.f \geq v.f > u.f \) (note: \( r \) could be just \( v \));
• as \( u \rightarrow v \) in \( G \), \( v \rightarrow u \) in \( G_T \);
• now consider the 2nd DFS; there are 2 situations:
  (1) searching from some \( w \) with \( w.f \geq v.f \) (note: \( w \) could be \( v \)) finds \( v \) first; then it finds \( u \);
  (2) the search finds \( u \) first; because \( v \rightarrow u \) in \( G \), \( u \rightarrow v \) is in \( G_T \), it finds also \( v \).

In both situations, \( u \) and \( v \) belong to the same search tree in the 2nd DFS search. Therefore, \( u \) and \( u \) belong to the same component.
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**Proof:**

1. If $v \xrightarrow{} u$ and $u \xrightarrow{} v$ in $G$, then $u$ and $v$ belong to the same component produced by the algorithm.

   **Sketch of proof:**
   - Assume in 1st DFS, $v$ was discovered before $u$ (or opposite);
   - As $v \xrightarrow{} u$ in $G$, $u$ and $v$ belong to the same search tree rooted at $r$ with $r.f \geq v.f > u.f$ (note: $r$ could be just $v$);
   - As $u \xrightarrow{} v$ in $G$, $v \xrightarrow{} u$ in $G_T$;
   - Now consider the 2nd DFS; there are 2 situations:
     1. Searching from some $w$ with $w.f \geq v.f$ (note: $w$ could be $v$) finds $v$ first; then it finds $u$;
     2. The search finds $u$ first; because $v \xrightarrow{} u$ in $G$, $u \xrightarrow{} v$ is in $G_T$, it finds also $v$.

   In both situations, $u$ and $v$ belong to the same search tree in the 2nd DFS search. Therefore, $u$ and $u$ belong to the same component.
Proof:

(1) If \( v \leadsto u \) and \( u \leadsto v \) in \( G \), then \( u \) and \( v \) belong to the same component \( C \) produced by the algorithm.
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(1) If $v \sim u$ and $u \sim v$ in $G$ in $G$, then $u$ and $v$ belong to the same component $C$ produced by the algorithm.

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Proof:

(1) If \( v \sim u \) and \( u \sim v \) in \( G \) in \( G \), then \( u \) and \( v \) belong to the same component \( C \) produced by the algorithm.

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In both situations, \( u \) and \( v \) belongs to the same search tree in the 2nd DFS search. Therefore, \( u \) and \( u \) belong to the same component.
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(2) If \( u, v \in C \), then we have \( v \mapsto u \) and \( u \mapsto v \) in \( G \). Sketch of proof:

(1) assume \( u, v \) belong to the same tree of root \( r \) in 2nd DFS;

(2) then \( r.f > u.f \) and \( r.f > v.f \) in 1st DFS;

(3) the assumption in (1) also implies:
   • \( r \mapsto u \) and \( r \mapsto v \) in \( G_T \);
   • that is, \( u \mapsto r \) and \( v \mapsto r \) in \( G \);
   • then \( u.f > r.f \) and \( v.f > r.f \) in 1st DFS, which conflict with conclusions in (2), UNLESS \( r \mapsto u \) and \( r \mapsto v \) in \( G \) also.

(4) This means: through \( r \), \( v \mapsto u \) and \( u \mapsto v \) in \( G \).
(2) If \( u, v \in C \), then we have \( v \leadsto u \) and \( u \leadsto v \) in \( G \).
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(2) If $u, v \in C$, then we have $v \leadsto u$ and $u \leadsto v$ in $G$.

Sketch of proof:

(1) assume $u$, $v$ belong to the same tree of root $r$ in 2nd DFS;
(2) then $r.f > u.f$; and $r.f > v.f$ in 1st DFS;
(2) If $u, v \in C$, then we have $v \Rightarrow u$ and $u \Rightarrow v$ in $G$.

Sketch of proof:

(1) assume $u, v$ belong to the same tree of root $r$ in 2nd DFS;
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(3) the assumption in (1) also implies:

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   • that is, $u \rightsquigarrow r$ and $v \rightsquigarrow r$ in $G$;
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Chapter 22. Elementary graph algorithms

Reachability Problem

Input: $G = (V, E)$, and $s, t \in V$;
Output: YES if and only there is a path $s \rightarrow t$ in $G$.

- The problem can be solved with DFS and BFS by search on the graph from $s$ until $t$ shows up.
  Linear time $O(|E| + |V|)$.
  Can we do better?

- But first answer the following question: Can you write an SQL program to solve Reachability?

- It appears that a loop is needed to solve Reachability.

  Why?
  Inherent difficulty in parallel computation. P-complete, it cannot be solved in time $O(\log n)$ even if $\Theta(n)$ CPUs are used.
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Chapter 23. Minimum Spanning Trees

• A spanning tree of a graph $G = (V,E)$ is a tree as a subgraph in $G$ which contains all vertices in $V$.

• A minimum spanning tree (MST) of an edge-weighted graph $G$ is a spanning tree with the least edge weight sum.
Chapter 23. Minimum Spanning Trees

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![Graph with a spanning tree highlighted]
Chapter 23. Minimum Spanning Trees

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![Diagram of a graph with a spanning tree highlighted in blue]
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Chapter 23. Minimum Spanning Trees

The MST problem

Input: connected, undirected graph \( G = (V,E) \) with weight \( w: E \rightarrow \mathbb{R} \).
Output: a spanning tree \( T = (V,E') \) such that \( W(T) = \sum_{(u,v) \in E'} w(u,v) \) is the minimum.

We will introduce two greedy algorithms: (1) Kruskal's and (2) Prim's.

• They have the same generic process to grow a spanning tree;
• but differ in which edge to add the partially grown tree.
Chapter 23. Minimum Spanning Trees

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Growing an MST

A generic process to grow an MST.

\[
\begin{align*}
\text{Generic MST} & \quad \text{\( (G, w) \)} \\
& \quad \text{given graph \( G \) and weight function \( w \)} \\
1. & \quad A = \emptyset \\
2. & \quad \text{while } A \text{ does not form a spanning tree} \\
3. & \quad \text{find an edge } (u, v) \text{ that is safe for } A \\
4. & \quad A = A \cup \{ (u, v) \} \\
5. & \quad \text{return } (A) \\
\end{align*}
\]

Loop invariant: \( A \) is always a subset of some MST; Note: when the loop terminates, \( A \) is a MST.

safe edge: edge \( (u, v) \) is safe for \( A \) if does not violate the loop invariant, i.e, \( A \cup \{ (u, v) \} \) is a subset of some MST.
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**GENERIC MST**\( (G, w) \) \{ given graph \( G \) and weight function \( w \) \}

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Chapter 23. Minimum Spanning Trees

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Chapter 23. Minimum Spanning Trees

Growing an MST

A generic process to grow an MST.

**Generic MST** \((G, w)\)  \{ given graph \(G\) and weight function \(w\) \}

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Chapter 23. Minimum Spanning Trees

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Chapter 23. Minimum Spanning Trees

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Chapter 23. Minimum Spanning Trees

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Chapter 23. Minimum Spanning Trees

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Chapter 23. Minimum Spanning Trees

We first need some terminologies
Chapter 23. Minimum Spanning Trees

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- **cut** : \((S, V - S)\), a partition of \(V\)
Chapter 23. Minimum Spanning Trees

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- **cut**: \((S, V-S)\), a partition of \(V\)
Chapter 23. Minimum Spanning Trees

We first need some terminologies

- **cut**: \((S, V - S)\), a partition of \(V\)

- **crossing**: \((u, v)\) crosses cut \((S, V - S)\)
  if \(u\) and \(v\) are in \(S\) and \(V - S\), respectively
Chapter 23. Minimum Spanning Trees

Some more terminologies
Some more terminologies

- **respect**: a cut respects a set $A$ of edges if no edge in $A$ crosses the cut.
Chapter 23. Minimum Spanning Trees

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- **respect**: a cut respects a set $A$ of edges if no edge in $A$ crosses the cut.

- **light edge**: an edge is a light edge crossing a cut if its weight is the minimum of any edge that crosses the cut.
Chapter 23. Minimum Spanning Trees

**Theorem 23.1** Let $G = (V, E)$. 

**Sketch of proof**:

(1) If $A \cup \{(u,v)\}$ forms a cycle there must have been another edge in $A$ that crosses cut $(S, V - S)$ (WHY?), implying the cut did not respect $A$. Contradicts.

(2) Assume that some MST $T$, $A \subset T$. First, $T \cup \{(u,v)\}$ forms a circle! Why? There must be another edge $(x,y)$ that crosses the cut $(S, V - S)$. Since $(u,v)$ is a light edge, $T' = T - \{(x,y)\} \cup \{(u,v)\}$ is an MST.

Now $A \cup \{(u,v)\} \subseteq T'$ because $(x,y) \not\in A$ (otherwise, the cut would not respect $A$).
Chapter 23. Minimum Spanning Trees

**Theorem 23.1** Let $G = (V, E)$. Let $A \subseteq E$, contained in some MST for $G$. Let $(S, V \setminus S)$ be any cut of $G$ that respects $A$. Let $(u, v)$ be a light edge crossing the cut. Then edge $(u, v)$ is safe for $A$. For the theorem, we need to prove:

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Chapter 23. Minimum Spanning Trees

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Theorem 23.1 gives a sufficient information for how to identify a safe edge to make the **Generic MST** algorithm work.

\begin{itemize}
\item Specific algorithms can be produced from **Generic MST** based on how the set $A$ is grown.
\item $A$ may always be a tree (Prim's algorithm) or could be a forest (Kruskal's algorithm).
\end{itemize}

**MST-Kruskal** $(G,w)$

1. $A = \emptyset$;
2. for each vertex $v \in G.V$;
3. Make-Set ($v$);
4. sort edges in $E$ into non-decreasing order by their weight $w$;
5. for each edge $(u,v) \in E$, taken in the order;
6. if $\text{Find Set}(u) \neq \text{Find Set}(v)$;
7. $A = A \cup \{(u,v)\}$;
8. Union ($u,v$);
9. return ($A$).
Theorem 23.1 gives a sufficient information for how to identify a safe edge to make the \textsc{Generic MST} algorithm work.

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Chapter 23. Minimum Spanning Trees

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Chapter 23. Minimum Spanning Trees

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$\text{MST-Kruskal}(G, w)$
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Chapter 23. Minimum Spanning Trees

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**MST-Kruskal\( (G, w) \)**
1. \( A = \emptyset \);
2. **for** each vertex \( v \in G.V \)
3. **Make-Set** \( (v) \)
4. **sort** edges in \( E \) into non-decreasing order by their weight \( w \)
5. **for** each edge \( (u, v) \in E \), taken in the order
6. **if** \( \text{Find Set} \ (u) \neq \text{Find Set} \ (v) \)
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\textbf{MST-Kruskal}(\(G, w\))

1. \(A = \emptyset;\)
2. \textbf{for} each vertex \(v \in G.V\)
3. \hspace{1em} \textsc{Make-Set}(v)
4. \hspace{1em} \textbf{sort} edges in \(E\) into non-decreasing order by their weight \(w\)
5. \textbf{for} each edge \((u, v) \in E\), taken in the order
6. \hspace{1em} \textbf{if} \textsc{Find Set}(u) \neq \textsc{Find Set}(v)
7. \hspace{1em} \hspace{1em} \(A = A \cup \{(u, v)\}\)
Chapter 23. Minimum Spanning Trees

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   3. Make-Set($v$)
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5. for each edge $(u,v) \in E$, taken in the order
6. if Find Set $(u) \neq$ Find Set$(v)$
7. $A = A \cup \{(u, v)\}$
8. Union$(u, v)$
Chapter 23. Minimum Spanning Trees

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MST-Kruskal $(G, w)$$\quad$
1. \hspace{2cm} $A = \emptyset$;
2. for each vertex $v \in G.V$
   3. \hspace{2cm} Make-Set ($v$)
4. sort edges in $E$ into non-decreasing order by their weight $w$
5. for each edge $(u, v) \in E$, taken in the order
   6. if Find Set ($u$) $\neq$ Find Set ($v$)
      7. \hspace{2cm} $A = A \cup \{(u, v)\}$
   8. \hspace{2cm} Union ($u, v$)
9. return ($A$)
Chapter 23. Minimum Spanning Trees

Execution of Kruskal's algorithm for MST

disjoint sets
[A] [B] [C] [D] [E] [F] [G] [H]
[A] [B] [C] [D] [E] [F] [G] [H]

[A] [B] [C] [D] [E] [F, G] [H]
[A] [B] [D] [E] [C, F, G] [H]

[A] [D] [E] [B, C, F, G] [H]

[D] [E] [A, B, C, F, G] [H]
[D, E] [A, B, C, F, G] [H]

[D, E, H] [A, B, C, F, G]
[A, B, C, D, E, F, G, H]
At each iteration of the for loop, e.g., identify

\[ \{A, F\}, \{B, F\}, \{C, G\}, \{F, G\}, \{D, E\} \]

S = \{A, B, C, D, F, G\}, V - S = \{E, H\}, light edge (D, E) crosses the cut;

\[ \{D, E\}, \{A, B, C, F, G\}, \{H\} \]

\[ \{D, E\}, \{A, B, C, F, G\}, \{H\} \]
Chapter 23. Minimum Spanning Trees

At each iteration of the for loop, e.g., identify

\[ A = \{(A,F), (B,F), (C,G), (F,G), (D,E)\} \]

cut that respects \( A \):

\[ S = \{A, B, C, D, F, G\} \]
\[ V - S = \{E, H\} \]
light edge \((D,E)\) crosses the cut;

\[ A = \{(A,F), (B,F), (C,G), (F,G), (D,E)\} \]

cut that respects \( A \):

\[ S = \{A, B, C, F, G, H\} \]
\[ V - S = \{D, E\} \]
light edge \((E,H)\) crosses the cut;

\[ \mathcal{A} = \]
At each iteration of the for loop, e.g., identify

\[ A = \{(A, F), (B, F), (C, G), (F, G)\}, \]

\[ S = \{A, B, C, F, G\}, \]
\[ V - S = \{E, H\} \]
\[ \text{light edge (D, E) crosses the cut}; \]

\[ A = \{(A, F), (B, F), (C, G), (F, G), (D, E)\}, \]
\[ S = \{A, B, C, F, G, H\}, \]
\[ V - S = \{D, E\} \]
\[ \text{light edge (E, H) crosses the cut}; \]
At each iteration of the **for** loop, e.g., identify

- $A = \{(A, F), (B, F), (C, G), (F, G)\}$,
  cut that respects $A$: $S = \{A, B, C, D, F, G\}$, $V - S = \{E, H\}$,
Chapter 23. Minimum Spanning Trees

At each iteration of the for loop, e.g., identify

- $A = \{(A, F), (B, F), (C, G), (F, G)\}$,
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Chapter 23. Minimum Spanning Trees

At each iteration of the for loop, e.g., identify

\[ A = \{(A, F), (B, F), (C, G), (F, G)\}, \]

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light edge \( (D, E) \) crosses the cut;

\[ A = \]
Chapter 23. Minimum Spanning Trees

At each iteration of the for loop, e.g., identify

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Chapter 23. Minimum Spanning Trees

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Cut that respect \( A \): \( S = \{A, B, C, F, G, H\} \), \( V-S = \{D, E\} \),
Chapter 23. Minimum Spanning Trees

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  cut that respect $A$: $S = \{A, B, C, F, G, H\}$, $V-S = \{D, E\}$,
  light edge $(E, H)$ crosses the cut;
Chapter 23. Minimum Spanning Trees

The Kruskal’s algorithm uses disjoint-set data structures (where elements are partitioned into disjoint sets)
Chapter 23. Minimum Spanning Trees

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- **Make Set** \( x \): create a set of single element \( x \);
Chapter 23. Minimum Spanning Trees

The Kruskal’s algorithm uses disjoint-set data structures (where elements are partitioned into disjoint sets)

- **Make Set** \((x)\): create a set of single element \(x\);
- **Find Set** \((x)\): identify the set that contains element \(x\);

Implementations (left: linked lists, Right: disjoint-set forest)

Time complexity: \(O(\log n)\) for **Make Set** \((x)\), **Find Set** \((x)\), **Union** \((x,y)\) with disjoint-set forest implementation.

Time complexity of Kruskal’s algorithm: \(O(|E| \log |V|) + |V|)\).
Chapter 23. Minimum Spanning Trees

The Kruskal’s algorithm uses disjoint-set data structures
(where elements are partitioned into disjoint sets)

- **Make Set**($x$): create a set of single element $x$;
- **Find Set**($x$): identify the set that contains element $x$;
- **Union**($x, y$): union the two sets containing $x$ and $y$ into one;
Chapter 23. Minimum Spanning Trees

The Kruskal’s algorithm uses disjoint-set data structures (where elements are partitioned into disjoint sets)

- **MAKE SET(x)**: create a set of single element \( x \);
- **FIND SET(x)**: identify the set that contains element \( x \);
- **UNION(x, y)**: union the two sets containing \( x \) and \( y \) into one;

Implementations (left: linked lists, Right: disjoint-set forest)
Chapter 23. Minimum Spanning Trees

The Kruskal’s algorithm uses disjoint-set data structures (where elements are partitioned into disjoint sets)

- **Make Set**(x): create a set of single element x;
- **Find Set**(x): identify the set that contains element x;
- **Union**(x, y): union the two sets containing x and y into one;

Implementations (left: linked lists, Right: disjoint-set forest)

![Disjoint-Set Data Structures](image)

Time complexity:
Chapter 23. Minimum Spanning Trees

The Kruskal’s algorithm uses disjoint-set data structures (where elements are partitioned into disjoint sets)

- **Make Set**\((x)\): create a set of single element \(x\);
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Chapter 23. Minimum Spanning Trees

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Time complexity of Kruskal’s algorithm: $O(|E| \log |V| + |V|)$. 
MST-Prim\((G, w, r)\)
Chapter 23. Minimum Spanning Trees

\text{MST-Prim}(G, w, r)
1. \textbf{for} each \( u \in G.V \)
Chapter 23. Minimum Spanning Trees

**MST-Prim** \((G, w, r)\)

1. **for** each \(u \in G.V\)
2. \(u.key = \infty\) \{ \(u.key\) is the \(u\)'s shortest distance to set \(A = V - Q\)\}

usage of Priority queue: \(Q, Extract\ Min\) takes \(O(\log n)\) time.

running time \(O(|E| + |V| \log |V|)\).
Chapter 23. Minimum Spanning Trees

MST-Prim\((G, w, r)\)
1. \textbf{for} each \(u \in G.V\)
2. \(u.key = \infty\) \{ \(u.key\) is the \(u\)'s shortest distance to set \(A = V - Q\)\}
3. \(u.\pi = NULL\)
Chapter 23. Minimum Spanning Trees

MST-Prim \((G, w, r)\)

1. **for each** \(u \in G.V\)
2. \(u.key = \infty\) \hspace{1cm} \{ \text{u.key is the u's shortest distance to set } A = V - Q\}\)
3. \(u.\pi = NULL\)
4. \(r.key = 0\) \hspace{1cm} \{ \text{start from vertex } r \} \}

usage of Priority queue: \(Q, \text{Extract Min}\) takes \(O(\log n)\) time.

running time \(O(|E| + |V| \log |V|)\).
Chapter 23. Minimum Spanning Trees

**MST-Prim** \((G, w, r)\)

1. **for each** \(u \in G.V\)  

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5. \(Q = G.V\) \{ establish priority queue *Q* wit *key* values\)
Chapter 23. Minimum Spanning Trees

**MST-Prim**\((G, w, r)\)

1. \textbf{for} each \(u \in G.V\)
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3. \(u.\pi = NULL\)
4. \(r.key = 0\) \{ \textit{start from vertex} \(r\) \}
5. \(Q = G.V\) \{ \textit{establish priority queue} \(Q\) \textit{with key values} \}
6. \textbf{while} \(Q \neq \emptyset\)

**Running time** \(O(|E| + |V| \log |V|)\).
Chapter 23. Minimum Spanning Trees

MST-Prim($G$, $w$, $r$)
1. for each $u \in G.V$
2. $u.key = \infty$ \{ $u.key$ is the $u$'s shortest distance to set $A = V - Q$ \}
3. $u.\pi = NULL$
4. $r.key = 0$ \{ start from vertex $r$ \}
5. $Q = G.V$ \{ establish priority queue $Q$ wit key values \}
6. while $Q \neq \emptyset$
7. $u = \text{Extract Min}(Q)$

running time $O(|E| + |V| \log |V|)$. 

usage of Priority queue: $Q$, $\text{Extract Min}$ takes $O(\log n)$ time.
Chapter 23. Minimum Spanning Trees

MST-Prim\((G, w, r)\)
1. \textbf{for} each \(u \in G.V\) \{ \(u\).\textit{key} is the \(u\)'s shortest distance to set \(A = V-Q\) \}
2. \(u\).\textit{key} = \(\infty\)
3. \(u\).\pi = \textit{NULL}
4. \(r\).\textit{key} = 0 \{ start from vertex \(r\) \}
5. \(Q = G.V\) \{ establish priority queue \(Q\) wit key values \}
6. \textbf{while} \(Q \neq \emptyset\)
7. \(u = \text{Extract Min}(Q)\)
8. \textbf{for} each \(v \in Adj[u]\)
Chapter 23. Minimum Spanning Trees

MST-Prím(\(G, w, r\))
1. for each \(u \in G.V\)  
2. \(u.key = \infty\) { \(u.key\) is the \(u\)'s shortest distance to set \(A = V-Q\)} 
3. \(u.\pi = NULL\) 
4. \(r.key = 0\) { start from vertex \(r\) } 
5. \(Q = G.V\) { establish priority queue \(Q\) wit key values} 
6. while \(Q \neq \emptyset\) 
7. \(u = \text{Extract Min}(Q)\) 
8. for each \(v \in \text{Adj}[u]\) 
9. if \(v \in Q\) and \(w(u, v) < v.key\) { for those not in A, update distances}
Chapter 23. Minimum Spanning Trees

MST-Prim(G, w, r)
1. for each \( u \in G.V \)  \{ \( u.key \) is the \( u \)'s shortest distance to set \( A = V-Q \)\}
2. \( u.key = \infty \)
3. \( u.\pi = NULL \)
4. \( r.key = 0 \)  \{ start from vertex \( r \) \}
5. \( Q = G.V \)  \{ establish priority queue \( Q \) wit key values\}
6. while \( Q \neq \emptyset \)
7. \( u = \text{Extract Min}(Q) \)
8. for each \( v \in \text{Adj}[u] \)
9. if \( v \in Q \) and \( w(u, v) < v.key \)  \{ for those not in \( A \), update distances\}
10. then \( v.\pi = u \)

running time \( O(|E| + |V| \log |V|) \).

return \( \pi \) usage of Priority queue: \( Q \), \( \text{Extract Min} \) takes \( O(\log n) \) time.
Chapter 23. Minimum Spanning Trees

MST-Prim($G, w, r$)
1. for each $u \in G.V$
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5.   $Q = G.V$ \{ establish priority queue $Q$ wit key values \}
6. while $Q \neq \emptyset$
7.   $u = \text{Extract Min}(Q)$
8.   for each $v \in Adj[u]$
9.       if $v \in Q$ and $w(u, v) < v.key$ \{ for those not in $A$, update distances \}
10.          then $v.\pi = u$
11.         $v.key = w(u, v)$
Chapter 23. Minimum Spanning Trees

MST-Prim \((G, w, r)\)
1. \textbf{for each} \(u \in G.V\) \{ \(u.key\) is the \(u\)'s shortest distance to set \(A = V-Q\) \}
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6. \textbf{while} \(Q \neq \emptyset\)
7. \(u = \text{Extract Min}(Q)\)
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9. \textbf{if} \(v \in Q\) \textbf{and} \(w(u, v) < v.key\) \{ for those not in \(A\), update distances\}
10. \textbf{then} \(v.\pi = u\)
11. \(v.key = w(u, v)\)
12. \textbf{return} \(\pi\)
Chapter 23. Minimum Spanning Trees

MST-Prim($G, w, r$)

1. for each $u \in G.V$

2. $u.key = \infty$ \{ $u.key$ is the $u$'s shortest distance to set $A = V-Q$ \}

3. $u.\pi = NULL$

4. $r.key = 0$ \{ start from vertex $r$ \}

5. $Q = G.V$ \{ establish priority queue $Q$ with key values \}

6. while $Q \neq \emptyset$

7. $u =$Extract Min$(Q)$

8. for each $v \in Adj[u]$

9. if $v \in Q$ and $w(u, v) < v.key$ \{ for those not in $A$, update distances \}

10. then $v.\pi = u$

11. $v.key = w(u, v)$

12. return $\pi$

usage of Priority queue: $Q$, Extract Min takes $O(\log n)$ time.
Chapter 23. Minimum Spanning Trees

MST-Prim(G, w, r)
1. for each \( u \in G.V \) \{ \( u.key \) is the \( u \)'s shortest distance to set \( A = V-Q \) \}
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4. \( r.key = 0 \) \{ start from vertex \( r \) \}
5. \( Q = G.V \) \{ establish priority queue \( Q \) wit key values \}
6. while \( Q \neq \emptyset \)
7. \( u = \text{Extract Min}(Q) \)
8. for each \( v \in Adj[u] \)
9. if \( v \in Q \) and \( w(u, v) < v.key \) \{ for those not in \( A \), update distances \}
10. then \( v.\pi = u \)
11. \( v.key = w(u, v) \)
12. return \( \pi \)

usage of Priority queue: \( Q \), \text{Extract Min} takes \( O(\log n) \) time.

running time \( O(|E| + |V| \log |V|) \).
Chapter 23. Minimum Spanning Trees
Chapter 23. Minimum Spanning Trees

Summary of Kruskal’s and Prim’s algorithms:
Chapter 23. Minimum Spanning Trees

Summary of Kruskal’s and Prim’s algorithms:

- initialize parent array
Summary of Kruskal’s and Prim’s algorithms:

- initialize parent array
  initialize $A = \emptyset$ or initial vertex $r$;
Chapter 23. Minimum Spanning Trees

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• initialize parent array
  initialize $A = \emptyset$ or initial vertex $r$;
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Summary of Kruskal’s and Prim’s algorithms:

- initialize parent array
  
  initialize $A = \emptyset$ or initial vertex $r$;

- repeatedly choosing from the remaining edges;
  
  pick a light edge that respects a cut
  add it to $A$
Chapter 23. Minimum Spanning Trees

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  initialize \( A = \emptyset \) or initial vertex \( r \);

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  pick a light edge that respects a cut
  
  add it to \( A \),
Chapter 23. Minimum Spanning Trees

Summary of Kruskal’s and Prim’s algorithms:

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  ensure that $A$ is a subset of some MST
Chapter 23. Minimum Spanning Trees

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- initialize parent array
  
  initialize $A = \emptyset$ or initial vertex $r$;

- repeatedly choosing from the remaining edges;

  pick a light edge that respects a cut
  
  add it to $A$,

  ensure that $A$ is a subset of some MST

- until $A$ forms a spanning tree.
Some questions about MST
Chapter 23. Minimum Spanning Trees

Some questions about MST

• What are the “cuts” implied in Kruskal’s algorithm and in Prim’s algorithm, respectively?
Chapter 23. Minimum Spanning Trees

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• Can we develop a DP algorithm for the MST problem?
Chapter 23. Minimum Spanning Trees

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  the main issue: how solutions to subproblems help build solution for the problem
Some questions about MST

• What are the "cuts" implied in Kruskal’s algorithm and in Prim’s algorithm, respectively?

• Can we develop a DP algorithm for the MST problem?
  
  the main issue: how solutions to subproblems help build solution for the problem
  
  what are subproblems, or what do subsolutions look like?
Chapter 24. Single Source Shortest Paths

Chapter 24. Single-source shortest paths

Given a graph $G = (V,E)$, with weight $w: E \to \mathbb{R}$; and single vertex $s \in V$; for each vertex $v \in V$, find a shortest path $s \Rightarrow v$.

- Shortest path is a simple path.
- "Distance" is measured by the total edge weight on the path, i.e., if the path $v_0 p \Rightarrow v_k$ is $p = (v_0, v_1, ..., v_k)$ then the path weight is $w(p) = \sum_{i=1}^{k} w(v_{i-1}, v_i)$

- Shortest distance between $u$ and $v$ is $\delta(u,v) = \min_{u \Rightarrow v} \{w(p)\}$
Chapter 24. Single Source Shortest Paths

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Given a graph $G = (V, E)$, with weight $w : E \rightarrow R$; and single vertex $s \in V$;
Chapter 24. Single Source Shortest Paths

Chapter 24. Single-source shortest paths

Given a graph $G = (V, E)$, with weight $w : E \rightarrow \mathbb{R}$; and single vertex $s \in V$;

for each vertex $v \in V$, find a shortest path $s \rightsquigarrow v$. 

• Shortest path is a simple path.
• "Distance" is measured by the total edge weight on the path
  $i.e.,$ if the path $v_0 p \rightarrow v_k$ is $p = (v_0, v_1, \ldots, v_k)$
  then the path weight is $w(p) = \sum_{i=1}^{k} w(v_{i-1}, v_i)$.

• Shortest distance between $u$ and $v$ is $\delta(u, v) = \min_u p \rightsquigarrow v \{ w(p) \}$. 
Chapter 24. Single Source Shortest Paths

Chapter 24. Single-source shortest paths

Given a graph $G = (V, E)$, with weight $w : E \rightarrow R$; and single vertex $s \in V$;

for each vertex $v \in V$, find a shortest path $s \leadsto v$.

- **Shortest path is a simple path.**
Chapter 24. Single Source Shortest Paths

Chapter 24. Single-source shortest paths

Given a graph \( G = (V, E) \), with weight \( w : E \to \mathbb{R} \); and single vertex \( s \in V \);
for each vertex \( v \in V \), find a shortest path \( s \leadsto v \).

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- “distance” is measured by the total edge weight on the path.
Chapter 24. Single-source shortest paths

Given a graph $G = (V, E)$, with weight $w : E \rightarrow \mathbb{R}$; and single vertex $s \in V$;
for each vertex $v \in V$, find a shortest path $s \leadsto v$.

- Shortest path is a simple path.
- “distance” is measured by the total edge weight on the path
  i.e., if the path $v_0 \xrightarrow{p} v_k$ is $p = (v_0, v_1, \ldots, v_k)$
  then the path weight is $w(p) = \sum_{i=1}^{k} w(v_{i-1}, v_i)$
Chapter 24. Single-source shortest paths

Given a graph $G = (V, E)$, with weight $w : E \to \mathbb{R}$; and single vertex $s \in V$;

for each vertex $v \in V$, find a shortest path $s \rightsquigarrow v$.

- **Shortest path is a simple path.**
- “distance” is measured by the total edge weight on the path
  i.e., if the path $v_0 \rightsquigarrow v_k$ is $p = (v_0, v_1, \ldots, v_k)$
  then the path weight is $w(p) = \sum_{i=1}^{k} w(v_{i-1}, v_i)$
- **Shortest distance between** $u$ and $v$ is
  \[
  \delta(u, v) = \min_{u \rightsquigarrow v} \{ w(p) \} \]
Chapter 24. Single Source Shortest Paths

- **Single-source shortest paths**: from $s$ to each vertex $v \in V$
Chapter 24. Single Source Shortest Paths

- **Single-source shortest paths**: from $s$ to each vertex $v \in V$
- a special case: **Single-pair shortest path**: from $s$ to $t$
Chapter 24. Single Source Shortest Paths

- **Single-source shortest paths**: from $s$ to each vertex $v \in V$
- a special case: **Single-pair shortest path**: from $s$ to $t$
- **All-pairs shortest paths**: from $s$ to $t$ for all pairs $s, t \in V$. 
Lemma 24.1 (a subpath of a shortest path is a shortest path)

Proof idea: (proof by contradiction)
Assume that \( p_{i,j} \) is not the shortest path from \( v_i \) to \( v_j \). Then there is a shorter path \( q_{i,j} \) from \( v_i \) to \( v_j \).

Define path \( q = (v_0, \ldots, v_i, q_{i,j}, v_j, \ldots, v_k) \) has weight

\[
\begin{align*}
    w(q) &= \sum_{t=1}^{i} w(v_t - 1, v_t) + w(q_{i,j}) + \sum_{t=j}^{k-1} w(v_t, v_{t+1}) \\
    &< \sum_{t=1}^{i} w(v_t - 1, v_t) + w(p_{i,j}) + \sum_{t=j}^{k-1} w(v_t, v_{t+1}) \\
    &= w(p)
\end{align*}
\]

contradicts to the assumption that \( p \) is the shortest path from \( v_0 \) to \( v_k \).
Lemma 24.1 (a subpath of a shortest path is a shortest path)

Given a weighted directed graph $G = (V, E)$ with edge weight function $w$. Let $p = (v_0, v_1, \ldots, v_k)$ be a shortest path $v_0 \xrightarrow{p} v_k$. Then $p_{i,j} = (v_i, \ldots, v_j)$ is a shortest path $v_i \xrightarrow{p_{i,j}} v_j$. 

Proof idea: (proof by contradiction)
Assume that $p_{i,j}$ is not the shortest path from $v_i$ to $v_j$. Then there is a shorter path $q_{i,j}$ from $v_i$ to $v_j$.

Define path $q = (v_0, \ldots, v_i, q_{i,j}, v_j, \ldots, v_k)$ has weight $w(q) = i \sum_{t=1}^{i} w(v_{t-1}, v_t) + w(q_{i,j}) + k-1 \sum_{t=j}^{k} w(v_t, v_{t+1}) < i \sum_{t=1}^{i} w(v_{t-1}, v_t) + w(p_{i,j}) + k-1 \sum_{t=j}^{k} w(v_t, v_{t+1}) = w(p)$
contradicts to the assumption that $p$ is the shortest path from $v_0$ to $v_k$. 


Lemma 24.1 (a subpath of a shortest path is a shortest path)

Given a weighted directed graph $G = (V, E)$ with edge weight function $w$. Let $p = (v_0, v_1, \ldots, v_k)$ be a shortest path $v_0 \overset{p}{\rightarrow} v_k$. Then $p_{i,j} = (v_i, \ldots, v_j)$ is a shortest path $v_i \overset{p_{i,j}}{\rightarrow} v_j$.

Proof idea: (proof by contradiction)
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Proof idea: (proof by contradiction) Assume that \( p_{i,j} \) is not the shortest path from \( v_i \) to \( v_j \). Then there is a shorter path \( q_{i,j} \) from \( v_i \) to \( v_j \).
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Define path

\[ q = (v_0, \ldots, v_i, q_{i,j}, v_j, \ldots, v_k) \]

has weight

\[ w(q) = \]
Lemma 24.1 (a subpath of a shortest path is a shortest path)

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Define path

$$q = (v_0, \ldots, v_i, q_{i,j}, v_j, \ldots, v_k)$$

has weight

$$w(q) = \sum_{t=1}^{i} w(v_{t-1}, v_t)$$
Chapter 24. Single Source Shortest Paths

**Lemma 24.1** (a subpath of a shortest path is a shortest path)

Given a weighted directed graph $G = (V, E)$ with edge weight function $w$. Let $p = (v_0, v_1, \ldots, v_k)$ be a shortest path $v_0 \leadsto v_k$. Then $p_{i,j} = (v_i, \ldots, v_j)$ is a shortest path $v_i \leadsto v_j$.

**Proof idea:** (proof by contradiction) Assume that $p_{i,j}$ is not the shortest path from $v_i$ to $v_j$. Then there is a shorter path $q_{i,j}$ from $v_i$ to $v_j$.

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$q = (v_0, \ldots, v_i, q_{i,j}, v_j, \ldots, v_k)$

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$$w(q) = \sum_{t=1}^{i} w(v_{t-1}, v_t) + w(q_{i,j}) +$$
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Proof idea: (proof by contradiction) Assume that $p_{i,j}$ is not the shortest path from $v_i$ to $v_j$. Then there is a shorter path $q_{i,j}$ from $v_i$ to $v_j$.

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$$q = (v_0, \ldots, v_i, q_{i,j}, v_j, \ldots, v_k)$$

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<
Chapter 24. Single Source Shortest Paths

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Chapter 24. Single Source Shortest Paths

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\]

\[
< \sum_{t=1}^{i} w(v_{t-1}, v_t) + w(p_{i,j}) + \]

Chapter 24. Single Source Shortest Paths

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$$q = (v_0, \ldots, v_i, q_{i,j}, v_j, \ldots, v_k)$$

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$$w(q) = \sum_{t=1}^{i} w(v_{t-1}, v_t) + w(q_{i,j}) + \sum_{t=j}^{k-1} w(v_t, v_{t+1})$$

$$< \sum_{t=1}^{i} w(v_{t-1}, v_t) + w(p_{i,j}) + \sum_{t=j}^{k-1} w(v_t, v_{t+1})$$

which contradicts the assumption that $p$ is the shortest path from $v_0$ to $v_k$. 


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Given a weighted directed graph $G = (V, E)$ with edge weight function $w$. Let $p = (v_0, v_1, \ldots, v_k)$ be a shortest path $v_0 \xrightarrow{p} v_k$. Then $p_{i,j} = (v_i, \ldots, v_j)$ is a shortest path $v_i \xrightarrow{p_{i,j}} v_j$.

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$$< \sum_{t=1}^{i} w(v_{t-1}, v_t) + w(p_{i,j}) + \sum_{t=j}^{k-1} w(v_t, v_{t+1}) = w(p)$$
Chapter 24. Single Source Shortest Paths

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w(q) = \sum_{t=1}^{i} w(v_{t-1}, v_t) + w(q_{i,j}) + \sum_{t=j}^{k-1} w(v_t, v_{t+1})
\]

\[
< \sum_{t=1}^{i} w(v_{t-1}, v_t) + w(p_{i,j}) + \sum_{t=j}^{k-1} w(v_t, v_{t+1}) = w(p)
\]

contradicts to the assumption that \( p \) is the shortest path from \( v_0 \) to \( v_k \).
 Chapter 24. Single Source Shortest Paths

Some terminologies:

• negative weights are allowed;
• cycles on a path: not a simple path;
• negative weight cycles, 0 weight cycles
• representing shortest paths: predecessor
• shortest path tree: http://graphserver.sourceforge.net/gallery.html

(width $\propto \frac{1}{\text{distance}}$)
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Chapter 24. Single Source Shortest Paths

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  http://graphserver.sourceforge.net/gallery.html
  (width $\propto 1/$distance)
Technique: relaxation

- Intuition:
  
  if $s \xrightarrow{p} v$ has distance $v.d$ (computed so far),
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Technique: relaxation

- Intuition:
  
  if \( s \xrightarrow{p} v \) has distance \( v.d \) (computed so far),
  
  \( s \xrightarrow{q} u \) is newly discovered.
Chapter 24. Single Source Shortest Paths

Technique: relaxation

- Intuition:

  if \( s \xrightarrow{p} v \) has distance \( v.d \) (computed so far),

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Chapter 24. Single Source Shortest Paths

**Technique: relaxation**

- **Intuition:**
  
  if $s \xrightarrow{p} v$ has distance $v.d$ (computed so far),
  
  $s \xrightarrow{q} u$ is newly discovered. Then

  $$v.d = \min\{v.d, u.d + w(u,v)\}$$
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Technique: relaxation

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  if \( s \xrightarrow{p} v \) has distance \( v.d \) (computed so far),
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• In other words:
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Technique: relaxation

- Intuition:
  
  if $s \xrightarrow{p} v$ has distance $v.d$ (computed so far),
  $s \xrightarrow{q} u$ is newly discovered. Then

  \[ v.d = \min\{v.d, u.d + w(u,v)\} \]

- In other words:
  
  Let $v.d$ be an weight upper bound of a shortest path from $s$ to $v$, 

Chapter 24. Single Source Shortest Paths

Technique: relaxation

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  if \( s \xrightarrow{p} v \) has distance \( v.d \) (computed so far),
  \( s \xrightarrow{q} u \) is newly discovered. Then

  \[
  v.d = \min\{v.d, u.d + w(u,v)\}
  \]

• In other words:

  Let \( v.d \) be an weight upper bound of a shortest path from \( s \) to \( v \),
  initialized \( \infty \).
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**Technique: relaxation**

- **Intuition:**
  
  If \( s \xrightarrow{p} v \) has distance \( v.d \) (computed so far),
  \( s \xrightarrow{q} u \) is newly discovered. Then
  
  \[ v.d = \min\{v.d, u.d + w(u,v)\} \]

- **In other words:**

  Let \( v.d \) be an weight upper bound of a shortest path from \( s \) to \( v \),
  initialized \( \infty \).

  The process of relaxing edge \((u,v)\): improves \( v.d \) by taking the path
  through \( u \), and update \( v.d \) and \( v.\pi \).
Bellman-Ford algorithm

Bellman-Ford algorithm

1. for each vertex \( v \) ∈ \( G.V \) initialization
2. \( v.d = \infty \)
3. \( v.\pi = \text{NULL} \)
4. \( s.d = 0 \)
5. for \( i = 1 \) to \( |V| - 1 \) relaxation
6. for each edge \((u,v)\) ∈ \( G.E \)
7. if \( v.d > u.d + w(u,v) \)
8. \( v.d = u.d + w(u,v) \)
9. \( v.\pi = u \)
10. for each edge \((u,v)\) ∈ \( G.E \) checking negative weight cycle
11. if \( v.d > u.d + w(u,v) \)
12. return (FALSE)
13. return (TRUE)

Running time: \( O(|V||E|) \)
Chapter 24. Single Source Shortest Paths

Bellman-Ford algorithm

BELLMAN-FORD(G, w, s)
Bellman-Ford algorithm

\textsc{Bellman-Ford}(G, w, s)

1. \textbf{for} each vertex \( v \in G.V \) \hspace{1cm} \text{initialization}

\text{Running time :} \[ O(|V| |E|) \]
Chapter 24. Single Source Shortest Paths

Bellman-Ford algorithm

\textbf{Bellman-Ford}(G, w, s)

1. \textbf{for} each vertex \( v \in G.V \) \textbf{ initialization}
2. \( v.d = \infty \)

\text{Running time: } O(\mid V \mid \mid E \mid)
Chapter 24. Single Source Shortest Paths

Bellman-Ford algorithm

\textbf{BELLMAN-\textsc{FORD}}(G, w, s)

1. \textbf{for} each vertex \( v \in G.V \) \hspace{1cm} \textbf{initialization}
2. \hspace{1cm} \( v.d = \infty \)
3. \hspace{1cm} \( v.\pi = NULL \)
4. \hspace{1cm} \( s.d = 0 \)
5. \textbf{for} \( i = 1 \) to \(|V| - 1\)
6. \hspace{1cm} \textbf{relaxation}
7. \hspace{1cm} \textbf{for} each edge \( (u, v) \in G.E \)
8. \hspace{1cm} \textbf{if} \( v.d > u.d + w(u,v) \)
9. \hspace{1cm} \hspace{1cm} \( v.d = u.d + w(u,v) \)
10. \hspace{1cm} \hspace{1cm} \( v.\pi = u \)
11. \hspace{1cm} \textbf{for} each edge \( (u, v) \in G.E \) \hspace{1cm} \textbf{checking negative weight cycle}
12. \hspace{1cm} \textbf{if} \( v.d > u.d + w(u,v) \)
13. \hspace{1cm} \hspace{1cm} \textbf{return} (FALSE)
14. \hspace{1cm} \textbf{return} (TRUE)

Running time: \( O(|V| \cdot |E|) \)
Bellman-Ford algorithm

\textbf{Bellman-Ford}(G, w, s)

1. \textbf{for} each vertex \( v \in G.V \) \hspace{1cm} \text{initialization}
2. \hspace{1cm} \( v.d = \infty \)
3. \hspace{1cm} \( v.\pi = NULL \)
4. \hspace{1cm} \( s.d = 0 \)

5. \textbf{for} \( i = 1 \) to \( |V| - 1 \) \hspace{1cm} \text{relaxation}
6. \hspace{2cm} \textbf{for} each edge \((u,v) \in G.E\) 
7. \hspace{3cm} \textbf{if} \( v.d > u.d + w(u,v) \)
8. \hspace{4cm} \( v.d = u.d + w(u,v) \)
9. \hspace{4cm} v.\pi = u
10. \hspace{2cm} \textbf{for} each edge \((u,v) \in G.E\) \hspace{1cm} \text{checking negative weight cycle}
11. \hspace{3cm} \textbf{if} \( v.d > u.d + w(u,v) \)
12. \hspace{4cm} \textbf{return} (FALSE)
13. \hspace{4cm} \textbf{return} (TRUE)

\textbf{Running time:} \( O(|V| \cdot |E|) \)
Chapter 24. Single Source Shortest Paths

Bellman-Ford algorithm

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Running time : \( O(|V| |E|) \)
Chapter 24. Single Source Shortest Paths

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Bellman-Ford algorithm

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6. \textbf{for} each edge \( (u, v) \in G.E \)
7. \textbf{if} \( v.d > u.d + w(u, v) \)
Bellman-Ford algorithm

**Bellman-Ford** \((G, w, s)\)

1. **for** each vertex \(v \in G.V\)  
   \(v.d = \infty\)  
   \(v.\pi = NULL\)  
   \(s.d = 0\)

2. **for** \(i = 1\) to \(|V| - 1\)  
   **relaxation**

3. for each edge \((u, v) \in G.E\)

4. if \(v.d > u.d + w(u, v)\)

5. \(v.d = u.d + w(u, v)\)

6. **return** \((FALSE)\)

7. **return** \((TRUE)\)

Running time: \(O(|V||E|)\)
Chapter 24. Single Source Shortest Paths

Bellman-Ford algorithm

\textsc{Bellman-Ford}(G, w, s)
1. \textbf{for} each vertex \( v \in G.V \) \hspace{1cm} \text{initialization}
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6. \hspace{1cm} \textbf{for} each edge \( (u, v) \in G.E \)
7. \hspace{1cm} \hspace{1cm} \textbf{if} \( v.d > u.d + w(u, v) \)
8. \hspace{1cm} \hspace{1cm} \hspace{1cm} \( v.d = u.d + w(u, v) \)
9. \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \( v.\pi = u \)
10. \hspace{1cm} \textbf{for} each edge \( (u, v) \in G.E \) \hspace{1cm} \text{checking negative weight cycle}
11. \hspace{1cm} \hspace{1cm} \textbf{if} \( v.d > u.d + w(u, v) \)
12. \hspace{1cm} \hspace{1cm} \hspace{1cm} \text{return} \( \text{FALSE} \)
13. \hspace{1cm} \hspace{1cm} \hspace{1cm} \text{return} \( \text{TRUE} \)

Running time: \( O(|V||E|) \)
Chapter 24. Single Source Shortest Paths

Bellman-Ford algorithm

\begin{align*}
\text{Bellman-Ford}(G, w, s) \\
1. & \quad \text{for each vertex } v \in G.V \quad \text{initialization} \\
2. & \quad v.d = \infty \\
3. & \quad v.\pi = NULL \\
4. & \quad s.d = 0 \\
5. & \quad \text{for } i = 1 \text{ to } |V| - 1 \quad \text{relaxation} \\
6. & \quad \text{for each edge } (u, v) \in G.E \\
7. & \quad \text{if } v.d > u.d + w(u, v) \\
8. & \quad \quad v.d = u.d + w(u, v) \\
9. & \quad \quad v.\pi = u \\
10. & \quad \text{for each edge } (u, v) \in G.E \quad \text{checking negative weight cycle}
\end{align*}
Bellman-Ford algorithm

**Bellman-Ford**($G, w, s$)

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2. $v.d = \infty$
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4. $s.d = 0$
5. for $i = 1$ to $|V| - 1$ relaxation
6. for each edge $(u, v) \in G.E$
7. if $v.d > u.d + w(u, v)$
8. $v.d = u.d + w(u, v)$
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Running time: $O(|V||E|)$
Bellman-Ford algorithm

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4. \hspace{1cm} \( s.d = 0 \)
5. \textbf{for} \( i = 1 \) \textbf{to} \( |V| - 1 \) \hspace{1cm} \text{relaxation}
6. \hspace{1cm} \textbf{for} each edge \((u, v) \in G.E\)
7. \hspace{2cm} \textbf{if} \( v.d > u.d + w(u, v) \)
8. \hspace{2cm} \hspace{1cm} \( v.d = u.d + w(u, v) \)
9. \hspace{2cm} \hspace{1cm} \( v.\pi = u \)
10. \textbf{for} each edge \((u, v) \in G.E\) \hspace{1cm} \text{checking negative weight cycle}
11. \hspace{2cm} \textbf{if} \( v.d > u.d + w(u, v) \)
12. \hspace{2cm} \hspace{1cm} \textbf{return} (FALSE)

\text{Running time :} \( O(|V| |E|) \)
Chapter 24. Single Source Shortest Paths

Bellman-Ford algorithm

\textbf{Bellman-Ford}(G, w, s)

1. \textbf{for} each vertex \( v \in G.V \) \quad \text{initialization}
2. \hspace{2cm} \( v.d = \infty \)
3. \hspace{2cm} \( v.\pi = NULL \)
4. \hspace{2cm} \( s.d = 0 \)
5. \textbf{for} \( i = 1 \) to \( |V| - 1 \) \quad \text{relaxation}
6. \hspace{2cm} \textbf{for} each edge \((u, v) \in G.E\)
7. \hspace{4cm} \textbf{if} \( v.d > u.d + w(u, v) \)
8. \hspace{4cm} \hspace{1cm} \( v.d = u.d + w(u, v) \)
9. \hspace{4cm} \hspace{1cm} \( v.\pi = u \)
10. \hspace{2cm} \textbf{for} each edge \((u, v) \in G.E\) \quad \text{checking negative weight cycle}
11. \hspace{4cm} \textbf{if} \( v.d > u.d + w(u, v) \)
12. \hspace{5cm} \textbf{return} (FALSE)
13. \hspace{2cm} \textbf{return} (TRUE)

\textbf{Running time :} \( O( |V| |E| ) \)
Bellman-Ford algorithm

\textsc{Bellman-Ford}(G, w, s)

1. \textbf{for} each vertex \( v \in G.V \) \hspace{1cm} \text{initialization}
2. \hspace{0.5cm} \( v.d = \infty \)
3. \hspace{0.5cm} \( v.\pi = NULL \)
4. \hspace{0.5cm} \( s.d = 0 \)
5. \hspace{0.5cm} \textbf{for} \( i = 1 \) to \(|V| - 1\) \hspace{1cm} \text{relaxation}
6. \hspace{1cm} \textbf{for} each edge \((u, v) \in G.E\)
7. \hspace{2cm} \textbf{if} \( v.d > u.d + w(u, v) \)
8. \hspace{2cm} \hspace{0.5cm} \( v.d = u.d + w(u, v) \)
9. \hspace{2cm} \hspace{0.5cm} \( v.\pi = u \)
10. \hspace{1cm} \textbf{for} each edge \((u, v) \in G.E\) \hspace{1cm} \text{checking negative weight cycle}
11. \hspace{2cm} \textbf{if} \( v.d > u.d + w(u, v) \)
12. \hspace{2cm} \hspace{0.5cm} \textbf{return} (FALSE)
13. \hspace{2cm} \hspace{0.5cm} \textbf{return} (TRUE)

Running time : \( O(|V||E|) \)
Chapter 24. Single Source Shortest Paths

(a) [Diagram of a network with labels on edges and vertices]
(b) [Diagram of a network with labels on edges and vertices]
(c) [Diagram of a network with labels on edges and vertices]
(d) [Diagram of a network with labels on edges and vertices]
(e) [Diagram of a network with labels on edges and vertices]
Chapter 24. Single Source Shortest Paths
Properties of shortest paths and relaxation

Lemma 24.14, Convergence property: Let \( s \Rightarrow u \rightarrow v \) is a shortest path. If \( u.d = \delta(s,u) \) holds before \( \text{Relax}(u,v,w) \) is called, then \( v.d = \delta(s,v) \) after the call.

Proof: \( v.d \leq u.d + w(u,v) = \delta(s,u) + w(u,v) = \delta(s,v) \). So \( v.d = \delta(s,v) \).
Properties of shortest paths and relaxation

\textsc{Relax}(u, v, w)
Properties of shortest paths and relaxation

\textsc{Relax}(u, v, w)

1.  \textbf{if} \ v.d > u.d + w(u, v)
Chapter 24. Single Source Shortest Paths

Properties of shortest paths and relaxation

\textbf{RELAX}(u, v, w)
1. if \( v.d > u.d + w(u, v) \)
2. \( v.d = u.d + w(u, v) \)
Properties of shortest paths and relaxation

\texttt{RELAX}(u, v, w)

1. \textbf{if} \( v.d > u.d + w(u, v) \)
2. \( v.d = u.d + w(u, v) \)
3. \( v.\pi = u \)
Properties of shortest paths and relaxation

\texttt{RELAX}(u, v, w)

1. if \( v.d > u.d + w(u, v) \)
2. \( v.d = u.d + w(u, v) \)
3. \( v.\pi = u \)

\textbf{Lemma 24.14, Convergence property}: Let \( s \leadsto u \rightarrow v \) is a shortest path. If \( u.d = \delta(s, u) \) holds before \texttt{RELAX}(u, v, w) is called, then \( v.d = \delta(s, v) \) after the call.
Properties of shortest paths and relaxation

RELAX\((u, v, w)\)

1. \(\text{if } v.d > u.d + w(u, v)\)
2. \(v.d = u.d + w(u, v)\)
3. \(v.\pi = u\)

**Lemma 24.14, Convergence property**: Let \(s \rightsquigarrow u \rightarrow v\) is a shortest path. If \(u.d = \delta(s, u)\) holds before RELAX\((u, v, w)\) is called, then \(v.d = \delta(s, v)\) after the call.

**Proof:**
Properties of shortest paths and relaxation

`RELAX(u, v, w)`
1. if $v.d > u.d + w(u, v)$
2. $v.d = u.d + w(u, v)$
3. $v.\pi = u$

**Lemma 24.14, Convergence property**: Let $s \leadsto u \rightarrow v$ is a shortest path. If $u.d = \delta(s, u)$ holds before `RELAX(u, v, w)` is called, then $v.d = \delta(s, v)$ after the call.

**Proof**: $v.d \leq u.d + w(u, v)$
Properties of shortest paths and relaxation

**RELAX**($u, v, w$)
1. if $v.d > u.d + w(u, v)$
2. $v.d = u.d + w(u, v)$
3. $v.\pi = u$

**Lemma 24.14, Convergence property:** Let $s \leadsto u \rightarrow v$ is a shortest path. If $u.d = \delta(s, u)$ holds before **RELAX**($u, v, w$) is called, then $v.d = \delta(s, v)$ after the call.

**Proof:** $v.d \leq u.d + w(u, v) = \delta(s, u) + w(u, v)$
Chapter 24. Single Source Shortest Paths

Properties of shortest paths and relaxation

\textsc{Relax}(u, v, w)

1. if \( v.d > u.d + w(u, v) \)
2. \( v.d = u.d + w(u, v) \)
3. \( v.\pi = u \)

\textbf{Lemma 24.14, Convergence property}: Let \( s \leadsto u \rightarrow v \) is a shortest path. If \( u.d = \delta(s, u) \) holds before \textsc{Relax}(u, v, w) is called, then \( v.d = \delta(s, v) \) after the call.

\textbf{Proof}: \( v.d \leq u.d + w(u, v) = \delta(s, u) + w(u, v) = \delta(s, v) \).
Properties of shortest paths and relaxation

\textsc{Relax}(u, v, w)
1. if $v.d > u.d + w(u, v)$
2. $v.d = u.d + w(u, v)$
3. $v.\pi = u$

\textbf{Lemma 24.14, Convergence property}: Let $s \sim u \rightarrow v$ is a shortest path. If $u.d = \delta(s, u)$ holds before \textsc{Relax}(u, v, w) is called, then $v.d = \delta(s, v)$ after the call.

\textbf{Proof}: $v.d \leq u.d + w(u, v) = \delta(s, u) + w(u, v) = \delta(s, v)$. So $v.d = \delta(s, v)$. 
We want to prove that, if a shortest path \( s \rightarrow v \) consists of \( k \) edges, Bellman-Fold obtains value \( v.d = \delta(s, v) \) after the \( k \)th round of relaxation (assuming there is no negative cycle).
Chapter 24. Single Source Shortest Paths

We want to prove that, if a shortest path $s \leadsto v$ consists of $k$ edges, Bellman-Fold obtains value $v.d = \delta(s, v)$ after the $k$th round of relaxation (assuming there is no negative cycle).

Proof idea: Induction on $k$. 
Chapter 24. Single Source Shortest Paths

We want to prove that, if a shortest path \( s \leadsto v \) consists of \( k \) edges, Bellman-Fold obtains value \( v.d = \delta(s, v) \) after the \( k \)th round of relaxation (assuming there is no negative cycle).

Proof idea: Induction on \( k \).

- \( k = 0 \), \( v \) can only be \( s \). Proved!
We want to prove that, if a shortest path $s \rightsquigarrow v$ consists of $k$ edges, Bellman-Fold obtains value $v.d = \delta(s, v)$ after the $k$th round of relaxation (assuming there is no negative cycle).

Proof idea: Induction on $k$.

• $k = 0$, $v$ can only be $s$. Proved!

• Assume the claim is proved for all vertices $v$ that have a shortest path of length $k$.
Chapter 24. Single Source Shortest Paths

We want to prove that, if a shortest path \( s \to v \) consists of \( k \) edges, Bellman-Fold obtains value \( v.d = \delta(s,v) \) after the \( k \)th round of relaxation (assuming there is no negative cycle).

**Proof idea:** Induction on \( k \).

- \( k = 0 \), \( v \) can only be \( s \). Proved!
- Assume the **claim** is proved for all vertices \( v \) that have a shortest path of length \( k \). What claim again??
We want to prove that, if a shortest path \( s \Rightarrow v \) consists of \( k \) edges, Bellman-Fold obtains value \( v.d = \delta(s, v) \) after the \( k \)th round of relaxation (assuming there is no negative cycle).

Proof idea: Induction on \( k \).

• \( k = 0 \), \( v \) can only be \( s \). Proved!

• Assume the claim is proved for all vertices \( v \) that have a shortest path of length \( k \). What claim again??

• Let \( v \) be any vertex that has a shortest path \( s \Rightarrow u \Rightarrow v \), consisting of \( k + 1 \) edges;
Chapter 24. Single Source Shortest Paths

We want to prove that, if a shortest path $s \leadsto v$ consists of $k$ edges, Bellman-Fold obtains value $v.d = \delta(s, v)$ after the $k$th round of relaxation (assuming there is no negative cycle).

Proof idea: Induction on $k$.

- $k = 0$, $v$ can only be $s$. Proved!
- Assume the claim is proved for all vertices $v$ that have a shortest path of length $k$. What claim again??
- Let $v$ be any vertex that has a shortest path $s \leadsto u \rightarrow v$, consisting of $k + 1$ edges;
  Then $s \leadsto u$ is a shortest path for $u$ consisting of $k$ edges;
Chapter 24. Single Source Shortest Paths

We want to prove that, if a shortest path $s \rightsquigarrow v$ consists of $k$ edges, Bellman-Fold obtains value $v.d = \delta(s, v)$ after the $k$th round of relaxation (assuming there is no negative cycle).

**Proof idea:** Induction on $k$.

- $k = 0$, $v$ can only be $s$. Proved!

- Assume the claim is proved for all vertices $v$ that have a shortest path of length $k$. What claim again??

- Let $v$ be any vertex that has a shortest path $s \rightsquigarrow u \rightarrow v$, consisting of $k + 1$ edges;

  Then $s \rightsquigarrow u$ is a shortest path for $u$ consisting of $k$ edges;

  Now by assumption, $u.d = \delta(s, u)$ after $k$ round of relaxation.
Chapter 24. Single Source Shortest Paths

We want to prove that, if a shortest path \( s \leadsto v \) consists of \( k \) edges, Bellman-Fold obtains value \( v.d = \delta(s, v) \) after the \( k \)th round of relaxation (assuming there is no negative cycle).

Proof idea: Induction on \( k \).

- \( k = 0 \), \( v \) can only be \( s \). Proved!
- Assume the claim is proved for all vertices \( v \) that have a shortest path of length \( k \). What claim again??
- Let \( v \) be any vertex that has a shortest path \( s \leadsto u \rightarrow v \), consisting of \( k + 1 \) edges;

Then \( s \leadsto u \) is a shortest path for \( u \) consisting of \( k \) edges;

Now by assumption, \( u.d = \delta(s, u) \) after \( k \) round of relaxation.

By Convergence property Lemma, \( v.d = \delta(s, v) \) after another round of relaxation.
Lemma 24.15, Path-relaxation property: Let $p = (v_0, v_1, \ldots, v_k)$ be a shortest path from $s = v_0$ to $v_k$. If a sequence relaxation steps occur that includes, in order, relaxing the edges $(v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k)$, then $v_k.d = \delta(s, v_k)$ after these relaxations and at all times afterward. This property holds no matter what other edge relaxations occur, including relaxations that are intermixed with relaxations of these edges.
Lemma 24.15, Path-relaxation property: Let $p = (v_0, v_1, \ldots, v_k)$ be a shortest path from $s = v_0$ to $v_k$. If a sequence relaxation steps occur that includes, in order, relaxing the edges $(v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k)$, then $v_k.d = \delta(s, v_k)$ after these relaxations and at all times afterward. This property holds no matter what other edge relaxations occur, including relaxations that are intermixed with relaxations of these edges.

Proof:
Lemma 24.15, Path-relaxation property: Let $p = (v_0, v_1, \ldots, v_k)$ be a shortest path from $s = v_0$ to $v_k$. If a sequence relaxation steps occur that includes, in order, relaxing the edges $(v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k)$, then $v_k.d = \delta(s, v_k)$ after these relaxations and at all times afterward. This property holds no matter what other edge relaxations occur, including relaxations that are intermixed with relaxations of these edges.

Proof: We prove by induction on $i$ that after the $i$th edge $(v_{i-1}, v_i)$ on path $p$ is relaxed, $v_i.d = \delta(s, v_i)$. 
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Lemma 24.15, Path-relaxation property: Let \( p = (v_0, v_1, \ldots, v_k) \) be a shortest path from \( s = v_0 \) to \( v_k \). If a sequence relaxation steps occur that includes, in order, relaxing the edges \((v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k)\), then \( v_k.d = \delta(s, v_k) \) after these relaxations and at all times afterward. This property holds no matter what other edge relaxations occur, including relaxations that are intermixed with relaxations of these edges.

Proof: We prove by induction on \( i \) that after the \( i \)th edge \((v_{i-1}, v_i)\) on path \( p \) is relaxed, \( v_i.d = \delta(s, v_i) \).

basis: \( i = 0 \). \( v_0 = s \), \( s.d = 0 = \delta(s, s) \).
Lemma 24.15, Path-relaxation property: Let \( p = (v_0, v_1, \ldots, v_k) \) be a shortest path from \( s = v_0 \) to \( v_k \). If a sequence relaxation steps occur that includes, in order, relaxing the edges \((v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k)\), then \( v_k.d = \delta(s, v_k) \) after these relaxations and at all times afterward. This property holds no matter what other edge relaxations occur, including relaxations that are intermixed with relaxations of these edges.

Proof: We prove by induction on \( i \) that after the \( i \)th edge \((v_{i-1}, v_i)\) on path \( p \) is relaxed, \( v_i.d = \delta(s, v_i) \).

basis: \( i = 0. \) \( v_0 = s, \) \( s.d = 0 = \delta(s, s) \)!
Assume: \( v_{i-1}.d = \delta(s, v_{i-1}) \).
Lemma 24.15, Path-relaxation property: Let $p = (v_0, v_1, \ldots, v_k)$ be a shortest path from $s = v_0$ to $v_k$. If a sequence relaxation steps occur that includes, in order, relaxing the edges $(v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k)$, then $v_k.d = \delta(s, v_k)$ after these relaxations and at all times afterward. This property holds no matter what other edge relaxations occur, including relaxations that are intermixed with relaxations of these edges.

Proof: We prove by induction on $i$ that after the $i$th edge $(v_{i-1}, v_i)$ on path $p$ is relaxed, $v_i.d = \delta(s, v_i)$.

basis: $i = 0$. $v_0 = s$, $s.d = 0 = \delta(s, s)$!
Assume: $v_{i-1}.d = \delta(s, v_{i-1})$.
Induction: After we relax edge $(v_{i-1}, v_i)$, by convergence property, we have $v_i.d = \delta(s, v_i)$. 
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Lemma 24.15, Path-relaxation property: Let \( p = (v_0, v_1, \ldots, v_k) \) be a shortest path from \( s = v_0 \) to \( v_k \). If a sequence relaxation steps occur that includes, in order, relaxing the edges \((v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k)\), then \( v_k.d = \delta(s, v_k) \) after these relaxations and at all times afterward. This property holds no matter what other edge relaxations occur, including relaxations that are intermixed with relaxations of these edges.

Proof: We prove by induction on \( i \) that after the \( i \)th edge \((v_{i-1}, v_i)\) on path \( p \) is relaxed, \( v_i.d = \delta(s, v_i) \).

**basis:** \( i = 0 \). \( v_0 = s \), \( s.d = 0 = \delta(s, s) \)!

**Assume:** \( v_{i-1}.d = \delta(s, v_{i-1}) \).

**Induction:** After we relax edge \((v_{i-1}, v_i)\), by convergence property, we have \( v_i.d = \delta(s, v_i) \). And this holds for all times afterward.
Chapter 24. Single Source Shortest Paths

Correctness of Bellman-Ford algorithm
Correctness of \textsc{Bellman-Ford} algorithm

1. On graphs without negative cycles)
Chapter 24. Single Source Shortest Paths

Correctness of **Bellman-Ford** algorithm

1. **On graphs without negative cycles**

**Lemma 24.2** Let $G = (V, E)$ be a weighted, directed graph with source $s$ and weight function $w : E \rightarrow \mathbb{R}$ and assume that $G$ contains no negative weight cycles that can be reached from $s$. Then after $|V| - 1$ iterations of line 5 in the algorithm, $v.d = \delta(s, v)$ for all vertices $v$ that are reachable from $s$. 
Chapter 24. Single Source Shortest Paths

Correctness of Bellman-Ford algorithm

1. On graphs without negative cycles)

Lemma 24.2 Let $G = (V, E)$ be a weighted, directed graph with source $s$ and weight function $w : E \rightarrow R$ and assume that $G$ contains no negative weight cycles that can be reached from $s$. Then after $|V| - 1$ iterations of line 5 in the algorithm, $v.d = \delta(s, v)$ for all vertices $v$ that are reachable from $s$.

Proof: (By induction on $k$, \(\delta(s,v)\)
Chapter 24. Single Source Shortest Paths

Correctness of \textbf{Bellman-Ford} algorithm

1. On graphs without negative cycles

\textbf{Lemma 24.2} Let $G = (V, E)$ be a weighted, directed graph with source $s$ and weight function $w : E \rightarrow R$ and assume that $G$ contains no negative weight cycles that can be reached from $s$. Then after $|V| - 1$ iterations of line 5 in the algorithm, $v.d = \delta(s, v)$ for all vertices $v$ that are reachable from $s$.

\textbf{Proof}: (By induction on $k$, the number of edges on the computed path $p : s \xymatrix{ & v}$, to prove the claim to be true).
Chapter 24. Single Source Shortest Paths

Correctness of \textsc{Bellman-Ford} algorithm

1. On graphs without negative cycles)

\textbf{Lemma 24.2} Let $G = (V, E)$ be a weighted, directed graph with source $s$ and weight function $w : E \rightarrow R$ and assume that $G$ contains no negative weight cycles that can be reached from $s$. Then after $|V| - 1$ iterations of line 5 in the algorithm, $v.d = \delta(s, v)$ for all vertices $v$ that are reachable from $s$.

\textbf{Proof:} (By induction on $k$, the number of edges on the computed path $p: s \xrightarrow{P} v$, to prove the claim to be true).
Chapter 24. Single Source Shortest Paths

Correctness of Bellman-Ford algorithm

1. On graphs without negative cycles)

Lemma 24.2 Let $G = (V, E)$ be a weighted, directed graph with source $s$ and weight function $w : E \to R$ and assume that $G$ contains no negative weight cycles that can be reached from $s$. Then after $|V| - 1$ iterations of line 5 in the algorithm, $v.d = \delta(s, v)$ for all vertices $v$ that are reachable from $s$.

Proof: (By induction on $k$, the number of edges on the computed path $p: s \rightarrow v$, to prove the claim to be true).

Base: $k = 0$. $v = s$. It is true.
Chapter 24. Single Source Shortest Paths

Correctness of \texttt{Bellman-Ford} algorithm

1. On graphs without negative cycles)

\textbf{Lemma 24.2} Let $G = (V, E)$ be a weighted, directed graph with source $s$ and weight function $w : E \to \mathbb{R}$ and assume that \textit{G contains no negative weight cycles that can be reached from s}. Then after $|V| - 1$ iterations of line 5 in the algorithm, $v.d = \delta(s, v)$ for all vertices $v$ that are reachable from $s$.

\textbf{Proof}: (By induction on $k$, the number of edges on the computed path $p: s \xrightarrow{P} v$, to prove the claim to be true).

\textbf{Base}: $k = 0. \ v = s. \ \text{It is true.}$

\textbf{Assume}: the claim is true for $k - 1$. 
Correctness of Bellman-Ford algorithm

1. On graphs without negative cycles)

**Lemma 24.2** Let $G = (V, E)$ be a weighted, directed graph with source $s$ and weight function $w : E \rightarrow R$ and assume that $G$ contains no negative weight cycles that can be reached from $s$. Then after $|V| - 1$ iterations of line 5 in the algorithm, $v.d = \delta(s, v)$ for all vertices $v$ that are reachable from $s$.

**Proof:** (By induction on $k$, the number of edges on the computed path $p$: $s \xrightarrow{p} v$, to prove the claim to be true).

**Base:** $k = 0$. $v = s$. It is true.

**Assume:** the claim is true for $k - 1$.

**Induction:** computed path $p$: $s \xrightarrow{p} v$ has $k$ edges and $p$ arrives at $x$ before reaching $v$ via $(x, v)$. 


Chapter 24. Single Source Shortest Paths

Correctness of Bellman-Ford algorithm

1. On graphs without negative cycles)

Lemma 24.2 Let $G = (V, E)$ be a weighted, directed graph with source $s$ and weight function $w : E \rightarrow R$ and assume that $G$ contains no negative weight cycles that can be reached from $s$. Then after $|V| - 1$ iterations of line 5 in the algorithm, $v.d = \delta(s, v)$ for all vertices $v$ that are reachable from $s$.

Proof: (By induction on $k$, the number of edges on the computed path $p$: $s \xrightarrow{p} v$, to prove the claim to be true).

Base: $k = 0$. $v = s$. It is true.

Assume: the claim is true for $k - 1$.

Induction: computed path $p$: $s \xrightarrow{p} v$ has $k$ edges and $p$ arrives at $x$ before reaching $v$ via $(x, v)$. So $v.d = x.d + w(x, v)$
Chapter 24. Single Source Shortest Paths

Correctness of Bellman-Ford algorithm

1. On graphs without negative cycles)

Lemma 24.2 Let $G = (V, E)$ be a weighted, directed graph with source $s$ and weight function $w : E \rightarrow R$ and assume that $G$ contains no negative weight cycles that can be reached from $s$. Then after $|V| - 1$ iterations of line 5 in the algorithm, $v.d = \delta(s, v)$ for all vertices $v$ that are reachable from $s$.

Proof: (By induction on $k$, the number of edges on the computed path $p$: $s \xrightarrow{p} v$, to prove the claim to be true).

Base: $k = 0$. $v = s$. It is true.

Assume: the claim is true for $k - 1$.

Induction: computed path $p$: $s \xrightarrow{p} v$ has $k$ edges and $p$ arrives at $x$ before reaching $v$ via $(x, v)$. So $v.d = x.d + w(x, v)$

By Lemma 24.1, $\delta(s, v) = \delta(s, y) + w(y, v)$ for some $y$. 

Chapter 24. Single Source Shortest Paths

Correctness of \textbf{Bellman-Ford} algorithm

1. On graphs without negative cycles)

\textbf{Lemma 24.2} Let $G = (V, E)$ be a weighted, directed graph with source $s$ and weight function $w : E \to \mathbb{R}$ and assume that $G$ contains no negative weight cycles that can be reached from $s$. Then after $|V| - 1$ iterations of line 5 in the algorithm, $v.d = \delta(s, v)$ for all vertices $v$ that are reachable from $s$.

\textbf{Proof}: (By induction on $k$, the number of edges on the computed path $p: s \xrightarrow{p} v$, to prove the claim to be true).

\textbf{Base}: $k = 0$. $v = s$. It is true.

\textbf{Assume}: the claim is true for $k - 1$.

\textbf{Induction}: computed path $p: s \xrightarrow{p} v$ has $k$ edges and $p$ arrives at $x$ before reaching $v$ via $(x, v)$. So $v.d = x.d + w(x, v)$

By Lemma 24.1, $\delta(s, v) = \delta(s, y) + w(y, v)$ for some $y$.

Since after $k$ iterations, $v.d$ has been updated with the statement \textbf{if} $v.d > u.d + w(u, v)$ \textbf{then} $v.d = u.d + w(u, v)$, for all $u$, including $x, y$.
Chapter 24. Single Source Shortest Paths

Correctness of Bellman-Ford algorithm

1. On graphs without negative cycles)

**Lemma 24.2** Let \( G = (V, E) \) be a weighted, directed graph with source \( s \) and weight function \( w : E \to \mathbb{R} \) and assume that \( G \) contains no negative weight cycles that can be reached from \( s \). Then after \(|V| - 1\) iterations of line 5 in the algorithm, \( v.d = \delta(s, v) \) for all vertices \( v \) that are reachable from \( s \).

**Proof:** (By induction on \( k \), the number of edges on the computed path \( p: s \xrightarrow{p} v \), to prove the claim to be true).

**Base:** \( k = 0. \) \( v = s \). It is true.

**Assume:** the claim is true for \( k - 1 \).

**Induction:** computed path \( p: s \xrightarrow{p} v \) has \( k \) edges and
\( p \) arrives at \( x \) before reaching \( v \) via \((x, v)\). So \( v.d = x.d + w(x, v) \)

By Lemma 24.1, \( \delta(s, v) = \delta(s, y) + w(y, v) \) for some \( y \).

Since after \( k \) iterations, \( v.d \) has been updated with the statement
if \( v.d > u.d + w(u, v) \) then \( v.d = u.d + w(u, v) \), for all \( u \), including \( x, y \)
By the assumption, for every \( u \), including \( x \) and \( y \), \( u.d = \delta(s, u) \) because
the computed path \( s \xrightarrow{p} u \) contains \( k - 1 \) edges. So we have
Chapter 24. Single Source Shortest Paths

Correctness of Bellman-Ford algorithm

1. On graphs without negative cycles)

Lemma 24.2 Let $G = (V, E)$ be a weighted, directed graph with source $s$ and weight function $w : E \to R$ and assume that $G$ contains no negative weight cycles that can be reached from $s$. Then after $|V| − 1$ iterations of line 5 in the algorithm, $v.d = \delta(s, v)$ for all vertices $v$ that are reachable from $s$.

Proof: (By induction on $k$, the number of edges on the computed path $p$: $s \xrightarrow{P} v$, to prove the claim to be true).

Base: $k = 0$. $v = s$. It is true.
Assume: the claim is true for $k − 1$.
Induction: computed path $p$: $s \xrightarrow{P} v$ has $k$ edges and $p$ arrives at $x$ before reaching $v$ via $(x, v)$. So $v.d = x.d + w(x, v)$

By Lemma 24.1, $\delta(s, v) = \delta(s, y) + w(y, v)$ for some $y$.

Since after $k$ iterations, $v.d$ has been updated with the statement if $v.d > u.d + w(u, v)$ then $v.d = u.d + w(u, v)$, for all $u$, including $x$, $y$ By the assumption, for every $u$, including $x$ and $y$, $u.d = \delta(s, u)$ because the computed path $s \xrightarrow{} u$ contains $k − 1$ edges. So we have

$v.d = x.d + w(x, v)$
Chapter 24. Single Source Shortest Paths

Correctness of Bellman-Ford algorithm

1. On graphs without negative cycles)

Lemma 24.2 Let $G = (V, E)$ be a weighted, directed graph with source $s$ and weight function $w : E \rightarrow \mathbb{R}$ and assume that $G$ contains no negative weight cycles that can be reached from $s$. Then after $|V| - 1$ iterations of line 5 in the algorithm, $v.d = \delta(s, v)$ for all vertices $v$ that are reachable from $s$.

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Induction: computed path $p$: $s \xrightarrow{P} v$ has $k$ edges and $p$ arrives at $x$ before reaching $v$ via $(x, v)$. So $v.d = x.d + w(x, v)$

By Lemma 24.1, $\delta(s, v) = \delta(s, y) + w(y, v)$ for some $y$.

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By the assumption, for every $u$, including $x$ and $y$, $u.d = \delta(s, u)$ because the computed path $s \xrightarrow{} u$ contains $k - 1$ edges. So we have

$v.d = x.d + w(x, v) \leq y.d + w(y, v)$
Chapter 24. Single Source Shortest Paths

Correctness of Bellman-Ford algorithm

1. On graphs without negative cycles)

Lemma 24.2 Let \( G = (V, E) \) be a weighted, directed graph with source \( s \) and weight function \( w : E \to R \) and assume that \( G \) contains no negative weight cycles that can be reached from \( s \). Then after \( |V| - 1 \) iterations of line 5 in the algorithm, \( v.d = \delta(s, v) \) for all vertices \( v \) that are reachable from \( s \).

Proof: (By induction on \( k \), the number of edges on the computed path \( p: s \leadsto v \), to prove the claim to be true).

- **Base:** \( k = 0 \). \( v = s \). It is true.
- **Assume:** the claim is true for \( k - 1 \).
- **Induction:** computed path \( p: s \leadsto v \) has \( k \) edges and \( p \) arrives at \( x \) before reaching \( v \) via \((x, v)\). So \( v.d = x.d + w(x, v) \)
  
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  By the assumption, for every \( u \), including \( x \) and \( y \), \( u.d = \delta(s, u) \) because the computed path \( s \leadsto u \) contains \( k - 1 \) edges. So we have

  \[
  v.d = x.d + w(x, v) \leq y.d + w(y, v) = \delta(s, y) + w(y, v) = \delta(s, v)
  \]
Chapter 24. Single Source Shortest Paths

**Theorem 24.4** Bellman-Ford algorithm is correct on weighted, directed graphs.

Proof: By Lemma 24.2, we only need to show, when $G$ contains a negative weight cycle reachable from $s$, the algorithm returns FALSE. Let the cycle to be $c = (v_0, v_1, \ldots, v_k)$, where $v_0 = v_k$ and $k \sum_{i=1}^{k} w(v_{i-1}, v_i) < 0$.

Assume for all $i$, $v_i.d \leq v_{i-1}.d + w(v_{i-1}, v_i)$.

$k \sum_{i=1}^{k} v_i.d \leq k \sum_{i=1}^{k} v_{i-1}.d + w(v_{i-1}, v_i)$, implying $k \sum_{i=1}^{k} w(v_{i-1}, v_i) \geq 0$, contradicting to $c$ being a negative cycle where $k \sum_{i=1}^{k} w(v_{i-1}, v_i) < 0$. 
Chapter 24. Single Source Shortest Paths

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Chapter 24. Single Source Shortest Paths

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\sum_{i=1}^{k} w(v_{i-1}, v_i) < 0
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Assume for all $i$, $v_i.d \leq v_{i-1}.d + w(v_{i-1}, v_i)$. 
Chapter 24. Single Source Shortest Paths

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$$\sum_{i=1}^{k} v_i.d \leq \sum_{i=1}^{k} v_{i-1}.d + w(v_{i-1}, v_i),$$
Chapter 24. Single Source Shortest Paths

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Assume for all $i$, $v_i.d \leq v_{i-1}.d + w(v_{i-1}, v_i)$.

$$ \sum_{i=1}^{k} v_i.d \leq \sum_{i=1}^{k} v_{i-1}.d + w(v_{i-1}, v_i), $$
**Theorem 24.4** Bellman-Ford algorithm is correct on weighted, directed graphs.

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Assume for all $i$, $v_i.d \leq v_{i-1}.d + w(v_{i-1}, v_i)$.

$$\sum_{i=1}^{k} v_i.d \leq \sum_{i=1}^{k} v_{i-1}.d + w(v_{i-1}, v_i),$$

But

$$\sum_{i=1}^{k} v_i.d = \sum_{i=1}^{k} v_{i-1}.d,$$
Theorem 24.4 Bellman-Ford algorithm is correct on weighted, directed graphs.

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$$\sum_{i=1}^{k} v_i.d \leq \sum_{i=1}^{k} v_{i-1}.d + w(v_{i-1}, v_i),$$

But

$$\sum_{i=1}^{k} v_i.d = \sum_{i=1}^{k} v_{i-1}.d, \quad \text{implying} \quad \sum_{i=1}^{k} w(v_{i-1}, v_i) \geq 0$$
**Chapter 24. Single Source Shortest Paths**

**Theorem 24.4** Bellman-Ford algorithm is correct on weighted, directed graphs.

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$$
\sum_{i=1}^{k} w(v_{i-1}, v_i) < 0
$$

Assume for all $i$, $v_i.d \leq v_{i-1}.d + w(v_{i-1}, v_i)$.

$$
\sum_{i=1}^{k} v_i.d \leq \sum_{i=1}^{k} v_{i-1}.d + w(v_{i-1}, v_i),
$$

But

$$
\sum_{i=1}^{k} v_i.d = \sum_{i=1}^{k} v_{i-1}.d, \quad \text{implying} \quad \sum_{i=1}^{k} w(v_{i-1}, v_i) \geq 0
$$

contradicting to $c$ being a negative cycle where $\sum_{i=1}^{k} w(v_{i-1}, v_i) < 0$.
Finding shortest paths on DAGs (directed acyclic graphs)
Finding shortest paths on DAGs (directed acyclic graphs)

- Algorithms can take the advantage of the non-cyclicity.
Chapter 24. Single Source Shortest Paths

Finding shortest paths on DAGs (directed acyclic graphs)

- Algorithms can take the advantage of the non-cyclicity.
- How would your algorithm be?
Chapter 24. Single Source Shortest Paths

Finding shortest paths on DAGs (directed acyclic graphs)

- Algorithms can take the advantage of the non-cyclicity.
- How would your algorithm be?

**topological order of vertices!**
Dag-Shortest Paths($G, w, s$)
Chapter 24. Single Source Shortest Paths

**Dag-Shortest Paths** ($G, w, s$)

1. topologically sort the vertices of $G.V$
Chapter 24. Single Source Shortest Paths

\textbf{DAG-Shortest Paths}(G, w, s)
1. topologically sort the vertices of $G.V$
2. \textbf{for} each vertex $v \in G.V$

\begin{itemize}
  \item Should we improve lines 6-7?
  \item Running time: ?
\end{itemize}
Chapter 24. Single Source Shortest Paths

**DAG-Shortest Paths**($G, w, s$)
1. topologically sort the vertices of $G.V$
2. for each vertex $v \in G.V$
3. $v.d = \infty$

---

Should we improve lines 6-7?

Running time: ?
Chapter 24. Single Source Shortest Paths

**DAG-Shortest Paths**$(G, w, s)$
1. topologically sort the vertices of $G.V$
2. for each vertex $v \in G.V$
3. $v.d = \infty$
4. $v.\pi = NULL$

• Should we improve lines 6-7?
• Running time: ?
Chapter 24. Single Source Shortest Paths

DAG-SHORTEST PATHS \((G, w, s)\)
1. topologically sort the vertices of \(G.V\)
2. for each vertex \(v \in G.V\)
3. \(v.d = \infty\)
4. \(v.\pi = NULL\)
5. \(s.d = 0\)

• Should we improve lines 6-7?
• Running time: ?
Chapter 24. Single Source Shortest Paths

Dag-Shortest Paths \((G, w, s)\)
1. topologically sort the vertices of \(G.V\)
2. for each vertex \(v \in G.V\)
3. \(v.d = \infty\)
4. \(v.\pi = NULL\)
5. \(s.d = 0\)
6. for each \(u \in G.V\), in the topologically sorted order

• Should we improve lines 6-7?
• Running time: ?
Chapter 24. Single Source Shortest Paths

**Dag-Shortest Paths**($G, w, s$)

1. topologically sort the vertices of $G.V$
2. for each vertex $v \in G.V$
3. $v.d = \infty$
4. $v.\pi = NULL$
5. $s.d = 0$
6. for each $u \in G.V$, in the topologically sorted order
7. for each vertex $v \in Adj[u]$
DAG-SHORTEST PATHS($G, w, s$)
1. topologically sort the vertices of $G.V$
2. for each vertex $v \in G.V$
3. \hspace{0.5cm} $v.d = \infty$
4. \hspace{0.5cm} $v.\pi = NULL$
5. \hspace{0.5cm} $s.d = 0$
6. for each $u \in G.V$, in the topologically sorted order
7. \hspace{0.5cm} for each vertex $v \in Adj[u]$
8. \hspace{0.5cm} if $v.d > u.d + w(u, v)$
Chapter 24. Single Source Shortest Paths

**Dag-Shortest Paths** \((G, w, s)\)

1. topologically sort the vertices of \(G.V\)
2. for each vertex \(v \in G.V\)
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4. \(v.\pi = NULL\)
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6. for each \(u \in G.V\), in the topologically sorted order
7. for each vertex \(v \in Adj[u]\)
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9. \(v.d = u.d + w(u,v)\)
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8. if \(v.d > u.d + w(u, v)\)
9. \(v.d = u.d + w(u, v)\)
10. \(v.\pi = u\)
Chapter 24. Single Source Shortest Paths

**DAG-Shortest Paths**\((G, w, s)\)
1. topologically sort the vertices of \(G.V\)
2. for each vertex \(v \in G.V\)
   3. \(v.d = \infty\)
   4. \(v.\pi = NULL\)
3. \(s.d = 0\)
4. for each \(u \in G.V\), in the topologically sorted order
   5. for each vertex \(v \in Adj[u]\)
      6. if \(v.d > u.d + w(u, v)\)
          7. \(v.d = u.d + w(u, v)\)
          8. \(v.\pi = u\)
5. return \((d, \pi)\)
Chapter 24. Single Source Shortest Paths

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- Should we improve lines 6-7?

Running time: ?
Chapter 24. Single Source Shortest Paths

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- Should we improve lines 6-7?
- Running time: ?
Chapter 24. Single Source Shortest Paths

note: the root is s.
Chapter 24. Single Source Shortest Paths

Dijkstra’s algorithm

On weighted, directed graphs in which each edge has non-negative weight.
Dijkstra’s algorithm

On weighted, directed graphs in which each edge has non-negative weight.

\textsc{Dijkstra}(G, w, s)
Chapter 24. Single Source Shortest Paths

Dijkstra’s algorithm

On weighted, directed graphs in which each edge has non-negative weight.

\textsc{Dijkstra}(G, w, s)

1. for each vertex \( v \in G.V \)
Chapter 24. Single Source Shortest Paths

Dijkstra’s algorithm

On weighted, directed graphs in which each edge has non-negative weight.

\[ \text{DIJKSTRA}(G, w, s) \]
1. for each vertex \( v \in G.V \)
2. \( v.d = \infty \)
Dijkstra’s algorithm

On weighted, directed graphs in which each edge has non-negative weight.

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3. \( v.\pi = NULL \)

Running time: ?
Chapter 24. Single Source Shortest Paths

Dijkstra’s algorithm

On weighted, directed graphs in which each edge has non-negative weight.

Dijkstra(G, w, s)
1. for each vertex v ∈ G.V
2. v.d = ∞
3. v.π = NULL
4. s.d = 0

5. S = ∅
6. Q = G.V
7. while Q is not empty
8. u = Extract Min(Q)
9. S = S ∪ {u}
10. for each vertex v ∈ Adj[u]
11. if v.d > u.d + w(u,v)
12. v.d = u.d + w(u,v)
13. v.π = u
14. return (d,π)
Dijkstra's algorithm

On weighted, directed graphs in which each edge has non-negative weight.

\texttt{Dijkstra}(G, w, s)
1. for each vertex \( v \in G.V \)
2. \( v.d = \infty \)
3. \( v.\pi = \text{NULL} \)
4. \( s.d = 0 \)
5. \( S = \emptyset \)

Running time: ?
Dijkstra’s algorithm

On weighted, directed graphs in which each edge has non-negative weight.

\[
\text{DIJKSTRA}(G, w, s)
\]

1. for each vertex \( v \in G.V \)
2. \( v.d = \infty \)
3. \( v.\pi = NULL \)
4. \( s.d = 0 \)
5. \( S = \emptyset \)
6. \( Q = G.V \)
Chapter 24. Single Source Shortest Paths

Dijkstra’s algorithm

On weighted, directed graphs in which each edge has non-negative weight.

$\text{Dijkstra}(G, w, s)$
1. for each vertex $v \in G.V$
2. $v.d = \infty$
3. $v.\pi = NULL$
4. $s.d = 0$
5. $S = \emptyset$
6. $Q = G.V$
7. while $Q$ is not empty
Dijkstra’s algorithm

On weighted, directed graphs in which each edge has non-negative weight.

**Dijkstra** \((G, w, s)\)

1. for each vertex \(v \in G.V\)
2. \(v.d = \infty\)
3. \(v.\pi = NULL\)
4. \(s.d = 0\)
5. \(S = \emptyset\)
6. \(Q = G.V\)
7. while \(Q\) is not empty
8. \(u = \text{Extract Min} (Q)\)
Chapter 24. Single Source Shortest Paths

Dijkstra’s algorithm

On weighted, directed graphs in which each edge has non-negative weight.

\textsc{Dijkstra}(G, w, s)
1. for each vertex \( v \in G.V \)
2. \( v.d = \infty \)
3. \( v.\pi = \text{NULL} \)
4. \( s.d = 0 \)
5. \( S = \emptyset \)
6. \( Q = G.V \)
7. \textbf{while} \( Q \) is not empty
8. \( u = \text{Extract Min} (Q) \)
9. \( S = S \cup \{u\} \)
Chapter 24. Single Source Shortest Paths

Dijkstra’s algorithm

On weighted, directed graphs in which each edge has non-negative weight.

\[
\text{DIJKSTRA}(G, w, s)
\]

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3. \( v.\pi = NULL \)
4. \( s.d = 0 \)
5. \( S = \emptyset \)
6. \( Q = G.V \)
7. while \( Q \) is not empty
8. \( u = \text{EXTRACT MIN} (Q) \)
9. \( S = S \cup \{u\} \)
10. for each vertex \( v \in Adj[u] \)
Dijkstra’s algorithm

On weighted, directed graphs in which each edge has non-negative weight.

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9. \( S = S \cup \{ u \} \)
10. for each vertex \( v \in \text{Adj}[u] \)
11. if \( v.d > u.d + w(u, v) \)
Chapter 24. Single Source Shortest Paths

Dijkstra’s algorithm

On weighted, directed graphs in which each edge has non-negative weight.

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5. \( S = \emptyset \)
6. \( Q = G.V \)
7. while \( Q \) is not empty
8. \( u = \textsc{Extract Min}(Q) \)
9. \( S = S \cup \{u\} \)
10. for each vertex \( v \in Adj[u] \)
11. \( \text{if } v.d > u.d + w(u, v) \)
12. \( v.d = u.d + w(u, v) \)

Running time: ?
Chapter 24. Single Source Shortest Paths

Dijkstra’s algorithm

On weighted, directed graphs in which each edge has non-negative weight.

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8. \( u = \text{Extract Min} (Q) \)
9. \( S = S \cup \{u\} \)
10. \textbf{for} each vertex \( v \in \text{Adj}[u] \)
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Dijkstra’s algorithm

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6. Q = G.V
7. while Q is not empty
8. u = Extract Min (Q)
9. S = S \cup \{u\}
10. for each vertex v \in Adj[u]
11. if v.d > u.d + w(u, v)
12. v.d = u.d + w(u, v)
13. v.\pi = u
14. return (d, \pi)
Chapter 24. Single Source Shortest Paths

Dijkstra’s algorithm

On weighted, directed graphs in which each edge has non-negative weight.

**Dijkstra**\((G, w, s)\)
1. for each vertex \(v \in G.V\)
2. \(v.d = \infty\)
3. \(v.\pi = NULL\)
4. \(s.d = 0\)
5. \(S = \emptyset\)
6. \(Q = G.V\)
7. while \(Q\) is not empty
8. \(u = \text{Extract Min} (Q)\)
9. \(S = S \cup \{u\}\)
10. for each vertex \(v \in Adj[u]\)
11. if \(v.d > u.d + w(u, v)\)
12. \(v.d = u.d + w(u, v)\)
13. \(v.\pi = u\)
14. return \((d, \pi)\)

Running time:?
Chapter 24. Single Source Shortest Paths

Note: the black-colored vertices are in set $S$. 
Chapter 24. Single Source Shortest Paths

Correctness of algorithm Dijkstra
Correctness of algorithm Dijkstra

**Theorem 24.6** Dijkstra’s algorithm, run on a weighted, directed graph $G = (V, E)$ with non-negative weight function $w$ and source $s$, terminates with $u.d = \delta(s, u)$ for all vertices $u \in V$. 

Proof: We need to show the while loop has loop invariant:

$u.d = \delta(s, u)$ for each $u \in S$

Assume $u$ to be the first such vertex that $u.d > \delta(s, u)$ when it is being added to $S$ then there must be a shortest path $p: s \rightarrow x \rightarrow y \rightarrow u$, for some $x \in S$ and some $y \neq S$.

$y.d = \delta(s, y)$ when $u$ is being added to $S$. This is because $x \in S$, $x.d = \delta(s, x)$ when $x$ was added to $S$.

Edge $(x, y)$ was related at that time, and $y.d = \delta(s, y)$ by Convergence-property. 

So when $u$ was chosen, $u.d \leq y.d = \delta(s, y) \leq \delta(s, u)$. Contradicts the choice of $u$. 

So $u.d = \delta(s, u)$ when it is being included to $S$. 


Correctness of algorithm \textbf{Dijkstra}

\textbf{Theorem 24.6} Dijkstra’s algorithm, run on a weighted, directed graph $G = (V, E)$ with non-negative weight function $w$ and source $s$, terminates with $u.d = \delta(s, u)$ for all vertices $u \in V$.

\textbf{Proof}: We need to show the \textbf{while} loop has \textbf{loop invariant}:
Chapter 24. Single Source Shortest Paths

**Correctness of algorithm Dijkstra**

**Theorem 24.6** Dijkstra’s algorithm, run on a weighted, directed graph $G = (V, E)$ with non-negative weight function $w$ and source $s$, terminates with $u.d = \delta(s, u)$ for all vertices $u \in V$.

**Proof**: We need to show the **while** loop has **loop invariant**: $u.d = \delta(s, u)$ for each $u \in S$. 

Edge $(x, y)$ was related at that time, and $y.d = \delta(s, y)$ by **Convergence-property**. So when $u$ was chosen, $u.d \leq y.d = \delta(s, y) \leq \delta(s, u)$. Contradicts the choice of $u$. So $u.d = \delta(s, u)$ when it is being included to $S$. 


Correctness of algorithm Dijkstra

Theorem 24.6 Dijkstra’s algorithm, run on a weighted, directed graph $G = (V, E)$ with non-negative weight function $w$ and source $s$, terminates with $u.d = \delta(s, u)$ for all vertices $u \in V$.

Proof: We need to show the while loop has loop invariant:

$u.d = \delta(s, u)$ for each $u \in S$

Assume $u$ to be the first such vertex that $u.d > \delta(s, u)$ when it is being added to $S$.
Chapter 24. Single Source Shortest Paths

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**Proof**: We need to show the while loop has loop invariant: 

$$u.d = \delta(s, u) \text{ for each } u \in S$$

Assume $u$ to be the first such vertex that $u.d > \delta(s, u)$ when it is being added to $S$ then there must be a shortest path $p$: $s \leadsto x \rightarrow y \leadsto u$, for some $x \in S$ and some $y \not\in S$. 


Chapter 24. Single Source Shortest Paths

Correctness of algorithm Dijkstra

Theorem 24.6 Dijkstra’s algorithm, run on a weighted, directed graph \( G = (V, E) \) with non-negative weight function \( w \) and source \( s \), terminates with \( u.d = \delta(s, u) \) for all vertices \( u \in V \).

Proof: We need to show the while loop has loop invariant:
\[ u.d = \delta(s, u) \] for each \( u \in S \)

Assume \( u \) to be the first such vertex that \( u.d > \delta(s, u) \) when it is being added to \( S \) then there must be a shortest path \( p: s \leadsto x \to y \leadsto u \), for some \( x \in S \) and some \( y \notin S \).

\( y.d = \delta(s, y) \) when \( u \) is being added to \( S \). This is because \( x \in S \), \( x.d = \delta(s, x) \) when \( x \) was added to \( S \).
Chapter 24. Single Source Shortest Paths

Correctness of algorithm **Dijkstra**

**Theorem 24.6** Dijkstra's algorithm, run on a weighted, directed graph $G = (V, E)$ with non-negative weight function $w$ and source $s$, terminates with $u.d = \delta(s, u)$ for all vertices $u \in V$.

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Assume $u$ to be the first such vertex that $u.d > \delta(s, u)$ when it is being added to $S$ then there must be a shortest path $p$: $s \rightsquigarrow x \rightarrow y \rightsquigarrow u$, for some $x \in S$ and some $y \notin S$.

$y.d = \delta(s, y)$ when $u$ is being added to $S$. This is because $x \in S$, $x.d = \delta(s, x)$ when $x$ was added to $S$. Edge $(x, y)$ was related at that time, and $y.d = \delta(s, y)$ by Convergence-property. So
Chapter 24. Single Source Shortest Paths

Correctness of algorithm \textbf{Dijkstra}

\textbf{Theorem 24.6} Dijkstra's algorithm, run on a weighted, directed graph $G = (V, E)$ with non-negative weight function $w$ and source $s$, terminates with $u.d = \delta(s, u)$ for all vertices $u \in V$.

\textbf{Proof:} We need to show the \textbf{while} loop has loop invariant: $u.d = \delta(s, u)$ for each $u \in S$

Assume $u$ to be the first such vertex that $u.d > \delta(s, u)$ when it is being added to $S$ then there must be a shortest path $p: s \leadsto x \rightarrow y \leadsto u$, for some $x \in S$ and some $y \not\in S$.

$y.d = \delta(s, y)$ when $u$ is being added to $S$. This is because $x \in S$, $x.d = \delta(s, x)$ when $x$ was added to $S$. Edge $(x, y)$ was related at that time, and $y.d = \delta(s, y)$ by \textbf{Convergence-property}. So

When $u$ was chosen, $u.d \leq y.d = \delta(s, y) \leq \delta(s, u)$. \textbf{Contradicts} the choice of $u$. 
Chapter 24. Single Source Shortest Paths

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$u.d = \delta(s, u)$ for each $u \in S$

Assume $u$ to be the first such vertex that $u.d > \delta(s, u)$ when it is being added to $S$ then there must be a shortest path $p$: $s \leadsto x \rightarrow y \leadsto u$, for some $x \in S$ and some $y \not\in S$.

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When $u$ was chosen, $u.d \leq y.d = \delta(s, y) \leq \delta(s, u)$. Contradicts the choice of $u$. So $u.d = \delta(s, u)$ when it is being included to $S$. 
Chapter 24. Single Source Shortest Paths

• Running time of Dijkstra?
Chapter 24. Single Source Shortest Paths

- Running time of DIJKSTRA?
- Can DIJKSTRA deals with negative edges or cycles?
Chapter 24. Single Source Shortest Paths

- Running time of Dijkstra?
- Can Dijkstra deal with negative edges or cycles?
- Fundamental differences between Bellman-Ford and Dijkstra?
Chapter 24. Single Source Shortest Paths

- Running time of Dijkstra?
- Can Dijkstra deals with negative edges or cycles?
- Fundamental differences between Bellman-Ford and Dijkstra?

**Dijkstra(G, w, s)**
1. for each vertex \( v \in G.V \)
2. \( v.d = \infty \)
3. \( v.\pi = NULL \)
4. \( s.d = 0 \)
5. \( S = \emptyset \)
6. \( Q = G.V \)
7. while \( Q \) is not empty
8. \( u = \text{EXTRACT MIN}(Q) \)
9. \( S = S \cup \{u\} \)
10. for each vertex \( v \in \text{Adj}[u] \)
11. if \( v.d > u.d + w(u, v) \)
12. \( v.d = u.d + w(u, v) \)
13. \( v.\pi = u \)
14. return \((d, \pi)\)

**Bellman-Ford(G, w, s)**
1. for each vertex \( v \in G.V \)
2. \( v.d = \infty \)
3. \( v.\pi = NULL \)
4. \( s.d = 0 \)
5. \( S = \emptyset \)
6. for \( i = 1 \) to \(|V| - 1\)
7. for each edge \((u, v) \in G.E\)
8. if \( v.d > u.d + w(u, v) \)
9. \( v.d = u.d + w(u, v) \)
10. \( v.\pi = u \)
11. return \((\text{FALSE})\)
12. return \((\text{TRUE})\)
Chapter 24. Single Source Shortest Paths
Chapter 24. Single Source Shortest Paths

- Fundamental differences between Dijkstra and MST-Prim?
Chapter 24. Single Source Shortest Paths

- Fundamental differences between Dijkstra and MST-Prim?

Dijkstra($G, w, s$)
1. for each vertex $v \in G.V$
2. $v.d = \infty$
3. $v.\pi = Null$
4. $s.d = 0$
5. $S = \emptyset$
6. $Q = G.V$
7. while $Q$ is not empty
8. $u = \text{EXTRACT MIN}(Q)$
9. $S = S \cup \{u\}$
10. for each vertex $v \in Adj[u]$
11. if $v.d > u.d + w(u,v)$
12. $v.d = u.d + w(u,v)$
13. $v.\pi = u$
14. return $(d, \pi)$

MST-Prim($G, w, r$)
1. for each $u \in G.V$
2. $u.key = \infty$ \{ $u.key$ is true \}
3. $u.\pi = Null$ \{ start from vertex \}
4. $r.key = 0$
5. $Q = G.V$
6. while $Q \neq \emptyset$
7. $u = \text{EXTRACT MIN}(Q)$
8. for each $v \in Adj[u]$
9. if $v \in Q$ and $w(u,v) < v.key$
10. then $v.\pi = u$
11. $v.key = w(u,v)$
12. return $\pi$
Chapter 25. All-pairs shortest paths

All Pair Shortest Paths Problem

Input: A weighted graph \( G = (V,E) \) with edge weight function \( w \);

Output: Shortest paths between every pair of vertices in \( G \).

- Dijkstra would run in time \( O(|V|^2 \log |V| + |V||E|) \) on non-negative edges.
- Bellman-Ford would run in time \( O(|V|^2 |E|) \) for general graphs, but \( O(|V|^4) \) on "dense" graphs.

New algorithms:
- A dynamic programming algorithm \( O(|V|^4) \), improved to \( O(|V|^3 \log |V|) \).
- Floyd-Warshall algorithm: \( O(|V|^3) \).

Graph representation: adjacency matrix \( W = (w_{ij}) \).
Chapter 25. All-pairs shortest paths

All Pair Shortest Paths Problem
Chapter 25. All-pairs shortest paths

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All Pair Shortest Paths Problem

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Chapter 25. All-pairs shortest paths

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Chapter 25. All-pairs shortest paths

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New algorithms
Chapter 25. All-pairs shortest paths

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Chapter 25. All-pairs shortest paths

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New algorithms

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- **Floyd-Warshall** algorithm: $O(|V|^3)$. 
Chapter 25. All-pairs shortest paths

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New algorithms

- A dynamic programming algorithm $O(|V|^4)$, improved to $O(|V|^3 \log |V|)$.
- **Floyd-Warshall algorithm:** $O(|V|^3)$.

**Graph representation:** adjacency matrix $W = (w_{ij})$
Chapter 25. All-pairs shortest paths

A dynamic programming approach
A dynamic programming approach

• Optimal substructure
• Objective function
Chapter 25. All-pairs shortest paths

A dynamic programming approach

- Optimal substructure
- Objective function

Define $l_{ij}$ be the minimum weight of any path from $v_i$ to $v_j$. 

or alternatively, Define $l_{ikj}$ be the minimum weight of any path from $v_i$ to $v_j$ in which intermediate vertices have indexes $\leq k$. 

does not work! having a data dependency issue.
Chapter 25. All-pairs shortest paths

A dynamic programming approach

- Optimal substructure
- Objective function

Define $l_{ij}$ be the minimum weight of any path from $v_i$ to $v_j$ does not work! having a data dependency issue.
Chapter 25. All-pairs shortest paths

A dynamic programming approach

• Optimal substructure
• Objective function

Define \( l_{ij} \) be the minimum weight of any path from \( v_i \) to \( v_j \) does not work! having a data dependency issue.

Define \( l_{ij}^m \) be the minimum weight of any path from \( v_i \) to \( v_j \) that contains at most \( m \) edges.
A dynamic programming approach

- Optimal substructure
- Objective function

Define $l_{ij}$ be the minimum weight of any path from $v_i$ to $v_j$ does not work! having a data dependency issue.

Define $l_{ij}^m$ be the minimum weight of any path from $v_i$ to $v_j$ that contains at most $m$ edges.

or alternatively,
Chapter 25. All-pairs shortest paths

A dynamic programming approach

- Optimal substructure
- Objective function

Define \( l_{ij} \) be the minimum weight of any path from \( v_i \) to \( v_j \) does not work! having a data dependency issue.

Define \( l^m_{ij} \) be the minimum weight of any path from \( v_i \) to \( v_j \) that contains at most \( m \) edges.

or alternatively,

Define \( l^k_{ij} \) be the minimum weight of any path from \( v_i \) to \( v_j \) in which intermediate vertices have indexes \( \leq k \).
Define $l^m_{ij}$ be the minimum weight of any path from $v_i$ to $v_j$ that contains at most $m$ edges.
Define $l^m_{ij}$ be the minimum weight of any path from $v_i$ to $v_j$ that contains at most $m$ edges.

$$l^m_{ij} = \min(l^{m-1}_{ij}, \min_{1 \leq k \leq n} \{l^{m-1}_{ik} + w_{kj}\})$$
Define $l^m_{ij}$ be the minimum weight of any path from $v_i$ to $v_j$ that contains at most $m$ edges.

$$l^m_{ij} = \min(l^{m-1}_{ij}, \min_{1 \leq k \leq n} \{l^{m-1}_{ik} + w_{kj}\})$$

If $w_{jj} = 0$, we can rewrite

$$l^m_{ij} = \min_{1 \leq k \leq n} \{l^{m-1}_{ik} + w_{kj}\}$$
Chapter 25. All-pairs shortest paths

Define $l^m_{ij}$ be the minimum weight of any path from $v_i$ to $v_j$ that contains at most $m$ edges.

$$l^m_{ij} = \min(l_{ij}^{m-1}, \min_{1 \leq k \leq n} \{l_{ik}^{m-1} + w_{kj}\})$$

If $w_{jj} = 0$, we can rewrite

$$l^m_{ij} = \min_{1 \leq k \leq n} \{l_{ik}^{m-1} + w_{kj}\}$$

and base cases:

$$l^1_{ij} = w_{ij}$$
Chapter 25. All-pairs shortest paths

Define $l_{ij}^m$ be the minimum weight of any path from $v_i$ to $v_j$ that contains at most $m$ edges.

\[
l_{ij}^m = \min(l_{ij}^{m-1}, \min_{1 \leq k \leq n} \{l_{ik}^{m-1} + w_{kj}\})
\]

If $w_{jj} = 0$, we can rewrite

\[
l_{ij}^m = \min_{1 \leq k \leq n} \{l_{ik}^{m-1} + w_{kj}\}
\]

and base cases:

\[
l_{ij}^1 = w_{ij}
\]

Adjacency matrix $W = (w_{ij})$ is the default.
Chapter 25. All-pairs shortest paths

DP table filling algorithm:

1. \( n \) = rows \([L]\);
2. let \( L' \) be an \( n \times n \) table;
3. for \( i = 1 \) to \( n \);
4. for \( j = 1 \) to \( n \);
5. \( L'[i,j] = \infty \) (\( L'[i,j] = L[i,j] \) in case \( w_{a,a} \neq 0 \));
6. for \( k = 1 \) to \( n \);
7. \( L'[i,j] = \min \{L'[i,j], L[i,k] + w[k,j]\} \);
8. return \( (L') \);
Chapter 25. All-pairs shortest paths

DP table filling algorithm:

For $L^1 = W$ and $m = 2, \ldots, n - 1$, compute table $L^m$ from table $L^{m-1}$;
Chapter 25. All-pairs shortest paths

DP table filling algorithm:

For $L^1 = W$ and $m = 2, \ldots, n - 1$, compute table $L^m$ from table $L^{m-1}$; technically two tables are enough.
Chapter 25. All-pairs shortest paths

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For $L^1 = W$ and $m = 2, \ldots, n - 1$, compute table $L^m$ from table $L^{m-1}$; technically two tables are enough.

**Extended Shortest Paths($L, W$)**
Chapter 25. All-pairs shortest paths

DP table filling algorithm:

For \( L^1 = W \) and \( m = 2, \ldots, n - 1 \), compute table \( L^m \) from table \( L^{m-1} \);

technically two tables are enough.

**Extended Shortest Paths** \((L, W)\)

1. \( n = \text{rows}[L] \);
Chapter 25. All-pairs shortest paths

DP table filling algorithm:

For $L^1 = W$ and $m = 2, \ldots, n - 1$, compute table $L^m$ from table $L^{m-1}$; technically two tables are enough.

**Extended Shortest Paths**$(L, W)$
1. $n = \text{rows}[L]$;
2. let $L'$ be an $n \times n$ table;
Chapter 25. All-pairs shortest paths

DP table filling algorithm:

For \( L^1 = W \) and \( m = 2, \ldots, n - 1 \), compute table \( L^m \) from table \( L^{m-1} \);

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**Extended Shortest Paths**(\( L, W \))

1. \( n = \text{rows}[L] \);
2. let \( L' \) be an \( n \times n \) table;
3. \textbf{for} \( i = 1 \) \textbf{to} \( n \)
Chapter 25. All-pairs shortest paths

DP table filling algorithm:

For $L^1 = W$ and $m = 2, \ldots, n - 1$, compute table $L^m$ from table $L^{m-1}$; technically two tables are enough.

**Extended Shortest Paths**($L, W$)
1. $n = \text{rows}[L]$;
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4. for $j = 1$ to $n$
Chapter 25. All-pairs shortest paths

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For $L^1 = W$ and $m = 2, \ldots, n - 1$, compute table $L^m$ from table $L^{m-1}$; technically two tables are enough.

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1. $n = \text{rows}[L]$;
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4.     for $j = 1$ to $n$
5.         $L'[i,j] = \infty$  
         ($L'[i,j] = L[i,j]$ in case $w_{a,a} \neq 0$)
Chapter 25. All-pairs shortest paths

DP table filling algorithm:

For \( L^1 = W \) and \( m = 2, \ldots, n - 1 \), compute table \( L^m \) from table \( L^{m-1} \);

technically two tables are enough.

**Extended Shortest Paths** \((L,W)\)
1. \( n = \text{rows}[L]; \)
2. let \( L' \) be an \( n \times n \) table;
3. for \( i = 1 \) to \( n \)
4.   for \( j = 1 \) to \( n \)
5.     \( L'[i,j] = \infty \quad (L'[i,j] = L[i,j] \text{ in case } w_{a,a} \neq 0) \)
6.     for \( k = 1 \) to \( n \)
Chapter 25. All-pairs shortest paths

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For $L^1 = W$ and $m = 2, \ldots, n - 1$, compute table $L^m$ from table $L^{m-1}$; technically two tables are enough.

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1. $n = \text{rows}[L]$;
2. let $L'$ be an $n \times n$ table;
3. for $i = 1$ to $n$
4. for $j = 1$ to $n$
5. $L'[i, j] = \infty$ (in case $w_{a,a} \neq 0$)
6. for $k = 1$ to $n$
7. $L'[i, j] = \min\{ L'[i, j], L[i, k] + w[k, j] \}$
Chapter 25. All-pairs shortest paths

DP table filling algorithm:

For \( L^1 = W \) and \( m = 2, \ldots, n - 1 \), compute table \( L^m \) from table \( L^{m-1} \); technically two tables are enough.

Extended Shortest Paths \((L, W)\)

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2. let \( L' \) be an \( n \times n \) table;
3. for \( i = 1 \) to \( n \)
4. \hspace{1em} for \( j = 1 \) to \( n \)
5. \hspace{2em} \( L'[i, j] = \infty \) \( (L'[i, j] = L[i, j] \text{ in case } w_{a,a} \neq 0) \)
6. \hspace{1em} for \( k = 1 \) to \( n \)
7. \hspace{2em} \( L'[i, j] = \min\{L'[i, j], L[i, k] + w[k, j]\} \)
8. return \((L')\)
Chapter 25. All-pairs shortest paths

DP table filling algorithm:

For \( L^1 = W \) and \( m = 2, \ldots, n - 1 \), compute table \( L^m \) from table \( L^{m-1} \);
   technically two tables are enough.

**Extended Shortest Paths** \((L, W)\)
1. \( n = \text{rows}[L] \);
2. let \( L' \) be an \( n \times n \) table;
3. for \( i = 1 \) to \( n \)
4.   for \( j = 1 \) to \( n \)
5.     \( L'[i, j] = \infty \quad (L'[i, j] = L[i, j] \text{ in case } w_{a,a} \neq 0) \)
6.     for \( k = 1 \) to \( n \)
7.       \( L'[i, j] = \min\{L'[i, j], L[i, k] + w[k, j]\} \)
8.   return \((L')\)

Call **Extended Shortest Paths** for \( m = 2, 3, \ldots, n - 1 \)
Chapter 25. All-pairs shortest paths

DP table filling algorithm:

For $L^1 = W$ and $m = 2, \ldots, n - 1$, compute table $L^m$ from table $L^{m-1}$; technically two tables are enough.

**EXTENDED SHORTEST PATHS**$(L, W)$

1. $n = \text{rows}[L]$;
2. let $L'$ be an $n \times n$ table;
3. for $i = 1$ to $n$
4. \hspace{1em} for $j = 1$ to $n$
5. \hspace{2em} $L'[i, j] = \infty$ $\quad (L'[i, j] = L[i, j]$ in case $w_{a,a} \neq 0)$
6. \hspace{1em} for $k = 1$ to $n$
7. \hspace{2em} $L'[i, j] = \min\{L'[i, j], L[i, k] + w[k, j]\}$
8. return $(L')$

Call **EXTENDED SHORTEST PATHS** for $m = 2, 3, \ldots, n - 1$

\[ L^m \leftarrow \text{EXTENDED SHORTEST PATHS}(L^{m-1}, W) \]
Chapter 25. All-pairs shortest paths

Running on an example:
Chapter 25. All-pairs shortest paths

Running on an example:

\[ W = L^1 = \text{the first matrix.} \]

\[ l_{0,0}^2 = \min \begin{cases} 
  l_{0,0}^1 \\
  l_{0,0}^1 + l_{0,0}^1 \\
  l_{0,1}^1 + l_{1,0}^1 \\
  l_{0,2}^1 + l_{2,0}^1 
\end{cases} \]

\[ k = 0, \text{value} = 8 + 8 = 16 \]

\[ k = 1, \text{value} = 1 + 6 = 7 \]

\[ k = 2, \text{value} = 1 + 3 = 4^* \]
Chapter 25. All-pairs shortest paths

- running time: $\Theta(n^4)$.

- Faster All Pair Shortest Paths

1. $n = \text{rows}[W]$;
2. $L = W$;
3. $m = 1$;
4. while $m < n - 1$
   5. $L = \text{Extended Shortest Paths}(L, L)$
   6. $m = 2 \times m$
   7. return $(L)$
Chapter 25. All-pairs shortest paths

- running time: $\Theta(n^4)$.
- improving the running time by repeatedly squaring:

Let $2^k = n - 1$. Then $k = \lceil \log_2 (n - 1) \rceil$.

Faster All Pair Shortest Paths

1. $n =$ rows $[W]$;
2. $L = W$;
3. $m = 1$;
4. while $m < n - 1$
5. $L = \text{Extended Shortest Paths}(L, L)$
6. $m = 2 \times m$
7. return $(L)$
Chapter 25. All-pairs shortest paths

- running time: $\Theta(n^4)$.
- improving the running time by repeatedly squaring:
  compute: $L^1, L^2, L^4, \ldots, L^{2^k}$.

Let $2^k = n - 1$. Then $k = \lceil \log_2 (n - 1) \rceil$. Faster All Pair Shortest Paths (W):

1. $n =$ rows $[W];$
2. $L =$ $W;$
3. $m = 1;$
4. while $m < n - 1$
5. $L =$ Extended Shortest Paths ($L, L$)
6. $m = \times 2 m$
7. return ($L$)
Chapter 25. All-pairs shortest paths

- running time: $\Theta(n^4)$.
- improving the running time by repeatedly squaring:
  compute: $L^1, L^2, L^4, \ldots, L^{2^k}$.
  what is $k$: here?
Chapter 25. All-pairs shortest paths

- running time: $\Theta(n^4)$.
- improving the running time by repeatedly squaring:
  
  compute: $L^1, L^2, L^4, \ldots, L^{2^k}$.

  what is $k$ here? Let $2^k = n - 1$. Then $k = \lceil \log_2(n - 1) \rceil$. 

Chapter 25. All-pairs shortest paths

- running time: \( \Theta(n^4) \).
- improving the running time by repeatedly squaring:
  - compute: \( L^1, L^2, L^4, \ldots, L^{2^k} \).

  what is \( k \) here? Let \( 2^k = n - 1 \). Then \( k = \lceil \log_2(n - 1) \rceil \).

**Faster All Pair Shortest Paths** \((W)\)
Chapter 25. All-pairs shortest paths

- running time: $\Theta(n^4)$.
- improving the running time by repeatedly squaring:
  
  compute: $L^1, L^2, L^4, \cdots, L^{2^k}$.

  what is $k$ here? Let $2^k = n - 1$. Then $k = \lceil \log_2(n - 1) \rceil$.

Faster All Pair Shortest Paths($W$)
1. $n = rows[W]$;
Chapter 25. All-pairs shortest paths

- running time: $\Theta(n^4)$.
- improving the running time by repeatedly squaring:
  
  compute: $L^1, L^2, L^4, \ldots, L^{2^k}$.

  what is $k$ here? Let $2^k = n - 1$. Then $k = \lceil \log_2(n - 1) \rceil$.

Faster All Pair Shortest Paths($W$)
1. $n = \text{rows}[W]$;
2. $L = W$;
Chapter 25. All-pairs shortest paths

• running time: $\Theta(n^4)$.

• improving the running time by repeatedly squaring:

  compute: $L^1, L^2, L^4, \ldots, L^{2^k}$.

  what is $k$ here? Let $2^k = n - 1$. Then $k = \lceil \log_2(n - 1) \rceil$.

Faster All Pair Shortest Paths($W$)
1. $n = \text{rows}[W]$;
2. $L = W$;
3. $m = 1$;
Chapter 25. All-pairs shortest paths

- running time: $\Theta(n^4)$.
- improving the running time by repeatedly squaring:
  
  compute: $L^1, L^2, L^4, \ldots, L^{2^k}$.

  what is $k$ here? Let $2^k = n - 1$. Then $k = \lceil \log_2(n - 1) \rceil$.

**Faster All Pair Shortest Paths** ($W$)

1. $n = \text{rows}[W]$;
2. $L = W$;
3. $m = 1$;
4. \textbf{while} $m < n - 1$
Chapter 25. All-pairs shortest paths

- running time: $\Theta(n^4)$.
- improving the running time by repeatedly squaring:
  
  compute: $L^1, L^2, L^4, \ldots, L^{2^k}$.
  
  what is $k$ here? Let $2^k = n - 1$. Then $k = \lceil \log_2(n - 1) \rceil$.

Faster All Pair Shortest Paths($W$)

1. $n = \text{rows}[W]$;
2. $L = W$;
3. $m = 1$;
4. while $m < n - 1$
5. $L = \text{Extended Shortest Paths}(L, L)$
Chapter 25. All-pairs shortest paths

- running time: $\Theta(n^4)$.  
- improving the running time by repeatedly squaring:
  
  compute: $L^1, L^2, L^4, \ldots, L^{2^k}$.

  what is $k$ here? Let $2^k = n - 1$. Then $k = \lceil \log_2(n - 1) \rceil$.

Faster All Pair Shortest Paths($W$)
1. $n = \text{rows}[W]$;
2. $L = W$;
3. $m = 1$;
4. while $m < n - 1$
5. $L = \text{Extended Shortest Paths}(L, L)$
6. $m = 2 \times m$
• running time: $\Theta(n^4)$.

• improving the running time by repeatedly squaring:

  compute: $L^1, L^2, L^4, \cdots, L^{2^k}$.

  what is $k$ here? Let $2^k = n - 1$. Then $k = \lceil \log_2(n - 1) \rceil$.

Faster All Pair Shortest Paths (W)
1. $n = \text{rows}[W]$;
2. $L = W$;
3. $m = 1$;
4. while $m < n - 1$
5. $L = \text{Extended Shortest Paths}(L, L)$
6. $m = 2 \times m$
7. return $(L)$
Chapter 25. All-pairs shortest paths

Floyd-Warshall algorithm
Chapter 25. All-pairs shortest paths

**Floyd-Warshall algorithm**

Intermediate vertices on a path $v_i \leadsto v_j$: those other than $v_i$ and $v_j$. 

Define: $d(k)_{ij}$ to be the shortest path distance from $v_i$ to $v_j$ with no intermediate vertices of indexes higher than $k$.

Thus $d(k)_{ij} = \min\{d(k-1)_{ij}, d(k-1)_{ik} + d(k-1)_{kj}\}$ with base case: $d(0)_{ij} = w_{ij}$.

Floyd-Warshall ($W$)

1. $n = \text{rows}[W]$
2. $D(0) = W$
3. for $k = 1$ to $n$
4. for $i = 1$ to $n$
5. for $j = 1$ to $n$
6. $D(k)_{ij} = \min\{D(k-1)_{ij}, D(k-1)_{ik} + D(k-1)_{kj}\}$
7. return ($D(n)$)
Chapter 25. All-pairs shortest paths

Floyd-Warshall algorithm

Intermediate vertices on a path $v_i \leadsto v_j$: those other than $v_i$ and $v_j$.

Define: $d_{ij}^{(k)}$ to be the shortest path distance from $v_i$ to $v_j$
Floyd-Warshall algorithm

Intermediate vertices on a path $v_i \leadsto v_j$: those other than $v_i$ and $v_j$.

Define: $d_{ij}^{(k)}$ to be the shortest path distance from $v_i$ to $v_j$ with no intermediate vertices of indexes higher than $k$.  

Floyd-Warshall ($W$)

1. $n = \text{rows}[W]$
2. $D(0) = W$
3. for $k = 1$ to $n$
4.   for $i = 1$ to $n$
5.     for $j = 1$ to $n$
6.       $D(k)[i,j] = \min\{D(k-1)[i,j], D(k-1)[i,k] + D(k-1)[k,j]\}$
7. return ($D(n)$)
Chapter 25. All-pairs shortest paths

Floyd-Warshall algorithm

intermediate vertices on a path \( v_i \leadsto v_j \): those other than \( v_i \) and \( v_j \).

Define: \( d_{i,j}^{(k)} \) to be the shortest path distance from \( v_i \) to \( v_j \) with no intermediate vertices of indexes higher than \( k \). Thus

\[
d_{i,j}^{(k)} = \min(d_{i,j}^{(k-1)}, d_{i,k}^{(k-1)} + d_{k,j}^{(k-1)})
\]

with base case: \( d_{i,j}^{(0)} = w_{i,j} \).
Chapter 25. All-pairs shortest paths

Floyd-Warshall algorithm

Intermediate vertices on a path $v_i \rightsquigarrow v_j$: those other than $v_i$ and $v_j$.

Define: $d_{ij}^{(k)}$ to be the shortest path distance from $v_i$ to $v_j$ with no intermediate vertices of indexes higher than $k$. Thus

$$d_{ij}^{(k)} = \min(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)})$$

with base case: $d_{ij}^{(0)} = w_{ij}$.

Floyd-Warshall($W$)
Chapter 25. All-pairs shortest paths

Floyd-Warshall algorithm

intermediate vertices on a path \( v_i \leadsto v_j \): those other than \( v_i \) and \( v_j \).

Define: \( d_{ij}^{(k)} \) to be the shortest path distance from \( v_i \) to \( v_j \) with no intermediate vertices of indexes higher than \( k \). Thus

\[
d_{ij}^{(k)} = \min(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)})
\]

with base case: \( d_{ij}^{(0)} = w_{ij} \).

Floyd-Warshall(\( W \))

1. \( n = \text{rows}[W] \)
Chapter 25. All-pairs shortest paths

Floyd-Warshall algorithm

intermediate vertices on a path $v_i \leadsto v_j$: those other than $v_i$ and $v_j$.

Define: $d_{ij}^{(k)}$ to be the shortest path distance from $v_i$ to $v_j$ with no intermediate vertices of indexes higher than $k$. Thus

$$d_{ij}^{(k)} = \min(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)})$$

with base case: $d_{ij}^{(0)} = w_{ij}$.

Floyd-Warshall($W$)
1. $n = rows[W]$
2. $D^{(0)} = W$
Chapter 25. All-pairs shortest paths

Floyd-Warshall algorithm

Intermediate vertices on a path $v_i \leadsto v_j$: those other than $v_i$ and $v_j$.

Define: $d_{ij}^{(k)}$ to be the shortest path distance from $v_i$ to $v_j$ with no intermediate vertices of indexes higher than $k$. Thus

$$d_{ij}^{(k)} = \min(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)})$$

with base case: $d_{ij}^{(0)} = w_{ij}$.

Floyd-Warshall($W$)
1. $n = \text{rows}[W]$
2. $D^{(0)} = W$
3. for $k = 1$ to $n$
Chapter 25. All-pairs shortest paths

Floyd-Warshall algorithm

intermediate vertices on a path $v_i \leadsto v_j$: those other than $v_i$ and $v_j$.

Define: $d_{ij}^{(k)}$ to be the shortest path distance from $v_i$ to $v_j$ with no intermediate vertices of indexes higher than $k$. Thus

$$d_{ij}^{(k)} = \min(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)})$$

with base case: $d_{ij}^{(0)} = w_{ij}$.

\begin{algorithm}
\textsc{Floyd-Warshall}(W)
\begin{enumerate}
\item $n = \text{rows}[W]$
\item $D^{(0)} = W$
\item \textbf{for} $k = 1$ \textbf{to} $n$
\item \hspace{1em} \textbf{for} $i = 1$ \textbf{to} $n$
\end{enumerate}
\end{algorithm}
Chapter 25. All-pairs shortest paths

Floyd-Warshall algorithm

Intermediate vertices on a path $v_i \rightsquigarrow v_j$: those other than $v_i$ and $v_j$.

Define: $d_{ij}^{(k)}$ to be the shortest path distance from $v_i$ to $v_j$ with no intermediate vertices of indexes higher than $k$. Thus

$$d_{ij}^{(k)} = \min(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)})$$

with base case: $d_{ij}^{(0)} = w_{ij}$.

Floyd-Warshall($W$)

1. $n = \text{rows}[W]$
2. $D^{(0)} = W$
3. for $k = 1$ to $n$
4. for $i = 1$ to $n$
5. for $j = 1$ to $n$
Chapter 25. All-pairs shortest paths

Floyd-Warshall algorithm

intermediate vertices on a path $v_i \leadsto v_j$: those other than $v_i$ and $v_j$.

Define: $d_{ij}^{(k)}$ to be the shortest path distance from $v_i$ to $v_j$ with no intermediate vertices of indexes higher than $k$. Thus

$$d_{ij}^{(k)} = \min(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)})$$

with base case: $d_{ij}^{(0)} = w_{ij}$.

Floyd-Warshall($W$)

1. $n = \text{rows}[W]$  
2. $D^{(0)} = W$  
3. for $k = 1$ to $n$  
4. for $i = 1$ to $n$  
5. for $j = 1$ to $n$  
6. $D^{(k)}[i,j] = \min\{D^{(k-1)}[i,j], D^{(k-1)}[i,k] + D^{(k-1)}[k,j]\}$
Chapter 25. All-pairs shortest paths

Floyd-Warshall algorithm

intermediate vertices on a path $v_i \rightsquigarrow v_j$: those other than $v_i$ and $v_j$.

Define: $d_{ij}^{(k)}$ to be the shortest path distance from $v_i$ to $v_j$ with no intermediate vertices of indexes higher than $k$. Thus

$$d_{ij}^{(k)} = \min(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)})$$

with base case: $d_{ij}^{(0)} = w_{ij}$.

**Floyd-Warshall**($W$)
1. $n = \text{rows}[W]$
2. $D^{(0)} = W$
3. for $k = 1$ to $n$
4. for $i = 1$ to $n$
5. for $j = 1$ to $n$
6. $D^{(k)}[i, j] = \min\{D^{(k-1)}[i, j], D^{(k-1)}[i, k] + D^{(k-1)}[k, j]\}$
7. return $(D^{(n)})$
### Chapter 25. All-pairs shortest paths

\[
D^{(0)} = \begin{pmatrix}
0 & 3 & 8 & \infty & -4 \\
\infty & 0 & \infty & 1 & 7 \\
\infty & 4 & 0 & \infty & \infty \\
2 & \infty & -5 & 0 & \infty \\
\infty & \infty & \infty & 6 & 0 \\
\end{pmatrix} \quad \Pi^{(0)} = \begin{pmatrix}
\text{NIL} & 1 & 1 & \text{NIL} & 1 \\
\text{NIL} & \text{NIL} & \text{NIL} & 2 & 2 \\
\text{NIL} & 3 & \text{NIL} & \text{NIL} & \text{NIL} \\
4 & \text{NIL} & 4 & \text{NIL} & \text{NIL} \\
\text{NIL} & \text{NIL} & \text{NIL} & 5 & \text{NIL} \\
\end{pmatrix}
\]

\[
D^{(1)} = \begin{pmatrix}
0 & 3 & 8 & \infty & -4 \\
\infty & 0 & \infty & 1 & 7 \\
\infty & 4 & 0 & \infty & \infty \\
2 & 5 & -5 & 0 & -2 \\
\infty & \infty & \infty & 6 & 0 \\
\end{pmatrix} \quad \Pi^{(1)} = \begin{pmatrix}
\text{NIL} & 1 & 1 & \text{NIL} & 1 \\
\text{NIL} & \text{NIL} & \text{NIL} & 2 & 2 \\
\text{NIL} & 3 & \text{NIL} & \text{NIL} & \text{NIL} \\
4 & 1 & 4 & \text{NIL} & 1 \\
\text{NIL} & \text{NIL} & \text{NIL} & 5 & \text{NIL} \\
\end{pmatrix}
\]

\[
D^{(2)} = \begin{pmatrix}
0 & 3 & 8 & 4 & -4 \\
\infty & 0 & \infty & 1 & 7 \\
\infty & 4 & 0 & 5 & 11 \\
2 & 5 & -5 & 0 & -2 \\
\infty & \infty & \infty & 6 & 0 \\
\end{pmatrix} \quad \Pi^{(2)} = \begin{pmatrix}
\text{NIL} & 1 & 1 & 2 & 1 \\
\text{NIL} & \text{NIL} & \text{NIL} & 2 & 2 \\
\text{NIL} & 3 & \text{NIL} & 2 & 2 \\
4 & 1 & 4 & \text{NIL} & 1 \\
\text{NIL} & \text{NIL} & \text{NIL} & 5 & \text{NIL} \\
\end{pmatrix}
\]

\[
D^{(3)} = \begin{pmatrix}
0 & 3 & 8 & 4 & -4 \\
\infty & 0 & \infty & 1 & 7 \\
\infty & 4 & 0 & 5 & 11 \\
2 & 5 & -5 & 0 & -2 \\
\infty & \infty & \infty & 6 & 0 \\
\end{pmatrix} \quad \Pi^{(3)} = \begin{pmatrix}
\text{NIL} & 1 & 1 & 2 & 1 \\
\text{NIL} & \text{NIL} & \text{NIL} & 2 & 2 \\
\text{NIL} & 3 & \text{NIL} & 2 & 2 \\
4 & 3 & 4 & \text{NIL} & 1 \\
\text{NIL} & \text{NIL} & \text{NIL} & 5 & \text{NIL} \\
\end{pmatrix}
\]

\[
D^{(4)} = \begin{pmatrix}
0 & 3 & 1 & 4 & -4 \\
2 & 0 & -4 & 1 & -1 \\
7 & 4 & 0 & 5 & 3 \\
2 & -1 & -5 & 0 & -2 \\
8 & 5 & 1 & 6 & 0 \\
\end{pmatrix} \quad \Pi^{(4)} = \begin{pmatrix}
\text{NIL} & 1 & 4 & 2 & 1 \\
4 & \text{NIL} & 4 & 2 & 1 \\
4 & 3 & \text{NIL} & 2 & 1 \\
4 & 3 & 4 & \text{NIL} & 1 \\
4 & 3 & 4 & 5 & \text{NIL} \\
\end{pmatrix}
\]

\[
D^{(5)} = \begin{pmatrix}
0 & 1 & 3 & 2 & -4 \\
3 & 0 & -4 & 1 & -1 \\
7 & 4 & 0 & 5 & 3 \\
2 & -1 & -5 & 0 & -2 \\
8 & 5 & 1 & 6 & 0 \\
\end{pmatrix} \quad \Pi^{(5)} = \begin{pmatrix}
\text{NIL} & 3 & 4 & 5 & 1 \\
4 & \text{NIL} & 4 & 2 & 1 \\
4 & 3 & \text{NIL} & 2 & 1 \\
4 & 3 & 4 & \text{NIL} & 1 \\
4 & 3 & 4 & 5 & \text{NIL} \\
\end{pmatrix}
\]
Chapter 25. All-pairs shortest paths

- Constructing a shortest path

• Constructing a shortest path
Chapter 25. All-pairs shortest paths

- Constructing a shortest path
- for each $v_i$ and each $v_j$, to remember the last step to reach $j$. 

\[
\pi(0)_{ij} = \text{NULL} \quad \text{if} \quad i = j \text{ or } w_{ij} = \infty, \quad \text{or} \quad \\
\pi(0)_{ij} = i \quad \text{if} \quad i \neq j \text{ and } w_{ij} < \infty.
\]

\[
\pi(k)_{ij} = \pi(k-1)_{ij} \quad \text{if} \quad d(k-1)_{ij} \leq d(k-1)_{ik} + d(k-1)_{kj}, \quad \text{or} \quad \\
\pi(k)_{ij} = \pi(k-1)_{kj} \quad \text{if} \quad d(k-1)_{ij} > d(k-1)_{ik} + d(k-1)_{kj}.
\]
Chapter 25. All-pairs shortest paths

- Constructing a shortest path
- for each \( v_i \) and each \( v_j \), to remember the last step to reach \( j \).
  predecessor matrix \( \pi \), recursively defined as

\[
\pi(0)_{ij} = \text{null} \quad \text{if} \quad i = j \quad \text{or} \quad w_{ij} = \infty,
\]
\[
\pi(k)_{ij} = \pi(k-1)_{ij} \quad \text{if} \quad d(k-1)_{ij} \leq d(k-1)_{ik} + d(k-1)_{kj},
\]
\[
\pi(k)_{ij} = \pi(k-1)_{kj} \quad \text{if} \quad d(k-1)_{ij} > d(k-1)_{ik} + d(k-1)_{kj}.
\]
Chapter 25. All-pairs shortest paths

- Constructing a shortest path
- for each \( v_i \) and each \( v_j \), to remember the last step to reach \( j \).

predecessor matrix \( \pi \), recursively defined as

\[
\pi_{ij}^{(0)} = \text{NULL} \quad \text{if} \quad i = j \quad \text{or} \quad w_{ij} = \infty, \quad \text{or}
\]

\[
\pi_{ij}^{(k)} = \pi_{ij}^{(k-1)} \quad \text{if} \quad d_{i,j}^{(k-1)} \leq d_{i,k}^{(k-1)} + d_{k,j}^{(k-1)}, \quad \text{or}
\]

\[
\pi_{ij}^{(k)} = \pi_{kj}^{(k-1)} \quad \text{if} \quad d_{i,j}^{(k-1)} > d_{i,k}^{(k-1)} + d_{k,j}^{(k-1)}.
\]
Chapter 25. All-pairs shortest paths

- Constructing a shortest path
- for each \(v_i\) and each \(v_j\), to remember the last step to reach \(j\).
  predecessor matrix \(\pi\), recursively defined as

\[
\pi_{ij}^{(0)} = \begin{cases} 
  NULL & \text{if } i = j \text{ or } w_{ij} = \infty, \\
  i & \text{if } i \neq j \text{ and } w_{ij} < \infty.
\end{cases}
\]
Chapter 25. All-pairs shortest paths

- Constructing a shortest path
- for each \( v_i \) and each \( v_j \), to remember the last step to reach \( j \).

predecessor matrix \( \pi \), recursively defined as

\[
\pi_{ij}^{(0)} = \text{NULL} \text{ if } i = j \text{ or } w_{ij} = \infty, \text{ or }
\]

\[
\pi_{ij}^{(0)} = i \text{ if } i \neq j \text{ and } w_{ij} < \infty.
\]

\[
\pi_{ij}^{(k)} = \pi_{ij}^{(k-1)} \text{ if } d_{ij}^{(k-1)} \leq d_{ik}^{(k-1)} + d_{kj}^{(k-1)}, \text{ or }
\]
Chapter 25. All-pairs shortest paths

• Constructing a shortest path

• for each $v_i$ and each $v_j$, to remember the last step to reach $j$.

predecessor matrix $\pi$, recursively defined as

$$\pi^{(0)}_{ij} = NULL \text{ if } i = j \text{ or } w_{ij} = \infty, \text{ or}$$

$$\pi^{(0)}_{ij} = i \text{ if } i \neq j \text{ and } w_{ij} < \infty.$$

$$\pi^{(k)}_{ij} = \pi^{(k-1)}_{ij} \text{ if } d^{(k-1)}_{ij} \leq d^{(k-1)}_{ik} + d^{(k-1)}_{kj}, \text{ or}$$

$$\pi^{(k)}_{ij} = \pi^{(k-1)}_{kj} \text{ if } d^{(k-1)}_{ij} > d^{(k-1)}_{ik} + d^{(k-1)}_{kj}$$
Chapter 25. All-pairs shortest paths

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