Part VII. Selected Topics
Chapter 33.9 Exhaustive Search (not in the text!)
Part VII. Selected Topics

Chapter 33.9 Exhaustive Search (not in the text!)

Chapter 34 NP-Completeness
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Chapter 33.9 Exhaustive Search (not in the text!)

Chapter 34 NP-Completeness
Chapter 33.9. Exhaustive Search

To enumerate all possible solutions to the problem instance

- systematic examining all solutions
- without repeating solutions that have been examined
- stop when a satisfactory solution is found
Chapter 33.9. Exhaustive Search

To enumerate all possible solutions to the problem instance
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How?
Chapter 33.9. Exhaustive Search

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Chapter 33.9. Exhaustive Search

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Chapter 33.9. Exhaustive Search

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Chapter 33.9. Exhaustive Search

First, we need to be able to count total number of “things” to be enumerated.
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- without missing one (correctness)
- without over-counting (efficiency)
- A sophisticated counting often has recursive solution.
Chapter 33.9. Exhaustive Search

Examples of counting:

(1) total number of permutations of \(1, 2, \ldots, n\) is
\[
P(n) = n \times P(n-1)
\]
with base case \(P(1) = 1\).

(2) total number of ways to choose \(k\) from \(n\) items is
\[
\binom{n}{k} = \frac{n(n-1)\ldots(n-k+1)}{k!} = \frac{n!}{(n-k)!k!}
\]
or, alternatively,
\[
\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}
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Chapter 33.9. Exhaustive Search

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Chapter 33.9. Exhaustive Search

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Chapter 33.9. Exhaustive Search

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Chapter 33.9. Exhaustive Search

Example: Boolean Formula Satisfiability problem (SAT)
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**INPUT:** boolean formula \( f(x_1, x_2, \ldots, x_n) \),
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\( f(x_1, x_2, \ldots, x_n) \) is satisfiable if there is an *assignment* to boolean variables

\[ x_i \in \{T, F\}, \ i = 1, 2, \ldots, n, \]
Chapter 33.9. Exhaustive Search

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\( f(x_1, x_2, \ldots, x_n) \) is **satisfiable** if there is an **assignment** to boolean variables

\( \forall i \in \{1, 2, \ldots, n\}, x_i \in \{T, F\} \), such that \( f \) is evaluated to \( T \).

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f(x_1, x_2, x_3) = (x_1 \lor x_2 \lor \neg x_3) \land (\neg x_1 \lor x_3) \text{ is satisfiable}
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\[ g(x_1, x_2) = (x_1 \lor \neg x_2) \land (\neg x_1 \lor x_2) \land (x_1 \lor x_2) \land (\neg x_1 \lor \neg x_2) \text{ is not!} \]
Chapter 33.9. Exhaustive Search

Use exhaustive search to solve the SAT problem.
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How? What will you exhaustively search on?
Chapter 33.9. Exhaustive Search

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- **Enumerate all combinations of T and F for $x_1, \ldots, x_n$.**
Chapter 33.9. Exhaustive Search

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- Can you solve it with a recursive algorithm?
Chapter 33.9. Exhaustive Search

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- Can you solve it with an iterative algorithm?
Chapter 33.9. Exhaustive Search

Solve SAT problem with a recursive algorithm:

what data will the recursion be applied to?
boolean formula \( f(x_1, \ldots, x_n) \)

what is the terminating (base) case?
n=0, formula without variables

what is the recursive case?
\[
f(x_1, \ldots, x_{n-1}, x_n) = f(x_1, \ldots, x_{n-1}, T) \lor f(x_1, \ldots, x_{n-1}, F)
\]

\[
f(x_1, \ldots, x_{n-1}, T) \implies g(x_1, \ldots, x_{n-1})
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Chapter 33.9. Exhaustive Search

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Chapter 33.9. Exhaustive Search

Algorithm $\text{SAT Solver}(f(x_1, \ldots, x_{n-1}, x_n))$
Chapter 33.9. Exhaustive Search

Algorithm SAT Solver($f(x_1, \ldots, x_{n-1}, x_n)$)

1. if $n = 0$, return ($f$);
Algorithm SAT SOLVER\( (f(x_1, \ldots, x_{n-1}, x_n)) \)

1. if \( n = 0 \), return \( f \);
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Algorithm \textsc{Sat Solver}\left(f(x_1, \ldots, x_{n-1}, x_n)\right)

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2. \textbf{else } g(x_1, \ldots, x_{n-1}) = f(x_1, \ldots, x_{n-1}, T)
3. \hspace{1cm} h(x_1, \ldots, x_{n-1}) = f(x_1, \ldots, x_{n-1}, F)
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4. return (SAT Solver($g(x_1, \ldots, x_{n-1})$) $\lor$ SAT Solver($h(x_1, \ldots, x_{n-1})$))

Does this algorithm exhaustively search all assignments to the variables?
Chapter 33.9. Exhaustive Search

Algorithm \textsc{SAT Solver} \(f(x_1, \ldots, x_{n-1}, x_n)\)

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Chapter 33.9. Exhaustive Search

Algorithm $\text{SAT Solver}(f(x_1, \ldots, x_{n-1}, x_n))$

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3. \hspace{1em} $h(x_1, \ldots, x_{n-1}) = f(x_1, \ldots, x_{n-1}, F)$
4. \hspace{1em} return ($\text{SAT Solver}(g(x_1, \ldots, x_{n-1})) \lor$
\hspace{2em} $\text{SAT Solver}(h(x_1, \ldots, x_{n-1})))$

Does this algorithm exhaustively search all assignments to the variables?

- draw a search tree based on the algorithm.
- what does the tree look like?
Chapter 33.9. Exhaustive Search

Algorithm \texttt{SAT Solver}(f(x_1, \ldots, x_{n-1}, x_n))

1. \textbf{if} \ n = 0, \textbf{return} \ f;
2. \textbf{else} \ g(x_1, \ldots, x_{n-1}) = f(x_1, \ldots, x_{n-1}, \text{T})
3. \quad h(x_1, \ldots, x_{n-1}) = f(x_1, \ldots, x_{n-1}, \text{F})
4. \quad \textbf{return} \ (\texttt{SAT Solver}(g(x_1, \ldots, x_{n-1})) \lor \texttt{SAT Solver}(h(x_1, \ldots, x_{n-1})))

Does this algorithm exhaustively search all assignments to the variables?

- draw a \textit{search tree} based on the algorithm.
- what does the tree look like?
- what does each path mean?

\[ T(n) = 2T(n-1) + cn, \quad T(0) = c, \Rightarrow T(n) = \Theta(2^n) \]
Chapter 33.9. Exhaustive Search

Algorithm SAT Solver\( (f(x_1, \ldots, x_{n-1}, x_n)) \)

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2. \textbf{else} \( g(x_1, \ldots, x_{n-1}) = f(x_1, \ldots, x_{n-1}, T) \)
3. \( h(x_1, \ldots, x_{n-1}) = f(x_1, \ldots, x_{n-1}, F) \)
4. \textbf{return} (SAT Solver\( (g(x_1, \ldots, x_{n-1})) \lor \) SAT Solver\( (h(x_1, \ldots, x_{n-1})) \))

Does this algorithm exhaustively search all assignments to the variables?

- draw a \textit{search tree} based on the algorithm.
- what does the tree look like?
- what does each path mean? how many paths?
Algorithm \textsc{SAT Solver}(f(x_1, \ldots, x_{n-1}, x_n))

1. \textbf{if} \ n = 0, \textbf{return} (f);
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3. \quad h(x_1, \ldots, x_{n-1}) = f(x_1, \ldots, x_{n-1}, F)
4. \textbf{return} (\textsc{SAT Solver}(g(x_1, \ldots, x_{n-1})) \lor \\
\quad \textsc{SAT Solver}(h(x_1, \ldots, x_{n-1})))

Does this algorithm exhaustively search all assignments to the variables?

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Chapter 33.9. Exhaustive Search

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Chapter 33.9. Exhaustive Search

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Chapter 33.9. Exhaustive Search

Solve SAT problem with iterative algorithms
Chapter 33.9. Exhaustive Search

Solve SAT problem with iterative algorithms

- How? what to iterate on?
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Solve SAT problem with iterative algorithms

- How? what to iterate on?

  assignments
Chapter 33.9. Exhaustive Search

Solve SAT problem with iterative algorithms

- How? what to iterate on?
  assignments

- what is the initial value?
Chapter 33.9. Exhaustive Search

Solve SAT problem with iterative algorithms

- How? what to iterate on?
  assignments
- what is the initial value?

\[ x_1 = F, x_2 = F, \ldots, x_n = F, \]
Chapter 33.9. Exhaustive Search

Solve SAT problem with iterative algorithms

- How? what to iterate on?
  
  assignments

- what is the initial value?
  
  \(x_1 = F, x_2 = F, \ldots, x_n = F\), or simply \((F, F, \ldots, F)\) always flip the last bit.
Chapter 33.9. Exhaustive Search

Solve SAT problem with iterative algorithms

- How? what to iterate on?
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- what is the initial value?
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- what to increment
Chapter 33.9. Exhaustive Search

Solve SAT problem with iterative algorithms

- How? what to iterate on?
  assignments
- what is the initial value?
  \[ x_1 = F, x_2 = F, \ldots, x_n = F, \text{ or simply } (F, F, \ldots, F) \]
- what to increment
  \[ (\ldots, F, T, \ldots, T) \]
Chapter 33.9. Exhaustive Search

Solve SAT problem with iterative algorithms

- How? what to iterate on?
  
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- what to increment
  
  \[ (\ldots, F, T, \ldots, T) \rightarrow (\ldots, T, F, \ldots, F) \]
Solve SAT problem with iterative algorithms

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- what to increment
  \[ (\ldots, F, T, \ldots, T) \rightarrow (\ldots, T, F, \ldots, F) \]
  always flip the last bit.
Algorithm SAT Solver-Enum

1. for $\langle x_1, \ldots, x_{n-1}, x_n \rangle =$ $\langle F, \ldots, F \rangle$ to $\langle T, \ldots, T \rangle$

2. $V =$ Evaluate($f, x_1, \ldots, x_n$)

3. if $V =$ $T$, return $\langle T \rangle$

4. return $\langle F \rangle$

• for loop can be implemented by encoding vectors $\langle F, \ldots, F \rangle$, \ldots, $\langle T, \ldots, T \rangle$ with binary numbers then further with integers

• a decoding process is needed to converting integers back to vectors
Chapter 33.9. Exhaustive Search

Algorithm $\text{SAT Solver-Enum}(f(x_1, \ldots, x_{n-1}, x_n))$
Algorithm SAT Solver-Enum($f(x_1, \ldots, x_{n-1}, x_n)$)

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Algorithm SAT SOLVER-ENUM\( (f(x_1, \ldots, x_{n-1}, x_n)) \)

1. \textbf{for} \( \langle x_1, \ldots, x_n \rangle = \langle F, \ldots, F \rangle \) \textbf{to} \( \langle T, \ldots, T \rangle \)

2. \( V = \text{Evaluate}(f, x_1, \ldots, x_n) \)
Algorithm SAT Solver-Enum($f(x_1, \ldots, x_{n-1}, x_n)$)

1. for $\langle x_1, \ldots, x_n \rangle = \langle F, \ldots, F \rangle$ to $\langle T, \ldots, T \rangle$
2. $V = Evaluate(f, x_1, \ldots, x_n)$
3. if $V = T$, return (T)
Chapter 33.9. Exhaustive Search

Algorithm SAT-SOLVER-ENUM\( (f(x_1, \ldots, x_{n-1}, x_n)) \)

1. for \( \langle x_1, \ldots, x_n \rangle = \langle F, \ldots, F \rangle \) to \( \langle T, \ldots, T \rangle \)
2. \( V = \text{Evaluate}(f, x_1, \ldots, x_n) \)
3. if \( V = T \), return \( (T) \)
4. return \( (F) \)
Chapter 33.9. Exhaustive Search

Algorithm SAT SOLVER-ENUM \( f(x_1, \ldots, x_{n-1}, x_n) \)

1. \[
\text{for } \langle x_1, \ldots, x_n \rangle = \langle F, \ldots, F \rangle \ \text{to} \ \langle T, \ldots, T \rangle
\]
2. \[ V = \text{Evaluate}(f, x_1, \ldots, x_n) \]
3. \[ \text{if } V = T, \ \text{return} \ (T) \]
4. \[ \text{return} \ (F) \]

- \textbf{for} loop can be implemented by encoding vectors \( \langle F, \ldots, F \rangle, \ldots, \langle T, \ldots, T \rangle \) with
Chapter 33.9. Exhaustive Search

Algorithm SAT SOLVER-ENUM(\( f(x_1, \ldots, x_{n-1}, x_n) \))

1. for \( \langle x_1, \ldots, x_n \rangle = \langle F, \ldots, F \rangle \) to \( \langle T, \ldots, T \rangle \)
2. \( V = \text{Evaluate}(f, x_1, \ldots, x_n) \)
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Chapter 33.9. Exhaustive Search

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4. \hspace{1em} \textbf{return} \(F\)

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Chapter 33.9. Exhaustive Search

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3. \hspace{1em} \textbf{if} \( \textbf{V} = T, \textbf{return} \) (T)
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- a decoding process is needed to converting integers back to vectors
Chapter 33.9. Exhaustive Search

Iterative exhaustive search seems to be more convenient. Another example: Travel Salesman Problem (TSP) Related problem: Hamiltonian Cycle

Input: a graph \( G = (V,E) \)

Output: yes if and only if \( G \) contains a Hamiltonian cycle (Hamiltonian path is a cycle going through every vertex exactly once.)

How to enumerate all cycles and validate?

- enumerate all permutations of \((1, 2, \ldots, n)\)
- how to encode these permutations as integers?
Iterative exhaustive search seems to be more convenient.
Chapter 33.9. Exhaustive Search

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Chapter 33.9. Exhaustive Search

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Chapter 33.9. Exhaustive Search

Exhaustive search could be non-trivial
Chapter 33.9. Exhaustive Search

Exhaustive search could be non-trivial

Maximum Independent Set

- **Input**: a graph $G = (V, E)$
- **Output**: a subset $I \subseteq V$ such that
  
  - $(u, v) \notin E$ for all $u, v \in I$;
  - $|I|$ is the maximum.

  • trivial exhaustive search: check every subset of $V$ and verify
  • non-trivial: use a search tree, achieving a better time upper bound.

  taking advantage of the independent set
Chapter 33.9. Exhaustive Search

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Maximum Independent Set

**INPUT:** a graph $G = (V, E)$

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Chapter 33.9. Exhaustive Search

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Chapter 33.9. Exhaustive Search

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Chapter 33.9. Exhaustive Search

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  taking advantage of the independent set
Chapter 33.9. Exhaustive Search

The algorithm follows a logical search tree:

- Given a graph \( G \), it picks an arbitrary vertex \( v \) from \( G \);
- Exhaustively, there are two cases to consider:
  1. To include \( v \) in the independent set;
  2. To exclude \( v \) from the independent set;
- Resulting in two subgraphs \( G_1 \) and \( G_2 \) to be recursively considered,
  1. \( G_1 \) is the result of \( G \) after \( v \) and all its neighbors are removed;
  2. \( G_2 \) is the result of \( G \) after \( v \) is removed.
- The algorithm terminates when the considered graph is empty.
Chapter 33.9. Exhaustive Search

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  1. $G_1$ is the result of $G$ after $v$ and all its neighbors are removed;
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Chapter 33.9. Exhaustive Search

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  (1) to **include** $v$ in the independent set;
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- resulting in two subgraphs $G_1$ and $G_2$ to be recursively considered,
  (1) $G_1$ is the result of $G$ after $v$ and all its neighbors are removed;
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Chapter 33.9. Exhaustive Search

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Chapter 33.9. Exhaustive Search

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Chapter 33.9. Exhaustive Search

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  1. $G_1$ is the result of $G$ after $v$
Chapter 33.9. Exhaustive Search

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• given a graph $G$, it picks an arbitrary vertex $v$ from $G$;

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Chapter 33.9. Exhaustive Search

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Chapter 33.9. Exhaustive Search

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Chapter 33.9. Exhaustive Search

Algorithm \texttt{MaxIndSet} \((G)\)
Chapter 33.9. Exhaustive Search

Algorithm MaxIndSet \((G)\)

1. \textbf{if} \( G = \emptyset \) \textbf{return} \((\emptyset)\)
Algorithm \textsc{MaxIndSet} \((G')\)

1. \textbf{if} \(G = \emptyset\) \textbf{return} \((\emptyset)\)
2. \textbf{else} pick an arbitrary vertex \(v\) in \(G\)

\textbullet\ the algorithm is a search tree
\textbullet\ the time complexity:
\[T(n) = T(n - 1 - m) + T(n - 1) + cn^2\]
where \(m\) is the number of neighbors of \(v\)'s
Chapter 33.9. Exhaustive Search

Algorithm `MaxIndSet (G)`

1. if $G = \emptyset$ return ($\emptyset$)
2. else pick an arbitrary vertex $v$ in $G$
3. let $G_1$ be $G$ with $v$ and all its neighbors removed

...
Chapter 33.9. Exhaustive Search

Algorithm $\text{MaxIndSet}(G)$

1. \textbf{if} $G = \emptyset$ \textbf{return} $(\emptyset)$
2. \textbf{else} pick an arbitrary vertex $v$ in $G$
3. \textbf{let} $G_1$ be $G$ with $v$ and all its neighbors removed
4. $I_1 = \{v\} \cup \text{MaxIndSet}(G_1)$
Algorithm $\text{MaxIndSet} \ (G)$

1. if $G = \emptyset$ return $(\emptyset)$
2. else pick an arbitrary vertex $v$ in $G$
3. let $G_1$ be $G$ with $v$ and all its neighbors removed
4. $I_1 = \{v\} \cup \text{MaxIndSet} \ (G_1)$
5. let $G_2$ be $G$ with $v$ removed
6. if $|I_1| \geq |I_2|$ return $(I_1)$
7. else return $(I_2)$
Chapter 33.9. Exhaustive Search

Algorithm $\text{MaxIndSet} \ (G)$

1. if $G = \emptyset$ return $(\emptyset)$
2. else pick an arbitrary vertex $v$ in $G$
3. let $G_1$ be $G$ with $v$ and all its neighbors removed
4. $I_1 = \{v\} \cup \text{MaxIndSet} \ (G_1)$
5. let $G_2$ be $G$ with $v$ removed
6. $I_2 = \text{MaxIndSet} \ (G_2)$
Algorithm \textsc{MaxIndSet} \((G)\)

1. \textbf{if} \(G = \emptyset\) \textbf{return} \((\emptyset)\)
2. \textbf{else} pick an arbitrary vertex \(v\) in \(G\)
3. \hspace{1em} let \(G_1\) be \(G\) with \(v\) and all its neighbors removed
4. \hspace{1em} \(I_1 = \{v\} \cup \text{MaxIndSet} \((G_1)\)\)
5. \hspace{1em} let \(G_2\) be \(G\) with \(v\) removed
6. \hspace{1em} \(I_2 = \text{MaxIndSet} \((G_2)\)\)
7. \hspace{1em} \textbf{if} \(|I_1| \geq |I_2|\) \textbf{return} \((I_1)\)
Algorithm \texttt{MaxIndSet} \((G)\)

1. \textbf{if} \(G = \emptyset\) return (\(\emptyset\))
2. \textbf{else} pick an arbitrary vertex \(v\) in \(G\)
3. \hspace{1em} let \(G_1\) be \(G\) with \(v\) and all its neighbors removed
4. \hspace{1em} \(I_1 = \{v\} \cup \text{MaxIndSet} \ (G_1)\)
5. \hspace{1em} let \(G_2\) be \(G\) with \(v\) removed
6. \hspace{1em} \(I_2 = \text{MaxIndSet} \ (G_2)\)
7. \hspace{1em} \textbf{if} \(|I_1| \geq |I_2|\) return (\(I_1\))
8. \hspace{1em} \textbf{else} return (\(I_2\))
Algorithm \textsc{MaxIndSet} \((G)\)

1. \textbf{if} \(G = \emptyset\) \textbf{return} \((\emptyset)\)
2. \textbf{else} pick an arbitrary vertex \(v\) in \(G\)
3. let \(G_1\) be \(G\) with \(v\) and all its neighbors removed
4. \(I_1 = \{v\} \cup \textsc{MaxIndSet} \((G_1)\)\)
5. let \(G_2\) be \(G\) with \(v\) removed
6. \(I_2 = \textsc{MaxIndSet} \((G_2)\)\)
7. \textbf{if} \(|I_1| \geq |I_2|\) \textbf{return} \((I_1)\)
8. \textbf{else return} \((I_2)\)

• the algorithm is a search tree
Chapter 33.9. Exhaustive Search

Algorithm `MaxIndSet (G)`

1. if \( G = \emptyset \) return \( (\emptyset) \)
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3. let \( G_1 \) be \( G \) with \( v \) and all its neighbors removed
4. \( I_1 = \{v\} \cup \text{MaxIndSet} (G_1) \)
5. let \( G_2 \) be \( G \) with \( v \) removed
6. \( I_2 = \text{MaxIndSet} (G_2) \)
7. if \( |I_1| \geq |I_2| \) return \( (I_1) \)
8. else return \( (I_2) \)

- the algorithm is a search tree
- the time complexity:
Algorithm $\text{MaxIndSet}(G)$

1. if $G = \emptyset$ return ($\emptyset$)
2. else pick an arbitrary vertex $v$ in $G$
3. let $G_1$ be $G$ with $v$ and all its neighbors removed
4. $I_1 = \{v\} \cup \text{MaxIndSet}(G_1)$
5. let $G_2$ be $G$ with $v$ removed
6. $I_2 = \text{MaxIndSet}(G_2)$
7. if $|I_1| \geq |I_2|$ return ($I_1$)
8. else return ($I_2$)

- the algorithm is a search tree
- the time complexity: $T(|G|) = cn^2 + T(|G_1|) + T(|G_2|)$
Chapter 33.9. Exhaustive Search

Algorithm \texttt{MaxIndSet}(G)

1. \textbf{if } \ G = \emptyset \ \textbf{return } (\emptyset)
2. \textbf{else} \ pick an arbitrary vertex \ v \ in \ G
3. \ \ \ let \ G_1 \ be \ G \ with \ v \ and \ all \ its \ neighbors \ removed
4. \ \ \ \ \ I_1 = \{v\} \cup \texttt{MaxIndSet}(G_1)
5. \ \ \ let \ G_2 \ be \ G \ with \ v \ removed
6. \ \ \ \ \ I_2 = \texttt{MaxIndSet}(G_2)
7. \ \ \ \ \ \textbf{if } |I_1| \geq |I_2| \ \textbf{return } (I_1)
8. \ \ \ \ \ \textbf{else} \ \textbf{return } (I_2)

- the algorithm is a search tree
- the time complexity: \ \ T(|G|) = cn^2 + T(|G_1|) + T(|G_2|)

\[ T(n) = T(n - 1 - m) + T(n - 1) + cn^2 \]

where \ m \ is the number of neighbors of \ v's
Chapter 33.9. Exhaustive Search

\[ T(n) = T(n - 1 - m) + T(n - 1) + cn^2 \]
Chapter 33.9. Exhaustive Search

\[ T(n) = T(n - 1 - m) + T(n - 1) + cn^2 \]

- \( m \geq 0 \),
Chapter 33.9. Exhaustive Search

\[ T(n) = T(n - 1 - m) + T(n - 1) + cn^2 \]

- \( m \geq 0, \ T(n) \leq T(n - 1) + T(n - 1) + cn^2, \)
Chapter 33.9. Exhaustive Search

\[ T(n) = T(n - 1 - m) + T(n - 1) + cn^2 \]

- \( m \geq 0, T(n) \leq T(n - 1) + T(n - 1) + cn^2, \implies T(n) = O(2^n) \)
Chapter 33.9. Exhaustive Search

\[ T(n) = T(n - 1 - m) + T(n - 1) + cn^2 \]

- \( m \geq 0, T(n) \leq T(n - 1) + T(n - 1) + cn^2 \), \( \implies T(n) = O(2^n) \)
- Can we guarantee \( m \geq 1 \)
Chapter 33.9. Exhaustive Search

\[ T(n) = T(n - 1 - m) + T(n - 1) + cn^2 \]

- \( m \geq 0, T(n) \leq T(n - 1) + T(n - 1) + cn^2, \implies T(n) = O(2^n) \)

- Can we guarantee \( m \geq 1 \) so we have \( T(n) \leq T(n - 2) + T(n - 1) + cn^2 \),
Chapter 33.9. Exhaustive Search

\[ T(n) = T(n - 1 - m) + T(n - 1) + cn^2 \]

- \( m \geq 0, T(n) \leq T(n - 1) + T(n - 1) + cn^2, \implies T(n) = O(2^n) \)

- Can we guarantee \( m \geq 1 \) so we have
  \[ T(n) \leq T(n - 2) + T(n - 1) + cn^2, \implies T(n) = O(1.6181^n) \]
Chapter 33.9. Exhaustive Search

\[ T(n) = T(n - 1 - m) + T(n - 1) + cn^2 \]

- \( m \geq 0, \ T(n) \leq T(n - 1) + T(n - 1) + cn^2, \implies T(n) = O(2^n) \)
- Can we guarantee \( m \geq 1 \) so we have
  \[ T(n) \leq T(n - 2) + T(n - 1) + cn^2, \implies T(n) = O(1.6181^n) \]
- Or even better, to guarantee \( m \geq 2? \)
Chapter 33.9. Exhaustive Search

\[ T(n) = T(n - 1 - m) + T(n - 1) + cn^2 \]

- \( m \geq 0, \ T(n) \leq T(n - 1) + T(n - 1) + cn^2, \implies T(n) = O(2^n) \)
- Can we guarantee \( m \geq 1 \) so we have
  \[ T(n) \leq T(n - 2) + T(n - 1) + cn^2, \implies T(n) = O(1.6181^n) \]
- Or even better, to guarantee \( m \geq 2 \) if we can,
  \[ T(n) \leq T(n - 3) + T(n - 1) + cn^2, \]
Chapter 33.9. Exhaustive Search

\[ T(n) = T(n - 1 - m) + T(n - 1) + cn^2 \]

- \( m \geq 0, \ T(n) \leq T(n - 1) + T(n - 1) + cn^2, \implies T(n) = O(2^n) \)

- Can we guarantee \( m \geq 1 \) so we have
  \( T(n) \leq T(n - 2) + T(n - 1) + cn^2, \implies T(n) = O(1.6181^n) \)

- Or even better, to guarantee \( m \geq 2 \)? if we can,
  \( T(n) \leq T(n - 3) + T(n - 1) + cn^2, \implies T(n) = O(1.5^n) \)
Chapter 33.9. Exhaustive Search

\[ T(n) = T(n - 1 - m) + T(n - 1) + cn^2 \]

- \( m \geq 0, \ T(n) \leq T(n - 1) + T(n - 1) + cn^2, \implies T(n) = O(2^n) \)
- Can we guarantee \( m \geq 1 \) so we have
  \[ T(n) \leq T(n - 2) + T(n - 1) + cn^2, \implies T(n) = O(1.6181^n) \]
- Or even better, to guarantee \( m \geq 2 \)? if we can,
  \[ T(n) \leq T(n - 3) + T(n - 1) + cn^2, \implies T(n) = O(1.5^n) \]
  use the substitution method to prove \( T(n) = O(1.5^n) \).
Chapter 33.9. Exhaustive Search

\[ T(n) = T(n - 1 - m) + T(n - 1) + cn^2 \]

- \( m \geq 0, T(n) \leq T(n - 1) + T(n - 1) + cn^2, \implies T(n) = O(2^n) \)
- Can we guarantee \( m \geq 1 \) so we have
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- Or even better, to guarantee \( m \geq 2 \)? if we can,
  \[ T(n) \leq T(n - 3) + T(n - 1) + cn^2, \implies T(n) = O(1.5^n) \]
  use the substitution method to prove \( T(n) = O(1.5^n) \).
- Can we guarantee \( m \geq 3 \)?
Chapter 33.9. Exhaustive Search

\[ T(n) = T(n - 1 - m) + T(n - 1) + cn^2 \]

- \( m \geq 0 \), \( T(n) \leq T(n - 1) + T(n - 1) + cn^2 \), \( \implies T(n) = O(2^n) \)

- Can we guarantee \( m \geq 1 \) so we have
  \( T(n) \leq T(n - 2) + T(n - 1) + cn^2 \), \( \implies T(n) = O(1.6181^n) \)

- Or even better, to guarantee \( m \geq 2 \)? if we can,
  \( T(n) \leq T(n - 3) + T(n - 1) + cn^2 \), \( \implies T(n) = O(1.5^n) \)

use the substitution method to prove \( T(n) = O(1.5^n) \).

- Can we guarantee \( m \geq 3 \)? possible but a little more complicated.
Chapter 33.9. Exhaustive Search

Let $T(n) \leq T(n - 3) + T(n - 1) + cn^2$, with $T(1) = O(1)$
Chapter 33.9. Exhaustive Search

Let $T(n) \leq T(n - 3) + T(n - 1) + cn^2$, with $T(1) = O(1)$

**Claim:** $T(n) = O(1.5^n)$
Chapter 33.9. Exhaustive Search

Let $T(n) \leq T(n-3) + T(n-1) + cn^2$, with $T(1) = O(1)$

Claim: $T(n) = O(1.5^n)$

Proof (use the substitution method)
Let \( T(n) \leq T(n - 3) + T(n - 1) + cn^2 \), with \( T(1) = O(1) \)

**Claim:** \( T(n) = O(1.5^n) \)

**Proof** (use the substitution method)

Assume that \( T(k) \leq 1.5^k \) for all \( k < n \).
Chapter 33.9. Exhaustive Search

Let \( T(n) \leq T(n - 3) + T(n - 1) + cn^2 \), with \( T(1) = O(1) \)

Claim: \( T(n) = O(1.5^n) \)

Proof (use the substitution method)

Assume that \( T(k) \leq 1.5^k \) for all \( k < n \). Then

\[
T(n) \leq T(n - 3) + T(n - 1) + n^2
\]
Let $T(n) \leq T(n - 3) + T(n - 1) + cn^2$, with $T(1) = O(1)$

**Claim:** $T(n) = O(1.5^n)$

**Proof** (use the substitution method)

Assume that $T(k) \leq 1.5^k$ for all $k < n$. Then

$$T(n) \leq T(n - 3) + T(n - 1) + n^2 \leq 1.5^{n-3} + 1.5^{n-1} + n^2$$
Let $T(n) \leq T(n - 3) + T(n - 1) + cn^2$, with $T(1) = O(1)$

**Claim:** $T(n) = O(1.5^n)$

**Proof** (use the substitution method)

Assume that $T(k) \leq 1.5^k$ for all $k < n$. Then

$$T(n) \leq T(n - 3) + T(n - 1) + n^2 \leq 1.5^{n-3} + 1.5^{n-1} + n^2$$

when $n > n_0$ ($n_0$ to be determined)
Chapter 33.9. Exhaustive Search

Let $T(n) \leq T(n - 3) + T(n - 1) + cn^2$, with $T(1) = O(1)$

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when $n > n_0$ ($n_0$ to be determined)

$$\leq 1.5^{n-3}(1 + 1.5^2 + \frac{n^2}{1.5^{n-3}})$$
Chapter 33.9. Exhaustive Search

Let \( T(n) \leq T(n - 3) + T(n - 1) + cn^2 \), with \( T(1) = O(1) \)

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\]

when \( n > n_0 \) (\( n_0 \) to be determined)

\[
\leq 1.5^{n-3}(1 + 1.5^2 + \frac{n^2}{1.5^{n-3}}) \leq 1.5^{n-3}(1 + 1.5^2 + 0.1)
\]
Chapter 33.9 Exhaustive Search

Let $T(n) \leq T(n - 3) + T(n - 1) + cn^2$, with $T(1) = O(1)$

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when $n > n_0$ ($n_0$ to be determined)

$$\leq 1.5^{n-3}(1 + 1.5^2 + \frac{n^2}{1.5^{n-3}}) \leq 1.5^{n-3}(1 + 1.5^2 + 0.1) = 1.5^{n-3}(1 + 2.25 + 0.1)$$
Let $T(n) \leq T(n - 3) + T(n - 1) + cn^2$, with $T(1) = O(1)$

**Claim:** $T(n) = O(1.5^n)$

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\]

\[
= 1.5^{n-3} \times 3.35
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Chapter 33.9. Exhaustive Search

Let \( T(n) \leq T(n - 3) + T(n - 1) + cn^2 \), with \( T(1) = O(1) \)

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when \( n > n_0 \) (\( n_0 \) to be determined)

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\]

\[
= 1.5^{n-3} \times 3.35 \leq 1.5^{n-3} \times 3.375
\]
Chapter 33.9. Exhaustive Search

Let $T(n) \leq T(n - 3) + T(n - 1) + cn^2$, with $T(1) = O(1)$

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$$= 1.5^{n-3} \times 3.35 \leq 1.5^{n-3} \times 3.375 = 1.5^{n-3} \times 1.5^3$$
Chapter 33.9. Exhaustive Search

Let \( T(n) \leq T(n - 3) + T(n - 1) + cn^2 \), with \( T(1) = O(1) \)

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\]

when \( n > n_0 \) (\( n_0 \) to be determined)

\[
\leq 1.5^{n-3}(1 + 1.5^2 + \frac{n^2}{1.5^{n-3}}) \leq 1.5^{n-3}(1 + 1.5^2 + 0.1) = 1.5^{n-3}(1 + 2.25 + 0.1)
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= 1.5^{n-3} \times 3.35 \leq 1.5^{n-3} \times 3.375 = 1.5^{n-3} \times 1.5^3 = 1.5^n
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Chapter 33.9. Exhaustive Search

Let \( T(n) \leq T(n - 3) + T(n - 1) + cn^2 \), with \( T(1) = O(1) \)

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\]

Now we decide \( n_0 \):

\[
\frac{n^2}{1.5^{n-3}} \leq 0.1
\]
Chapter 33.9. Exhaustive Search

Let \( T(n) \leq T(n - 3) + T(n - 1) + cn^2 \), with \( T(1) = O(1) \)

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\leq 1.5^{n-3} \left( 1 + 1.5^2 + \frac{n^2}{1.5^{n-3}} \right) \leq 1.5^{n-3} (1 + 1.5^2 + 0.1) = 1.5^{n-3} (1 + 2.25 + 0.1)
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\frac{n^2}{1.5^{n-3}} \leq 0.1 \implies n^2 \leq 0.1 \times 1.5^{n-3}
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Chapter 33.9. Exhaustive Search

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\]

\[= 1.5^{n-3} \times 3.35 \leq 1.5^{n-3} \times 3.375 = 1.5^{n-3} \times 1.5^3 = 1.5^n\]

Now we decide \( n_0 \):

\[
\frac{n^2}{1.5^{n-3}} \leq 0.1 \implies n^2 \leq 0.1 \times 1.5^{n-3} \text{ holds when roughly } n \geq n_0 = 29
\]
Chapter 33.9. Exhaustive Search

- Algorithms for SAT and $\text{MAXINDSET}$ run in exponential time $O(2^n)$ or $O(\gamma^n)$ for $1 < \gamma < 2$
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- Algorithms for SAT and \textsc{MaxIndSet} run in exponential time $O(2^n)$
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- Search tree (solution search space) is large, inherently large
Algorithms for SAT and MAXINDSET run in exponential time $O(2^n)$ or $O(\gamma^n)$ for $1 < \gamma < 2$.

- search tree (solution search space) is large, inherently large
- search tree does not have obvious overlapping subproblems,
Chapter 33.9. Exhaustive Search

- Algorithms for SAT and MAXINDSET run in exponential time $O(2^n)$ or $O(\gamma^n)$ for $1 < \gamma < 2$

- Search tree (solution search space) is large, inherently large

- Search tree does not have obvious overlapping subproblems, which otherwise would incur dynamic programming approaches.
Chapter 34. NP-Completeness

1. Intractable problems
   - decision versions of optimization problems

2. Nondeterministic computational models
   - nondeterministic computation = certificate + verification

3. NP-completeness framework
   - reduction, polynomial-time reduction

4. NP-completeness proof
   - NP-complete problems, reduction techniques.
Chapter 34. NP-Completeness

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- we have seen many problems solvable in polynomial time
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- we have seen many problems solvable in polynomial time
  e.g., sorting, SCC, MST
Chapter 34. NP-Completeness

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   • there are problems that do not seem to have polynomial time algorithms
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- there are problems that do not seem to have polynomial time algorithms
  i.e., not solvable in time $O(n \log n)$,
1. Intractable problems

- we have seen many problems solvable in polynomial time
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- there are problems that do not seem to have polynomial time algorithms
  i.e., not solvable in time $O(n \log n)$, $O(n^3)$, or
1. Intractable problems

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- there are problems that do not seem to have polynomial time algorithms
  i.e., not solvable in time $O(n \log n)$, $O(n^3)$, or $O(n^{100})$. 
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1. Intractable problems

- we have seen many problems solvable in polynomial time
e.g., sorting, SCC, MST

- there are problems that do not seem to have polynomial time algorithms
  i.e., not solvable in time $O(n \log n)$, $O(n^3)$, or $O(n^{100})$.

- why would a time $O(n^{100})$-time algorithm be attractive?
1. Intractable problems

- we have seen many problems solvable in polynomial time
e.g., sorting, SCC, MST

- there are problems that do not seem to have polynomial time algorithms
  i.e., not solvable in time $O(n \log n)$, $O(n^3)$, or $O(n^{100})$.

- why would a time $O(n^{100})$-time algorithm be attractive?
  only theoretical?
Chapter 34. NP-Completeness

1. Intractable problems

- we have seen many problems solvable in polynomial time
  e.g., sorting, SCC, MST

- there are problems that do not seem to have polynomial time algorithms
  i.e., not solvable in time $O(n \log n)$, $O(n^3)$, or $O(n^{100})$.

- why would a time $O(n^{100})$-time algorithm be attractive?
  only theoretical? practical significance as well
Chapter 34. NP-Completeness

Define: A Hamiltonian cycle in a graph is a circular path going through every vertex exactly once. Different from an Eulerian cycle that goes through every edge exactly once.
Chapter 34. NP-Completeness

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Chapter 34. NP-Completeness

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Chapter 34. NP-Completeness

Travel Salesman Problem (TSP)

Input: an edge-weighted graph $G = (V, E)$;
Output: a Hamiltonian cycle of the minimum weight sum.

- intuitively, a circular path is a permutation of $(v_1, v_2, ..., v_n)$ or simply a permutation of $(1, 2, ..., n)$, where $|V| = n$.

so the problem has time upper bound $O(n! |E|)$, exponential time.

$n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1 \geq n \times (n-1) \times \cdots \times n/2 \geq (n/2)^n$.

- all known algorithms (solving TSP) are of exponential time.
Chapter 34. NP-Completeness

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Chapter 34. NP-Completeness

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Chapter 34. NP-Completeness

Instead of considering the Travel Salesman Problem (TSP)
Input: an edge-weighted graph $G = (V,E)$;
Output: a Hamiltonian cycle of the minimum weight sum.

We may consider a related problem: H-Cycle Weight Decision (HCW)
Input: an edge-weighted graph $G = (V,E)$, a weight value $K$;
Output: "YES" if and only if there is a Hamiltonian cycle of weight $\leq K$ in $G$.

• HCW appears to "easier" than TSP as an H-cycle is not produced in the answer.
• However, HCW may not be "easier".

Theorem 1: HCW is solvable in P-time if and only if TSP is solvable in P-time.
Chapter 34. NP-Completeness

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Chapter 34. NP-Completeness

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Trivially,
Chapter 34. NP-Completeness

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Trivially, P-time algorithms for TSP
Chapter 34. NP-Completeness

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Trivially, P-time algorithms for TSP $\implies$ P-time algorithms for HCW,
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Trivially, P-time algorithms for TSP $\implies$ P-time algorithms for HCW, why?
Chapter 34. NP-Completeness

**Theorem 1**: \( \text{HCW} \) is solvable in P-time if and only if \( \text{TSP} \) is solvable in P-time.

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How to prove:
Chapter 34. NP-Completeness

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How to prove: P-time algorithms for TSP
Chapter 34. NP-Completeness

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How to prove: P-time algorithms for TSP $\iff$ P-time algorithms for HCW?
Chapter 34. NP-Completeness

**Theorem 1:** HCW is solvable in P-time if and only if TSP is solvable in P-time.

**Proof:** (P-time algorithms for TSP $\iff$ P-time algorithms for HCW)
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**Proof**: (P-time algorithms for TSP $\iff$ P-time algorithms for HCW)

- assume P-time algorithm $A$ for HCW such that $A(G, K) = "YES/"NO$
Chapter 34. NP-Completeness

**Theorem 1**: HCW is solvable in P-time if and only if TSP is solvable in P-time.

**Proof**: (P-time algorithms for TSP $\iff$ P-time algorithms for HCW)

- assume P-time algorithm $A$ for HCW such that $A(G, K) = \text{"YES/"NO"}$
- construct a P-time algorithm $B(G)$ for TSP to behave as follows:
Chapter 34. NP-Completeness

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**Proof**: (P-time algorithms for TSP $\iff$ P-time algorithms for HCW)

- Assume P-time algorithm $A$ for HCW such that $A(G, K) \equiv \text{"YES/"NO}$.  
- Construct a P-time algorithm $B(G)$ for TSP to behave as follows:
  1. on input $G$, for every possible values of $K$, call $A(G, K)$.
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- assume P-time algorithm \( A \) for HCW such that \( A(G, K) = \text{"YES/NO"} \)
- construct a P-time algorithm \( B(G) \) for TSP to behave as follows:

  1. on input \( G \), for every possible values of \( K \), call \( A(G, K) \);
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Chapter 34. NP-Completeness

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  2. mark all edges in $G$ as “unvisited”;

How to make Step 1 P-time?
Chapter 34. NP-Completeness

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Chapter 34. NP-Completeness

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- assume P-time algorithm \( A \) for HCW such that \( A(G, K) = \text{"YES"/"NO"} \)
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1. on input \( G \), for every possible values of \( K \), call \( A(G, K) \);
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2. mark all edges in \( G \) as “unvisited”;
   while there are “unvisited” edges in \( G \)
   pick an “unvisited” edge \((u, v)\), mark it “visited”;

How to make Step 1 P-time?

Theorem 1 says problems HCW and TSP are “polynomially equivalent.”
Theorem 1: HCW is solvable in P-time if and only if TSP is solvable in P-time.

Proof: (P-time algorithms for TSP $\iff$ P-time algorithms for HCW)

- assume P-time algorithm $A$ for HCW such that $A(G, K) =$ “YES/“NO”
- construct a P-time algorithm $B(G)$ for TSP to behave as follows:

1. on input $G$, for every possible values of $K$, call $A(G, K)$; remember the smallest $k_{min}$ such that $A(G, k_{min}) =$ “YES”.

2. mark all edges in $G$ as “unvisited”; while there are “unvisited” edges in $G$
   - pick an “unvisited” edge $(u, v)$, mark it “visited”;
   - let $G' = G - \{ (u, v) \}$;
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**Theorem 1:** HCW is solvable in P-time if and only if TSP is solvable in P-time.

**Proof:** (P-time algorithms for TSP ⇐ P-time algorithms for HCW)

- assume P-time algorithm $A$ for HCW such that $A(G, K) = \text{"YES/NO"}$
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  1. on input $G$, for every possible values of $K$, call $A(G, K)$;
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  2. mark all edges in $G$ as “unvisited”;
     while there are “unvisited” edges in $G$
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     if $A(G', k_{\min}) = \text{"YES"}$

How to make Step 1 P-time?

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Chapter 34. NP-Completeness

**Theorem 1:** HCW is solvable in P-time if and only if TSP is solvable in P-time.

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- assume P-time algorithm $A$ for HCW such that $A(G, K) = \text{"YES/"NO}$
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2. mark all edges in $G$ as "unvisited";
   while there are "unvisited" edges in $G$
      pick an "unvisited" edge $(u, v)$, mark it "visited";
      let $G' = G - \{(u, v)\}$;
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         then $G = G'$;
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How to make Step 1 P-time?

Theorem 1 says problems HCW and TSP are "polynomially equivalent".
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**Theorem 1:** HCW is solvable in P-time **if and only if** TSP is solvable in P-time.

**Proof:** (P-time algorithms for TSP $\iff$ P-time algorithms for HCW)

- assume P-time algorithm $A$ for HCW such that $A(G, K) = \text{"YES/"NO}.$
- construct a P-time algorithm $B(G)$ for TSP to behave as follows:

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   then $G = G'$;
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return (all “critical” edges)
Chapter 34. NP-Completeness

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- show algorithm $B$ runs in P-time.
Chapter 34. NP-Completeness

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Chapter 34. NP-Completeness

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- show algorithm $B$ runs in P-time. **How to make Step 1 P-time?**

*Theorem 1 says problems HCW and TSP are “polynomially equivalent”.*
Consider another related problem:

**H-Cycle Decision (HC)**

Input: an edge-weighted graph $G = (V,E)$;  
Output: "YES" if and only there is a Hamiltonian cycle in $G$.

**H-Cycle Weight Decision (HCW)**

Input: an edge-weighted graph $G = (V,E)$, a weight value $K$;  
Output: "YES" if and only there is a Hamiltonian cycle of weight $\leq K$ in $G$.

• Which problem is seemingly "easier"?

**Theorem 2**: $HCW$ is P-time solvable if and only if $HC$ is P-time solvable.

Can you prove it? Theorem 2 says problems $HCW$ and $HC$ are "polynomially equivalent".
Consider another related problem:

**H-Cycle Decision (HC)**

**Input:** an edge-weighted graph $G = (V, E)$;

Compared with **H-Cycle Weight Decision (HCW)**

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Compared with

**H-Cycle Weight Decision (HCW)**

- **Input:** an edge-weighted graph \( G = (V, E) \), a weight value \( K \);
- **Output:** “YES” if and only if there is a Hamiltonian cycle of weight \( \leq K \) in \( G \).

Theorem 2: \( \text{HCW} \) is P-time solvable if and only if \( \text{HC} \) is P-time solvable.

Can you prove it?

Theorem 2 says problems \( \text{HCW} \) and \( \text{HC} \) are “polynomially equivalent.”
Chapter 34. NP-Completeness

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Compared with

**H-Cycle Weight Decision (HCW)**
- **Input:** an edge-weighted graph $G = (V, E)$, a weight value $K$;
- **Output:** “YES” if and only there is a Hamiltonian cycle of weight $\leq K$ in $G$.

- Which problem is seemingly “easier”?

**Theorem 2:** HCW is P-time solvable if and only if HC is P-time solvable.

*Can you prove it?*

Theorem 2 says problems HCW and HC are “polynomially equivalent”.
Chapter 34. NP-Completeness

**Corollary 3:** Problems TSP, HCW, and HC are all “polynomially equivalent”.

Max Independent Set (MaxIS)

**Input:** graph $G = (V,E)$

**Output:** an independent set of vertices of the maximum size;

Independent Set (IS)

**Input:** graph $G = (V,E)$, integer $k$

**Output:** “YES” if and only if $G$ has an independent set of size $\geq k$.

**Theorem 4:** MaxIS is P-time solvable if and only if IS is P-time solvable.

Can you prove the theorem?
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Similarly,

**Min Vertex Cover** ($\text{MinVC}$)

**Input**: graph $G = (V,E)$;

**Output**: a vertex cover set of vertices of the minimum size;

**Vertex Cover** ($\text{VC}$)

**Input**: graph $G = (V,E)$, integer $k$;

**Output**: "YES" if and only if $G$ has a vertex cover of size $\leq k$.

**Theorem 5**: $\text{MinVC}$ is P-time solvable if and only if $\text{VC}$ is P-time solvable.

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Chapter 34. NP-Completeness

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Chapter 34. NP-Completeness

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Chapter 34. NP-Completeness

Conclusions:

1. "Polynomial equivalency" can be established between optimization problems and decision problems. To study tractability of optimization problems, often it suffices to investigate decision problems. (Decision problems are also called languages.)

2. "Polynomial equivalency" can also be established between different decision problems, e.g., Corollary 6: VC is P-time solvable if and only if IS is P-time solvable.

3. However, "Polynomial equivalency" does not tell us the tractability of the problems.

4. We need a rigorous framework to study tractability via the notion "Polynomial equivalency".
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Chapter 34. NP-Completeness

2. Nondeterministic algorithms
Chapter 34. NP-Completeness

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Deterministic algorithms

• Given input data, a deterministic algorithm has its every step completely determined by the algorithm and data.
• All algorithms we have seen so far are deterministic.
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```
M = -\infty
n = 3
i = 1
check 1 \leq 3
check -\infty < 10
M = 10
i = 2
check 2 \leq 3
check 10 < 30
M = 30
i = 3
check 3 \leq 3
check 30 < 20
i = 4
check 4 \leq 3
return (30)
```

MaxOfList($L$)

1. $M = -\infty$
2. $n = \text{length}(L)$
3. for $i = 1$ to $n$
   4. if $M < L[i]$
      5. $M = L[i]$
6. return ($M$)

Unfolded when input $L = (10, 30, 20)$
Chapter 34. NP-Completeness

A deterministic algorithm can be thought of a linear path of steps;
Chapter 34. NP-Completeness

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Chapter 34. NP-Completeness

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Let us call this tree model of nondeterministic algorithms.
Chapter 34. NP-Completeness
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Use nondeterministic algorithms to solve problem Hamiltonian Cycle
Use nondeterministic algorithms to solve problem Hamiltonian Cycle in polynomial time.
Chapter 34. NP-Completeness

Use nondeterministic algorithms to solve problem **Hamiltonian Cycle** in polynomial time.

1. starting from any vertex \( v \) in the graph;

![Graph](image.png)
Chapter 34. NP-Completeness

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Chapter 34. NP-Completeness

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Chapter 34. NP-Completeness

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\[ \text{Graph representation} \]

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Chapter 34. NP-Completeness

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Chapter 34. NP-Completeness

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- The algorithm runs in polynomial time as each path takes \( O(n) \) steps.
Problems like Independent Set, Vertex Cover, HCW can all be solved with nondeterministic algorithms in polynomial time.
Problems like **Independent Set**, **Vertex Cover**, **HCW** can all be solved with **nondeterministic algorithms** in **polynomial time**.

**Can you prove the claim?**
Chapter 34. NP-Completeness

**Definition**: \( P \) is the class of languages (i.e., decision problems) that can be solved by **deterministic polynomial-time algorithms**.
Chapter 34. NP-Completeness

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Chapter 34. NP-Completeness

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Chapter 34. NP-Completeness

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Chapter 34. NP-Completeness

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- class \( NP \) contains problems like \textsc{VC}, \textsc{HC}, \textsc{IS} and many others.

Because every deterministic algorithm is a special case of a nondeterministic algorithm, 

\[ P \subseteq NP \]
Chapter 34. NP-Completeness

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Chapter 34. NP-Completeness

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Chapter 34. NP-Completeness
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Chapter 34. NP-Completeness

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The binary string is called certificate or witness; the deterministic computation part is called verification.

Deterministic algorithms are when the certificate is empty.
Chapter 34. NP-Completeness

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Chapter 34. NP-Completeness

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Chapter 34. NP-Completeness

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The binary string is called **certificate** or **witness**;
The deterministic computation part is called **verification**.
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Chapter 34. NP-Completeness

Alternative view of nondeterministic polynomial-time computation
Chapter 34. NP-Completeness

Alternative view of nondeterministic polynomial-time computation

Every nondeterministic polynomial time computation is

• to nondeterministically choose a binary string of a polynomial length,
• then to compute deterministically in polynomial time.

Let $\Pi \in \text{NP}$. Then there is a deterministic algorithm $A_{\Pi}$, and a constant $c > 0$, such that

1. if $x$ is a positive instance of $\Pi$, there is a binary string $y$ of length $n^c$, $A_{\Pi}(x, y) = \text{"YES"}$;
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We call $y$ a certificate/witness and $A_{\Pi}$ the verification algorithm.

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Chapter 34. NP-Completeness

Alternative view of nondeterministic polynomial-time computation

Every nondeterministic polynomial time computation is

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Chapter 34. NP-Completeness

Definition of $\mathcal{NP}$ in terms of languages:

Let $L \subseteq \{0, 1\}^*$ be a language in the class $\mathcal{NP}$. Then there is a deterministic algorithm $A_L$, and a constant $c > 0$, such that, for every $x \in \{0, 1\}^*$, $x \in L \iff \exists y, |y| \leq |x|^{c}$, $A_L(x, y) = 1$ and $A_L$ runs in polynomial time.

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Chapter 34. NP-Completeness

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The diagram illustrates a nondeterministic algorithm with possible moves labeled as 'det. moves' and 'nondet. moves'. The algorithm involves making choices that can lead to different outcomes, as indicated by the arrows leading to 'Y' and 'N' states. The process continues with deterministic steps, where the decision is verified. The text on the right side of the diagram suggests choosing a certificate $y = 10...11...$ for verification.
Proof that $HC \in NP$.

• Certificate $y$ represents a sequence of ordered vertices;
• Algorithm $A$ is to verify that $y$ does form a H-cycle.

Details:
• $y = B_1 B_2 \ldots B_n$, where $B_i$ is a binary representation of some vertex in $G$;
• How many bits does $B_i$ need? $\lceil \log_2 n \rceil$;
• Whether $y$ forms a H-cycle can be verified in time $O(|E|)$. 
Proof that $HC \in \mathcal{NP}$.

We need to show there is a deterministic algorithm $A$ and a constant $c > 0$, such that for any $G$,

$$G \in HC \iff \exists y, |y| \leq |G|^c, A(G, y) = \text{"YES"}$$

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Chapter 34. NP-Completeness

exercises:

Proof that Independent Set $\in \text{NP}$.

Proof that Vertex Cover $\in \text{NP}$.

Notes

1. To prove a language is in the class $\text{NP}$ by no mean to prove that the language can be solved in polynomial time. Instead, it only shows the language is in the class $\text{NP}$.

2. There is a difference between deciding $x \in L$ and checking $A_L(x, y) = 1$.

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3. NP-Completeness Framework
Chapter 34. NP-Completeness

3. NP-Completeness Framework

The notion of reduction (i.e., transformation) between languages
3. NP-Completeness Framework

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- We use languages for decision problems.
3. NP-Completeness Framework

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Chapter 34. NP-Completeness

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$$ L \cup \overline{L} = \{0, 1\}^* = \mathcal{U}, $$
3. NP-Completeness Framework

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\[ L = \{ x : x \not\in L \} \] called complement of \( L \)

\[ L \cup \overline{L} = \{0, 1\}^* = \mathcal{U}, \] called universe
Chapter 34. NP-Completeness

Let $L_1$ and $L_2$ are two languages over the alphabet $\{0, 1\}$. 
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$$x \in L_1 \implies f(x) \in L_2; \quad x \in \overline{L_1} \implies f(x) \in \overline{L_2};$$
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Let $L_1$ and $L_2$ are two languages over the alphabet $\{0, 1\}$.

A reduction from $L_1$ to $L_2$, denoted as $L_1 \leq L_2$, is some mapping function $f : \{0, 1\}^* \rightarrow \{0, 1\}^*$, such that for any $x \in \{0, 1\}^*$,

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Two example problems:

Independent Set (IS)
Input: graph \( G = (V,E) \), integer \( k \);
Output: "YES" if and only if \( G \) has an independent set of size \( \geq k \).

Vertex Cover (VC)
Input: graph \( G = (V,E) \), integer \( k \);
Output: "YES" if and only if \( G \) has a vertex cover of size \( \leq k \).
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Because the two problems are very relevant to each other, we have:

**Theorem:** \[ L_{IS} \leq L_{VC} \]

**Proof:** we use the fact that complement set of an independent set is a vertex cover in the same graph.

We construct a mapping \( f \) that maps instance \( \langle G, k \rangle \) to instance \( \langle G, |G| - k \rangle \), i.e., \( f(\langle G, k \rangle) = \langle G, |G| - k \rangle \).

This is a reduction from \( L_{IS} \) to \( L_{VC} \).

**Claim:** \( G \) has an i.s. of size \( \geq k \) \( \iff \) \( G \) has an v.c. of size \( \leq |G| - k \)

(proof of the claim is on the next slide)

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(proof of the claim is on the next slide)

So \( L_{IS} \leq L_{VC} \).
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**Proof:** “⇒”
(to prove that $G$ has i.s. of size $\geq k$ implies $G$ has v.c of size $\geq k$)

Let $G$ be such that has vertices $V = \{v_1,\ldots,v_n\}$. Assume that $G$ has a i.s. of size $k_0$ for some $k_0 \geq k$. We further assume, without loss of generality, the i.s include vertices $\{v_1,\ldots,v_{k_0}\}$. Then vertices $\{v_{k_0+1},\ldots,v_n\}$ form a v.c. for $G$. Suppose otherwise, $\exists$ edge $(u,v)$ that is not covered, i.e., neither $u \in \{v_{k_0+1},\ldots,v_n\}$ nor $v \in \{v_{k_0+1},\ldots,v_n\}$. Thus, $u,v \in \{v_1,\ldots,v_{k_0}\}$, the independent set. But $(u,v)$ is an edge, contradicts that $\{v_1,\ldots,v_{k_0}\}$ is an i.s.

Can you prove “⇐”??
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Thus, $u, v \in \{v_1, \ldots, v_{k_0}\}$, the independent set.
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An important motivation for reduction:
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- a reduction transforms instances of the first problem to the instances of the second problem;
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So the combined algorithm (gray-color box) solves for $L_1$. 
Formally,
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A polynomial-time reduction from $L_1$ to $L_2$, denoted as $L_1 \leq_p L_2$, is some mapping function $f : \{0, 1\}^* \rightarrow \{0, 1\}^*$,
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where $f$ can be computed in time $O(|x|^c)$ for some fixed $c > 0$. 
Formally,

A polynomial-time reduction from \( L_1 \) to \( L_2 \), denoted as \( L_1 \leq_p L_2 \), is some mapping function \( f : \{0, 1\}^* \rightarrow \{0, 1\}^* \), such that for any \( x \in \{0, 1\}^* \),

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For example, \( L_{IS} \leq_p L_V \).
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**Theorem:** Let $L_1 \leq_p L_2$. If $L_2 \in \mathcal{P}$, then $L_1 \in \mathcal{P}$. 

Proof: Assume algorithm $F$ computes $f$, and algorithm $A_2$ solves for $L_2$. We need to show that the gray box runs in polynomial time if both $F$ and $A_2$ runs in polynomial time: Total time is the sum of time for $F$ and time for $A_2$.

Now, what is the length of $f(x)$? Because $F$ runs in time $O(|x|^c)$, the number of bits outputted by $F$ is $O(|x|^c)$. So $O(|x|^c) + O(|f(x)|^d) = O(|x|^c + O((|x|^c)^d)) = O(|x|^{cd})$.
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![Diagram showing the flow of inputs through algorithms $F$, $A_2$, and $A_1$.]

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![Diagram of the computation process]

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Because $F$ runs in time $O(|x|^c)$, the number of bits outputted by $F$ is $O(|x|^c)$.

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$$O(|x|^c) + O(|f(x)|^d) = O(|x|^c + O((|x|^c)^d))$$
**Theorem:** Let $L_1 \leq_p L_2$. If $L_2 \in \mathcal{P}$, then $L_1 \in \mathcal{P}$.

**Proof:** Assume algorithm $F$ computes $f$, and algorithm $A_2$ solves for $L_2$.

We need to show that the gray box runs in polynomial time if both $F$ and $A_2$ runs in polynomial time:

![Diagram](image)

Total time is the sum of time for $F$ and time for $A_2$.

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**Theorem**: Polynomial-time reductions compose (are transitive). That is, if $L_1 \leq_p L_2$ and $L_2 \leq_p L_3$, then $L_1 \leq_p L_3$.

**Proof**. Assume functions $f$ for $L_1 \leq_p L_2$; function $h$ for $L_2 \leq_p L_3$. For every $x \in \{0, 1\}^*$, $x \in L_1 \iff f(x) \in L_2 \iff h(f(x)) \in L_3$.

That is $x \in L_1 \iff h(f(x)) \in L_3$. So composite function $(h \circ f)$ realizes reduction $L_1 \leq_p L_3$.

But we need to show the reduction is $\leq_p$, i.e., a polynomial time reduction. Assume that algorithm $F$ computes $f$: $F(x) = f(x)$ in time $O(|x|^c)$ and algorithm $H$ computes $h$: $H(y) = h(y)$ in time $O(|y|^d)$.

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• If those languages at the end of a $\leq_p$ chain have polynomial-time algorithms, so does every language on the chain.

• Informally, those at the end of a $\leq_p$ chain are called NP-hard.
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Definition 1: \( L \) is NP-hard

Definition 2: \( L \) is NP-complete if (1) \( L \) is NP-hard and (2) \( L \) \( \in \) NP.

Properties of NP-hard problems

• If \( L \) is NP-hard and \( L \) \( \in \) P, then P = NP.
  Proof?

• If \( L \) is NP-hard and \( L \leq_p L' \), then \( L' \) is NP-hard.
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Definition 1: $L$ is **NP-hard** if for every language $L' \in \mathcal{NP}$, $L' \leq_p L$. 

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Properties of NP-hard problems
- If $L$ is NP-hard and $L \in \mathcal{P}$, then $\mathcal{P} = \mathcal{NP}$.
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How to prove a language is NP-hard?
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How to prove a language is NP-hard?
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4. NP-Completeness Proofs
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- Instead, formulate a generic language that represents all languages in NP and prove that every language in $\mathcal{NP}$ can be reduced to the generic language in polynomial time.

- To obtain such a generic language, we need to consider the definition of languages in NP.
Recall the definition of languages in $\mathcal{NP}$:

Let $L \subseteq \{0,1\}^*$ be any language in the class $\mathcal{NP}$. Then there is a deterministic algorithm $A_L$, and a constant $c > 0$, such that, for every $x \in \{0,1\}^*$, $x \in L$ $\iff \exists y, |y| \leq |x|^c, A_L(x,y) = 1$ and $A_L$ runs in polynomial time.

The "iff" relationship looks a little like the relationship in a reduction $x \in L \iff \exists y, |y| \leq |x|^c, A_L(x,y) = 1 \leftrightarrow x \in L \iff f(x) \in L_{tbd}$ where $L_{tbd}$ is a language to be defined.

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Chapter 34. NP-Completeness

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Let $L \subseteq \{0,1\}^*$ be any language in the class $\mathcal{NP}$. Then there is a deterministic algorithm $A_L$, and a constant $c > 0$, such that, for every $x \in \{0,1\}^*$,

$$x \in L$$
Chapter 34. NP-Completeness

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$$x \in L \iff \exists y, |y| \leq |x|^c, A_L(x, y) = 1$$

and $A_L$ runs in polynomial time.

The "iff" relationship looks a little like the relationship in a reduction:

$$x \in L \iff \exists y, |y| \leq |x|^c, A_L(x, y) = 1$$

$\Leftrightarrow$

$$x \in L \iff f(x) \in L_{\text{tbd}}$$

where $L_{\text{tbd}}$ is a language to be defined.

Can we identify $L_{\text{tbd}}$ and $f$?
Chapter 34. NP-Completeness

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Chapter 34. NP-Completeness

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Chapter 34. NP-Completeness

Again we examine

\[ x \in L \iff \exists y, |y| \leq |x|^c, A_L(x, y) = 1 \]  

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Chapter 34. NP-Completeness

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\[ x \in L \iff \exists y, |y| \leq |x|^c, A_L(x, y) = 1 \]  \hspace{1cm} (1)

- \( A_L \) is a deterministic algorithm can be implemented with a boolean circuit \( B_L \) with two sets of input gates \( x = x_1 x_2 \ldots x_n \) and \( y = y_1 y_2 \ldots y_m \) such that

\[ A_L(x, y) = 1 \text{ if and only if } B_L(x, y) = 1 \]  \hspace{1cm} (2)
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Because \( x \) is given, circuit \( B_L \) can be made into circuit \( C^x_L \) such that

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(3)

From (1), (2), and (3), we have

\[ x \in L \iff \exists y, C^x_L(y) = 1 \]

(4)
Chapter 34. NP-Completeness

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Chapter 34. NP-Completeness

Now we have

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• Define: a boolean circuit \( C \) is satisfiable if there exists at least one set of values \( y \) to its input gates such that \( C(y) = 1 \).

  e.g., \( C(x_1, x_2) = (\neg x_1 \lor \neg x_2) \land (x_1 \lor x_2) \) is satisfiable; but \( D(x_1, x_2) = (x_1 \lor x_2) \land (x_1 \lor \neg x_2) \land (\neg x_1 \lor x_2) \land (\neg x_1 \lor \neg x_2) \) is not!

• Define the following language:

\[ CSAT = \{ C : \text{circuit } C \text{ is satisfiable} \} \]

From (4), we have

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It remains to be shown

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Chapter 34. NP-Completeness

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Chapter 34. NP-Completeness

Unfold deterministic polynomial-time algorithm $A(x, y)$ with input $\langle x, y \rangle$
Chapter 34. NP-Completeness

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Chapter 34. NP-Completeness

The algorithm is physically implemented with a boolean circuit
Chapter 34. NP-Completeness

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\[ x_1 \ x_2 \ x_3 \ \ldots \ x_n \quad y_1 \ y_2 \ y_3 \ \ldots \ y_m \]
Chapter 34. NP-Completeness

The algorithm is physically implemented with a boolean circuit

And the circuit can be built from the algorithm in polynomial time.
Chapter 34. NP-Completeness

The above discussion shows that \( L_{CSAT} \) is NP-hard.
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**Theorem**: Language $CSAT$ is NP-complete.
Chapter 34. NP-Completeness

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Actually, the following language $SAT$ was first proved to be NP-complete [Cook'71]

$$SAT = \{\phi : \text{CNF boolean formula } \phi \text{ is satisfiable}\}$$
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Chapter 34. NP-Completeness

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Actually, the following language SAT was first proved to be NP-complete [Cook’71] https://www.cs.toronto.edu/~sacook/homepage/1971.pdf

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SAT = \{ \phi : \text{CNF boolean formula } \phi \text{ is satisfiable} \}
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**Cook’s Theorem:** SAT is NP-complete.

Cook’s reduction: characterizing a polynomial-time computation on nondeterministic Turing machine with a boolean formula, such that a nondeterministic path leading to the accept state **corresponds** to an assignment to the variables making the the formula TRUE.
Chapter 34. NP-Completeness

It is very easy to convert a boolean formula to a boolean circuit. So
It is very easy to convert a boolean formula to a boolean circuit. So

**Theorem:** $SAT \leq_p CSAT$. 

On the other hand, $CSAT \leq_p SAT$. 

*How to convert a circuit to a boolean formula (from network to tree)?* Simply replicating gates may blow-up the size of formula to exponential!
Chapter 34. NP-Completeness

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Chapter 34. NP-Completeness

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Chapter 34. NP-Completeness

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Chapter 34. NP-Completeness

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is satisfiable if and only if formula $\phi$ is satisfiable:
Chapter 34. NP-Completeness

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is satisfiable if and only if formula $\phi$ is satisfiable:

$$
\phi = x_{10} \land (x_4 \leftrightarrow \neg x_3)
\land (x_5 \leftrightarrow (x_1 \lor x_2))
\land (x_6 \leftrightarrow \neg x_4)
\land (x_7 \leftrightarrow (x_1 \land x_2 \land x_4))
\land (x_8 \leftrightarrow (x_5 \lor x_6))
\land (x_9 \leftrightarrow (x_6 \lor x_7))
\land (x_{10} \leftrightarrow (x_7 \land x_8 \land x_9)).
$$
Chapter 34. NP-Completeness

**Theorem:** $CSAT \leq_p SAT$.

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\land (x_{10} \leftrightarrow (x_7 \land x_8 \land x_9)) .
\]

$\phi$ can be transformed to an equivalent CNF formula.
Chapter 34. NP-Completeness

Landscape of NP problems and beyond
Chapter 34. NP-Completeness

Landscape of NP problems and beyond
Chapter 34. NP-Completeness

Many problems/languages have been proved NP-complete (Karp70s)
Examples of reduction techniques
Chapter 34. NP-Completeness

Examples of reduction techniques

Example 1: \( SAT \leq_p 3SAT \)
Examples of reduction techniques

Example 1: SAT $\leq_p$ 3SAT

($z$)
Examples of reduction techniques

Example 1: \( \text{SAT} \leq_p \text{3SAT} \)

\[ (z) \implies (z, x_1, x_2) \land (z, x_1, \neg x_2) \land (z, \neg x_1, x_2) \land (z, \neg x_1, \neg x_2) \]
Examples of reduction techniques

Example 1: $\text{SAT} \leq_p \text{3SAT}$

$$(z) \iff (z, x_1, x_2) \land (z, x_1, \neg x_2) \land (z, \neg x_1, x_2) \land (z, \neg x_1, \neg x_2)$$

$$(y, z)$$
Examples of reduction techniques

Example 1: SAT $\leq_p$ 3SAT

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$$(y, z) \implies (y, z, x_1) \land (y, z, \neg x_1)$$
Examples of reduction techniques

Example 1: SAT $\leq_p$ 3SAT

$(z) \implies (z, x_1, x_2) \land (z, x_1, \neg x_2) \land (z, \neg x_1, x_2) \land (z, \neg x_1, \neg x_2)$

$(y, z) \implies (y, z, x_1) \land (y, z, \neg x_1)$

$(x, y, z)$
Chapter 34. NP-Completeness

Examples of reduction techniques

Example 1: SAT \( \leq_p 3\text{SAT} \)

\[
\begin{align*}
(z) & \implies (z, x_1, x_2) \land (z, x_1, \neg x_2) \land (z, \neg x_1, x_2) \land (z, \neg x_1, \neg x_2) \\
(y, z) & \implies (y, z, x_1) \land (y, z, \neg x_1) \\
(x, y, z) & \implies (x, y, z)
\end{align*}
\]
Examples of reduction techniques

Example 1: SAT $\leq_p$ 3SAT

$$(z) \implies (z, x_1, x_2) \land (z, x_1, \neg x_2) \land (z, \neg x_1, x_2) \land (z, \neg x_1, \neg x_2)$$

$$(y, z) \implies (y, z, x_1) \land (y, z, \neg x_1)$$

$$(x, y, z) \implies (x, y, z)$$

$$(y, z, u, v)$$
Chapter 34. NP-Completeness

Examples of reduction techniques

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\]

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\[
(y, z, u, v) \implies (y, z, x_1) \land (\neg x_1, u, v)
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Example 1: SAT \((\leq_p)\) 3SAT

\[
(z) \Rightarrow (z, x_1, x_2) \land (z, x_1, \neg x_2) \land (z, \neg x_1, x_2) \land (z, \neg x_1, \neg x_2)
\]

\[
y, z \Rightarrow (y, z, x_1) \land (y, z, \neg x_1)
\]

\[
x, y, z \Rightarrow (x, y, z)
\]

\[
y, z, u, v \Rightarrow (y, z, x_1) \land (\neg x_1, u, v)
\]

\[
y, z, u, v, w
\]
Examples of reduction techniques

**Example 1: SAT \( \leq_p \) 3SAT**

\[
\begin{align*}
(z) & \implies (z, x_1, x_2) \land (z, x_1, \neg x_2) \land (z, \neg x_1, x_2) \land (z, \neg x_1, \neg x_2) \\
(y, z) & \implies (y, z, x_1) \land (y, z, \neg x_1) \\
(x, y, z) & \implies (x, y, z) \\
(y, z, u, v) & \implies (y, z, x_1) \land (\neg x_1, u, v) \\
(y, z, u, v, w) & \implies (y, z, x_1) \land (\neg x_1, u, x_2) \land (\neg x_2, v, w)
\end{align*}
\]
Example 2: 3SAT \leq_p IS

An assignment TRUE to one literal in each clause corresponds to an independent set in the transformed graph.
Example 2: $3\text{SAT} \leq_p \text{IS}$
Chapter 34. NP-Completeness

Example 2: \(3\text{SAT} \leq_p \text{IS}\)

\[(x_1 \vee x_2 \vee \overline{x_3}) \land (x_2 \vee x_3 \vee \overline{x_4}) \land (x_1 \vee \overline{x_2} \vee x_4)\]
Example 2: $3\text{SAT} \leq_p \text{IS}$

\[(x_1 \lor x_2 \lor \overline{x_3}) \land (x_2 \lor x_3 \lor \overline{x_4}) \land (x_1 \lor \overline{x_2} \lor x_4)\]

An assignment TRUE to one literal in each clause
Example 2: $3\text{SAT} \leq_p \text{IS}$

\[(x_1 \lor x_2 \lor \overline{x_3}) \land (x_2 \lor x_3 \lor \overline{x_4}) \land (x_1 \lor \overline{x_2} \lor x_4)\]

An assignment TRUE to one literal in each clause corresponds to an independent set in the transformed graph.
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Scope of the Final Exam

- Greedy algorithms (greedy choice property and proof)
- Depth-First-Search algorithm, DFS search tree, time stamps
- Applications: topological sort
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Scope of the Final Exam (cont’)

Minimum spanning tree concept/properties of MST
generic, Kruskal’s and Prim’s

Shortest path (single source and all pairs) concept/properties of shortest path, greedy algorithms, relaxation technique
single source: Bellman-Ford’s, Dijkstra’s
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Scope of the Final Exam (cont’)

▶ Exhaustive search, recursive, enumerative search tree method to reduce complexity

▶ NP-completeness theory
  - definitions of NP class (certificate + verification)
  - proof that a language is in NP
  - reduction, polynomial-time reduction, properties
  - definitions of NP-hard, NP-complete languages, properties
  - NP-completeness proofs (simple, limited to previously known reductions)
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ABET survey for CSCI 4470:

https://ugeorgia.qualtrics.com/jfe/form/SV_3NKSJMO6f6HJMavb