Part VI. Graph Algorithms
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- Chapter 22 Elementary graph algorithms
- Chapter 23. Minimum spanning trees
- Chapter 24. Single-source shortest paths
- Chapter 25. All-pairs shortest paths
Chapter 22. Elementary graph algorithms

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- Representations of graphs
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- Traverse graphs:
  - breadth-first-search (BFS)
  - depth-first-search (DFS)
- Applications:
  - topological sort
  - strongly connected components
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- traverse graphs:
  - (1) breadth-first-search (BFS);
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Chapter 22. Elementary graph algorithms

Graph: \( G = (V, E) \)
Chapter 22. Elementary graph algorithms

Terminologies and notations:

• graph $G = (V,E)$, where $V = \{v_1, \ldots, v_n\}$ and $E \subseteq V \times V$

$V = \{1, 2, 3, 4, 5, 6, 7\}$

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• weight $w: E \rightarrow \mathbb{R}$, e.g., $w(1, 2) = 4$, $w(5, 6) = 3$, etc.

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• path: there is a path $a \leadsto b$, if $(v_1, v_2), \ldots, (v_{k-1}, v_k) \in E$ and $v_1 = a$ and $v_k = b$. The path is a simple path if $v_1, \ldots, v_k$ are all different.

• cycle: when $v_1 = v_k$. It is a self-loop, if when $k = 1$ and $(v_1, v_k) \in E$. 
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Chapter 22. Elementary graph algorithms

- **digraphs**: directed graphs

- **complete graphs**: $K_n$, e.g., $K_6$

- **bipartite graphs**: $G = (V_1 \cup V_2, E)$, $V_1 \cap V_2 = \emptyset$, $K_3, 3$

- **planar graphs**: embedded in the plane without crossing edges: However, $K_5$ is not planar, neither is $K_{3,3}$
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![Diagram of a digraph](image-url)
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  1-tree is tree;

  2-tree is a graph but with **tree width** = 2
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Representations of graphs

adjacency-matrix
adjacency-list
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Representations of graphs

adjacency-matrix
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adjacency-matrix for a weighted graph
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Traverse graphs

basic ideas of depth-first-search (DFS) and breadth-first-search (BFS)

Both methods yield "search trees"
  or "search forest" (if the graph is not connected)
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DFS on directed graphs, search tree
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DFS on directed graphs, search tree

\[
\begin{align*}
\text{Directed graph:} & \quad \begin{array}{c}
A \\
B \\
C \\
D \\
E \\
F \\
G \\
H
\end{array} \\
\text{DFS search tree:} & \quad \begin{array}{c}
A \\
B \\
C
\end{array}
\end{align*}
\]
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DFS on directed graphs, search tree

DFS on non-directed graphs, search tree
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Traversal on graphs is an important task:

- Navigating the whole graph;
- For connectivity check;
- For circle check;
- Etc.

DFS and BFS are two fundamental algorithms for graph traversal!
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First recursive DFS algorithm, assuming $G$ is connected.
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First recursive DFS algorithm, assuming $G$ is connected.

`Recursive-DFS(G, u);`

How does the algorithm start?

- initially set $u.visit = false$ for every vertex $u \in G.V$
- $s.\pi = \text{NULL}$ for some specific $s \in G.V$
- call `Recursive-DFS(G, s)`

But if $G$ is not connected, what should we do?
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Recursively-DFS($G$, $u$);
1. if not $u$.visit

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**RECURSIVE-DFS**$(G, u);$  
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**To-Start-DFS**\((G)\)
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**To-Start-DFS**($G$)
1. **for** each $s \in G.V$  
   \begin{itemize}
   \item {initialize visit values}
   \end{itemize}
2. $s.visit = \text{false}$;
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To-Start-DFS($G$)
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Chapter 22. Elementary graph algorithms

**TO-START-DFS(G)**
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2. \hfill $s.visit = \text{false}$;
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3. \textbf{for} each $v \in \text{Adj}[u]$ and \textbf{not} $v.visit$;
4. \hfill $v.\pi = u$; \{ set $v$’s parent to be $u$ \}
5. \hfill \text{RECURSIVE-DFS}(G, v);
6. \textbf{return} ( );
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DFS (from the textbook) computes discover and finish time stamps ($u.d$ and $u.f$) for every visited vertex $u$.

\begin{verbatim}
DFS(G)
1   for each vertex $u \in G.V$
2       $u.color = \text{WHITE}$
3       $u.\pi = \text{NIL}$
4   \textbf{time} = 0
5   for each vertex $u \in G.V$
6       if $u.color == \text{WHITE}$
7           DFS-VISIT($G, u$)

DFS-VISIT($G, u$)
1   $\textbf{time} = \textbf{time} + 1$   // white vertex $u$ has just been discovered
2   $u.d = \textbf{time}$
3   $u.color = \text{GRAY}$
4   for each $v \in G.Adj[u]$   // explore edge $(u, v)$
5       if $v.color == \text{WHITE}$
6           $v.\pi = u$
7           DFS-VISIT($G, v$)
8   $u.color = \text{BLACK}$   // blacken $u$; it is finished
9   $\textbf{time} = \textbf{time} + 1$
10  $u.f = \textbf{time}$
\end{verbatim}
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DFS-\textsc{Visit}(G, u)
1. \textit{time} = \textit{time} + 1 \quad \text{ // white vertex } u \text{ has just been discovered}
2. \textit{u.d} = \textit{time}
3. \textit{u.color} = \text{GRAY}
4. \textbf{for} each \( v \in G.\text{Adj}[u] \) \quad \text{ // explore edge } (u, v)
5. \quad \textbf{if} \ v.\text{color} = \text{WHITEx}
6. \quad \quad v.\pi = u
7. \quad \text{DFS-\textsc{Visit}}(G, v)
8. \quad \textit{u.color} = \text{BLACK} \quad \text{ // blacken } u; \text{ it is finished}
9. \quad \textit{time} = \textit{time} + 1
10. \quad \textit{u.f} = \textit{time}

→: edge being explored;
→: edge path taken by DFS
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DFS-VISIT(G, u)
1 \(time = time + 1\) \hspace{1cm} \text{// white vertex } u \text{ has just been discovered}
2 \(u.d = time\)
3 \(u.color = \text{GRAY}\)
4 \textbf{for each } v \in G.\text{Adj}[u] \hspace{1cm} \text{// explore edge } (u, v)
5 \hspace{1cm} \textbf{if } v.color == \text{WHITE}
6 \hspace{2cm} v.\pi = u
7 \hspace{3cm} \text{DFS-VISIT}(G, v)
8 \hspace{1cm} u.color = \text{BLACK} \hspace{1cm} \text{// blacken } u; \text{ it is finished}
9 \hspace{1cm} time = time + 1
10 \hspace{1cm} u.f = time

\rightarrow: \text{edge being explored;}
\rightarrow: \text{edge path taken by DFS}
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DFS-VISIT(G, u)

1. time = time + 1          // white vertex u has just been discovered
2. u.d = time
3. u.color = GRAY
4. for each v ∈ G.Adj[u]     // explore edge (u, v)
   5. if v.color == WHITE
      6. v.π = u
      7. DFS-VISIT(G, v)
6. u.color = BLACK           // blacken u; it is finished
7. time = time + 1
8. u.f = time

→: edge being explored;
→: edge path taken by DFS
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DFS-Visit \((G, u)\)

1. \(time = time + 1\)  // white vertex \(u\) has just been discovered
2. \(u.d = time\)
3. \(u.color = GRAY\)
4. for each \(v \in G.Adj[u]\)  // explore edge \((u, v)\)
5. \(\text{if } v.color == WHITE\)
6. \(\quad v.\pi = u\)
7. \(\quad \text{DFS-Visit}(G, v)\)
8. \(u.color = BLACK\)  // blacken \(u\); it is finished
9. \(time = time + 1\)
10. \(u.f = time\)

→: edge being explored;
→: edge path taken by DFS
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Chapter 22. Elementary graph algorithms

Another example of DFS execution (page 605)
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Time complexity of DFS algorithm
Time complexity of DFS algorithm

\[
\Theta(|E| + |V|), \text{ where } |E| \text{ is the number of edges in } G.
\]

DFS(G)
1. for each vertex \( u \in G.V \)
2. \( u.color = \text{WHITE} \)
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4. \( time = 0 \)
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6. if \( u.color == \text{WHITE} \)
7. \( \text{DFS-VISIT}(G, u) \)

DFS-VISIT(G, u)
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Time complexity of DFS algorithm

DFS(G)
1 for each vertex u ∈ G.V
2 u.color = WHITE
3 u.π = NIL
4 time = 0
5 for each vertex u ∈ G.V
6 if u.color == WHITE
7 DFS-VISIT(G, u)

DFS-VISIT(G, u)
1 time = time + 1 // white vertex u has just been discovered
2 u.d = time
3 u.color = GRAY
4 for each v ∈ G.Adj[u] // explore edge (u, v)
5 if v.color == WHITE
6 v.π = u
7 DFS-VISIT(G, v)
8 u.color = BLACK // blacken u; it is finished
9 time = time + 1
10 u.f = time

Θ(|E| + |V|), where |E| is the number of edges in G.
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Properties of depth-first-search:

1. $u = v.\pi$ if DFS-Visit($G,v$) is called.

Theorem 22.7 (Parenthesis Theorem): for any $u,v$, exactly one of the following three conditions holds:

- $[u.d,u.f]$ and $[v.d,v.f]$ are entirely disjoint, and neither $u$ nor $v$ is a descendant of the other in the search tree.
- $[u.d,u.f]$ is contained entirely within $[v.d,v.f]$ and $u$ is a descendant of $v$.
- $[v.d,v.f]$ is contained entirely within $[u.d,u.f]$ and $v$ is a descendant of $u$.

Corollary 22.8 (Nesting of descendants’ intervals) Vertex $v$ is a proper descendant of $u$ in the depth-first search forest if and only if $u.d < v.d < v.f < u.f$. 
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Properties of depth-first-search:

(1) \( u = v.\pi \)
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Properties of depth-first-search:

(1) \( u = v.\pi \) iff DFS-\textsc{Visit}(G, v) is called.
Properties of depth-first-search:

(1) \( u = v.\pi \) iff DFS-Visit\((G, v)\) is called.

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Properties of depth-first-search:

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2. **Theorem 22.7 (Parenthesis Theorem):** for any \( u, v \), exactly one of the following three conditions holds:
   - \([u.d, u.f]\) and \([v.d, v.f]\) are entirely disjoint, and neither \( u \) nor \( v \) is a descendant of the other in the search tree.
   - \([u.d, u.f]\) is contained entirely within \([v.d, v.f]\) and \( u \) is a descendant of \( v \), or
   - \([v.d, v.f]\) is contained entirely within \([u.d, u.f]\) and \( v \) is a descendant of \( u \).

**Corollary 22.8 (Nesting of descendants' intervals):** Vertex \( v \) is a proper descendant of \( u \) in the depth-first search forest if and only if \( u.d < v.d < v.f < u.f \).
Properties of depth-first-search:

(1) $u = v.\pi$ iff $\text{DFS-Visit}(G, v)$ is called.

(2) **Theorem 22.7 (Parenthesis Theorem):** for any $u, v$, exactly one of the following three conditions holds:

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Properties of depth-first-search:

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(2) **Theorem 22.7 (Parenthesis Theorem):** for any $u, v$, exactly one of the following three conditions holds:

- $[u.d, u.f]$ and $[v.d, v.f]$ are entirely disjoint, and neither $u$ nor $v$ is a descendant of the other in the search tree.
- $[u.d, u.f]$ is contained entirely within $[v.d, v.f]$ and $u$ is a descendant of $v$, or
- $[v.d, v.f]$ is contained entirely within $[u.d, u.f]$ and $v$ is a descendant of $u$. 

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Properties of depth-first-search:

(1) \( u = v.\pi \) iff DFS-\textsc{Visit}(G,v) is called.

(2) \textbf{Theorem 22.7 (Parenthesis Theorem)}: for any \( u, v \), exactly one of the following three conditions holds:

- \([u.d, u.f]\) and \([v.d, v.f]\) are entirely disjoint, and neither \( u \) nor \( v \) is a descendant of the other in the search tree.

- \([u.d, u.f]\) is contained entirely within \([v.d, v.f]\) and \( u \) is a descendant of \( v \), or

- \([v.d, v.f]\) is contained entirely within \([u.d, u.f]\) and \( v \) is a descendant of \( u \).

\textbf{Corollary 22.8 (Nesting of descendants' intervals)} Vertex \( v \) is a proper descendant of \( u \) in the depth-first search forest if and only if \( u.d < v.d < v.f < u.f \).
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**Chapter 22. Elementary graph algorithms**

**Theorem 22.9 (White-path Theorem)** In a depth-first search forest of graph $G$, vertex $v$ is a descendant of $u$ if and only if at the time $u.d$ that the search discovers $u$, there is a path from $u$ to $v$ consisting of entirely of white vertices.

**Proof:**

$\Rightarrow$

- **Case 1:** $u = v$, apparently the claim is true;
- **Case 2:** $v$ is a proper descendant of $u$, use Corollary 22.8 on every vertex on the path from $u$ to $v$; the claim is true;

$\Leftarrow$

Assume that at the time $u.d$, there is a path from $u$ to $v$ as stated in the theorem but $v$ is not descendant of $u$. Assume $(w,x)$ be an edge on the path and $x$ is the first vertex on the path which is not descendant of $u$. Note that $w$ is a descendant of $u$ (or just $u$).

Because at time $u.d$, $x$ is of WHITE color, $u.d < x.d$.

Because $(w,x)$ is an edge, there are two possible scenarios:

1. When $(w,x)$ is being explored, $x$ has already been discovered; we thus have $x.d < w.f$;
2. When $(w,x)$ is being explored, $x$ has WHITE color but will then be discovered, we thus also have $x.d < w.f$;

According to Corollary 22.8, $u.d < x.d < w.f < u.f$.

Thus, $u.d < x.d < u.f$.

By Theorem 22.7, the interval $[x.d, x.f]$ is entirely contained within interval $[u.d, u.f]$. Therefore, $x$ is a descendant of $u$. Contradicts to the earlier assumption. $v$ should be a descendant of $u$. 

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Theorem 22.9 (White-path Theorem) In a depth-first search forest of graph $G$, vertex $v$ is a descendant of $u$ if and only if at the time $u.d$ that the search discovers $u$, there is a path from $u$ to $v$ consisting of entirely of white vertices.

Proof: $\Rightarrow$
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**Proof:** $\Rightarrow$
- case 1: $u = v$, apparently the claim is true;
- case 2: $v$ is a proper descendant of $u$, use Corollary 22.8 on every vertex on the path from $u$ to $v$; the claim is true;

$\Leftarrow$
Assume that at the time $u.d$, there is a path from $u$ to $v$ as stated in the theorem but $v$ is not descendant of $u$.

Assume $(w,x)$ be an edge on the path and $x$ is the first vertex on the path which is not descendant of $u$.

Note that $w$ is a descendant of $u$ (or just $u$).

Because at time $u.d$ $x$ is of WHITE color, $u.d < x.d$.

Because $(w,x)$ is an edge, there are two possible scenarios:
1. when $(w,x)$ is being explored, $x$ has already been discovered; we thus have $x.d < w.f$;
2. when $(w,x)$ is being explored, $x$ has WHITE color but will then be discovered; we thus also have $x.d < w.f$;

According to Corollary 22.8, $u.d < x.d < w.f < u.f$.

Thus, $u.d < x.d < u.f$.

By Theorem 22.7, the interval $[x.d, x.f]$ is entirely contained within interval $[u.d, u.f]$.

Therefore, $x$ is a descendant of $u$. Contradicts to the earlier assumption.

$v$ should be a descendant of $u$. 
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Theorem 22.9 (White-path Theorem) In a depth-first search forest of graph $G$, vertex $v$ is a descendant of $u$ if and only if at the time $u.d$ that the search discovers $u$, there is a path from $u$ to $v$ consisting of entirely of white vertices.

Proof: $\Rightarrow$

- case 1: $u = v$, apparently the claim is true;
- case 2: $v$ is a proper descendant of $u$, use Corollary 22.8 on every vertex on the path from $u$ to $v$; the claim is true;

$\Leftarrow$
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$\Leftarrow$

Assume that at the time $u.d$, there is a path from $u$ to $v$ as stated in the theorem but $v$ is not descendant of $u$. 
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**Theorem 22.9 (White-path Theorem)** In a depth-first search forest of graph $G$, vertex $v$ is a descendant of $u$ if and only if at the time $u.d$ that the search discovers $u$, there is a path from $u$ to $v$ consisting of entirely of white vertices.

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- case 1: $u = v$, apparently the claim is true;
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$\Leftarrow$

Assume that at the time $u.d$, there is a path from $u$ to $v$ as stated in the theorem but $v$ is not descendant of $u$.

Assume $(w, x)$ be an edge on the path and $x$ is the first vertex on the path which is not descendant of $u$. 
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Proof: \(\Rightarrow\)
- case 1: $u = v$, apparently the claim is true;
- case 2: $v$ is a proper descendant of $u$, use Corollary 22.8 on every vertex on the path from $u$ to $v$; the claim is true;

\(\Leftarrow\)
- Assume that at the time $u.d$, there is a path from $u$ to $v$ as stated in the theorem but $v$ is not descendant of $u$.

Assume $(w, x)$ be an edge on the path and $x$ is the first vertex on the path which is not descendant of $u$. Note that $w$ is a descendant of $u$ (or just $u$)
Theorem 22.9 (White-path Theorem) In a depth-first search forest of graph $G$, vertex $v$ is a descendant of $u$ if and only if at the time $u.d$ that the search discovers $u$, there is a path from $u$ to $v$ consisting of entirely of white vertices.

Proof: $\Rightarrow$
- case 1: $u = v$, apparently the claim is true;
- case 2: $v$ is a proper descendant of $u$, use Corollary 22.8 on every vertex on the path from $u$ to $v$; the claim is true;

$\Leftarrow$
Assume that at the time $u.d$, there is a path from $u$ to $v$ as stated in the theorem but $v$ is not descendant of $u$.

Assume $(w, x)$ be an edge on the path and $x$ is the first vertex on the path which is not descendant of $u$. Note that $w$ is a descendant of $u$ (or just $u$). Because at time $u.d$ $x$ is of WHITE color, $u.d < x.d$. 
Theorem 22.9 (White-path Theorem) In a depth-first search forest of graph $G$, vertex $v$ is a descendant of $u$ if and only if at the time $u.d$ that the search discovers $u$, there is a path from $u$ to $v$ consisting of entirely of white vertices.

Proof: $\Rightarrow$
- case 1: $u = v$, apparently the claim is true;
- case 2: $v$ is a proper descendant of $u$, use Corollary 22.8 on every vertex on the path from $u$ to $v$; the claim is true;

$\Leftarrow$
Assume that at the time $u.d$, there is a path from $u$ to $v$ as stated in the theorem but $v$ is not descendant of $u$.

Assume $(w, x)$ be an edge on the path and $x$ is the first vertex on the path which is not descendant of $u$. Note that $w$ is a descendant of $u$ (or just $u$)
Because at time $u.d$ $x$ is of WHITE color, $u.d < x.d$.
Because $(w, x)$ is an edge, there are two possible scenarios:
- (1) when $(w, x)$ is being explored, $x$ has already been discovered;
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**Theorem 22.9 (White-path Theorem)** In a depth-first search forest of graph $G$, vertex $v$ is a descendant of $u$ if and only if at the time $u.d$ that the search discovers $u$, there is a path from $u$ to $v$ consisting of entirely of white vertices.

**Proof:** $\Rightarrow$

- case 1: $u = v$, apparently the claim is true;
- case 2: $v$ is a proper descendant of $u$, use **Corollary 22.8** on every vertex on the path from $u$ to $v$; the claim is true;

$\Leftarrow$

Assume that at the time $u.d$, there is a path from $u$ to $v$ as stated in the theorem but $v$ is not descendant of $u$.

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Chapter 22. Elementary graph algorithms

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- **case 1:** $u = v$, apparently the claim is true;
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$\Leftarrow$

Assume that at the time $u.d$, there is a path from $u$ to $v$ as stated in the theorem but $v$ is not descendant of $u$.

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**Theorem 22.9 (White-path Theorem)** In a depth-first search forest of graph \( G \), vertex \( v \) is a descendant of \( u \) if and only if at the time \( u.d \) that the search discovers \( u \), there is a path from \( u \) to \( v \) consisting of entirely of white vertices.

**Proof:** \( \Rightarrow \)
- case 1: \( u = v \), apparently the claim is true;
- case 2: \( v \) is a proper descendant of \( u \), use **Corollary 22.8** on every vertex on the path from \( u \) to \( v \); the claim is true;

\( \Leftarrow \)
Assume that at the time \( u.d \), there is a path from \( u \) to \( v \) as stated in the theorem but \( v \) is not descendant of \( u \).

Assume \((w, x)\) be an edge on the path and \( x \) is the first vertex on the path which is not descendant of \( u \). Note that \( w \) is a descendant of \( u \) (or just \( u \))

Because at time \( u.d \) \( x \) is of WHITE color, \( u.d < x.d \).

Because \((w, x)\) is an edge, there are two possible scenarios:
- (1) when \((w, x)\) is being explored, \( x \) has already been discovered;
  we thus have \( x.d < w.f \);
- (2) when \((w, x)\) is being explored, \( x \) has WHITE color but will then be discovered we thus also have \( x.d < w.f \);

According to **Corollary 22.8**, \( u.d < x.d < w.f < u.f \).
Chapter 22. Elementary graph algorithms

**Theorem 22.9 (White-path Theorem)** In a depth-first search forest of graph $G$, vertex $v$ is a descendant of $u$ if and only if at the time $u.d$ that the search discovers $u$, there is a path from $u$ to $v$ consisting of entirely of white vertices.

**Proof:** $\Rightarrow$

- case 1: $u = v$, apparently the claim is true;
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2. when $(w, x)$ is being explored, $x$ has WHITE color but will then be discovered we thus also have $x.d < w.f$;

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Chapter 22. Elementary graph algorithms

**Theorem 22.9 (White-path Theorem)** In a depth-first search forest of graph $G$, vertex $v$ is a descendant of $u$ if and only if at the time $u.d$ that the search discovers $u$, there is a path from $u$ to $v$ consisting of entirely of white vertices.

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Assume $(w, x)$ be an edge on the path and $x$ is the first vertex on the path which is not descendant of $u$. Note that $w$ is a descendant of $u$ (or just $u$).  
Because at time $u.d$ $x$ is of WHITE color, $u.d < x.d$.
Because $(w, x)$ is an edge, there are two possible scenarios:

- (1) when $(w, x)$ is being explored, $x$ has already been discovered; we thus have $x.d < w.f$;
- (2) when $(w, x)$ is being explored, $x$ has WHITE color but will then be discovered we thus also have $x.d < w.f$;

According to Corollary 22.8, $u.d < x.d < w.f < u.f$. Thus, $u.d < x.d < u.f$
By Theorem 22.7, the interval $[x.d, x.f]$ is entirely contained within interval $[u.d, u.f]$. 
Chapter 22. Elementary graph algorithms

**Theorem 22.9 (White-path Theorem)** In a depth-first search forest of graph $G$, vertex $v$ is a descendant of $u$ if and only if at the time $u.d$ that the search discovers $u$, there is a path from $u$ to $v$ consisting of entirely of white vertices.

**Proof:** $\Rightarrow$

- **case 1:** $u = v$, apparently the claim is true;
- **case 2:** $v$ is a proper descendant of $u$, use **Corollary 22.8** on every vertex on the path from $u$ to $v$; the claim is true;

$\Leftarrow$

Assume that at the time $u.d$, there is a path from $u$ to $v$ as stated in the theorem but $v$ is not descendant of $u$.

Assume $(w, x)$ be an edge on the path and $x$ is the first vertex on the path which is not descendant of $u$. Note that $w$ is a descendant of $u$ (or just $u$).

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According to **Corollary 22.8**, $u.d < x.d < w.f < u.f$. Thus, $u.d < x.d < u.f$

By **Theorem 22.7**, the interval $[x.d, x.f]$ is entirely contained within interval $[u.d, u.f]$. Therefore, $x$ is a descendant of $u$. Contradicts to the earlier assumption.
Chapter 22. Elementary graph algorithms

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Chapter 22. Elementary graph algorithms

Classification of edges (for directed graphs)
Chapter 22. Elementary graph algorithms

Classification of edges (for directed graphs)

- **tree edges**: those in the search tree (forest); $(u,v)$ is a tree edge if $v$ was discovered by exploring $(u,v)$;
- **back edges**: those connecting a vertex to an ancestor; a selfloop, in a directed graph, can be a back edge;
- **forward edges**: those connecting a vertex to a descendant;
- **cross edges**: all other edges;
Classification of edges (for directed graphs)

- tree edges: those in the search tree (forest);
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- forward edges: those connecting a vertex to a descendant;
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- DFS forest with back, forward, & cross edges.
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Chapter 22. Elementary graph algorithms

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Chapter 22. Elementary graph algorithms

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Chapter 22. Elementary graph algorithms

To identify the type of edge \((u,v)\) with the color of \(v\):
- WHITE: tree edge;
- GRAY: back edge;
- BLACK: forward or cross edge;

First stages of a Directed DFS, showing Edges, the DFS TREE, a Tree Edge, a Back Edge, a Forward Edge, and a Cross Edge.
Chapter 22. Elementary graph algorithms

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Chapter 22. Elementary graph algorithms

**Theorem 22.10** In a depth-first search of undirected graph $G$, every edge of $G$ is either tree edge or back edge.
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Theorem 22.10 In a depth-first search of undirected graph $G$, every edge of $G$ is either tree edge or back edge.

Proof. Let $(u, v)$ be an edge in $G$. Assume in the depth-first search of $G$, $u.d < v.d$. Then there are two scenarios:
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(1) $v$ is discovered by exploring edge $(u, v)$, then $(u, v)$ is a tree edge;
Chapter 22. Elementary graph algorithms

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1. $v$ is discovered by exploring edge $(u, v)$, then $(u, v)$ is a tree edge;
2. $v$ is discovered not through exploring edge $(u, v)$. 
Chapter 22. Elementary graph algorithms

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1. $v$ is discovered by exploring edge $(u, v)$, then $(u, v)$ is a tree edge;
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   Because $(u, v)$ is an edge, $v$ is discovered when $u$ is in gray color.
Theorem 22.10 In a depth-first search of undirected graph $G$, every edge of $G$ is either tree edge or back edge.

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Because $(u, v)$ is an edge, $v$ is discovered when $u$ is in gray color. Since $u$ is in the adjacency list of $v$, $(v, u)$ will eventually be explored and thus a back edge.
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Breadth First Search (BFS)
Chapter 22. Elementary graph algorithms

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Chapter 22. Elementary graph algorithms

Breadth First Search Algorithm (with a queue)

Time complexity of BFS: $O(|V| + |E|)$

Note: BFS can find a shortest path from $s$ to all other nodes (non-weighted). (Why?)
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Chapter 22. Elementary graph algorithms

Breadth First Search Algorithm (with a queue)

```
BFS(G, s)
1 for each vertex u ∈ G.V − {s}
2 u.color = WHITE
3 u.d = ∞
4 u.π = NIL
5 s.color = GRAY
6 s.d = 0
7 s.π = NIL
8 Q = ∅
9 ENQUEUE(Q, s)
10 while Q ≠ ∅
11 u = DEQUEUE(Q)
12 for each v ∈ G.Adj[u]
13 if v.color == WHITE
14 v.color = GRAY
15 v.d = u.d + 1
16 v.π = u
17 ENQUEUE(Q, v)
18 u.color = BLACK
```
Chapter 22. Elementary graph algorithms

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Time complexity of BFS:
Chapter 22. Elementary graph algorithms

Breadth First Search Algorithm (with a queue)

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2. \hspace{1em} $u.color = \text{WHITE}$
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8. \hspace{1em} $Q = \emptyset$
9. \hspace{1em} ENQUEUE($Q$, $s$)
10. while $Q \neq \emptyset$
11. \hspace{1em} $u = \text{DEQUEUE}(Q)$
12. \hspace{1em} for each $v \in G.Adj[u]$
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Chapter 22. Elementary graph algorithms

Applications

Reachability Problem

Input: \(G = (V, E)\), and \(s, t \in V\);

Output: YES if and only there is a path \(s \Rightarrow t\) in \(G\).

• The problem can be solved with DFS and BFS by search on the graph from \(s\) until \(t\) shows up.
Chapter 22. Elementary graph algorithms

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Chapter 22. Elementary graph algorithms

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Reachability Problem

Reachability \((G,u,t)\):

1. \(u.\text{visit} = \text{true}\);
2. for each \(v \in \text{Adj}[u]\) and not \(v.\text{visit}\);
3. if \(v = t\) then reachable = Yes; exit;
4. else \(v.\pi = u\);
5. Reachability \((G,v,t)\);
6. return ( ).

Main ()

reachable = No;
Reachability \((G,s,t)\);
print (reachable);
Reachability Problem

Reachability Problem
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5. \texttt{REACHABILITY}(G, v, t);
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\texttt{MAIN()}
\begin{verbatim}
    reachable = \texttt{No};
    \texttt{REACHABILITY}(G, s, t);
    \texttt{print}(reachable);
\end{verbatim}
Chapter 22. Elementary graph algorithms

Path Counting Problem

Input: \( G = (V, E) \), and \( s, t \in V \);

Output: the number of paths from \( s \Rightarrow t \) in \( G \).

- we modify Reachability to count paths.

PathCounting \( (G, u, t) \):

1. \( u.\text{visit} = \text{true} \);
2. for each \( v \in \text{Adj}[u] \);
3. if \( v.\text{visit} \) then \( u.c = u.c + v.c \);
4. else \( v.\pi = u \);
5. PathCounting \( (G, v, t) \);
6. \( u.c = u.c + v.c \);
7. return ( )

Main ()

1. for each \( u \in G \);
2. \( u.c = 0 \);
3. PathCounting \( (G, s, t) \);
4. print \( (s.c) \)
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Path Counting Problem
Path Counting Problem

**Input:** $G = (V, E)$, and $s, t \in V$;

**Output:** the number of paths from $s \rightarrow t$ in $G$. 

We modify Reachability to count paths.

PathCounting($G, u, t$);

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7. return $()$;

Main $()$;

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Chapter 22. Elementary graph algorithms

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```plaintext
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```plaintext
Main()
1. for each u ∈ G
2. u.c = 0;
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```
Chapter 22. Elementary graph algorithms

Topological sorting

• On directed acyclic graphs (DAGs)

A sorted order:
socks, shorts, pants, shoes, shirt, tie, belt, jacket, watch.
Chapter 22. Elementary graph algorithms

Topological sorting
Chapter 22. Elementary graph algorithms

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Chapter 22. Elementary graph algorithms

• apply DFS algorithm.
• reversed order of finish times: p,n,o,s,m,r,y,v,x,w,z,u,q,t

• Correctness proof?
Chapter 22. Elementary graph algorithms

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Chapter 22. Elementary graph algorithms

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Chapter 22. Elementary graph algorithms

Strongly connected components (SCC)

Let $G = (V, E)$ be a digraph. A strongly connected component is a maximal subgraph $H = (V_H, E_H)$ of $G$ such that for every two nodes $v, u \in V_H$,

1. there is a directed path $v \Rightarrow u$ consisting of edges in $E_H$; and
2. there is a directed path $u \Rightarrow v$ consisting of edges in $E_H$. 
Chapter 22. Elementary graph algorithms

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Strongly connected components (SCC)

Let $G = (V, E)$ be a digraph. A strongly connected component is a maximal subgraph $H = (V_H, E_H)$ of $G$ such that for every two nodes $v, u \in V_H$, there is a directed path $v \xrightarrow{} u$ consisting of edges in $E_H$ and there is a directed path $u \xrightarrow{} v$ consisting of edges in $E_H$. 
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Let $G = (V, E)$ be a digraph. A *strongly connected component* is a maximal subgraph $H = (V_H, E_H)$ of $G$ such that for every two nodes $v, u \in V_H$,

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Chapter 22. Elementary graph algorithms

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Chapter 22. Elementary graph algorithms

Idea of an algorithm to use DFS to solve SCC problem.

- use DFS to generate DFS forest;
  
  each search tree $T_u$ (rooted at $u$) consists of vertices $v$ such that $u \rightarrow v$;

- use DFS again on $T_u$;
  
  hope to search from every one $v$ within $T_u$ to make sure $v \rightarrow u$ as well.

- however, this may be difficult (proof is left as an exercise).
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• use DFS to generate DFS forest; each search tree $T_u$ (rooted at $u$) consists of vertices $v$ such that $u \xrightarrow{} v$;

• use DFS again on $T_u$; hope to search from every one $v$ within $T_u$ to make sure $v \xrightarrow{} u$ as well.

• however, this may be difficult (proof is left as an exercise).
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Algorithm

1. call DFS(G) to compute u.f for each u ∈ G.V
2. compute G\T the transpose of G {reverse all edges in G}
3. call DFS(G\T) (vertices are considered in the decreasing order of finish times computed in step 1)
4. output each tree in the depth-first forest produced by step 3.
Algorithm **Strongly Connected Components**($G$)

1. call DFS($G$) to compute $u.f$ for each $u \in G.V$
2. compute $G^T$: the transpose of $G$
   - reverse all edges in $G$
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• the first pass DFS results in a DFS forest, let $T$ be a tree with root $r$; for every vertex $u \in T$, $r \leadsto u$, so an SCC can only be produced from some tree in the forest;
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Ideas behind the algorithm:

• the first pass DFS results in a DFS forest, let $T$ be a tree with root $r$; for every vertex $u \in T$, $r \sim u$,
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• let vertex $v \in T$ but $v \neq r$, we are not sure $v \sim u$;

• that is the same as to use second-DFS (starting from $r$) to check if $r \sim v$ after edge directions are reversed;

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Properties from algorithm

Strongly Connected Components (G)

(1) Component graph: \( G_{SCC} = (V_{SCC}, E_{SCC}) \) is defined as follow:

- Let \( C_1, C_2, ..., C_k \) be \( k \) distinct SCCs for \( G \).
- Then \( V_{SCC} = \{ v_1, v_2, v_k \} \);
- \( E_{SCC} = \{ (v_i, v_j) : \exists u \in C_i, v \in C_j, (u, v) \in E \} \).

Then \( G_{SCC} \) is a DAG (directed acyclic graph).

Proof. Assume the opposite to the claim that, for some \( v_i, v_j \in V_{SCC} \), there is a path \( v_i \Rightarrow v_j \) and another path \( v_j \Rightarrow v_i \), forming a cycle in \( V_{SCC} \).

By the definition of \( G_{SCC} \), there must be a path in \( G \), from some vertex in \( C_i \) to some vertex in \( C_j \); at the same time, there is a path in \( G \), from some vertex in \( C_j \) to some vertex in \( C_i \). Then \( C_i \) and \( C_j \) should form a single SCC, not two distinct SCCs. Contradicts.
Properties from algorithm \textit{Strongly Connected Components}(G)
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Let $C$ be a SCC, define $f(C) = \max_{u \in C} \{u.f\}$, (with the finish times from the first DFS call).

Lemma 22.14: Let $C$ and $C'$ be distinct strongly connected components for directed graph $G$. If $(u,v) \in E$, where $u \in C$ and $v \in C'$, then $f(C) > f(C')$.

Proof: Assume the opposite, i.e., $f(C) < f(C')$. Then there must be vertices $x \in C$ and $y \in C'$ such that $x.f < y.f$.

Now consider the first DFS call, there are two situations:

(1) $y$ was searched before $x$: by property (1) there is no path from $y$ to $x$, $x.f > y.f$.

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[Proof text]

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The algorithm

Strongly Connected Components

(G)
correctly computes the strongly connected components for a directed graph G.

We need to prove two statements:

(1) If v⇝u and u⇝v in G, then u and v belong to the same component C produced by the algorithm.

(2) If u,v ∈ C, then we have v⇝u and u⇝v in G.
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(3) The algorithm \texttt{Strongly Connected Components}(G) correctly computes the strongly connected components for a directed graph \( G \).
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We need to prove two statements:

(1) If $v \leadsto u$ and $u \leadsto v$ in $G$, then $u$ and $v$ belong to the same component $C$ produced by the algorithm.

(2) If $u, v \in C$, then we have $v \leadsto u$ and $u \leadsto v$ in $G$. 
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Proof:
(1) If $v \xrightarrow{} u$ and $u \xrightarrow{} v$ in $G$, then $u$ and $v$ belong to the same component $C$ produced by the algorithm. Sketch of proof:
• assume in 1st DFS, $v$ was discovered before $u$ (or opposite);
• as $v \xrightarrow{} u$ in $G$, $u$ and $v$ belong to the same search tree rooted at $r$ with $r.f \geq v.f > u.f$ (note: $r$ could be just $v$);
• as $u \xrightarrow{} v$ in $G$, $v \xrightarrow{} u$ in $G_T$;
• now consider the 2nd DFS; there are 2 situations:
  (1) searching from some $w$ with $w.f \geq v.f$ (note: $w$ could be $v$) finds $v$ first; then it finds $u$;
  (2) the search finds $u$ first; because $v \xrightarrow{} u$ in $G$, $u \xrightarrow{} v$ is in $G_T$, it finds also $v$.
In both situations, $u$ and $v$ belong to the same search tree in the 2nd DFS search. Therefore, $u$ and $u$ belong to the same component.
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Proof:

• Assume in 1st DFS, \( v \) was discovered before \( u \) (or opposite);
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• In both situations, \( u \) and \( v \) belong to the same search tree in the 2nd DFS search. Therefore, \( u \) and \( v \) belong to the same component.
Proof:

(1) If \( v \leadsto u \) and \( u \leadsto v \) in \( G \), then \( u \) and \( v \) belong to the same component \( C \) produced by the algorithm.
Proof:

(1) If $v \sim u$ and $u \sim v$ in $G$, then $u$ and $v$ belong to the same component $C$ produced by the algorithm.

Sketch of proof:
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Proof:

(1) If $v \sim u$ and $u \sim v$ in $G$ in $G$, then $u$ and $v$ belong to the same component $C$ produced by the algorithm.

Sketch of proof:

• assume in 1st DFS, $v$ was discovered before $u$ (or opposite);
Proof:

(1) If \( v \leadsto u \) and \( u \leadsto v \) in \( G \) in \( G \), then \( u \) and \( v \) belong to the same component \( C \) produced by the algorithm.

Sketch of proof:

• assume in 1st DFS, \( v \) was discovered before \( u \) (or opposite);
• as \( v \leadsto u \) in \( G \), \( u \) and \( v \) belong to the same search tree rooted at \( r \) with \( r.f \geq v.f > u.f \) (note: \( r \) could be just \( v \)
Proof:

(1) If $v \leadsto u$ and $u \leadsto v$ in $G$, then $u$ and $v$ belong to the same component $C$ produced by the algorithm.

Sketch of proof:

- assume in 1st DFS, $v$ was discovered before $u$ (or opposite);
- as $v \leadsto u$ in $G$, $u$ and $v$ belong to the same search tree rooted at $r$ with $r.f \geq v.f > u.f$ (note: $r$ could be just $v$)
- as $u \leadsto v$ in $G$, $v \leadsto u$ in $G^T$;
Chapter 22. Elementary graph algorithms

Proof:

(1) If \( v \sim u \) and \( u \sim v \) in \( G \), then \( u \) and \( v \) belong to the same component \( C \) produced by the algorithm.

Sketch of proof:

• assume in 1st DFS, \( v \) was discovered before \( u \) (or opposite);
• as \( v \sim u \) in \( G \), \( u \) and \( v \) belong to the same search tree rooted at \( r \) with \( r.f \geq v.f > u.f \) (note: \( r \) could be just \( v \))
• as \( u \sim v \) in \( G \), \( v \sim u \) in \( G^T \);
• now consider the 2nd DFS; there are 2 situations:
Chapter 22. Elementary graph algorithms

Proof:

(1) If \( v \leadsto u \) and \( u \leadsto v \) in \( G \), then \( u \) and \( v \) belong to the same component \( C \) produced by the algorithm.

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- As \( u \leadsto v \) in \( G \), \( v \leadsto u \) in \( G^T \);
- Now consider the 2nd DFS; there are 2 situations:

  (1) searching from some \( w \) with \( w.f \geq v.f \) (note: \( w \) could be \( v \))
      finds \( v \) first; then it finds \( u \);
Proof:

(1) If $v \leadsto u$ and $u \leadsto v$ in $G$ in $G$, then $u$ and $v$ belong to the same component $C$ produced by the algorithm.

Sketch of proof:

- assume in 1st DFS, $v$ was discovered before $u$ (or opposite);
- as $v \leadsto u$ in $G$, $u$ and $v$ belong to the same search tree rooted at $r$ with $r.f \geq v.f > u.f$ (note: $r$ could be just $v$)
- as $u \leadsto v$ in $G$, $v \leadsto u$ in $G^T$;
- now consider the 2nd DFS; there are 2 situations:

  (1) searching from some $w$ with $w.f \geq v.f$ (note: $w$ could be $v$) finds $v$ first; then it finds $u$;

  (2) the search finds $u$ first; because $v \leadsto u$ in $G$, $u \leadsto v$ is in $G^T$, it finds also $v$. 
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Proof:

(1) If \( v \rightarrow u \) and \( u \rightarrow v \) in \( G \) in \( G \), then \( u \) and \( v \) belong to the same component \( C \) produced by the algorithm.

Sketch of proof:

• assume in 1st DFS, \( v \) was discovered before \( u \) (or opposite);
• as \( v \rightarrow u \) in \( G \), \( u \) and \( v \) belong to the same search tree rooted at \( r \) with \( r.f \geq v.f > u.f \) (note: \( r \) could be just \( v \))
• as \( u \rightarrow v \) in \( G \), \( v \rightarrow u \) in \( G^T \);
• now consider the 2nd DFS; there are 2 situations:
  (1) searching from some \( w \) with \( w.f \geq v.f \) (note: \( w \) could be \( v \)) finds \( v \) first; then it finds \( u \);
  (2) the search finds \( u \) first; because \( v \rightarrow u \) in \( G \), \( u \rightarrow v \) is in \( G^T \), it finds also \( v \).

In both situations, \( u \) and \( v \) belongs to the same search tree in the 2nd DFS search. Therefore, \( u \) and \( u \) belong to the same component.
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If \( u, v \in C \), then we have \( v \Rightarrow u \) and \( u \Rightarrow v \) in \( G \).

Sketch of proof:
1. Assume \( u, v \) belong to the same tree of root \( r \) in 2nd DFS;
2. Then \( r.f > u.f \) and \( r.f > v.f \) in 1st DFS;
3. The assumption in (1) also implies:
   - \( r \Rightarrow u \) and \( r \Rightarrow v \) in \( G_T \);
   - That is, \( u \Rightarrow r \) and \( v \Rightarrow r \) in \( G \);
   - Then \( u.f > r.f \) and \( v.f > r.f \) in 1st DFS, which conflict with conclusions in (2), UNLESS \( r \Rightarrow u \) and \( r \Rightarrow v \) in \( G \) also.
4. This means: through \( r \), \( v \Rightarrow u \) and \( u \Rightarrow v \) in \( G \).
(2) If $u, v \in C$, then we have $v \rightsquigarrow u$ and $u \rightsquigarrow v$ in $G$. 
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(2) If \( u, v \in C \), then we have \( v \leadsto u \) and \( u \leadsto v \) in \( G \).

Sketch of proof:
Chapter 22. Elementary graph algorithms

(2) If \( u, v \in C \), then we have \( v \sim u \) and \( u \sim v \) in \( G \).

Sketch of proof:

(1) assume \( u, v \) belong to the same tree of root \( r \) in 2nd DFS;

(2) then \( r.f > u.f \) and \( r.f > v.f \) in 1st DFS;

(3) the assumption in (1) also implies:
   - \( r \Rightarrow u \) and \( r \Rightarrow v \) in \( G_T \);
   - that is, \( u \Rightarrow r \) and \( v \Rightarrow r \) in \( G \);
   - then \( u.f > r.f \) and \( v.f > r.f \) in 1st DFS,
     which conflict with conclusions in (2), UNLESS
     \( r \Rightarrow u \) and \( r \Rightarrow v \) in \( G \) also.

(4) This means: through \( r \), \( v \Rightarrow u \) and \( u \Rightarrow v \) in \( G \).
(2) If \( u, v \in C \), then we have \( v \Rightarrow u \) and \( u \Rightarrow v \) in \( G \).

Sketch of proof:

(1) assume \( u, v \) belong to the same tree of root \( r \) in 2nd DFS;
(2) then \( r.f > u.f \); and \( r.f > v.f \) in 1st DFS;
(2) If \( u, v \in C \), then we have \( v \sim u \) and \( u \sim v \) in \( G \).

Sketch of proof:

(1) assume \( u, v \) belong to the same tree of root \( r \) in 2nd DFS;
(2) then \( r.f > u.f \); and \( r.f > v.f \) in 1st DFS;
(3) the assumption in (1) also implies:

• \( r \sim u \) and \( r \sim v \) in \( G_T \);
• that is, \( u \sim r \) and \( v \sim r \) in \( G \);
• then \( u.f > r.f \) and \( v.f > r.f \) in 1st DFS, which conflict with conclusions in (2), UNLESS \( r \sim u \) and \( r \sim v \) in \( G \) also.

(4) This means: through \( r \), \( v \sim u \) and \( u \sim v \) in \( G \).
(2) If \( u, v \in C \), then we have \( v \leadsto u \) and \( u \leadsto v \) in \( G \).

Sketch of proof:

(1) assume \( u, v \) belong to the same tree of root \( r \) in 2nd DFS;
(2) then \( r.f > u.f \); and \( r.f > v.f \) in 1st DFS;
(3) the assumption in (1) also implies:
  • \( r \leadsto u \) and \( r \leadsto v \) in \( G^T \);
(2) If $u, v \in C$, then we have $v \rightsquigarrow u$ and $u \rightsquigarrow v$ in $G$.

Sketch of proof:

(1) assume $u, v$ belong to the same tree of root $r$ in 2nd DFS;
(2) then $r.f > u.f$; and $r.f > v.f$ in 1st DFS;
(3) the assumption in (1) also implies:
   - $r \rightsquigarrow u$ and $r \rightsquigarrow v$ in $G^T$;
   - that is, $u \rightsquigarrow r$ and $v \rightsquigarrow r$ in $G$;
(2) If $u, v \in C$, then we have $v \rightsquigarrow u$ and $u \rightsquigarrow v$ in $G$.

Sketch of proof:

(1) assume $u, v$ belong to the same tree of root $r$ in 2nd DFS;
(2) then $r.f > u.f$; and $r.f > v.f$ in 1st DFS;
(3) the assumption in (1) also implies:
   • $r \rightsquigarrow u$ and $r \rightsquigarrow v$ in $G^T$;
   • that is, $u \rightsquigarrow r$ and $v \rightsquigarrow r$ in $G'$;
   • then $u.f > r.f$ and
(2) If \( u, v \in C \), then we have \( v \leadsto u \) and \( u \leadsto v \) in \( G \).

Sketch of proof:

(1) assume \( u, v \) belong to the same tree of root \( r \) in 2nd DFS;
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   - \( r \leadsto u \) and \( r \leadsto v \) in \( G^T \);
   - that is, \( u \leadsto r \) and \( v \leadsto r \) in \( G \);
   - then \( u.f > r.f \) and
     \( v.f > r.f \) in 1st DFS,
(2) If \( u, v \in C \), then we have \( v \leadsto u \) and \( u \leadsto v \) in \( G \).

Sketch of proof:

(1) assume \( u, v \) belong to the same tree of root \( r \) in 2nd DFS;
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   - \( r \leadsto u \) and \( r \leadsto v \) in \( G^T \);
   - that is, \( u \leadsto r \) and \( v \leadsto r \) in \( G \);
   - then \( u.f > r.f \) and
     \( v.f > r.f \) in 1st DFS,
   which conflict with conclusions in (2),
(2) If \( u, v \in C \), then we have \( v \preceq u \) and \( u \preceq v \) in \( G \).

Sketch of proof:

(1) assume \( u, v \) belong to the same tree of root \( r \) in 2nd DFS;
(2) then \( r.f > u.f \); and \( r.f > v.f \) in 1st DFS;
(3) the assumption in (1) also implies:
   - \( r \preceq u \) and \( r \preceq v \) in \( G^T \);
   - that is, \( u \preceq r \) and \( v \preceq r \) in \( G \);
   - then \( u.f > r.f \) and \( v.f > r.f \) in 1st DFS,
     which conflict with conclusions in (2), UNLESS
(2) If $u, v \in C$, then we have $v \rightsquigarrow u$ and $u \rightsquigarrow v$ in $G$.

Sketch of proof:

(1) assume $u, v$ belong to the same tree of root $r$ in 2nd DFS;
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   • $r \rightsquigarrow u$ and $r \rightsquigarrow v$ in $G^T$;
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     $r \rightsquigarrow u$ and $r \rightsquigarrow v$ in $G$ also.
(2) If \( u, v \in C \), then we have \( v \leadsto u \) and \( u \leadsto v \) in \( G \).

Sketch of proof:

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(4) This means: through \( r, v \leadsto u \) and \( u \leadsto v \) in \( G \).
Chapter 22. Elementary graph algorithms

Reachability Problem

Input: $G = (V, E)$, and $s, t \in V$;

Output: YES if and only there is a path $s \Rightarrow t$ in $G$.

• The problem can be solved with DFS and BFS by search on the graph from $s$ until $t$ shows up.

Linear time $O(|E| + |V|)$. Can we do better?

• But first answer the following question: Can you write an SQL program to solve Reachability?

• It appears that a loop is needed to solve Reachability. Why?

Inherent difficulty in parallel computation. P-complete, it cannot be solved in time $O(\log n)$ even if $\Theta(n)$ CPUs are used.
Chapter 22. Elementary graph algorithms

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- Inherent difficulty in parallel computation.
- P-complete, it cannot be solved in time $O(\log n)$ even if $\Theta(n)$ CPUs are used.
Reachability Problem

**Input:** $G = (V, E)$, and $s, t \in V$;

**Output:** YES if and only there is a path $s \sim t$ in $G$.

- The problem can be solved with DFS and BFS.
Chapter 22. Elementary graph algorithms

Reachability Problem

**Input:** \( G = (V, E) \), and \( s, t \in V \);  
**Output:** YES if and only there is a path \( s \rightarrow t \) in \( G \).

- The problem can be solved with DFS and BFS  
  by search on the graph from \( s \) until \( t \) shows up.

Linear time \( O(|E| + |V|) \). Can we do better?

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Reachability Problem

**Input:** $G = (V, E)$, and $s, t \in V$;

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**Input:** $G = (V, E)$, and $s, t \in V$;

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Chapter 22. Elementary graph algorithms

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Reachability Problem

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Chapter 22. Elementary graph algorithms

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  \textbf{P-complete}, it cannot be solved in time \( O(\log n) \) even if \( \Theta(n) \) CPUs are used.
Chapter 23. Minimum Spanning Trees

Chapter 23. Minimum Spanning Trees
A spanning tree of a graph $G = (V, E)$ is a tree as a subgraph in $G$ which contains all vertices in $V$. 

A minimum spanning tree (MST) of an edge-weighted graph $G$ is a spanning tree with the least edge weight sum.
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Chapter 23. Minimum Spanning Trees

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- A \textit{spanning tree} of a graph $G = (V, E)$ is a tree as a subgraph in $G$ which contains all vertices in $V$. 
Chapter 23. Minimum Spanning Trees

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Chapter 23. Minimum Spanning Trees

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\[\begin{align*}
\text{a} & \quad 2 \quad \text{b} \\
\text{d} & \quad 4 \quad \text{c} \\
\text{e} & \quad 5
\end{align*}\]
Chapter 23. Minimum Spanning Trees

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Chapter 23. Minimum Spanning Trees

The MST problem

Input: connected, undirected graph $G = (V,E)$ with weight $w: E \rightarrow \mathbb{R}$

Output: a spanning tree $T = (V,E')$ such that $W(T) = \sum_{(u,v) \in E'} w(u,v)$ is the minimum

We will introduce two greedy algorithms: (1) Kruskal's and (2) Prim's

• They have the same generic process to grow a spanning tree;
• but differ in which edge to add the partially grown tree.
The MST problem

**Input:** connected, undirected graph \( G = (V, E) \) with weight \( w : E \to \mathbb{R} \),
Chapter 23. Minimum Spanning Trees

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\[
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\]

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Chapter 23. Minimum Spanning Trees

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Chapter 23. Minimum Spanning Trees

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**Input:** connected, undirected graph $G = (V, E)$ with weight $w : E \to R$,

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Chapter 23. Minimum Spanning Trees

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Growing an MST

**Chapter 23. Minimum Spanning Trees**

**Growing an MST**
Growing an MST

A generic process to grow an MST.
Chapter 23. Minimum Spanning Trees

Growing an MST

A generic process to grow an MST.

\textbf{Generic MST}(G, w) \{ given graph \( G \) and weight function \( w \) \}
Chapter 23. Minimum Spanning Trees

Growing an MST

A generic process to grow an MST.

**Generic MST** \((G, w)\) \{ given graph \(G\) and weight function \(w\) \}

1. \(A = \emptyset\);
Growing an MST

A generic process to grow an MST.

\text{\textsc{Generic MST}}(G, w) \quad \{ \text{given graph } G \text{ and weight function } w \} 

1. \quad A = \emptyset; 
2. \quad \textbf{while} A does not form a spanning tree 
3. \quad \textbf{find an edge} \ (u, v) \ \text{that is safe for} \ A
Chapter 23. Minimum Spanning Trees

Growing an MST

A generic process to grow an MST.

**Generic MST**

\[ \text{given graph } G \text{ and weight function } w \]  

1. \( A = \emptyset \);  
2. \textbf{while} \( A \) does not form a spanning tree  
3. \textbf{find an edge} \((u, v)\) that is safe for \( A \)  
4. \( A = A \cup \{(u, v)\} \)
Chapter 23. Minimum Spanning Trees

Growing an MST

A generic process to grow an MST.

**Generic MST** \( (G, w) \) \{ given graph \( G \) and weight function \( w \) \}

1. \( A = \emptyset \);
2. **while** \( A \) does not form a spanning tree
3. find an edge \((u, v)\) that is **safe** for \( A \)
4. \( A = A \cup \{(u, v)\} \)
5. **return** \( A \)
Growing an MST

A generic process to grow an MST.

**Generic MST**\( (G, w) \) { given graph \( G \) and weight function \( w \) }

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3. find an edge \((u, v)\) that is safe for \( A \)
4. \( A = A \cup \{(u, v)\} \)
5. return \((A)\)

Loop invariant:
Chapter 23. Minimum Spanning Trees

Growing an MST

A generic process to grow an MST.

**Generic MST**\( (G, w) \) \{ given graph \( G \) and weight function \( w \) \}
1. \( A = \emptyset \);
2. while \( A \) does not form a spanning tree
3. find an edge \((u, v)\) that is safe for \( A \)
4. \( A = A \cup \{(u, v)\} \)
5. return \( (A) \)

Loop invariant: \( A \) is always a subset of some MST;
Chapter 23. Minimum Spanning Trees

Growing an MST
A generic process to grow an MST.

\textbf{Generic MST}(G, w) \quad \{ \text{ given graph } G \text{ and weight function } w \} 
1. \quad A = \emptyset; 
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3. \quad \text{find an edge } (u, v) \text{ that is safe for } A 
4. \quad A = A \cup \{(u, v)\} 
5. \quad \textbf{return} \ (A) 

Loop invariant: \( A \) is always a subset of some MST;
Chapter 23. Minimum Spanning Trees

Growing an MST

A generic process to grow an MST.

\[ \text{Generic MST}(G, w) \quad \{ \text{given graph } G \text{ and weight function } w \} \]

1. \( A = \emptyset \);
2. while \( A \) does not form a spanning tree
3. find an edge \((u, v)\) that is safe for \( A \)
4. \( A = A \cup \{ (u, v) \} \)
5. return \((A)\)

Loop invariant: \( A \) is always a subset of some MST;

Note: when the loop terminates, \( A \) is a MST.
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safe edge:
Growing an MST

A generic process to grow an MST.

**Generic MST** *(G, w)* \{ given graph G and weight function w \}

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**safe edge:**

edge \((u, v)\) is safe for \( A \) if **does not violate the loop invariant**,
Chapter 23. Minimum Spanning Trees

**Growing an MST**

A generic process to grow an MST.

** Generic MST** $(G, w)$ \{ given graph $G$ and weight function $w$ \}

1. $A = \emptyset$
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Loop invariant: $A$ is always a subset of some MST;

Note: when the loop terminates, $A$ is a MST.

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eedge $(u, v)$ is safe for $A$ if does not violate the loop invariant,
i.e, $A \cup \{(u, v)\}$ is a subset of some MST.
Chapter 23. Minimum Spanning Trees

We first need some terminologies
Chapter 23. Minimum Spanning Trees

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- **cut**: \((S, V-S)\), a partition of \(V\)
Chapter 23. Minimum Spanning Trees

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- **cut**: $(S, V-S)$, a partition of $V$
Chapter 23. Minimum Spanning Trees

We first need some terminologies

- **cut**: \((S, V - S)\), a partition of \(V\)

- **crossing**: \((u, v)\) crosses cut \((S, V - S)\) if \(u\) and \(v\) are in \(S\) and \(V - S\), respectively
Chapter 23. Minimum Spanning Trees

Some more terminologies
Chapter 23. Minimum Spanning Trees

Some more terminologies

- **respect**: a cut respects a set $A$ of edges if no edge in $A$ crosses the cut.
Some more terminologies

- **respect**: a cut respects a set $A$ of edges if no edge in $A$ crosses the cut.

- **light edge**: an edge is a light edge crossing a cut if its weight is the minimum of any edge that crosses the cut.
Theorem 23.1 Let $G = (V, E)$. 

Sketch of proof: 

(1) If $A \cup \{(u,v)\}$ forms a cycle, there must have been another edge in $A$ that crosses cut $(S, V - S)$, implying the cut did not respect $A$. Contradicts. 

(2) Assume that some MST $T$, $A \subset T$. First, $T \cup \{(u,v)\}$ forms a circle! Why? There must be another edge $(x,y)$ crossing the cut $(S, V - S)$. Since $(u,v)$ is light edge, $T' = T - \{(x,y)\} \cup \{(u,v)\}$ is an MST. Now, $A \cup \{(u,v)\} \subseteq T'$ because $(x,y) \not\in A$ (otherwise, the cut would not respect $A$).
Chapter 23. Minimum Spanning Trees

**Theorem 23.1** Let $G = (V, E)$.
Let $A \subseteq E$, contained in some MST for $G$.  

**Sketch of proof**: 
(1) If $A \cup \{(u, v)\}$ forms a cycle there must have been another edge in $A$ that crosses cut $(S, V - S)$ (WHY?), implying the cut did not respect $A$. Contradicts. 
(2) Assume that some MST $T$, $A \subset T$. First, $T \cup \{(u, v)\}$ forms a circle! Why? There must be another edge $(x, y)$ cross the cut $(S, V - S)$. Since $(u, v)$ is light edge, $T' = T - \{(x, y)\} \cup \{(u, v)\}$ is an MST. Now $A \cup \{(u, v)\} \subseteq T'$ because $(x, y) \not\in A$ (otherwise, the cut would not respect $A$).
Chapter 23. Minimum Spanning Trees

Theorem 23.1 Let $G = (V, E)$.
Let $A \subseteq E$, contained in some MST for $G$.
Let $(S, V - S)$ be any cut of $G$ that respects $A$. 

Sketch of proof:
(1) If $A \cup \{u,v\}$ forms a cycle there must have been another edge in $A$ that crosses cut $(S, V - S)$ (WHY?), implying the cut did not respect $A$.
Contradicts.
(2) Assume that some MST $T$, $A \subset T$.
First, $T \cup \{u,v\}$ forms a circle! Why?
There must be another edge $(x,y)$ cross the cut $(S, V - S)$.
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Chapter 23. Minimum Spanning Trees

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Chapter 23. Minimum Spanning Trees

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Chapter 23. Minimum Spanning Trees

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Chapter 23. Minimum Spanning Trees

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**Chapter 23. Minimum Spanning Trees**

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Chapter 23. Minimum Spanning Trees

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Contradicts.

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First, $T \cup \{(u, v)\}$ forms a circle!
Chapter 23. Minimum Spanning Trees

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   First, $T \cup \{(u, v)\}$ forms a circle! Why?
Chapter 23. Minimum Spanning Trees

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Chapter 23. Minimum Spanning Trees

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Chapter 23. Minimum Spanning Trees

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Since \((u, v)\) is light edge, \( T' = T - \{(x, y)\} \cup \{(u, v)\} \) is an MST.

Now \( A \cup \{(u, v)\} \subseteq T' \) because \((x, y) \notin A\)
(otherwise, the cut would not respect \( A \).)
Theorem 23.1 gives a sufficient information for how to identify a safe edge to make the **Generic MST** algorithm work.

---

**MST-Kruskal** \((G, w)\)

1. \(A = \emptyset\);
2. for each vertex \(v \in G.V\)
3. Make-Set \((v)\)
4. sort edges in \(E\) into non-decreasing order by their weight \(w\)
5. for each edge \((u, v)\) \(\in E\), taken in the order
6. if Find Set \((u)\) \(\neq\) Find Set \((v)\)
7. \(A = A \cup \{(u, v)\}\)
8. Union \((u, v)\)
9. return \((A)\)
Chapter 23. Minimum Spanning Trees

Theorem 23.1 gives a sufficient information for how to identify a safe edge to make the Generic MST algorithm work.

- specific algorithms can be produced from Generic MST based on how the set $A$ is grown.
Chapter 23. Minimum Spanning Trees

Theorem 23.1 gives a sufficient information for how to identify a safe edge to make the $\text{GENERIC MST}$ algorithm work.

- specific algorithms can be produced from $\text{GENERIC MST}$ based on how the set $A$ is grown.
- $A$ may always be a tree (Prim’s algorithm)
Chapter 23. Minimum Spanning Trees

Theorem 23.1 gives a sufficient information for how to identify a safe edge to make the Generic MST algorithm work.

- specific algorithms can be produced from Generic MST based on how the set $A$ is grown.

- $A$ may always be a tree (Prim’s algorithm) or could be a forest (Kruskal’s algorithm).
Chapter 23. Minimum Spanning Trees

Theorem 23.1 gives a sufficient information for how to identify a safe edge to make the **Generic MST** algorithm work.

- specific algorithms can be produced from **Generic MST** based on how the set $A$ is grown.
- $A$ may always be a tree (Prim’s algorithm) or could be a forest (Kruskal’s algorithm).

**MST-Kruskal** ($G, w$)
Theorem 23.1 gives a sufficient information for how to identify a safe edge to make the \textsc{Generic MST} algorithm work.

- specific algorithms can be produced from \textsc{Generic MST} based on how the set $A$ is grown.

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\begin{itemize}
  \item \textsc{MST-Kruskal}(G, w)
  \end{itemize}

1. $A = \emptyset$;
Chapter 23. Minimum Spanning Trees

Theorem 23.1 gives a sufficient information for how to identify a safe edge to make the GENERIC MST algorithm work.

- specific algorithms can be produced from GENERIC MST based on how the set $A$ is grown.

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MST-Kruskal$(G, w)$
1. $A = \emptyset$;
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Theorem 23.1 gives a sufficient information for how to identify a safe edge to make the **GENERIC MST** algorithm work.

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**MST-Kruskal**($G, w$)

1. $A = \emptyset$
2. for each vertex $v \in G.V$
3. MAKE-SET($v$)
Chapter 23. Minimum Spanning Trees

Theorem 23.1 gives a sufficient information for how to identify a safe edge to make the Generic MST algorithm work.

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\[ \text{MST-Kruskal}(G, w) \]
1. $A = \emptyset$;
2. for each vertex $v \in G.V$
3. \text{Make-Set}(v)
4. sort edges in $E$ into non-decreasing order by their weight $w$
Chapter 23. Minimum Spanning Trees

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**MST-Kruskal** $(G, w)$

1. $A = \emptyset$;
2. **for** each vertex $v \in G.V$
3. **Make-Set**$(v)$
4. **sort** edges in $E$ into non-decreasing order by their weight $w$
5. **for** each edge $(u, v) \in E$, taken in the order
Chapter 23. Minimum Spanning Trees

Theorem 23.1 gives a sufficient information for how to identify a safe edge to make the \texttt{Generic MST} algorithm work.

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\texttt{MST-Kruskal}(G, w)

1. $A = \emptyset$;
2. \texttt{for} each vertex $v \in G.V$
3. \texttt{Make-Set}(v)
4. \texttt{sort} edges in $E$ into non-decreasing order by their weight $w$
5. \texttt{for} each edge $(u, v) \in E$, taken in the order
6. \texttt{if} $\texttt{Find Set}(u) \neq \texttt{Find Set}(v)$

\texttt{return} $(A)$
Chapter 23. Minimum Spanning Trees

Theorem 23.1 gives a sufficient information for how to identify a safe edge to make the **Generic** MST algorithm work.

- specific algorithms can be produced from **Generic** MST based on how the set $A$ is grown.

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**MST-Kruskal**$(G, w)$

1. $A = \emptyset$
2. for each vertex $v \in G.V$
3. \hspace{1cm} Make-Set$(v)$
4. sort edges in $E$ into non-decreasing order by their weight $w$
5. for each edge $(u, v) \in E$, taken in the order
6. \hspace{1cm} if FIND SET $(u) \neq$ FIND SET$(v)$
7. \hspace{1cm} $A = A \cup \{(u, v)\}$
Chapter 23. Minimum Spanning Trees

Theorem 23.1 gives a sufficient information for how to identify a safe edge to make the Generic MST algorithm work.

- specific algorithms can be produced from Generic MST based on how the set $A$ is grown.

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MST-Kruskal($G, w$)
1. $A = \emptyset$
2. for each vertex $v \in G.V$
3.   Make-Set($v$)
4. sort edges in $E$ into non-decreasing order by their weight $w$
5. for each edge $(u, v) \in E$, taken in the order
6.   if $\text{Find Set}(u) \neq \text{Find Set}(v)$
7.     $A = A \cup \{(u, v)\}$
8.   Union($u, v$)
Chapter 23. Minimum Spanning Trees

Theorem 23.1 gives a sufficient information for how to identify a safe edge to make the Generic MST algorithm work.

- specific algorithms can be produced from Generic MST based on how the set \( A \) is grown.

- \( A \) may always be a tree (Prim’s algorithm) or could be a forest (Kruskal’s algorithm).

MST-Kruskal\((G, w)\)
1. \( A = \emptyset \);
2. for each vertex \( v \in G.V \)
3. \hspace{0.5cm} Make-Set\((v)\)
4. sort edges in \( E \) into non-decreasing order by their weight \( w \)
5. for each edge \((u, v) \in E\), taken in the order
6. \hspace{0.5cm} if Find Set \((u) \neq \text{Find Set}(v)\)
7. \hspace{1.5cm} \( A = A \cup \{(u, v)\} \)
8. \hspace{1.5cm} \text{Union}(u, v) \)
9. return \((A)\)
Chapter 23. Minimum Spanning Trees

Execution of Kruskal's algorithm for MST

disjoint sets

[A] [B] [C] [D] [E] [F] [G] [H]

[A] [B] [C] [D] [E] [F] [G] [H]

[A] [B] [C] [D] [E] [F, G] [H]

[A] [B] [D] [E] [C, F, G] [H]

[A] [D] [E] [B, C, F, G] [H]

[D] [E] [A, B, C, F, G] [H]

[D, E] [A, B, C, F, G] [H]

[D, E, H] [A, B, C, F, G]

[A, B, C, D, E, F, G, H]
Chapter 23. Minimum Spanning Trees

At each iteration of the for loop, e.g., identify

[D] [E] [A, B, C, F, G] [H]

[D, E] [A, B, C, F, G] [H]
Chapter 23. Minimum Spanning Trees

At each iteration of the for loop, e.g., identify

- $A = \{(A, F), (B, F), (C, G), (F, G), (D, E)\}$, cut that respects $A$: $S = \{A, B, C, D, F, G\}$, $V - S = \{E, H\}$, light edge $(D, E)$ crosses the cut;

- $A = \{(A, F), (B, F), (C, G), (F, G), (D, E)\}$, cut that respects $A$: $S = \{A, B, C, F, G, H\}$, $V - S = \{D, E\}$, light edge $(E, H)$ crosses the cut;
Chapter 23. Minimum Spanning Trees

At each iteration of the for loop, e.g., identify

- $A = \{(A, F), (B, F), (C, G), (F, G)\}$,
At each iteration of the `for` loop, e.g., identify

\[ A = \{ (A, F), (B, F), (C, G), (F, G) \} \]

Cut that respects \( A \): \( S = \{ A, B, C, D, F, G \} \), \( V - S = \{ E, H \} \),
Chapter 23. Minimum Spanning Trees

At each iteration of the for loop, e.g., identify

\[ A = \{(A, F), (B, F), (C, G), (F, G), (D, E)\} \]

cut that respects \( A \): \( S = \{A, B, C, D, F, G\} \), \( V - S = \{E, H\} \),
light edge \((D, E)\) crosses the cut;
Chapter 23. Minimum Spanning Trees

At each iteration of the for loop, e.g., identify

• $A = \{(A, F), (B, F), (C, G), (F, G)\}$,
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Chapter 23. Minimum Spanning Trees

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Chapter 23. Minimum Spanning Trees

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Chapter 23. Minimum Spanning Trees

The Kruskal’s algorithm uses disjoint-set data structures (where elements are partitioned into disjoint sets)
Chapter 23. Minimum Spanning Trees

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- **Make Set** \(x\): create a set of single element \(x\);
Chapter 23. Minimum Spanning Trees

The Kruskal’s algorithm uses disjoint-set data structures (where elements are partitioned into disjoint sets)

- **Make Set**(x): create a set of single element x;
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Implementations (left: linked lists, Right: disjoint-set forest)

Time complexity: \(O(\log n)\) for *Make Set* (x), *Find Set* (x), *Union* (x,y)

Time complexity of Kruskal’s algorithm: \(O(|E| \log |V| + |V|)\).
Chapter 23. Minimum Spanning Trees

The Kruskal’s algorithm uses disjoint-set data structures (where elements are partitioned into disjoint sets)

- **MAKE SET(x)**: create a set of single element $x$;
- **FIND SET(x)**: identify the set that contains element $x$;
- **UNION(x, y)**: union the two sets containing $x$ and $y$ into one;

Implementations (left: linked lists, Right: disjoint-set forest)

Time complexity: $O(\log n)$ for **MAKE SET(x)**, **FIND SET(x)**, **UNION(x, y)** with disjoint-set forest implementation.

Time complexity of Kruskal’s algorithm: $O(|E| \log |V|) + |V|$.
Chapter 23. Minimum Spanning Trees

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![Diagram A](image1.png)

![Diagram B](image2.png)

Time complexity:
Chapter 23. Minimum Spanning Trees

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Chapter 23. Minimum Spanning Trees

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Chapter 23. Minimum Spanning Trees

\textbf{MST-Prim}(G, w, r)
Chapter 23. Minimum Spanning Trees

MST-Prim\( (G, w, r) \)
1. for each \( u \in G.V \)
MST-Prim($G, w, r$)
1. for each $u \in G.V$
2. $u.key = \infty$ { $u.key$ is the $u$'s shortest distance to set $A = V-Q$}
Chapter 23. Minimum Spanning Trees

MST-Prim(G, w, r)
1. for each \( u \in G.V \)
2. \( u.key = \infty \) \{ \( u.key \) is the \( u \)'s shortest distance to set \( A = V-Q \) \}
3. \( u.\pi = NULL \)
Chapter 23. Minimum Spanning Trees

**MST-Prim** \((G, w, r)\)

1. **for** each \(u \in G.V\)  
2. \(u.key = \infty\)  
3. \(u.\pi = NULL\)  
4. \(r.key = 0\)  

\(\{ u.key \) is the \(u\)'s shortest distance to set \(A = V-Q\}\)  

\(\{\) start from vertex \(r\) \(\}\)
MST-Prim \((G, w, r)\)

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4. \(Q = G.V\) \{ establish priority queue \(Q\) wit \(key\) values\}
5. while \(Q \neq \emptyset\)
6. \(u = \text{Extract Min}(Q)\)
7. for each \(v \in \text{Adj}[u]\)
8. if \(v \in Q\) and \(w(u, v) < v.key\) \{ for those not in \(A\), update distances\}
9. then \(v.\pi = u\)
10. \(v.key = w(u, v)\)
11. return \(\pi\)

usage of Priority queue: \(Q\), \(\text{Extract Min}\) takes \(O(\log n)\) time.

running time \(O(\lvert E \rvert + \lvert V \rvert \log \lvert V \rvert)\).
Chapter 23. Minimum Spanning Trees

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usage of Priority queue: $Q$, Extract Min takes $O(\log n)$ time.
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Chapter 23. Minimum Spanning Trees

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MST-Prim($G, w, r$)
1. \textbf{for} each $u \in G.V$
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3. \hspace{1cm} $u.\pi = NULL$
4. \hspace{1cm} $r.key = 0$ \hspace{1cm} \{ start from vertex $r$ \}
5. \hspace{1cm} $Q = G.V$ \hspace{1cm} \{ establish priority queue $Q$ wit key values \}
6. \hspace{1cm} \textbf{while} $Q \neq \emptyset$
7. \hspace{2cm} $u = \text{Extract Min}(Q)$
8. \hspace{2cm} \textbf{for} each $v \in Adj[u]$

\text{running time} $O(|E| + |V| \log |V|)$. 

\text{usage of Priority queue: } $Q$, Extract Min takes $O(\log n)$ time.
Chapter 23. Minimum Spanning Trees

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Chapter 23. Minimum Spanning Trees

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10. then \(v\).\(\pi\) = \(u\)

running time \(O(|E| + |V| \log |V|)\).
Chapter 23. Minimum Spanning Trees

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Chapter 23. Minimum Spanning Trees

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8. for each $v \in Adj[u]$
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10. then $v.\pi = u$
11. $v.key = w(u, v)$
12. return $\pi$

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Chapter 23. Minimum Spanning Trees
Summary of Kruskal’s and Prim’s algorithms:
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Chapter 23. Minimum Spanning Trees

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Chapter 23. Minimum Spanning Trees

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Chapter 23. Minimum Spanning Trees

Summary of Kruskal’s and Prim’s algorithms:

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  add it to \( A \)
Chapter 23. Minimum Spanning Trees

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Chapter 23. Minimum Spanning Trees

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Summary of Kruskal’s and Prim’s algorithms:

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• repeatedly choosing from the remaining edges;
  pick a light edge that respects a cut
  add it to $A$,
  ensure that $A$ is a subset of some MST

• until $A$ forms a spanning tree.
Some questions about MST

- What are the "cuts" implied in Kruskal’s algorithm and in Prim’s algorithm, respectively?
- Can we develop a DP algorithm for the MST problem?
  - The main issue: how solutions to subproblems help build solution for the problem.
  - What are subproblems, or what do subsolutions look like?
Chapter 23. Minimum Spanning Trees

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Chapter 24. Single Source Shortest Paths

Chapter 24. Single-source shortest paths
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Chapter 24. Single-source shortest paths

Given a graph $G = (V, E)$, with weight $w : E \to \mathbb{R}$; and single vertex $s \in V$;
Chapter 24. Single Source Shortest Paths

Chapter 24. Single-source shortest paths

Given a graph $G = (V, E)$, with weight $w : E \rightarrow R$; and single vertex $s \in V$;

for each vertex $v \in V$, find a shortest path $s \rightsquigarrow v$. 
Chapter 24. Single Source Shortest Paths

Chapter 24. Single-source shortest paths

Given a graph $G = (V, E)$, with weight $w : E \rightarrow \mathbb{R}$; and single vertex $s \in V$;

for each vertex $v \in V$, find a shortest path $s \leadsto v$.

- Shortest path is a simple path.
Chapter 24. Single Source Shortest Paths

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Chapter 24. Single-source shortest paths

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- “distance” is measured by the total edge weight on the path
  i.e., if the path $v_0 \rightsquigglyrightarrow v_k$ is $p = (v_0, v_1, \ldots, v_k)$
  then the path weight is $w(p) = \sum_{i=1}^{k} w(v_{i-1}, v_i)$
Chapter 24. Single Source Shortest Paths

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Given a graph \( G = (V, E) \), with weight \( w : E \rightarrow \mathbb{R} \); and single vertex \( s \in V \);

for each vertex \( v \in V \), find a shortest path \( s \leadsto v \).

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  then the path weight is \( w(p) = \sum_{i=1}^{k} w(v_{i-1}, v_i) \)

- shortest distance between \( u \) and \( v \) is

\[
\delta(u, v) = \min_{u \leadsto v} \{w(p)\}
\]
Chapter 24. Single Source Shortest Paths

- **Single-source shortest paths**: from $s$ to each vertex $v \in V$
Chapter 24. Single Source Shortest Paths

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- a special case: **Single-pair shortest path**: from $s$ to $t$
• **Single-source shortest paths**: from $s$ to each vertex $v \in V$

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• **All-pairs shortest paths**: from $s$ to $t$ for all pairs $s, t \in V$. 
Chapter 24. Single Source Shortest Paths

Lemma 24.1 (a subpath of a shortest path is a shortest path)
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Given a weighted directed graph $G = (V, E)$ with edge weight function $w$. Let $p = (v_0, v_1, \ldots, v_k)$ be a shortest path $v_0 \overset{p}{\sim} v_k$. Then $p_{i,j} = (v_i, \ldots, v_j)$ is a shortest path $v_i \overset{p_{i,j}}{\sim} v_j$. 

Chapter 24. Single Source Shortest Paths

**Lemma 24.1 (a subpath of a shortest path is a shortest path)**

Given a weighted directed graph $G = (V, E)$ with edge weight function $w$. Let $p = (v_0, v_1, \ldots, v_k)$ be a shortest path $v_0 \xrightarrow{p} v_k$. Then $p_{i,j} = (v_i, \ldots, v_j)$ is a shortest path $v_i \xrightarrow{p_{i,j}} v_j$.

**Proof idea:** (proof by contradiction)
Lemma 24.1 (a subpath of a shortest path is a shortest path)

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Proof idea: (proof by contradiction) Assume that \( p_{i,j} \) is not the shortest path from \( v_i \) to \( v_j \). Then there is a shorter path \( q_{i,j} \) from \( v_i \) to \( v_j \).
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Define path $q = (v_0, \ldots, v_i, q_{i,j}, v_j, \ldots, v_k)$

has weight

$$w(q) =$$
Chapter 24. Single Source Shortest Paths

Lemma 24.1 (a subpath of a shortest path is a shortest path)

Given a weighted directed graph $G = (V, E)$ with edge weight function $w$. Let $p = (v_0, v_1, \ldots, v_k)$ be a shortest path $v_0 \overset{p}{\rightarrow} v_k$. Then $p_{i,j} = (v_i, \ldots, v_j)$ is a shortest path $v_i \overset{p_{i,j}}{\rightarrow} v_j$.

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w(q) = \sum_{t=1}^{i} w(v_{t-1}, v_t) + w(q_{i,j}) + \sum_{t=j}^{k-1} w(v_t, v_{t+1})
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Lemma 24.1 (a subpath of a shortest path is a shortest path)

Given a weighted directed graph $G = (V, E)$ with edge weight function $w$. Let $p = (v_0, v_1, \ldots, v_k)$ be a shortest path $v_0 \xrightarrow{p} v_k$. Then $p_{i,j} = (v_i, \ldots, v_j)$ is a shortest path $v_i \xrightarrow{p_{i,j}} v_j$.

Proof idea: (proof by contradiction) Assume that $p_{i,j}$ is not the shortest path from $v_i$ to $v_j$. Then there is a shorter path $q_{i,j}$ from $v_i$ to $v_j$.

Define path

$$q = (v_0, \ldots, v_i, q_{i,j}, v_j, \ldots, v_k)$$

has weight

$$w(q) = \sum_{t=1}^{i} w(v_{t-1}, v_t) + w(q_{i,j}) + \sum_{t=j}^{k-1} w(v_t, v_{t+1})$$

<
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Proof idea: (proof by contradiction) Assume that \( p_{i,j} \) is not the shortest path from \( v_i \) to \( v_j \). Then there is a shorter path \( q_{i,j} \) from \( v_i \) to \( v_j \).

Define path

\[
q = (v_0, \ldots, v_i, q_{i,j}, v_j, \ldots, v_k)
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has weight

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$$w(q) = \sum_{t=1}^{i} w(v_{t-1}, v_t) + w(q_{i,j}) + \sum_{t=j}^{k-1} w(v_t, v_{t+1})$$

$$< \sum_{t=1}^{i} w(v_{t-1}, v_t) + w(p_{i,j}) +$$
Chapter 24. Single Source Shortest Paths

**Lemma 24.1** (a subpath of a shortest path is a shortest path)

Given a weighted directed graph $G = (V, E)$ with edge weight function $w$. Let $p = (v_0, v_1, \ldots, v_k)$ be a shortest path $v_0 \xrightarrow{p} v_k$. Then $p_{i,j} = (v_i, \ldots, v_j)$ is a shortest path $v_i \xrightarrow{p_{i,j}} v_j$.

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$$q = (v_0, \ldots, v_i, q_{i,j}, v_j, \ldots, v_k)$$

has weight

$$w(q) = \sum_{t=1}^{i} w(v_{t-1}, v_t) + w(q_{i,j}) + \sum_{t=j}^{k-1} w(v_t, v_{t+1})$$

$$< \sum_{t=1}^{i} w(v_{t-1}, v_t) + w(p_{i,j}) + \sum_{t=j}^{k-1} w(v_t, v_{t+1})$$

contradicts to the assumption that $p$ is the shortest path from $v_0$ to $v_k$. 


Lemma 24.1 (a subpath of a shortest path is a shortest path)

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$$w(q) = \sum_{t=1}^{i} w(v_{t-1}, v_t) + w(q_{i,j}) + \sum_{t=j}^{k-1} w(v_t, v_{t+1})$$

$$< \sum_{t=1}^{i} w(v_{t-1}, v_t) + w(p_{i,j}) + \sum_{t=j}^{k-1} w(v_t, v_{t+1}) = w(p)$$
Chapter 24. Single Source Shortest Paths

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Define path
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\]
\[
< \sum_{t=1}^{i} w(v_{t-1}, v_t) + w(p_{i,j}) + \sum_{t=j}^{k-1} w(v_t, v_{t+1}) = w(p)
\]

contradicts to the assumption that \( p \) is the shortest path from \( v_0 \) to \( v_k \).
Some terminologies:

• negative weights are allowed;
• cycles on a path: not a simple path;
• negative weight cycles, 0 weight cycles representing shortest paths: predecessor $\pi$
• shortest path tree:
  
  [link]
  
  (width $\propto \frac{1}{\text{distance}}$)
Chapter 24. Single Source Shortest Paths

Some terminologies:

- **negative weights** are allowed;
Chapter 24. Single Source Shortest Paths

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Chapter 24. Single Source Shortest Paths

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Chapter 24. Single Source Shortest Paths

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Chapter 24. Single Source Shortest Paths

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• **negative weights** are allowed;

• cycles on a path: not a simple path;

• **negative weight cycles**, 0 weight cycles

• representing shortest paths: predecessor \( \pi \)
  shortest path tree:

  http://graphserver.sourceforge.net/gallery.html
  (width \( \propto 1 \text{/distance} \))
Chapter 24. Single Source Shortest Paths

Technique: relaxation

- Intuition:
  
  If \( s \xrightarrow{p} v \) has distance \( v.d \) (computed so far),
Chapter 24. Single Source Shortest Paths

Technique: relaxation

- Intuition:
  
  if $s \xrightarrow{p} v$ has distance $v.d$ (computed so far),
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Chapter 24. Single Source Shortest Paths

Technique: relaxation

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Technique: relaxation

- Intuition:

  if \( s \xrightarrow{p} v \) has distance \( v.d \) (computed so far),
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  \[
  v.d = \min\{v.d, u.d + w(u,v)\}
  \]
Chapter 24. Single Source Shortest Paths

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Technique: relaxation

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- In other words:

  Let \( v.d \) be an weight upper bound of a shortest path from \( s \) to \( v \),
Chapter 24. Single Source Shortest Paths

Technique: relaxation

• Intuition:

  if \( s \xrightarrow{p} v \) has distance \( v.d \) (computed so far),
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  v.d = \min\{v.d, u.d + w(u,v)\}
  \]

• In other words:

  Let \( v.d \) be an weight upper bound of a shortest path from \( s \) to \( v \),
  initialized \( \infty \).
Chapter 24. Single Source Shortest Paths

Technique: relaxation

• Intuition:

  if $s \xrightarrow{p} v$ has distance $v.d$ (computed so far),
  $s \xrightarrow{q} u$ is newly discovered. Then

  $$v.d = \min\{v.d, u.d + w(u, v)\}$$

• In other words:

  Let $v.d$ be an weight upper bound of a shortest path from $s$ to $v$,
  initialized $\infty$.

  The process of relaxing edge $(u, v)$: improves $v.d$ by taking the path
  through $u$, and update $v.d$ and $v.\pi$.
Bellman-Ford algorithm

1. for each vertex $v$ in $G.V$
2. \hspace{1em} initialization
3. \hspace{1em} $v.d = \infty$
4. \hspace{1em} $v.\pi = \text{NULL}$
5. \hspace{1em} $s.d = 0$
6. \hspace{1em} for $i = 1$ to $|V| - 1$
7. \hspace{2em} for each edge $\langle u, v \rangle$ in $G.E$
8. \hspace{3em} if $v.d > u.d + w(u, v)$
9. \hspace{4em} $v.d = u.d + w(u, v)$
10. \hspace{4em} $v.\pi = u$
11. \hspace{1em} for each edge $\langle u, v \rangle$ in $G.E$
12. \hspace{2em} checking negative weight cycle
13. \hspace{3em} if $v.d > u.d + w(u, v)$
14. \hspace{4em} return (FALSE)
15. \hspace{1em} return (TRUE)

Running time: $O(|V||E|)$
Chapter 24. Single Source Shortest Paths

Bellman-Ford algorithm

$\text{BELLMAN-FORD}(G, w, s)$
Bellman-Ford algorithm

\textsc{Bellman-Ford}(G, w, s)

1. \textbf{for} each vertex \( v \in G.V \) \quad \text{initialization}

\textbf{Running time:} \( O(|V|\cdot|E|) \)
Chapter 24. Single Source Shortest Paths

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1. \textbf{for} each vertex \( v \in G.V \) \textbf{initialization}
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Chapter 24. Single Source Shortest Paths

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1. \textbf{for} each vertex \( v \in G.V \) \hspace{1cm} \textbf{initialization}
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3. \( v.\pi = NULL \)

4. \( s.d = 0 \)

5. \textbf{for} \( i = 1 \) \textbf{to} \(|V| - 1\) \hspace{1cm} \textbf{relaxation}

6. \textbf{for} each edge \((u,v) \in G.E\)

7. \hspace{1cm} \textbf{if} \( v.d > u.d + w(u,v) \)

8. \hspace{1cm} \( v.d = u.d + w(u,v) \)

9. \hspace{1cm} \( v.\pi = u \)

10. \textbf{for} each edge \((u,v) \in G.E\) \hspace{1cm} \textbf{checking negative weight cycle}

11. \hspace{1cm} \textbf{if} \( v.d > u.d + w(u,v) \)

12. \hspace{1cm} \textbf{return} \( \text{FALSE} \)

13. \hspace{1cm} \textbf{return} \( \text{TRUE} \)

Running time : \( O(|V||E|) \)
Bellman-Ford algorithm

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Running time: \(O(|V| \cdot |E|)\)
Chapter 24. Single Source Shortest Paths

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Bellman-Ford algorithm

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Bellman-Ford algorithm

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6. \hspace{1cm} \textbf{for} each edge \( (u, v) \in G.E \)
7. \hspace{1cm} \hspace{1cm} \textbf{if} \( v.d > u.d + w(u, v) \)
8. \hspace{1cm} \hspace{1cm} \hspace{1cm} \( v.d = u.d + w(u, v) \)

\textbf{Running time} : \( O(|V||E|) \)
Bellman-Ford algorithm

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Running time: \(O(|V| \cdot |E|)\)
Bellman-Ford algorithm

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8. \(v.d = u.d + w(u,v)\)
9. \(v.\pi = u\)
10. for each edge \((u, v) \in G.E\) checking negative weight cycle

Running time: \(O(|V||E|)\)
Chapter 24. Single Source Shortest Paths

Bellman-Ford algorithm

**Bellman-Ford** \((G, w, s)\)

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4. \(s.d = 0\)
5. for \(i = 1\) to \(|V| - 1\) \hspace{1cm} relaxation
6. for each edge \((u, v) \in G.E\)
7. \hspace{1cm} if \(v.d > u.d + w(u, v)\)
8. \hspace{1cm} \(v.d = u.d + w(u, v)\)
9. \hspace{1cm} \(v.\pi = u\)
10. for each edge \((u, v) \in G.E\) \hspace{1cm} checking negative weight cycle
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Running time : \(O(|V|\cdot|E|)\)
Bellman-Ford algorithm

**Bellman-Ford**\((G, w, s)\)

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8. \(v.d = u.d + w(u, v)\)
9. \(v.\pi = u\)
10. for each edge \((u, v) \in G.E\) checking negative weight cycle
11. if \(v.d > u.d + w(u, v)\)
12. return (FALSE)

**Running time** \(O(|V||E|)\)
Chapter 24. Single Source Shortest Paths

Bellman-Ford algorithm

\textbf{Bellman-Ford}(G, w, s)

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11. \hspace{1cm} \textbf{if} \( v.d > u.d + w(u, v) \)
12. \hspace{1cm} \hspace{1cm} \textbf{return} (FALSE)
13. \hspace{1cm} \textbf{return} (TRUE)

Running time: \( O(|V||E|) \)
Bellman-Ford algorithm

\textsc{Bellman-Ford}(G, w, s)

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6. \textbf{for} each edge \( (u, v) \in G.E \)
7. \quad \textbf{if} \( v.d > u.d + w(u, v) \)
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9. \quad \quad \( v.\pi = u \)
10. \textbf{for} each edge \( (u, v) \in G.E \) \quad \text{checking negative weight cycle}
11. \quad \textbf{if} \( v.d > u.d + w(u, v) \)
12. \quad \quad \textbf{return} \ (\text{FALSE})
13. \quad \textbf{return} \ (\text{TRUE})

Running time: \( O(|V||E|) \)
Chapter 24. Single Source Shortest Paths
Chapter 24. Single Source Shortest Paths

1.

2.

3.

4.

5.
Chapter 24. Single Source Shortest Paths

Properties of shortest paths and relaxation
Chapter 24. Single Source Shortest Paths

Properties of shortest paths and relaxation

\textsc{Relax}(u,v,w)
Properties of shortest paths and relaxation

\texttt{RELAX}(u, v, w)

1. if \( v.d > u.d + w(u, v) \)
Properties of shortest paths and relaxation

Relax \( (u, v, w) \)
1. if \( v.d > u.d + w(u, v) \)
2. \( v.d = u.d + w(u, v) \)
Chapter 24. Single Source Shortest Paths

Properties of shortest paths and relaxation

\textbf{RELAX}(u, v, w)
1. \textbf{if} $v.d > u.d + w(u, v)$
2. $v.d = u.d + w(u, v)$
3. $v.\pi = u$

Lemma 24.14, Convergence property: Let $s \rightarrow u \rightarrow v$ is a shortest path. If $u.d = \delta(s, u)$ holds before \textbf{Relax}(u, v, w) is called, then $v.d = \delta(s, v)$ after the call.

Proof:
$v.d \leq u.d + w(u, v) = \delta(s, u) + w(u, v) = \delta(s, v)$. So $v.d = \delta(s, v)$.
Chapter 24. Single Source Shortest Paths

Properties of shortest paths and relaxation

\textsc{Relax}(u, v, w)

1. if \( v.d > u.d + w(u, v) \)
2. \( v.d = u.d + w(u, v) \)
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\textbf{Lemma 24.14, Convergence property}: Let \( s \leadsto u \rightarrow v \) is a shortest path. If \( u.d = \delta(s, u) \) holds before \textsc{Relax}(u, v, w) is called, then \( v.d = \delta(s, v) \) after the call.
Properties of shortest paths and relaxation

\( \text{RELAX}(u, v, w) \)

1. \( v.d > u.d + w(u, v) \)
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3. \( v.\pi = u \)

**Lemma 24.14, Convergence property**: Let \( s \leadsto u \rightarrow v \) is a shortest path. If \( u.d = \delta(s, u) \) holds before \( \text{RELAX}(u, v, w) \) is called, then \( v.d = \delta(s, v) \) after the call.

**Proof**: 
Properties of shortest paths and relaxation

\textbf{RELAX}(u, v, w)

1. \textbf{if} \( v.d > u.d + w(u, v) \)
2. \( v.d = u.d + w(u, v) \)
3. \( v.\pi = u \)

\textbf{Lemma 24.14, Convergence property}: Let \( s \sim u \rightarrow v \) is a shortest path. If \( u.d = \delta(s, u) \) holds before \textbf{RELAX}(u, v, w) is called, then \( v.d = \delta(s, v) \) after the call.

\textbf{Proof}: \( v.d \leq u.d + w(u, v) \)
Properties of shortest paths and relaxation

RELAX$(u, v, w)$
1. if $v.d > u.d + w(u, v)$
2. $v.d = u.d + w(u, v)$
3. $v.\pi = u$

Lemma 24.14, Convergence property: Let $s \rightsquigarrow u \rightarrow v$ is a shortest path. If $u.d = \delta(s, u)$ holds before RELAX$(u, v, w)$ is called, then $v.d = \delta(s, v)$ after the call.

Proof: $v.d \leq u.d + w(u, v) = \delta(s, u) + w(u, v)$
Properties of shortest paths and relaxation

\textsc{Relax}(u, v, w)

1. if \( v.d > u.d + w(u, v) \)
2. \( v.d = u.d + w(u, v) \)
3. \( v.\pi = u \)

\textbf{Lemma 24.14, Convergence property}: Let \( s \leadsto u \rightarrow v \) is a shortest path. If \( u.d = \delta(s, u) \) holds before \textsc{Relax}(u, v, w) is called, then \( v.d = \delta(s, v) \) after the call.

\textbf{Proof}: \( v.d \leq u.d + w(u, v) = \delta(s, u) + w(u, v) = \delta(s, v) \).
Properties of shortest paths and relaxation

\texttt{RELAX}(u, v, w)

1. if \( v.d > u.d + w(u, v) \)
2. \( v.d = u.d + w(u, v) \)
3. \( v.\pi = u \)

\textbf{Lemma 24.14, Convergence property}: Let \( s \leadsto u \rightarrow v \) is a shortest path. If \( u.d = \delta(s, u) \) holds before \texttt{RELAX}(u, v, w) is called, then \( v.d = \delta(s, v) \) after the call.

\textbf{Proof}: \( v.d \leq u.d + w(u, v) = \delta(s, u) + w(u, v) = \delta(s, v) \). So \( v.d = \delta(s, v) \).
Chapter 24. Single Source Shortest Paths

We want to prove that, if a shortest path \( s \leadsto v \) consists of \( k \) edges, Bellman-Fold obtains value \( v.d = \delta(s,v) \) after the \( k \)-th round of relaxation (assuming there is no negative cycle).
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Proof idea: Induction on \( k \).
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Proof idea: Induction on $k$.

- $k = 0$, $v$ can only be $s$. Proved!
We want to prove that, if a shortest path $s \leadsto v$ consists of $k$ edges, Bellman-Fold obtains value $v.d = \delta(s, v)$ after the $k$th round of relaxation (assuming there is no negative cycle).

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- Assume the claim is proved for all vertices $v$ that have a shortest path of length $k$. 

What claim again??
We want to prove that, if a shortest path \( s \leadsto v \) consists of \( k \) edges, Bellman-Fold obtains value \( v.d = \delta(s, v) \) after the \( k \)th round of relaxation (assuming there is no negative cycle).

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We want to prove that, if a shortest path $s \leadsto v$ consists of $k$ edges, Bellman-Fold obtains value $v.d = \delta(s, v)$ after the $k$th round of relaxation (assuming there is no negative cycle).

Proof idea: Induction on $k$.

- $k = 0$, $v$ can only be $s$. Proved!
- Assume the claim is proved for all vertices $v$ that have a shortest path of length $k$. What claim again??
- Let $v$ be any vertex that has a shortest path $s \leadsto u \rightarrow v$, consisting of $k + 1$ edges;
We want to prove that, if a shortest path $s \rightsquigarrow v$ consists of $k$ edges, Bellman-Fold obtains value $v.d = \delta(s, v)$ after the $k$th round of relaxation (assuming there is no negative cycle).

Proof idea: Induction on $k$.

• $k = 0$, $v$ can only be $s$. Proved!

• Assume the claim is proved for all vertices $v$ that have a shortest path of length $k$. What claim again??

• Let $v$ be any vertex that has a shortest path $s \rightsquigarrow u \rightarrow v$, consisting of $k + 1$ edges;

  Then $s \rightsquigarrow u$ is a shortest path for $u$ consisting of $k$ edges;
We want to prove that, if a shortest path $s \leadsto v$ consists of $k$ edges, Bellman-Fold obtains value $v.d = \delta(s, v)$ after the $k$th round of relaxation (assuming there is no negative cycle).

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- $k = 0$, $v$ can only be $s$. Proved!
- Assume the claim is proved for all vertices $v$ that have a shortest path of length $k$. What claim again??
- Let $v$ be any vertex that has a shortest path $s \leadsto u \rightarrow v$, consisting of $k + 1$ edges;
  
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  Now by assumption, $u.d = \delta(s, u)$ after $k$ round of relaxation.
Chapter 24. Single Source Shortest Paths

We want to prove that, if a shortest path \( s \leadsto v \) consists of \( k \) edges, Bellman-Fold obtains value \( v.d = \delta(s, v) \) after the \( k \)th round of relaxation (assuming there is no negative cycle).

Proof idea: Induction on \( k \).

- \( k = 0 \), \( v \) can only be \( s \). Proved!

- Assume the claim is proved for all vertices \( v \) that have a shortest path of length \( k \). What claim again??

- Let \( v \) be any vertex that has a shortest path \( s \leadsto u \rightarrow v \), consisting of \( k + 1 \) edges;

Then \( s \leadsto u \) is a shortest path for \( u \) consisting of \( k \) edges;

Now by assumption, \( u.d = \delta(s, u) \) after \( k \) round of relaxation.

By Convergence property Lemma, \( v.d = \delta(s, v) \) after another round of relaxation.
Lemma 24.15, Path-relaxation property: Let $p = (v_0, v_1, \ldots, v_k)$ be a shortest path from $s = v_0$ to $v_k$. If a sequence relaxation steps occur that includes, in order, relaxing the edges $(v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k)$, then $v_k.d = \delta(s, v_k)$ after these relaxations and at all times afterward. This property holds no matter what other edge relaxations occur, including relaxations that are intermixed with relaxations of these edges.
Lemma 24.15, Path-relaxation property: Let $p = (v_0, v_1, \ldots, v_k)$ be a shortest path from $s = v_0$ to $v_k$. If a sequence relaxation steps occur that includes, in order, relaxing the edges $(v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k)$, then $v_k.d = \delta(s, v_k)$ after these relaxations and at all times afterward. This property holds no matter what other edge relaxations occur, including relaxations that are intermixed with relaxations of these edges.

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Proof: We prove by induction on $i$ that after the $i$th edge $(v_{i-1}, v_i)$ on path $p$ is relaxed, $v_i.d = \delta(s, v_i)$. 
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**Proof:** We prove by induction on \( i \) that after the \( i \)th edge \((v_{i-1}, v_i)\) on path \( p \) is relaxed, \( v_i.d = \delta(s, v_i) \).

**basis:** \( i = 0. \) \( v_0 = s, \) \( s.d = 0 = \delta(s, s) \)!
Chapter 24. Single Source Shortest Paths

**Lemma 24.15, Path-relaxation property:** Let \( p = (v_0, v_1, \ldots, v_k) \) be a shortest path from \( s = v_0 \) to \( v_k \). If a sequence of relaxation steps occur that includes, in order, relaxing the edges \((v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k)\), then \( v_k.d = \delta(s, v_k) \) after these relaxations and at all times afterward. This property holds no matter what other edge relaxations occur, including relaxations that are intermixed with relaxations of these edges.

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**Assume:** \( v_{i-1}.d = \delta(s, v_{i-1}) \).
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**basis**: $i = 0$. $v_0 = s$, $s.d = 0 = \delta(s, s)$!

**Assume**: $v_{i-1}.d = \delta(s, v_{i-1})$.

**Induction**: After we relax edge $(v_{i-1}, v_i)$, by convergence property, we have $v_i.d = \delta(s, v_i)$. 
Lemma 24.15, Path-relaxation property: Let \( p = (v_0, v_1, \ldots, v_k) \) be a shortest path from \( s = v_0 \) to \( v_k \). If a sequence relaxation steps occur that includes, in order, relaxing the edges \( (v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k) \), then \( v_k.d = \delta(s, v_k) \) after these relaxations and at all times afterward. This property holds no matter what other edge relaxations occur, including relaxations that are intermixed with relaxations of these edges.

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Induction: After we relax edge \( (v_{i-1}, v_i) \), by convergence property, we have \( v_i.d = \delta(s, v_i) \). And this holds for all times afterward.
Chapter 24. Single Source Shortest Paths

Correctness of Bellman-Ford algorithm
Chapter 24. Single Source Shortest Paths

Correctness of Bellman-Ford algorithm

1. On graphs without negative cycles)
Correctness of Bellman-Ford algorithm

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Lemma 24.2 Let $G = (V, E)$ be a weighted, directed graph with source $s$ and weight function $w : E \rightarrow \mathbb{R}$ and assume that $G$ contains no negative weight cycles that can be reached from $s$. Then after $|V| - 1$ iterations of line 5 in the algorithm, $v.d = \delta(s, v)$ for all vertices $v$ that are reachable from $s$. 
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Correctness of **Bellman-Ford** algorithm

1. **On graphs without negative cycles**

**Lemma 24.2** Let $G = (V, E)$ be a weighted, directed graph with source $s$ and weight function $w : E \rightarrow R$ and assume that $G$ contains no negative weight cycles that can be reached from $s$. Then after $|V| - 1$ iterations of line 5 in the algorithm, $v.d = \delta(s, v)$ for all vertices $v$ that are reachable from $s$.

**Proof**: (By induction on $k$,
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Proof: (By induction on $k$, the number of edges on the computed path $p: s \xrightarrow{p} v$, to prove the claim to be true).
Chapter 24. Single Source Shortest Paths

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Chapter 24. Single Source Shortest Paths

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Proof: (By induction on $k$, the number of edges on the computed path $p$: $s \xrightarrow{p} v$, to prove the claim to be true).

   Base: $k = 0$. $v = s$. It is true.
Correctness of Bellman-Ford algorithm

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Proof: (By induction on $k$, the number of edges on the computed path $p: s \xrightarrow{P} v$, to prove the claim to be true).

Base: $k = 0. \ v = s$. It is true.
Assume: the claim is true for $k - 1$. 

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Correctness of Bellman-Ford algorithm

1. On graphs without negative cycles)

**Lemma 24.2** Let $G = (V, E)$ be a weighted, directed graph with source $s$ and weight function $w : E \to \mathbb{R}$ and assume that $G$ contains no negative weight cycles that can be reached from $s$. Then after $|V| - 1$ iterations of line 5 in the algorithm, $v.d = \delta(s, v)$ for all vertices $v$ that are reachable from $s$.

**Proof:** (By induction on $k$, the number of edges on the computed path $p : s \xrightarrow{p} v$, to prove the claim to be true).

- **Base:** $k = 0$. $v = s$. It is true.
- **Assume:** the claim is true for $k - 1$.
- **Induction:** computed path $p : s \xrightarrow{p} v$ has $k$ edges and $p$ arrives at $x$ before reaching $v$ via $(x, v)$. 


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- **Base**: $k = 0$. $v = s$. It is true.
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- **Induction**: computed path $p: s \xrightarrow{p} v$ has $k$ edges and
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Chapter 24. Single Source Shortest Paths

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Assume: the claim is true for $k - 1$.

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By Lemma 24.1, $\delta(s, v) = \delta(s, y) + w(y, v)$ for some $y$. 
Chapter 24. Single Source Shortest Paths

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\( p \) arrives at \( x \) before reaching \( v \) via \((x, v)\). So \( v.d = x.d + w(x, v) \)

By Lemma 24.1, \( \delta(s, v) = \delta(s, y) + w(y, v) \) for some \( y \).

Since after \( k \) iterations, \( v.d \) has been updated with the statement

**if** \( v.d > u.d + w(u, v) \) **then** \( v.d = u.d + w(u, v) \), for all \( u \), including \( x, y \)
Chapter 24. Single Source Shortest Paths

Correctness of Bellman-Ford algorithm

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Proof: (By induction on $k$, the number of edges on the computed path $p$: $s \xrightarrow{p} v$, to prove the claim to be true).

Base: $k = 0$. $v = s$. It is true.

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Induction: computed path $p$: $s \xrightarrow{p} v$ has $k$ edges and $p$ arrives at $x$ before reaching $v$ via $(x, v)$. So $v.d = x.d + w(x, v)$

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Since after $k$ iterations, $v.d$ has been updated with the statement if $v.d > u.d + w(u, v)$ then $v.d = u.d + w(u, v)$, for all $u$, including $x$, $y$.

By the assumption, for every $u$, including $x$ and $y$, $u.d = \delta(s, u)$ because the computed path $s \xrightarrow{} u$ contains $k - 1$ edges. So we have
Chapter 24. Single Source Shortest Paths

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Lemma 24.2 Let \( G = (V, E) \) be a weighted, directed graph with source \( s \) and weight function \( w : E \to \mathbb{R} \) and assume that \( G \) contains no negative weight cycles that can be reached from \( s \). Then after \( |V| - 1 \) iterations of line 5 in the algorithm, \( v.d = \delta(s, v) \) for all vertices \( v \) that are reachable from \( s \).

Proof: (By induction on \( k \), the number of edges on the computed path \( p: s \overset{p}{\to} v \), to prove the claim to be true).

Base: \( k = 0. \ v = s \). It is true.
Assume: the claim is true for \( k - 1 \).
Induction: computed path \( p: s \overset{p}{\to} v \) has \( k \) edges and

\( p \) arrives at \( x \) before reaching \( v \) via \( (x, v) \). So \( v.d = x.d + w(x, v) \)

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\( v.d = x.d + w(x, v) \)
Correctness of Bellman-Ford algorithm

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Proof: (By induction on $k$, the number of edges on the computed path $p$: $s \xrightarrow{p} v$, to prove the claim to be true).

Base: $k = 0$. $v = s$. It is true.

Assume: the claim is true for $k - 1$.

Induction: computed path $p$: $s \xrightarrow{p} v$ has $k$ edges and $p$ arrives at $x$ before reaching $v$ via $(x, v)$. So $v.d = x.d + w(x, v)$

By Lemma 24.1, $\delta(s, v) = \delta(s, y) + w(y, v)$ for some $y$.

Since after $k$ iterations, $v.d$ has been updated with the statement if $v.d > u.d + w(u, v)$ then $v.d = u.d + w(u, v)$, for all $u$, including $x$, $y$

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$$v.d = x.d + w(x, v) \leq y.d + w(y, v)$$
Correctness of Bellman-Ford algorithm

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Base: $k = 0$. $v = s$. It is true.
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Chapter 24. Single Source Shortest Paths

**Theorem 24.4** Bellman-Ford algorithm is correct on weighted, directed graphs.

Proof: By Lemma 24.2, we only need to show, when $G$ contains a negative weight cycle reachable from $s$, the algorithm returns FALSE. Let the cycle to be $c = (v_0, v_1, \ldots, v_k)$, where $v_0 = v_k$ and $k \sum_{i=1}^{k} w(v_{i-1}, v_i) < 0$. Assume for all $i$, $v_i.d \leq v_{i-1}.d + w(v_{i-1}, v_i)$. Then $k \sum_{i=1}^{k} v_i.d \leq k \sum_{i=1}^{k} v_{i-1}.d + w(v_{i-1}, v_i)$, implying $k \sum_{i=1}^{k} w(v_{i-1}, v_i) \geq 0$ contradicting $c$ being a negative cycle where $k \sum_{i=1}^{k} w(v_{i-1}, v_i) < 0$. 


Chapter 24. Single Source Shortest Paths

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$$\sum_{i=1}^{k} w(v_{i-1}, v_i) < 0$$

Assume for all $i$, $v_i.d \leq v_{i-1}.d + w(v_{i-1}, v_i)$. 
Chapter 24. Single Source Shortest Paths

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Let the cycle to be $c = (v_0, v_1, \ldots, v_k)$, where $v_0 = v_k$ and

$$\sum_{i=1}^{k} w(v_{i-1}, v_i) < 0$$

Assume for all $i$, $v_i.d \leq v_{i-1}.d + w(v_{i-1}, v_i)$.

$$\sum_{i=1}^{k} v_i.d \leq \sum_{i=1}^{k} v_{i-1}.d + w(v_{i-1}, v_i),$$
Theorem 24.4 Bellman-Ford algorithm is correct on weighted, directed graphs.

Proof: By Lemma 24.2, we only need to show, when $G$ contains a negative weight cycle reachable from $s$, the algorithm returns FALSE.

Let the cycle to be $c = (v_0, v_1, \cdots, v_k)$, where $v_0 = v_k$ and
\[
\sum_{i=1}^{k} w(v_{i-1}, v_i) < 0
\]

Assume for all $i$, $v_i.d \leq v_{i-1}.d + w(v_{i-1}, v_i)$.

\[
\sum_{i=1}^{k} v_i.d \leq \sum_{i=1}^{k} v_{i-1}.d + w(v_{i-1}, v_i),
\]
Chapter 24. Single Source Shortest Paths

Theorem 24.4 Bellman-Ford algorithm is correct on weighted, directed graphs.

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Let the cycle to be $c = (v_0, v_1, \cdots, v_k)$, where $v_0 = v_k$ and

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Assume for all $i$, $v_i.d \leq v_{i-1}.d + w(v_{i-1}, v_i)$.

$$\sum_{i=1}^{k} v_i.d \leq \sum_{i=1}^{k} v_{i-1}.d + w(v_{i-1}, v_i),$$

But

$$\sum_{i=1}^{k} v_i.d = \sum_{i=1}^{k} v_{i-1}.d,$$
Chapter 24. Single Source Shortest Paths

**Theorem 24.4** Bellman-Ford algorithm is correct on weighted, directed graphs.

**Proof:** By Lemma 24.2, we only need to show, when $G$ contains a negative weight cycle reachable from $s$, the algorithm returns FALSE.

Let the cycle to be $c = (v_0, v_1, \cdots, v_k)$, where $v_0 = v_k$ and

$$\sum_{i=1}^{k} w(v_{i-1}, v_i) < 0$$

**Assume for all** $i$, $v_i.d \leq v_{i-1}.d + w(v_{i-1}, v_i)$.

$$\sum_{i=1}^{k} v_i.d \leq \sum_{i=1}^{k} v_{i-1}.d + w(v_{i-1}, v_i),$$

But

$$\sum_{i=1}^{k} v_i.d = \sum_{i=1}^{k} v_{i-1}.d, \ \text{implying} \ \sum_{i=1}^{k} w(v_{i-1}, v_i) \geq 0$$
Chapter 24. Single Source Shortest Paths

**Theorem 24.4** Bellman-Ford algorithm is correct on weighted, directed graphs.

**Proof:** By Lemma 24.2, we only need to show, when $G$ contains a negative weight cycle reachable from $s$, the algorithm returns FALSE.

Let the cycle to be $c = (v_0, v_1, \ldots, v_k)$, where $v_0 = v_k$ and

$$
\sum_{i=1}^{k} w(v_{i-1}, v_i) < 0
$$

Assume for all $i$, $v_i.d \leq v_{i-1}.d + w(v_{i-1}, v_i)$.

$$
\sum_{i=1}^{k} v_i.d \leq \sum_{i=1}^{k} v_{i-1}.d + w(v_{i-1}, v_i),
$$

But

$$
\sum_{i=1}^{k} v_i.d = \sum_{i=1}^{k} v_{i-1}.d, \quad \text{implying} \quad \sum_{i=1}^{k} w(v_{i-1}, v_i) \geq 0
$$

contradicting to $c$ being a negative cycle where $\sum_{i=1}^{k} w(v_{i-1}, v_i) < 0$
Chapter 24. Single Source Shortest Paths

Finding shortest paths on DAGs (directed acyclic graphs)
Finding shortest paths on DAGs (directed acyclic graphs)

- Algorithms can take the advantage of the non-cyclicity.
Chapter 24. Single Source Shortest Paths

Finding shortest paths on DAGs (directed acyclic graphs)

- Algorithms can take the advantage of the non-cyclicity.
- How would your algorithm be?
Finding shortest paths on DAGs (directed acyclic graphs)

- Algorithms can take the advantage of the non-cyclicity.
- How would your algorithm be?

**topological order of vertices**
Chapter 24. Single Source Shortest Paths

**Dag-Shortest Paths**($G, w, s$)

1. topologically sort the vertices of $G.V$
2. for each vertex $v \in G.V$
3. $v.d = \infty$
4. $v.\pi = \text{NULL}$
5. $s.d = 0$
6. for each $u \in G.V$, in the topologically sorted order
7. for each vertex $v \in \text{Adj}[u]$
8. if $v.d > u.d + w(u, v)$
9. $v.d = u.d + w(u, v)$
10. $v.\pi = u$
11. return $(d, \pi)$

• Should we improve lines 6-7?
• Running time: ?
Chapter 24. Single Source Shortest Paths

\[
\text{Dag-Shortest Paths}(G, w, s)
\]
1. topologically sort the vertices of \( G.V \)
Dag-Shortest Paths \((G, w, s)\)
1. topologically sort the vertices of \(G.V\)
2. for each vertex \(v \in G.V\)
Chapter 24. Single Source Shortest Paths

**Dag-Shortest Paths** \((G, w, s)\)
1. topologically sort the vertices of \(G.V\)
2. for each vertex \(v \in G.V\)
3. \(v.d = \infty\)

**Dag-Shortest Paths**

1. topologically sort the vertices of \(G.V\)
2. for each vertex \(v \in G.V\)
3. \(v.d = \infty\)
Chapter 24. Single Source Shortest Paths

**DAG-Shortest Paths** ($G, w, s$)

1. topologically sort the vertices of $G.V$
2. for each vertex $v \in G.V$
3. $v.d = \infty$
4. $v.\pi = NULL$

Should we improve lines 6-7?

Running time: ?
Dag-Shortest Paths($G, w, s$)
1. topologically sort the vertices of $G.V$
2. for each vertex $v \in G.V$
3. \hspace{1cm} $v.d = \infty$
4. \hspace{1cm} $v.\pi = NULL$
5. \hspace{1cm} $s.d = 0$

Should we improve lines 6-7?

Running time: ?
Chapter 24. Single Source Shortest Paths

**DAG-Shortest Paths**\((G, w, s)\)
1. topologically sort the vertices of \(G.V\)
2. **for** each vertex \(v \in G.V\)
3. \(v.d = \infty\)
4. \(v.\pi = NULL\)
5. \(s.d = 0\)
6. **for** each \(u \in G.V\), in the topologically sorted order
Chapter 24. Single Source Shortest Paths

**Dag-Shortest Paths**($G, w, s$)
1. topologically sort the vertices of $G.V$
2. for each vertex $v \in G.V$
3. $v.d = \infty$
4. $v.\pi = NULL$
5. $s.d = 0$
6. for each $u \in G.V$, in the topologically sorted order
7. for each vertex $v \in Adj[u]$
Chapter 24. Single Source Shortest Paths

**Dag-Shortest Paths** $(G, w, s)$

1. topologically sort the vertices of $G.V$
2. for each vertex $v \in G.V$
3. \hspace{1em} $v.d = \infty$
4. \hspace{1em} $v.\pi = NULL$
5. \hspace{1em} $s.d = 0$
6. for each $u \in G.V$, in the topologically sorted order
7. \hspace{1em} for each vertex $v \in Adj[u]$
8. \hspace{1em} if $v.d > u.d + w(u, v)$

Should we improve lines 6-7?

Running time: ?
Chapter 24. Single Source Shortest Paths

**DAG-Shortest Paths**$(G, w, s)$
1. topologically sort the vertices of $G.V$
2. for each vertex $v \in G.V$
3. $v.d = \infty$
4. $v.\pi = NULL$
5. $s.d = 0$
6. for each $u \in G.V$, in the topologically sorted order
7. for each vertex $v \in Adj[u]$
8. if $v.d > u.d + w(u, v)$
9. $v.d = u.d + w(u, v)$
**Chapter 24. Single Source Shortest Paths**

**Dag-Shortest Paths** $(G, w, s)$

1. topologically sort the vertices of $G.V$
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8. if $v.d > u.d + w(u,v)$
9. $v.d = u.d + w(u,v)$
10. $v.\pi = u$

Should we improve lines 6-7?

Running time: ?
Dag-Shortest Paths($G, w, s$)
1. topologically sort the vertices of $G.V$
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9. $v.d = u.d + w(u, v)$
10. $v.\pi = u$
11. return $(d, \pi)$
Chapter 24. Single Source Shortest Paths

**Dag-Shortest Paths** \((G, w, s)\)
1. topologically sort the vertices of \(G.V\)
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9. \(v.d = u.d + w(u, v)\)
10. \(v.\pi = u\)
11. return \((d, \pi)\)

- Should we improve lines 6-7?

Running time: ?
Chapter 24. Single Source Shortest Paths

**Dag-Shortest Paths** \((G, w, s)\)

1. topologically sort the vertices of \(G.V\)
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7. for each vertex \(v \in Adj[u]\)
8. if \(v.d > u.d + w(u, v)\)
9. \(v.d = u.d + w(u, v)\)
10. \(v.\pi = u\)
11. return \((d, \pi)\)

- Should we improve lines 6-7?
- Running time: ?
Chapter 24. Single Source Shortest Paths

note: the root is $s$. 
Chapter 24. Single Source Shortest Paths

Dijkstra’s algorithm

On weighted, directed graphs in which each edge has non-negative weight.
Dijkstra’s algorithm
On weighted, directed graphs in which each edge has non-negative weight.
\textsc{Dijkstra}(G, w, s)
Chapter 24. Single Source Shortest Paths

**Dijkstra’s algorithm**

On weighted, directed graphs in which each edge has non-negative weight.

\[
\text{DIJKSTRA}(G, w, s)
\]

1. for each vertex \( v \in G.V \)
Chapter 24. Single Source Shortest Paths

Dijkstra’s algorithm

On weighted, directed graphs in which each edge has non-negative weight.

$\text{DIJKSTRA}(G, w, s)$
1. for each vertex $v \in G.V$
2. $v.d = \infty$

Running time:?
Chapter 24. Single Source Shortest Paths

**Dijkstra's algorithm**

On weighted, directed graphs in which each edge has non-negative weight.

\[
\text{Dijkstra}(G, w, s)
\]

1. for each vertex \( v \in G.V \)
2. \( v.d = \infty \)
3. \( v.\pi = NULL \)

Dijkstra's algorithm

On weighted, directed graphs in which each edge has non-negative weight.

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Dijkstra’s algorithm

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2. \( v.d = \infty \)
3. \( v.\pi = NULL \)
4. \( s.d = 0 \)
Dijkstra’s algorithm

On weighted, directed graphs in which each edge has non-negative weight.

\textbf{Dijkstra}(G, w, s)
1. for each vertex \( v \in G.V \)
2. \( v.d = \infty \)
3. \( v.\pi = NULL \)
4. \( s.d = 0 \)
5. \( S = \emptyset \)
Dijkstra’s algorithm

On weighted, directed graphs in which each edge has non-negative weight.

\[
\text{DIJKSTRA}(G, w, s)
\]

1. for each vertex \( v \in G.V \)
2. \( v.d = \infty \)
3. \( v.\pi = NULL \)
4. \( s.d = 0 \)
5. \( S = \emptyset \)
6. \( Q = G.V \)
Chapter 24. Single Source Shortest Paths

Dijkstra’s algorithm

On weighted, directed graphs in which each edge has non-negative weight.

\textsc{Dijkstra}(G, w, s)
1. for each vertex \( v \in G.V \)
2. \( v.d = \infty \)
3. \( v.\pi = NULL \)
4. \( s.d = 0 \)
5. \( S = \emptyset \)
6. \( Q = G.V \)
7. \textbf{while} \( Q \) is not empty

Running time: ?
Dijkstra’s algorithm

On weighted, directed graphs in which each edge has non-negative weight.

\text{Dijkstra}(G, w, s)
1. for each vertex \( v \in G.V \)
2. \( v.d = \infty \)
3. \( v.\pi = NULL \)
4. \( s.d = 0 \)
5. \( S = \emptyset \)
6. \( Q = G.V \)
7. while \( Q \) is not empty
8. \( u = \text{Extract Min} (Q) \)
Dijkstra’s algorithm

On weighted, directed graphs in which each edge has non-negative weight.

\[ \text{DIJKSTRA}(G, w, s) \]
1. for each vertex \( v \in G.V \)
2. \( v.d = \infty \)
3. \( v.\pi = NULL \)
4. \( s.d = 0 \)
5. \( S = \emptyset \)
6. \( Q = G.V \)
7. \textbf{while} \( Q \) is not empty
8. \( u = \text{EXTRACT MIN} (Q) \)
9. \( S = S \cup \{u\} \)
Chapter 24. Single Source Shortest Paths

Dijkstra’s algorithm

On weighted, directed graphs in which each edge has non-negative weight.

Dijkstra\((G, w, s)\)
1. for each vertex \(v \in G.V\)
2. \(v.d = \infty\)
3. \(v.\pi = NULL\)
4. \(s.d = 0\)
5. \(S = \emptyset\)
6. \(Q = G.V\)
7. while \(Q\) is not empty
8. \(u = \text{Extract Min} (Q)\)
9. \(S = S \cup \{u\}\)
10. for each vertex \(v \in Adj[u]\)
Dijkstra’s algorithm

On weighted, directed graphs in which each edge has non-negative weight.

\textbf{Dijkstra}(G, w, s)
1. for each vertex \( v \in G.V \)
2. \( v.d = \infty \)
3. \( v.\pi = NULL \)
4. \( s.d = 0 \)
5. \( S = \emptyset \)
6. \( Q = G.V \)
7. while \( Q \) is not empty
8. \( u = \text{ Extract Min } (Q) \)
9. \( S = S \cup \{u\} \)
10. for each vertex \( v \in Adj[u] \)
11. if \( v.d > u.d + w(u, v) \)
Chapter 24. Single Source Shortest Paths

**Dijkstra’s algorithm**

On weighted, directed graphs in which each edge has non-negative weight.

\[ \text{Dijkstra}(G, w, s) \]

1. for each vertex \( v \in G.V \)
2. \( v.d = \infty \)
3. \( v.\pi = NULL \)
4. \( s.d = 0 \)
5. \( S = \emptyset \)
6. \( Q = G.V \)
7. while \( Q \) is not empty
   8. \( u = \text{Extract Min} (Q) \)
   9. \( S = S \cup \{u\} \)
   10. for each vertex \( v \in Adj[u] \)
      11. if \( v.d > u.d + w(u, v) \)
      12. \( v.d = u.d + w(u, v) \)

Running time:?
Dijkstra’s algorithm

On weighted, directed graphs in which each edge has non-negative weight.

\textsc{Dijkstra}(G,w,s)
1. for each vertex \( v \in G.V \)
2. \( v.d = \infty \)
3. \( v.\pi = NULL \)
4. \( s.d = 0 \)
5. \( S = \emptyset \)
6. \( Q = G.V \)
7. while \( Q \) is not empty
8. \( u = \text{Extract Min} (Q) \)
9. \( S = S \cup \{u\} \)
10. for each vertex \( v \in \text{Adj}[u] \)
11. if \( v.d > u.d + w(u,v) \)
12. \( v.d = u.d + w(u,v) \)
13. \( v.\pi = u \)
Chapter 24. Single Source Shortest Paths

Dijkstra’s algorithm

On weighted, directed graphs in which each edge has non-negative weight.

Dijkstra\((G, w, s)\)
1. for each vertex \(v \in G.V\)
2. \(v.d = \infty\)
3. \(v.\pi = NULL\)
4. \(s.d = 0\)
5. \(S = \emptyset\)
6. \(Q = G.V\)
7. while \(Q\) is not empty
8. \(u = \text{EXTRACT MIN} (Q)\)
9. \(S = S \cup \{u\}\)
10. for each vertex \(v \in \text{Adj}[u]\)
11. if \(v.d > u.d + w(u, v)\)
12. \(v.d = u.d + w(u, v)\)
13. \(v.\pi = u\)
14. return \((d, \pi)\)
Dijkstra’s algorithm

On weighted, directed graphs in which each edge has non-negative weight.

**Dijkstra**(G, w, s)

1. for each vertex \( v \in G.V \)
2. \( v.d = \infty \)
3. \( v.\pi = NULL \)
4. \( s.d = 0 \)
5. \( S = \emptyset \)
6. \( Q = G.V \)
7. while \( Q \) is not empty
8. \( u = \text{Extract Min} (Q) \)
9. \( S = S \cup \{u\} \)
10. for each vertex \( v \in \text{Adj}[u] \)
11. if \( v.d > u.d + w(u, v) \)
12. \( v.d = u.d + w(u, v) \)
13. \( v.\pi = u \)
14. return \((d, \pi)\)

Running time:?
Chapter 24. Single Source Shortest Paths

Note: the black-colored vertices are in set $S$. 
Chapter 24. Single Source Shortest Paths

Correctness of algorithm Dijkstra
Chapter 24. Single Source Shortest Paths

**Correctness of algorithm Dijkstra**

**Theorem 24.6** Dijkstra’s algorithm, run on a weighted, directed graph $G = (V, E)$ with non-negative weight function $w$ and source $s$, terminates with $u.d = \delta(s, u)$ for all vertices $u \in V$. 

Proof: We need to show the while loop has loop invariant:

$u.d = \delta(s, u)$ for each $u \in S$

Assume $u$ to be the first such vertex that $u.d > \delta(s, u)$ when it is being added to $S$ then there must be a shortest path $p: s \rightarrow x \rightarrow y \rightarrow u$, for some $x \in S$ and some $y \notin S$.

$y.d = \delta(s, y)$ when $u$ is being added to $S$. This is because $x \in S$, $x.d = \delta(s, x)$ when $x$ was added to $S$.

Edge $(x, y)$ was related at that time, and $y.d = \delta(s, y)$ by Convergence-property.

So when $u$ was chosen, $u.d \leq y.d = \delta(s, y) \leq \delta(s, u)$. Contradicts the choice of $u$.

So $u.d = \delta(s, u)$ when it is being included to $S$. 
Correctness of algorithm \textsc{Dijkstra}

\textbf{Theorem 24.6} Dijkstra’s algorithm, run on a weighted, directed graph $G = (V, E)$ with non-negative weight function $w$ and source $s$, terminates with $u.d = \delta(s, u)$ for all vertices $u \in V$.

\textbf{Proof:} We need to show the \texttt{while} loop has loop invariant:
Correctness of algorithm Dijkstra

**Theorem 24.6** Dijkstra’s algorithm, run on a weighted, directed graph $G = (V, E)$ with non-negative weight function $w$ and source $s$, terminates with $u.d = \delta(s, u)$ for all vertices $u \in V$.

**Proof:** We need to show the **while** loop has **loop invariant**: $u.d = \delta(s, u)$ for each $u \in S$
Chapter 24. Single Source Shortest Paths

Correctness of algorithm Dijkstra

**Theorem 24.6** Dijkstra’s algorithm, run on a weighted, directed graph $G = (V, E)$ with non-negative weight function $w$ and source $s$, terminates with $u.d = \delta(s, u)$ for all vertices $u \in V$.

**Proof:** We need to show the while loop has loop invariant:

$u.d = \delta(s, u)$ for each $u \in S$

Assume $u$ to be the first such vertex that $u.d > \delta(s, u)$ when it is being added to $S$.
Chapter 24. Single Source Shortest Paths

Correctness of algorithm Dijkstra

**Theorem 24.6** Dijkstra's algorithm, run on a weighted, directed graph \( G = (V, E) \) with non-negative weight function \( w \) and source \( s \), terminates with \( u.d = \delta(s, u) \) for all vertices \( u \in V \).

**Proof:** We need to show the **while** loop has **loop invariant**: \( u.d = \delta(s, u) \) for each \( u \in S \)

Assume \( u \) to be the first such vertex that \( u.d > \delta(s, u) \) when it is being added to \( S \) then there must be a shortest path \( p: s \rightsquigarrow x \rightarrow y \rightsquigarrow u \), for some \( x \in S \) and some \( y \not\in S \).
Correctness of algorithm Dijkstra

**Theorem 24.6** Dijkstra’s algorithm, run on a weighted, directed graph $G = (V, E)$ with non-negative weight function $w$ and source $s$, terminates with $u.d = \delta(s, u)$ for all vertices $u \in V$.

**Proof:** We need to show the while loop has loop invariant:
$u.d = \delta(s, u)$ for each $u \in S$

Assume $u$ to be the first such vertex that $u.d > \delta(s, u)$ when it is being added to $S$ then there must be a shortest path $p: s \leadsto x \rightarrow y \leadsto u$, for some $x \in S$ and some $y \notin S$.

$y.d = \delta(s, y)$ when $u$ is being added to $S$. This is because $x \in S$, $x.d = \delta(s, x)$ when $x$ was added to $S$. 

Chapter 24. Single Source Shortest Paths

Correctness of algorithm \textbf{Dijkstra}

\textbf{Theorem 24.6} Dijkstra’s algorithm, run on a weighted, directed graph $G = (V, E)$ with non-negative weight function $w$ and source $s$, terminates with $u.d = \delta(s, u)$ for all vertices $u \in V$.

\textbf{Proof:} We need to show the \textbf{while} loop has \textbf{loop invariant}: 
$u.d = \delta(s, u)$ for each $u \in S$

Assume $u$ to be the first such vertex that $u.d > \delta(s, u)$ when it is being added to $S$ then there must be a shortest path $p$: $s \xrightarrow{} x \xrightarrow{} y \xrightarrow{} u$, for some $x \in S$ and some $y \notin S$.

$y.d = \delta(s, y)$ when $u$ is being added to $S$. This is because $x \in S$, $x.d = \delta(s, x)$ when $x$ was added to $S$. Edge $(x, y)$ was related at that time, and $y.d = \delta(s, y)$ by \textbf{Convergence-property}.
Chapter 24. Single Source Shortest Paths

Correctness of algorithm Dijkstra

**Theorem 24.6** Dijkstra’s algorithm, run on a weighted, directed graph $G = (V, E)$ with non-negative weight function $w$ and source $s$, terminates with $u.d = \delta(s, u)$ for all vertices $u \in V$.

**Proof:** We need to show the while loop has loop invariant:

$u.d = \delta(s, u)$ for each $u \in S$

Assume $u$ to be the first such vertex that $u.d > \delta(s, u)$ when it is being added to $S$ then there must be a shortest path $p$: $s \leadsto x \rightarrow y \leadsto u$, for some $x \in S$ and some $y \notin S$.

$y.d = \delta(s, y)$ when $u$ is being added to $S$. This is because $x \in S$, $x.d = \delta(s, x)$ when $x$ was added to $S$. Edge $(x, y)$ was related at that time, and $y.d = \delta(s, y)$ by Convergence-property. So
Chapter 24. Single Source Shortest Paths

Correctness of algorithm **Dijkstra**

**Theorem 24.6** Dijkstra’s algorithm, run on a weighted, directed graph \( G = (V, E) \) with non-negative weight function \( w \) and source \( s \), terminates with \( u.d = \delta(s, u) \) for all vertices \( u \in V \).

**Proof:** We need to show the **while** loop has **loop invariant:**
\[ u.d = \delta(s, u) \text{ for each } u \in S \]

Assume \( u \) to be the first such vertex that \( u.d > \delta(s, u) \) when it is being added to \( S \) then there must be a shortest path \( p: s \leadsto x \rightarrow y \leadsto u \), for some \( x \in S \) and some \( y \notin S \).

\( y.d = \delta(s, y) \text{ when } u \text{ is being added to } S \). This is because \( x \in S \), \( x.d = \delta(s, x) \) when \( x \) was added to \( S \). Edge \( (x, y) \) was related at that time, and \( y.d = \delta(s, y) \) by **Convergence-property**. So

When \( u \) was chosen, \( u.d \leq y.d = \delta(s, y) \leq \delta(s, u) \). **Contradicts** the choice of \( u \).
Chapter 24. Single Source Shortest Paths

Correctness of algorithm **Dijkstra**

**Theorem 24.6** Dijkstra’s algorithm, run on a weighted, directed graph $G = (V, E)$ with non-negative weight function $w$ and source $s$, terminates with $u.d = \delta(s, u)$ for all vertices $u \in V$.

**Proof:** We need to show the **while** loop has loop invariant: $u.d = \delta(s, u)$ for each $u \in S$.

Assume $u$ to be the first such vertex that $u.d > \delta(s, u)$ when it is being added to $S$ then there must be a shortest path $p$: $s \leadsto x \rightarrow y \leadsto u$, for some $x \in S$ and some $y \notin S$.

$y.d = \delta(s, y)$ when $u$ is being added to $S$. This is because $x \in S$, $x.d = \delta(s, x)$ when $x$ was added to $S$. Edge $(x, y)$ was related at that time, and $y.d = \delta(s, y)$ by **Convergence-property**. So

When $u$ was chosen, $u.d \leq y.d = \delta(s, y) \leq \delta(s, u)$. **Contradicts** the choice of $u$. So $u.d = \delta(s, u)$ when it is being included to $S$. 
Chapter 24. Single Source Shortest Paths

- Running time of Dijkstra?
Chapter 24. Single Source Shortest Paths

- Running time of Dijkstra?
- Can Dijkstra deals with negative edges or cycles?
Chapter 24. Single Source Shortest Paths

- Running time of Dijkstra?
- Can Dijkstra deals with negative edges or cycles?
- Fundamental differences between Bellman-Ford and Dijkstra?
Chapter 24. Single Source Shortest Paths

- Running time of Dijkstra?
- Can Dijkstra deals with negative edges or cycles?
- Fundamental differences between Bellman-Ford and Dijkstra?

**Dijkstra**($G, w, s$)
1. for each vertex $v \in G.V$
2. $v.d = \infty$
3. $v.\pi = \text{NULL}$
4. $s.d = 0$
5. $S = \emptyset$
6. $Q = G.V$
7. while $Q$ is not empty
   8. $u = \text{EXTRACT MIN} (Q)$
   9. $S = S \cup \{u\}$
   10. for each vertex $v \in \text{Adj}[u]$
       11. if $v.d > u.d + w(u, v)$
           12. $v.d = u.d + w(u, v)$
           13. $v.\pi = u$
14. return $(d, \pi)$

**Bellman-Ford**($G, w, s$)
1. for each vertex $v \in G.V$
2. $v.d = \infty$
3. $v.\pi = \text{NULL}$
4. $s.d = 0$
5. for $i = 1$ to $|V| - 1$
6. for each edge $(u, v) \in G.E$
7. if $v.d > u.d + w(u, v)$
8. $v.d = u.d + w(u, v)$
9. $v.\pi = u$
10. for each edge $(u, v) \in G.E$
11. if $v.d > u.d + w(u, v)$
12. return (FALSE)
13. return (TRUE)
Chapter 24. Single Source Shortest Paths
Chapter 24. Single Source Shortest Paths

- Fundamental differences between Dijkstra and MST-Prim?
Chapter 24. Single Source Shortest Paths

- Fundamental differences between **Dijkstra** and **MST-Prim**?

**Dijkstra** \( (G, w, s) \)
1. for each vertex \( v \in G.V \)
2. \( v.d = \infty \)
3. \( v.\pi = NULL \)
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5. \( S = \emptyset \)
6. \( Q = G.V \)
7. **while** \( Q \) is not empty
8. \( u = \text{EXTRACT MIN}(Q) \)
9. \( S = S \cup \{u\} \)
10. for each vertex \( v \in \text{Adj}[u] \)
11. \( \text{if} \ v.d > u.d + w(u, v) \)
12. \( v.d = u.d + w(u, v) \)
13. \( v.\pi = u \)
14. return \( (d, \pi) \)

**MST-Prim** \( (G, w, r) \)
1. for each \( u \in G.V \)
2. \( u.key = \infty \) \{ \( u.key \) is initially \( \infty \) \}
3. \( u.\pi = NULL \) \{ to establish priority \}
4. \( r.key = 0 \)
5. \( Q = G.V \)
6. **while** \( Q \neq \emptyset \)
7. \( u = \text{EXTRACT MIN}(Q) \)
8. for each \( v \in \text{Adj}[u] \)
9. \( \text{if} \ v \in Q \text{ and } w(u, v) < v.key \)
10. \( \text{then } v.\pi = u \)
11. \( v.key = w(u, v) \)
12. return \( \pi \)
Chapter 25. All-pairs shortest paths

Chapter 25. All-pairs shortest paths

All Pair Shortest Paths Problem

Input: A weighted graph $G = (V,E)$ with edge weight function $w$;
Output: Shortest paths between every pair of vertices in $G$.

- Dijkstra would run in time $O(|V|^2 \log |V| + |V||E|)$ on non-negative edges.
- Bellman-Ford would run in time $O(|V|^2 |E|)$ for general graphs, but $O(|V|^4)$ on "dense" graphs.

New algorithms:
- A dynamic programming algorithm $O(|V|^4)$, improved to $O(|V|^3 \log |V|)$.
- Floyd-Warshall algorithm: $O(|V|^3)$.

Graph representation: adjacency matrix $W = (w_{ij})$. 
Chapter 25. All-pairs shortest paths

All Pair Shortest Paths Problem
Chapter 25. All-pairs shortest paths

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Chapter 25. All-pairs shortest paths

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Chapter 25. All-pairs shortest paths

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Chapter 25. All-pairs shortest paths

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Chapter 25. All-pairs shortest paths

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New algorithms
Chapter 25. All-pairs shortest paths

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**Output**: Shortest paths between every pair of vertices in $G$.

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New algorithms

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Chapter 25. All-pairs shortest paths

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New algorithms

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Chapter 25. All-pairs shortest paths

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**Graph representation:** adjacency matrix $W = (w_{ij})$.
Chapter 25. All-pairs shortest paths

A dynamic programming approach
A dynamic programming approach

- Optimal substructure
- Objective function
Chapter 25. All-pairs shortest paths

A dynamic programming approach

• Optimal substructure
• Objective function

Define $l_{ij}$ be the minimum weight of any path from $v_i$ to $v_j$
Chapter 25. All-pairs shortest paths

A dynamic programming approach

• Optimal substructure
• Objective function

Define $l_{ij}$ be the minimum weight of any path from $v_i$ to $v_j$

**does not work!** having a data dependency issue.
Chapter 25. All-pairs shortest paths

A dynamic programming approach

- Optimal substructure
- Objective function

Define $l_{ij}$ be the minimum weight of any path from $v_i$ to $v_j$

or alternatively,

Define $l^m_{ij}$ be the minimum weight of any path from $v_i$ to $v_j$ that contains at most $m$ edges.
Chapter 25. All-pairs shortest paths

A dynamic programming approach

- Optimal substructure
- Objective function

Define $l_{ij}$ be the minimum weight of any path from $v_i$ to $v_j$ does not work! having a data dependency issue.

Define $l_{ij}^m$ be the minimum weight of any path from $v_i$ to $v_j$ that contains at most $m$ edges.

or alternatively,
Chapter 25. All-pairs shortest paths

A dynamic programming approach

- Optimal substructure
- Objective function

Define $l_{ij}$ be the minimum weight of any path from $v_i$ to $v_j$

\textbf{does not work!} having a data dependency issue.

Define $l^m_{ij}$ be the minimum weight of any path from $v_i$ to $v_j$ \textbf{that contains at most} $m$ \textbf{edges}.

or alternatively,

Define $l^k_{ij}$ be the minimum weight of any path from $v_i$ to $v_j$ \textbf{in which intermediate vertices have indexes} $\leq k$.  

Define $l^m_{ij}$ be the minimum weight of any path from $v_i$ to $v_j$ that contains at most $m$ edges.
Chapter 25. All-pairs shortest paths

Define $l^m_{ij}$ be the minimum weight of any path from $v_i$ to $v_j$ that contains at most $m$ edges.

$$l^m_{ij} = \min(l^m_{ij}, \min_{1 \leq k \leq n} \{l^m_{ik} + w_{kj}\})$$
Define $l_{ij}^m$ be the minimum weight of any path from $v_i$ to $v_j$ that contains at most $m$ edges.

$$l_{ij}^m = \min(l_{ij}^{m-1}, \min_{1 \leq k \leq n} \{l_{ik}^{m-1} + w_{kj}\})$$

If $w_{jj} = 0$, we can rewrite

$$l_{ij}^m = \min_{1 \leq k \leq n} \{l_{ik}^{m-1} + w_{kj}\}$$
Chapter 25. All-pairs shortest paths

Define $l_{ij}^m$ be the minimum weight of any path from $v_i$ to $v_j$ that contains at most $m$ edges.

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If $w_{jj} = 0$, we can rewrite

$$l_{ij}^m = \min_{1 \leq k \leq n} \{l_{ik}^{m-1} + w_{kj}\}$$

and base cases:

$$l_{ij}^1 = w_{ij}$$
Define \( l^m_{ij} \) be the minimum weight of any path from \( v_i \) to \( v_j \) that contains at most \( m \) edges.

\[
l^m_{ij} = \min(l^{m-1}_{ij}, \min_{1 \leq k \leq n} \{l^{m-1}_{ik} + w_{kj}\})
\]

If \( w_{jj} = 0 \), we can rewrite

\[
l^m_{ij} = \min_{1 \leq k \leq n} \{l^{m-1}_{ik} + w_{kj}\}
\]

and base cases:

\[
l^1_{ij} = w_{ij}
\]

Adjacency matrix \( W = (w_{ij}) \) is the default.
Chapter 25. All-pairs shortest paths

DP table filling algorithm:
Chapter 25. All-pairs shortest paths

DP table filling algorithm:
For $L^1 = W$ and $m = 2, \ldots, n - 1$, compute table $L^m$ from table $L^{m-1}$;
DP table filling algorithm:

For $L^1 = W$ and $m = 2, \ldots, n - 1$, compute table $L^m$ from table $L^{m-1}$; technically two tables are enough.
Chapter 25. All-pairs shortest paths

DP table filling algorithm:
For $L^1 = W$ and $m = 2, \ldots, n - 1$, compute table $L^m$ from table $L^{m-1}$; technically two tables are enough.

**Extended Shortest Paths**$(L, W)$

**Call** Extended Shortest Paths for $m = 2, 3, \ldots, n - 1$

$L^m \leftarrow$ Extended Shortest Paths$(L^{m-1}, W)$
Chapter 25. All-pairs shortest paths

DP table filling algorithm:

For $L^1 = W$ and $m = 2, \ldots, n - 1$, compute table $L^m$ from table $L^{m-1}$; technically two tables are enough.

\textbf{Extended Shortest Paths($L, W$)}

1. $n = \text{rows}[L]$;
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DP table filling algorithm:

For $L^1 = W$ and $m = 2, \ldots, n - 1$, compute table $L^m$ from table $L^{m-1}$; technically two tables are enough.

**Extended Shortest Paths** $(L, W)$
1. $n = \text{rows}[L]$;
2. let $L'$ be an $n \times n$ table;
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DP table filling algorithm:

For $L^1 = W$ and $m = 2, \ldots, n - 1$, compute table $L^m$ from table $L^{m-1}$; technically two tables are enough.

Extended Shortest Paths($L, W$)
1. $n = \text{rows}[L]$;
2. let $L'$ be an $n \times n$ table;
3. for $i = 1$ to $n$
Chapter 25. All-pairs shortest paths

DP table filling algorithm:

For \( L^1 = W \) and \( m = 2, \ldots, n - 1 \), compute table \( L^m \) from table \( L^{m-1} \);

technically two tables are enough.

**Extended Shortest Paths** \((L, W)\)
1. \( n = \text{rows}[L]; \)
2. let \( L' \) be an \( n \times n \) table;
3. \textbf{for} \( i = 1 \) \textbf{to} \( n \)
4. \textbf{for} \( j = 1 \) \textbf{to} \( n \)
Chapter 25. All-pairs shortest paths

DP table filling algorithm:

For $L^1 = W$ and $m = 2, \ldots, n - 1$, compute table $L^m$ from table $L^{m-1}$; technically two tables are enough.

Extended Shortest Paths $(L, W)$
1. $n = \text{rows}[L]$;
2. let $L'$ be an $n \times n$ table;
3. for $i = 1$ to $n$
4. for $j = 1$ to $n$
5. $L'[i, j] = \infty$  \quad ($L'[i, j] = L[i, j]$ in case $w_{a,a} \neq 0$)
Chapter 25. All-pairs shortest paths

DP table filling algorithm:

For $L^1 = W$ and $m = 2, \ldots , n - 1$, compute table $L^m$ from table $L^{m-1}$; technically two tables are enough.

**Extended Shortest Paths** $(L, W)$

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5.         $L'[i, j] = \infty$ \quad ($L'[i, j] = L[i, j]$ in case $w_{a,a} \neq 0$)
6.     for $k = 1$ to $n$
DP table filling algorithm:

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**Extended Shortest Paths** ($L, W$)
1. $n = \text{rows}[L]$;
2. let $L'$ be an $n \times n$ table;
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   4. for $j = 1$ to $n$
   5. $L'[i, j] = \infty$ ($L'[i, j] = L[i, j]$ in case $w_{a,a} \neq 0$)
6. for $k = 1$ to $n$
7. $L'[i, j] = \min\{L'[i, j], L[i, k] + w[k, j]\}$
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**DP table filling algorithm:**

For $L^1 = W$ and $m = 2, \ldots, n - 1$, compute table $L^m$ from table $L^{m-1}$;

- technically two tables are enough.

**Extended Shortest Paths** $(L, W)$

1. $n = \text{rows}[L]$;
2. let $L'$ be an $n \times n$ table;
3. for $i = 1$ to $n$
4.     for $j = 1$ to $n$
5.         $L'[i, j] = \infty$ \hskip 0.1cm ($L'[i, j] = L[i, j]$ in case $w_{a,a} \neq 0$)
6.     for $k = 1$ to $n$
7.         $L'[i, j] = \min\{L'[i, j], L[i, k] + w[k, j]\}$
8. return $(L')$
Chapter 25. All-pairs shortest paths

DP table filling algorithm:

For \( L^1 = W \) and \( m = 2, \ldots, n - 1 \), compute table \( L^m \) from table \( L^{m-1} \);
  technically two tables are enough.

**Extended Shortest Paths** \((L, W)\)
1. \( n = \text{rows}[L] \);
2. let \( L' \) be an \( n \times n \) table;
3. **for** \( i = 1 \) **to** \( n \)
4. \hspace{1em} **for** \( j = 1 \) **to** \( n \)
5. \hspace{2em} \( L'[i, j] = \infty \) \hspace{1em} \((L'[i, j] = L[i, j] \text{ in case } w_{a,a} \neq 0)\)
6. \hspace{1em} **for** \( k = 1 \) **to** \( n \)
7. \hspace{2em} \( L'[i, j] = \min\{L'[i, j], L[i, k] + w[k, j]\} \)
8. **return** \( (L') \)

Call **Extended Shortest Paths** for \( m = 2, 3, \ldots, n - 1 \)
Chapter 25. All-pairs shortest paths

DP table filling algorithm:

For $L^1 = W$ and $m = 2, \ldots, n - 1$, compute table $L^m$ from table $L^{m-1}$; technically two tables are enough.

**Extended Shortest Paths** $(L, W)$

1. $n = \text{rows}[L]$;
2. let $L'$ be an $n \times n$ table;
3. for $i = 1$ to $n$
4.   for $j = 1$ to $n$
5.     $L'[i, j] = \infty$ \hspace{1em} ($L'[i, j] = L[i, j]$ in case $w_{a,a} \neq 0$)
6.   for $k = 1$ to $n$
7.     $L'[i, j] = \min\{L'[i, j], L[i, k] + w[k, j]\}$
8. return ($L'$)

Call **Extended Shortest Paths** for $m = 2, 3, \ldots, n - 1$

$$L^m \leftarrow \text{Extended Shortest Paths}(L^{m-1}, W)$$
Chapter 25. All-pairs shortest paths

Running on an example:
Chapter 25. All-pairs shortest paths

Running on an example:

\[ W = L^1 = \text{the first matrix.} \]

\[
\begin{align*}
    l^2_{0,0} &= \min \left\{ \begin{array}{l}
        l^1_{0,0} \quad \text{value} = 8 \\
        l^1_{0,0} + l^1_{0,0} \quad k = 0, \text{value} = 8 + 8 = 16 \\
        l^1_{0,1} + l^1_{1,0} \quad k = 1, \text{value} = 1 + 6 = 7 \\
        l^1_{0,2} + l^1_{2,0} \quad k = 2, \text{value} = 1 + 3 = 4^* \\
    \end{array} \right. 
\end{align*}
\]
Chapter 25. All-pairs shortest paths

- running time: $\Theta(n^4)$. 

Faster All Pair Shortest Paths

1. $n = \text{rows}[W]$;
2. $L = W$;
3. $m = 1$;
4. while $m < n - 1$
5. $L = \text{Extended Shortest Paths}(L, L)$
6. $m = 2 \times m$
7. return $(L)$
Chapter 25. All-pairs shortest paths

- running time: $\Theta(n^4)$.
- improving the running time by repeatedly squaring:
Chapter 25. All-pairs shortest paths

- running time: $\Theta(n^4)$.
- improving the running time by repeatedly squaring:
  compute: $L^1, L^2, L^4, \ldots, L^{2^k}$.
Chapter 25. All-pairs shortest paths

- running time: $\Theta(n^4)$.
- improving the running time by repeatedly squaring:
  - compute: $L^1, L^2, L^4, \ldots, L^{2^k}$.
  - what is $k$ here?
Chapter 25. All-pairs shortest paths

- running time: $\Theta(n^4)$.
- improving the running time by repeatedly squaring:
  compute: $L^1, L^2, L^4, \ldots, L^{2^k}$.
  what is $k$ here? Let $2^k = n - 1$. Then $k = \lceil \log_2(n - 1) \rceil$. 
Chapter 25. All-pairs shortest paths

- running time: $\Theta(n^4)$.
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Faster All Pair Shortest Paths($W$)
Chapter 25. All-pairs shortest paths

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Faster All Pair Shortest Paths($W$)
1. $n = \text{rows}[W]$;
Chapter 25. All-pairs shortest paths

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Faster All Pair Shortest Paths($W$)

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2. $L = W$;
Chapter 25. All-pairs shortest paths

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Faster All Pair Shortest Paths($W$)

1. $n = \text{rows}[W]$;
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3. $m = 1$;
Chapter 25. All-pairs shortest paths

- running time: $\Theta(n^4)$.
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Faster All Pair Shortest Paths($W$)

1. $n = \text{rows}[W]$;
2. $L = W$;
3. $m = 1$;
4. while $m < n - 1$
Chapter 25. All-pairs shortest paths

- running time: $\Theta(n^4)$.
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Faster All Pair Shortest Paths($W$)

1. $n = \text{rows}[W]$;
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Chapter 25. All-pairs shortest paths

- running time: $\Theta(n^4)$.
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Faster All Pair Shortest Paths($W$)

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3. $m = 1$;
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6. $m = 2 \times m$
Chapter 25. All-pairs shortest paths

- running time: $\Theta(n^4)$.
- improving the running time by repeatedly squaring:
  
  compute: $L^1, L^2, L^4, \cdots, L^{2^k}$.

  what is $k$ here? Let $2^k = n - 1$. Then $k = \lceil \log_2(n - 1) \rceil$.

Faster All Pair Shortest Paths($W$)

1. $n = \text{rows}[W]$;
2. $L = W$;
3. $m = 1$;
4. while $m < n - 1$
5. $L = \text{Extended Shortest Paths}(L, L)$
6. $m = 2 \times m$
7. return $(L)$
Chapter 25. All-pairs shortest paths

Floyd-Warshall algorithm
Floyd-Warshall algorithm

intermediate vertices on a path \( v_i \leadsto v_j \): those other than \( v_i \) and \( v_j \).
Chapter 25. All-pairs shortest paths

Floyd-Warshall algorithm

intermediate vertices on a path $v_i \rightsquigarrow v_j$: those other than $v_i$ and $v_j$.

Define: $d^{(k)}_{i,j}$ to be the shortest path distance from $v_i$ to $v_j$
Chapter 25. All-pairs shortest paths

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Floyd-Warshall ($W$)

1. $n = \text{rows}[W]$
2. $D(0) = W$
3. for $k = 1$ to $n$
4. for $i = 1$ to $n$
5. for $j = 1$ to $n$
6. $D(k) [i,j] = \min \{ D(k-1) [i,j], D(k-1) [i,k] + D(k-1) [k,j] \}$
7. return ($D(n)$)
Chapter 25. All-pairs shortest paths

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Intermediate vertices on a path $v_i \leadsto v_j$: those other than $v_i$ and $v_j$.

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$$d_{ij}^{(k)} = \min(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)})$$

with base case: $d_{ij}^{(0)} = w_{ij}$. 

Floyd-Warshall algorithm

Intermediate vertices on a path $v_i \sim v_j$: those other than $v_i$ and $v_j$.

Define: $d^{(k)}_{ij}$ to be the shortest path distance from $v_i$ to $v_j$ with no intermediate vertices of indexes higher than $k$. Thus

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Floyd-Warshall($W$)
Chapter 25. All-pairs shortest paths

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Chapter 25. All-pairs shortest paths

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FLOYD-WARSHALL(\( W \))
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Chapter 25. All-pairs shortest paths

**Floyd-Warshall algorithm**

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7. return ($D^{(n)}$)
Chapter 25. All-pairs shortest paths

\[ D^{(0)} = \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix} \]

\[ \Pi^{(0)} = \begin{pmatrix} \text{NIL} & 1 & 1 & \text{NIL} & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 2 & 2 \\ \text{NIL} & 3 & \text{NIL} & \text{NIL} & \text{NIL} \\ 4 & \text{NIL} & 4 & \text{NIL} & \text{NIL} \\ \text{NIL} & \text{NIL} & \text{NIL} & 5 & \text{NIL} \end{pmatrix} \]

\[ D^{(1)} = \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix} \]

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\[ D^{(2)} = \begin{pmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix} \]

\[ \Pi^{(2)} = \begin{pmatrix} \text{NIL} & 1 & 1 & 2 & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 2 & 2 \\ \text{NIL} & 3 & \text{NIL} & 2 & 2 \\ 4 & 1 & 4 & \text{NIL} & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 5 & \text{NIL} \end{pmatrix} \]

\[ D^{(3)} = \begin{pmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & \textcircled{-1} & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix} \]

\[ \Pi^{(3)} = \begin{pmatrix} \text{NIL} & 1 & 1 & 2 & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 2 & 2 \\ \text{NIL} & 3 & \text{NIL} & 2 & 2 \\ 4 & 3 & 4 & \text{NIL} & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 5 & \text{NIL} \end{pmatrix} \]

\[ D^{(4)} = \begin{pmatrix} 0 & 3 & 1 & 4 & -4 \\ \textcircled{2} & 0 & -4 & 1 & -1 \\ \textcircled{7} & 4 & 0 & 5 & 3 \\ 2 & \textcircled{-1} & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix} \]

\[ \Pi^{(4)} = \begin{pmatrix} \text{NIL} & 1 & 4 & 2 & 1 \\ 4 & \text{NIL} & 4 & 2 & 1 \\ 4 & 3 & \text{NIL} & 2 & 1 \\ 4 & 3 & 4 & \text{NIL} & 1 \\ 4 & 3 & 4 & 5 & \text{NIL} \end{pmatrix} \]

\[ D^{(5)} = \begin{pmatrix} 0 & 1 & -3 & 2 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & \textcircled{-1} & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix} \]

\[ \Pi^{(5)} = \begin{pmatrix} \text{NIL} & 3 & 4 & 5 & 1 \\ 4 & \text{NIL} & 4 & 2 & 1 \\ 4 & 3 & \text{NIL} & 2 & 1 \\ 4 & 3 & 4 & \text{NIL} & 1 \\ 4 & 3 & 4 & 5 & \text{NIL} \end{pmatrix} \]
Chapter 25. All-pairs shortest paths

- Constructing a shortest path
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- Constructing a shortest path
- for each $v_i$ and each $v_j$, to remember the last step to reach $j$. 

π(0)_{ij} = NULL if $i = j$ or $w_{ij} = \infty$, or π(0)_{ij} = i if $i \neq j$ and $w_{ij} < \infty$.

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Chapter 25. All-pairs shortest paths

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Summary of shortest path algorithms:

1. Bellman-Ford's algorithm (able to detect negative weight cycles)
2. DAG Shortest Paths (use topological sorting) [Lawler]
3. Dijkstra's algorithm (assuming non-negative weights)
4. Matrix multiplication (DP) [Lawler, folklore]
5. Floyd-Warshall algorithm (DP)
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