CSCI 4470/6470 Algorithms, Fall 2019

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Department of Computer Science, UGA

Syllabus: http://cobweb.cs.uga.edu/~cai/courses/alg/2019Fall/

August 27, 2019
An Introduction to the Introduction

There are two ways of constructing a software design: One way is to make it so simple that there are obviously no deficiencies, and the other way is to make it so complicated that there are no obvious deficiencies.

C.A.R. Hoare (1980 Turing Award recipient)

Which way do we take in algorithm design?

Both correctness and efficiency are desired.
An Introduction to the Introduction

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C.A.R. Hoare (1980 Turing Award recipient)

Which way do we take in algorithm design?

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An Introduction to the Introduction

Example 1

Sequence Homology Reveals Functions

- Homology reveals evolution of structure/function
  - FOS_RAT: MMFSGFNADYEAASSRSSASASPAGDLSTYYRHSPDFDSSMGSPVNTQDFCADDLSVSSANF 60
  - FOS_MOUSE: MMFSGFNADYEAASSRSSASASPAGDLSTYYRHSPDFDSSMGSPVNTQDFCADDLSVSSANF 60
  - FOS_CHICK: MMYSQGFAGYEAAPSSRSSASASPAGDLSTYYPSPADFSMGGSPVNSQDFCADDLSVSSANF 60
  - FOSB_MOUSE: -MFQAFPGYDS-GSRS--SPSAGSQ—YLSVDSFGPSPTAASQ-E-CAGLMPGQS 54
  - FOSB_HUMAN: -MFQAFPGYDS-GSRS--SPSAGSQ—YLSVDSFGPSPTAASQ-E-CAGLMPGQS 54

- Homology reveals regulatory structure (E. Coli promoters)
  - tyrRNA: TCTCAAAGTAACTTTTTATATATATGACGGGG -- CGGCGGCTGCTGCCGATAAGG
  - rrm D1: GATCAAAAAATAATTGTGCAAAAAAAATTGGGATCTTATAAGACACGG
  - rrm X1: ATGCTTTTGCGTCTGGTGCTG
  - rrm iDXE2: CTTGAAAATTACCCGGGTTAGCTG
  - rrm C1: CTGAAATTTTGTTATGGCGGCTG
  - rrm A1: TTTTAAATTTCTCTTTCTGTTAGG
  - rrm A2: AAACACCGGCGCGGAAG--TGTTGATGTATACGC
  - α_Fr: TATCTCCGCGGCTGATTATAGA--CCACTGCCTGCTG
  - α_Fl: TATCTCCGCGGCTGATTATAGA--CCACTGCCTGCTG
  - T7 A3: GTGAACACACACCCTTAACGGTCAACATAG
  - T7 A1: TATCAAAGGATTATACCTGATACTATGCCAGCTG
  - T7 A2: ACGAAAACCGTATCAGGATGAACTTGAATAT
  - fd VIII: GATACAAATCTTCTCGTGTTAC

-35  -10  +1
An Introduction to the Introduction

Example 1

called **Multiple Sequence Alignment**.
An Introduction to the Introduction

Example 1

Problem **Multiple Sequence Alignment:**

- **Input:** $k$ sequences, each of length $\approx n$;
- **Output:** a biologically most "plausible" alignment
  - e.g., for $k = 2$ editing through substitutions, insertions, deletions
  - possible alignments $\geq 2^n$, naive methods may not be efficient

Algorithm design goal 1:

To craft algorithms as efficient as possible (to develop skills more than naive ideas!)
An Introduction to the Introduction

Example 1

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Input: \( k \) sequences, each of length \( \approx n \);
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An Introduction to the Introduction

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  FOSB_MOUSE  -MFQAFPGDYDS-GSRCSS-PSAESEQ--YLSSVDSFGSPPTAAASQE-CAGLEMPGSF 54
```
An Introduction to the Introduction

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Problem **Multiple Sequence Alignment:**

Input: $k$ sequences, each of length $\approx n$;
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- e.g., for $k = 2$

<table>
<thead>
<tr>
<th>FOS_CHICK</th>
<th>MMYQGFAGEYEAPSSRCSSASPAGDSLTYYPSPADSFSSMGSPVNSQDFCTDLAVSSANF 60</th>
</tr>
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<tbody>
<tr>
<td>FOSB_MOUSE</td>
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</tr>
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editing through **substitutions, insertions, deletions**
An Introduction to the Introduction

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editing through substitutions, insertions, deletions

possible alignments $\geq 2^n$, naive methods may not be efficient

Algorithm design goal 1:

To craft algorithms as efficient as possible
(to develop skills more than naive ideas!)
An Introduction to the Introduction

Example 2
## An Introduction to the Introduction

### Example 2

**US Open 2016 Men's Singles**

<table>
<thead>
<tr>
<th>Round</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>Quarterfinals</th>
<th>Semifinals</th>
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<td>A. Zverev</td>
<td>A. Zverev</td>
<td>A. Zverev</td>
<td>6-2 7-6(2) 6-4 6-3</td>
<td>6-2 7-6(2) 6-4 6-3</td>
</tr>
<tr>
<td>3</td>
<td>M. Youzhny</td>
<td>M. Youzhny</td>
<td>M. Youzhny</td>
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<td>6-2</td>
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<tr>
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<td>R. Nadal</td>
<td>R. Nadal</td>
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<td>7-6(1) 6-4 7-6(1)</td>
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<tr>
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</tr>
</tbody>
</table>

Champion: **Wawrinka, Stan SUI**

8-7(1) 6-4 7-5 6-3
An Introduction to the Introduction
An Introduction to the Introduction
An Introduction to the Introduction

• 128 players in total, single elimination
• 128 players in total, single elimination
• How many matches are played to determine a champion?
• 128 players in total, single elimination
• How many matches are played to determine a champion? 127
• 128 players in total, single elimination
• How many matches are played to determine a champion? 127
• Is it possible to just play fewer matches?
An Introduction to the Introduction

- 7 matches is sufficient to determine the champion.
  - What does it mean?
    - The specific arrangement of 7 matches finds the champion.

- Are 7 matches necessary to determine the champion?
  - What does that mean?
    - \( \leq 6 \) matches, regardless of arrangement, cannot find a champion.
An Introduction to the Introduction

• 7 matches is **sufficient** to determine the champion.
An Introduction to the Introduction

- 7 matches is **sufficient** to determine the champion.

**What does it mean?**
An Introduction to the Introduction

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An Introduction to the Introduction

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7 matches is sufficient to determine the champion.

What does it mean?
The specific arrangement of 7 matches finds the champion.

Are 7 matches necessary to determine the champion?

What does that mean?
$\leq 6$ matches, regardless arrangement, cannot find a champion.
Problem **Find Max:**

Input: a set of \( n \) numbers \( x_1, \ldots, x_n \);
Output: \( x_i \), for some \( i \) (\( 1 \leq i \leq n \)), such that for \( j = 1, 2, \ldots, n \), \( x_i \geq x_j \).

The problem can be solved with \( n-1 \) comparisons.

There is a way to find the maximum using \( n-1 \) comparisons.

Regardless how to compare these numbers.
Problem **Find Max:**

Input: a set of $n$ numbers $x_1, \ldots, x_n$;
An Introduction to the Introduction

Problem **FIND MAX:**

Input: a set of $n$ numbers $x_1, \ldots, x_n$;
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\[ x_i \geq x_j \]
An Introduction to the Introduction

Problem **Find Max:**

Input: a set of \( n \) numbers \( x_1, \ldots, x_n \);
Output: \( x_i \), for some \( i \) \( (1 \leq i \leq n) \), such that for \( j = 1, 2, \ldots, n \),
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- The problem can be solved with \( n - 1 \) comparisons.
Problem **FIND MAX:**

Input: a set of $n$ numbers $x_1, \ldots, x_n$;
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- The problem can be solved with $n - 1$ comparisons.
  There is a way to find the maximum using $n - 1$ comparisons.
Problem \textbf{FIND MAX}:

Input: a set of \( n \) numbers \( x_1, \ldots, x_n \);
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- The problem can be solved with \( n - 1 \) comparisons.
  There is a way to find the maximum using \( n - 1 \) comparisons.
- The problem requires at least \( n - 1 \) comparisons.
Problem **Find Max**: 

- Input: a set of $n$ numbers $x_1, \ldots, x_n$;  
- Output: $x_i$, for some $i$ ($1 \leq i \leq n$), such that for $j = 1, 2, \ldots, n$,
  
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- The problem can be solved with $n - 1$ comparisons. 
  There is a way to find the maximum using $n - 1$ comparisons. 

- The problem requires at least $n - 1$ comparisons. 
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• \( n - 1 \) comparisons are \textbf{sufficient} to find the maximum called an \textbf{upper bound};
An Introduction to the Introduction

Problem **Find Max**:

Input: a set of $n$ numbers $x_1, \ldots, x_n$;
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$$x_i \geq x_j$$

- $n - 1$ comparisons are **sufficient** to find the maximum called an **upper bound**;

You only need to show an **algorithm** to show an upper bound;
An Introduction to the Introduction

Problem FIND MAX:

Input: a set of $n$ numbers $x_1, \ldots, x_n$;
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• $n - 1$ comparisons are sufficient to find the maximum called an upper bound;

You only need to show an algorithm to show an upper bound;

• $n - 1$ comparisons are necessary to find the maximum called a lower bound;
An Introduction to the Introduction

Problem **Find Max:**

Input: a set of $n$ numbers $x_1, \ldots, x_n$;
Output: $x_i$, for some $i$ ($1 \leq i \leq n$), such that for $j = 1, 2, \ldots, n,
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- $n - 1$ comparisons are sufficient to find the maximum called an upper bound;

You only need to show an algorithm to show an upper bound;

- $n - 1$ comparisons are necessary to find the maximum called a lower bound;

you need a (mathematical) proof to show a lower bound.
An Introduction to the Introduction

Problem **Find Max**:  

Input: a set of \( n \) numbers \( x_1, \ldots, x_n \);  
Output: \( x_i \), for some \( i \) (\( 1 \leq i \leq n \)), such that for \( j = 1, 2, \ldots, n \),  
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x_i \geq x_j
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- Is \( n + 3 \) an upper bound for problem \textbf{Find Max}?
An Introduction to the Introduction

Problem $\text{Find Max}$:

Input: a set of $n$ numbers $x_1, \ldots, x_n$;
Output: $x_i$, for some $i$ ($1 \leq i \leq n$), such that for $j = 1, 2, \ldots, n$,

\[ x_i \geq x_j \]

- Is $n + 3$ an upper bound for problem $\text{Find Max}$?
- Is $n + 3$ a lower bound for the problem?
An Introduction to the Introduction

Problem \textsc{Find Max}:

Input: a set of \( n \) numbers \( x_1, \ldots, x_n \);
Output: \( x_i \), for some \( i \) (\( 1 \leq i \leq n \)), such that for \( j = 1, 2, \ldots, n \),

\[ x_i \geq x_j \]

- Is \( n + 3 \) an upper bound for problem \textsc{Find Max}?
- Is \( n + 3 \) a lower bound for the problem?
- Is \( n - 3 \) an upper bound? How about a lower bound?
Problem \texttt{Find Max}:

Input: a set of \( n \) numbers \( x_1, \ldots, x_n \);
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- Is \( n + 3 \) an upper bound for problem \texttt{Find Max}?
- Is \( n + 3 \) a lower bound for the problem?
- Is \( n - 3 \) an upper bound? How about a lower bound?

Algorithm design goal 2:
Problem \texttt{Find Max}:

\begin{quote}
Input: a set of \( n \) numbers \( x_1, \ldots, x_n \); 
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\end{quote}

- Is \( n + 3 \) an upper bound for problem \texttt{Find Max}?
- Is \( n + 3 \) a lower bound for the problem?
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\underline{Algorithm design goal 2:}

To understand challenges posed by a problem to be solved
Problem **Find Max**: 

Input: a set of \( n \) numbers \( x_1, \ldots, x_n \); 
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\[
    x_i \geq x_j
\]

- Is \( n + 3 \) an upper bound for problem **Find Max**? 
- Is \( n + 3 \) a lower bound for the problem? 
- Is \( n - 3 \) an upper bound? How about a lower bound? 

**Algorithm design goal 2:** 

To understand challenges posed by a problem to be solved (need to develop capability to observe and analyze!)
An Introduction to the Introduction

**Example 3** Toss coin over the phone
Example 3 Toss coin over the phone

- Two distant cities toss coin to decide who to a football match;
Example 3 Toss coin over the phone

• Two distant cities toss coin to decide who to a football match;
• Lack of visual communication tools, coin toss is done via phone calls;
Example 3 Toss coin over the phone

- Two distant cities toss coin to decide who to a football match;
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An Introduction to the Introduction

Example 3 Toss coin over the phone

- Two distant cities toss coin to decide who to a football match;
- Lack of visual communication tools, coin toss is done via phone calls;

- How would it work?
They decided the following protocol:
They decided the following protocol:

- A told B a huge Boolean circuit through the phone;
They decided the following protocol:

- A told B a huge Boolean circuit through the phone; along with a
They decided the following protocol:

- A told B a huge Boolean circuit through the phone; along with a binary string $Y$, the output produced from some binary string input $X$ (but the input string $X$ is not given to $B$);
They decided the following protocol:

- A told B a huge Boolean circuit through the phone; along with a binary string $Y$, the output produced from some binary string input $X$ (but the input string $X$ is not given to $B$);
- Then $B$ guesses if the number of 0's in $X$ is odd or even;
An Introduction to the Introduction

- From Y, it would be very difficult for B to figure out which X is used to produce Y (one-way function, an intractable problem);
- So the best B can do is to randomly guess (odd/even), which has the same effect as tossing a coin.

Algorithm design goal 3: To understand inherent difficulties of some prominent problems (need to study math underlying the intractability)
• From $Y$, it would be very difficult for $B$ to figure out which $X$ is used to produce $Y$ (one-way function, an intractable problem);
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An Introduction to the Introduction

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An Introduction to the Introduction

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To understand inherent difficulties of some prominent problems (need to study math underlying the intractability)
An Introduction to the Introduction

To design good algorithms for computational problems,
An Introduction to the Introduction

To design good algorithms for computational problems,
• need to be familiar with techniques for complexity analysis;
An Introduction to the Introduction

To design good algorithms for computational problems,
• need to be familiar with techniques for complexity analysis;
• need to master skills for efficient algorithm design;
An Introduction to the Introduction

To design good algorithms for computational problems,

• need to be familiar with techniques for complexity analysis;
• need to master skills for efficient algorithm design;
• need to be able to know if an algorithm is optimal (how?).
The Introduction

What is this course about (and why is it needed)?
• about basic yet indispensable skills for problem solving
• algorithm design leads to writing code, i.e., creative thinking without programming languages

How different is this course from other algorithm courses?
• design technique-oriented, not application-oriented
• emphasis on guaranteed performance (typically in efficiency)

Goals to achieve
• to learn to measure performance of algorithms
• to master some fundamental algorithmic techniques
• to study advanced algorithmic skills
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Part I. Foundations
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- Chapter 1. The role of algorithms in computing
- Chapter 2. Getting started
- Chapter 3. Growth of functions
- Chapter 4. Solving recurrences
- Chapter 5. Probabilistic analysis and randomized algorithms
Part I. Foundations

The theme of the course
Part I. Foundations

The theme of the course

• Goal: learning techniques to design efficient algorithms
Part I. Foundations

The theme of the course

• Goal: learning techniques to design efficient algorithms
• mean: through developing skills to analyze algorithms
Part I. Foundations

The theme of the course

- Goal: learning techniques to design efficient algorithms
- mean: through developing skills to analyze algorithms

Design and analysis of algorithms are closely related.
Example: the Fibonacci sequence.

\[
f(n) = \begin{cases} 
  f(n - 1) + f(n - 2) & \text{if } n \geq 3 \\
  1, & \text{otherwise}
\end{cases}
\]
Example: the Fibonacci sequence.

\[
f(n) = \begin{cases} 
  f(n - 1) + f(n - 2) & \text{if } n \geq 3 \\
  1, & \text{otherwise}
\end{cases}
\]

That is:

\[
\begin{array}{cccccccccc}
  n & : & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \ldots \\
  f(n) & : & 1 & 1 & 2 & 3 & 5 & 8 & 13 & 21 & 34 & \ldots 
\end{array}
\]
Part I. Foundations

Problem 1: Computing the $n$th Fibonacci number:

Input: $n \geq 1$; Output: the $n$th number in the Fibonacci sequence.

- recursive: task decomposition, top-down, recursive calls;
- iterative: more tightly coupled tasks, bottom-up approaches;
Part I. Foundations

Problem 1: Computing the $n$th Fibonacci number:

**Input:** $n \geq 1$;

**Output:** the $n$th number in the Fibonacci sequence.
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Two different types of algorithms: *recursive* and *iterative*
Part I. Foundations

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Two different types of algorithms: *recursive* and *iterative*

- recursive: task decomposition, top-down, recursive calls;
- iterative: more tightly coupled tasks, bottom-up approaches;
Part I. Foundations

Rec-Fibonacci($n$)

But how efficient is it? Or how slow is it? Its execution is via a run-time stack:

- Suitable for execution of subroutines
- But oblivious, cannot remember any completed subroutine.
Part I. Foundations

Rec-Fibonacci\((n)\)

if \(n = 1\) or \(n = 2\),
Rec-Fibonacci($n$)

if $n = 1$ or $n = 2$, return (1);
Part I. Foundations

\textbf{Rec-Fibonacci}(n)

\begin{itemize}
\item \textbf{if} \( n = 1 \) or \( n = 2 \), \textbf{return} (1);
\item \textbf{else}
\item \( T_1 = \textbf{Rec-Fibonacci}(n - 1) \);
\end{itemize}
Rec-Fibonacci(n)

if $n = 1$ or $n = 2$, return (1);
else
   $T_1 = \text{Rec-Fibonacci}(n - 1);
   T_2 = \text{Rec-Fibonacci}(n - 2);
Part I. Foundations

Rec-Fibonacci(n)

if $n = 1$ or $n = 2$, return (1);
else
    $T_1 = \text{Rec-Fibonacci}(n - 1)$;
    $T_2 = \text{Rec-Fibonacci}(n - 2)$;
return ($T_1 + T_2$);
Rec-Fibonacci(n)

if \( n = 1 \) or \( n = 2 \), return (1);
else
\[ T_1 = \text{Rec-Fibonacci}(n - 1); \]
\[ T_2 = \text{Rec-Fibonacci}(n - 2); \]
return \( (T_1 + T_2) \);

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Part I. Foundations

Rec-Fibonacci($n$)

if $n = 1$ or $n = 2$, return (1);
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    \text{return} (T_1 + T_2);

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Part I. Foundations

\textbf{Rec-Fibonacci}(n)

\begin{itemize}
  \item \textbf{if} \( n = 1 \) or \( n = 2 \), return (1);
  \item else
    \begin{itemize}
      \item \( T_1 = \text{Rec-Fibonacci}(n - 1) \);
      \item \( T_2 = \text{Rec-Fibonacci}(n - 2) \);
    \end{itemize}
    \item return \((T_1 + T_2)\);
\end{itemize}

But how efficient is it? Or how slow is it?

its execution is via a \textit{run-time stack}

\begin{itemize}
  \item suitable for execution of subroutines
\end{itemize}
**Part I. Foundations**

**Rec-Fibonacci**

\[
\text{Rec-Fibonacci}(n) = \begin{cases} 
1 & \text{if } n = 1 \text{ or } n = 2, \\
\text{return } (1); & \\
\text{else} & \\
T_1 = \text{Rec-Fibonacci}(n - 1); & \\
T_2 = \text{Rec-Fibonacci}(n - 2); & \\
\text{return } (T_1 + T_2); & 
\end{cases}
\]

But how efficient is it? Or how slow is it?

its execution is via a *run-time stack*

- suitable for execution of subroutines
- but oblivious, cannot remember any completed subroutine.
Part I. Foundations

```python
def REC_FIBONACCI(n):
    if n == 1 or n == 2:
        return 1
    else:
        T1 = REC_FIBONACCI(n-1)
        T2 = REC_FIBONACCI(n-2)
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        return T1 + T2
```
Repeated computations everywhere!
Part I. Foundations

The size of tree is the number of recursive calls;
The size of tree is the number of recursive calls; How big is it?
Part I. Foundations

\[ \text{small triangle} \leq \text{size of tree} \leq \text{large triangle} \approx 2^n \]
Part I. Foundations

small triangle $\leq$ size of tree $\leq$ large triangle
small triangle $\leq$ size of tree $\leq$ large triangle

roughly: $2^{n/2} \leq$ size of tree $\leq 2^n$
Part I. Foundations

\textbf{Iterative-Fibonacci}(n)

\textbf{How fast is it?}

\[ T_{\text{total}} = \max \{ T_{\text{if}}, T_{\text{else}} \} \]

where

\[ T_{\text{if}} = c_1, \quad T_{\text{else}} = c_2 + T_{\text{for}} = c_2 + d \times (n - 2) \]

\[ T_{\text{total}} \leq c_1 + c_2 + d \times (n - 2), \quad \text{a linear function in } n \]

\textbf{Iterative-Fibonacci}(n)\ is a simple dynamic programming algorithm.
Iterative-Fibonacci($n$)

if $n = 1$ or $n = 2$ return (1);
Part I. Foundations

**Iterative-Fibonacci**($n$)

```plaintext
if $n = 1$ or $n = 2$ return (1);
else
    $M[1] = 1$, $M[2] = 1$
```

**How fast is it?**

$$T_{\text{total}} = \max \{ T_{\text{if}}, T_{\text{else}} \}$$

where

- $T_{\text{if}} = c_1$
- $T_{\text{else}} = c_2 + T_{\text{for}}$
- $T_{\text{for}} = c_2 + d \times (n - 2)$

$$T_{\text{total}} \leq c_1 + c_2 + d (n - 2),$$

a linear function in $n$.
Iterative-Fibonacci(n)

if \( n = 1 \) or \( n = 2 \) return \((1)\);
else
    \( M[1] = 1, \ M[2] = 1 \)
    for \( i = 3 \) to \( n \) do
        \( M[i] = M[i - 1] + M[i - 2] \)

How fast is it?

\( T_{total} = \max\{T_{if}, T_{else}\} \)
where
\( T_{if} = c_1 \)
\( T_{else} = c_2 + T_{for} = c_2 + d \times (n - 2) \)

\( T_{total} \leq c_1 + c_2 + d \times (n - 2) \), a linear function in \( n \)

Iterative-Fibonacci(n) is a simple dynamic programming algorithm.
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    return \( M[n] \)
Part I. Foundations

Iterative-Fibonacci\((n)\)

\[
\begin{align*}
\text{if } n = 1 \text{ or } n = 2 & \text{ return } (1); \\
\text{else} & \\
& \text{for } i = 3 \text{ to } n \text{ do} \\
& \quad M[i] = M[i - 1] + M[i - 2] \\
& \text{return } (M[n])
\end{align*}
\]

How fast is it?
Part I. Foundations

**Iterative-Fibonacci**\(_n\)

\[
\text{if } n = 1 \text{ or } n = 2 \text{ return } (1); \\
\text{else} \\
\text{for } i = 3 \text{ to } n \text{ do} \\
M[i] = M[i - 1] + M[i - 2] \\
\text{return } (M[n])
\]

How fast is it?

\[
T_{total} = \max\{T_{if}, T_{else}\}
\]
Iterative-Fibonacci\( (n) \)

\[
\text{if } n = 1 \text{ or } n = 2 \text{ return } (1); \\
\text{else} \\
M[1] = 1, \ M[2] = 1 \\
\text{for } i = 3 \text{ to } n \text{ do} \\
\quad M[i] = M[i - 1] + M[i - 2] \\
\text{return } (M[n])
\]

How fast is it?

\[
T_{total} = \max\{T_{if}, T_{else}\}
\]

where \( T_{if} = c_1 \),
Part I. Foundations

**ITERATIVE-FIBONACCI**(\(n\))

\[
\text{if } n = 1 \text{ or } n = 2 \text{ return } (1);
\]

\[\text{else}\]

\[
M[1] = 1, \quad M[2] = 1
\]

\[\text{for } i = 3 \text{ to } n \text{ do}\]

\[
M[i] = M[i - 1] + M[i - 2]
\]

\[\text{return } (M[n])\]

How fast is it?

\[
T_{total} = \max\{T_{if}, T_{else}\}
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\[\text{where } T_{if} = c_1, \quad T_{else} = c_2 + T_{for}\]
Iterative-Fibonacci(n)

if \( n = 1 \) or \( n = 2 \) return (1);
else
    \( M[1] = 1, M[2] = 1 \)
    for \( i = 3 \) to \( n \) do
        \( M[i] = M[i - 1] + M[i - 2] \)
    return (\( M[n] \))

How fast is it?

\[ T_{total} = \max\{T_{if}, T_{else}\} \]

where \( T_{if} = c_1, \ T_{else} = c_2 + T_{for} = c_2 + d \times (n - 2) \)
Part I. Foundations

ITERATIVE-FIBONACCI($n$)

\[
\text{if } n = 1 \text{ or } n = 2 \text{ return (1);}
\]

\text{else}

\[M[1] = 1, \ M[2] = 1\]

\text{for } i = 3 \text{ to } n \text{ do}

\[M[i] = M[i - 1] + M[i - 2]\]

\text{return } (M[n])

How fast is it?

\[T_{total} = \max\{T_{if}, T_{else}\}\]

where \(T_{if} = c_1, \ T_{else} = c_2 + T_{for} = c_2 + d \times (n - 2)\)

\[T_{total} \leq c_1\]
Part I. Foundations

**Iterative-Fibonacci** \((n)\)

```plaintext
if \(n = 1\) or \(n = 2\) return \((1)\);
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        \(M[i] = M[i - 1] + M[i - 2]\)
    return \((M[n])\)
```

How fast is it?

\[
T_{total} = \max\{T_{if}, T_{else}\}
\]

where \(T_{if} = c_1\), \(T_{else} = c_2 + T_{for} = c_2 + d \times (n - 2)\)

\[
T_{total} \leq c_1 + c_2 + d(n - 2),
\]
Part I. Foundations

Iterative-Fibonacci($n$)

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    return ($M[n]$)

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T_{total} = \max\{T_{if}, T_{else}\}
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\[
T_{total} \leq c_1 + c_2 + d(n - 2), \text{ a linear function in } n
\]
Iterative-Fibonacci$(n)$

\[
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\[
T_{total} \leq c_1 + c_2 + d(n - 2), \text{ a linear function in } n
\]

Iterative-Fibonacci$(n)$ is a simple dynamic programming algorithm.
Actually, all the algorithm \texttt{ITERATIVE-FIBONACCI}(n) does is:
Part I. Foundations

Actually, all the algorithm $\textsc{Iterative-Fibonacci}(n)$ does is:

To fill out a table of size $n$, with
Part I. Foundations

Actually, all the algorithm \textsc{Iterative-Fibonacci}(n) does is:

To fill out a table of size \( n \), with

- each entry being filled out exactly once, and
Actually, all the algorithm \textsc{Iterative-Fibonacci}(n) does is:

To fill out a table of size $n$, with

- each entry being filled out exactly once, and
- filling out an entry takes a constant, say $c$ steps.
Actually, all the algorithm **ITERATIVE-FIBONACCI**(*n*) does is:

To fill out a table of size *n*, with

- each entry being filled out exactly once, and
- filling out an entry takes a constant, say *c* steps.

So the total time **ITERATIVE-FIBONACCI**(*n*) uses is
Part I. Foundations

Actually, all the algorithm \textsc{Iterative-Fibonacci}(n) does is:

To fill out a table of size \( n \), with

\begin{itemize}
  \item each entry being filled out exactly once, and
  \item filling out an entry takes a constant, say \( c \) steps.
\end{itemize}

So the total time \textsc{Iterative-Fibonacci}(n) uses is

\[ T(n) = c \times n \]
Chapter 1. The role of algorithms in computing

Chapter 1. The role of algorithms in computing
Chapter 1. The role of algorithms in computing

What is an Algorithm: a well-defined, finite procedure that takes an input and produces an output.
Chapter 1. The role of algorithms in computing

What is an Algorithm: a well-defined, finite procedure that takes an input and produces an output.

Example 2: An algorithm skeleton;

Algorithm Maximum;

INPUT: list \( X = \{a_1, \cdots, a_n\} \); 

Body that is a series of instructions; 

OUTPUT: \( y \), the maximum of \( a_1, \cdots, a_n \).
Chapter 1. The Role of Algorithms in Computing

Alternatively, an algorithm specifies a finite process to compute a function or a relation.
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e.g., algorithm \textsc{Maximum} computes the following function:

\[
  f_{\text{max}}(X) = y, \text{ where } \forall a \in X, y \geq a,
\]
Alternatively, an algorithm specifies a finite process to compute a function or a relation.

E.g., algorithm **MAXIMUM** computes the following function:

\[ f_{\text{max}}(X) = y, \text{ where } \forall a \in X, y \geq a, \]

For some problems, the functions computed are **predicates**, i.e., output \( y \in \{\text{TRUE, FALSE}\} \)
Chapter 1. The Role of Algorithms in Computing

Algorithms as a technology to resolve efficiency issues
Chapter 1. The Role of Algorithms in Computing

Algorithms as a technology to resolve efficiency issues

Efficient use of computer resources such as time and space is necessary.
Chapter 1. The Role of Algorithms in Computing

Algorithms as a technology to resolve efficiency issues

Efficient use of computer resources such as time and space is necessary.

Two typical situations:
Chapter 1. The Role of Algorithms in Computing

Algorithms as a technology to resolve efficiency issues

Efficient use of computer resources such as time and space is necessary.

Two typical situations:

- very large input data for “easy” problems;
Chapter 1. The Role of Algorithms in Computing

Algorithms as a technology to resolve efficiency issues

Efficient use of computer resources such as time and space is necessary.

Two typical situations:

- very large input data for “easy” problems;
- moderately large input data for “hard” problems.
The Sorting Problem

Input: \( n \) numbers \( \langle a_1, \cdots, a_n \rangle \);
Output: a reordering \( \langle a'_1, \cdots, a'_n \rangle \) of the input such that
\[ a'_1 \leq a'_2 \leq \cdots \leq a'_n. \]

Insertion Sort
idea: an iterative process to produce a new list such that
at each iteration, the new list consists of two sublists,
• a sorted sublist followed by an unsorted sublist, and
• the leftmost number of the unsorted is being inserted into the sorted.
As the process goes, the sorted sublist gets longer, the unsorted sublist
gets shorter, until the unsorted becomes empty.
Chapter 2. Getting Started

Chapter 2. Getting started

The Sorting Problem

**INPUT:** $n$ numbers $\langle a_1, \cdots, a_n \rangle$;
Chapter 2. Getting Started

The Sorting Problem

**Input:** $n$ numbers $\langle a_1, \ldots, a_n \rangle$;

**Output:** a reordering $\langle a'_1, \ldots, a'_n \rangle$ of the input such that

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**Idea:** an iterative process to produce a new list such that
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- at each iteration, the new list consists of two sublists,
  - a sorted sublist followed by an unsorted sublist, and
  - the leftmost number of the unsorted is being inserted into the sorted.

As the process goes, the sorted sublist gets longer, the unsorted sublist gets shorter, until the unsorted becomes empty.
Chapter 2. Getting Started

Algorithm **INSERTION-SORT**($A$)

```plaintext
for j = 2 to length[A] do
   key = A[j]
   { Insert A[j] into sorted A[1..j-1] }
   i = j - 1
   while i > 0 and A[i] > key do
      A[i+1] = A[i]
      i = i - 1
   A[i+1] = key
```

Analysis of the algorithm:

- **(correctness proof):** to show that the algorithm is as desired;
- **(efficiency proof):** to show a guaranteed efficiency of the algorithm.
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Algorithm INSERTION-SORT($A$)

1. for $j = 2$ to length[$A$] do
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Algorithm **INSERTION-SORT**(A)

1. for \( j = 2 \) to \( \text{length}[A] \) do
2. \( \text{key} = A[j] \)
Algorithm \textsc{Insertion-Sort}(A)

1. \textbf{for } \textit{j} = 2 \textbf{ to } length[A] \textbf{ do}
2. \hspace{1em} \textit{key} = A[j]
3. \hspace{1em} \{Insert \textit{A}[j] \text{ into sorted } A[1..j - 1]\}
Algorithm Insertion-Sort(A)

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Chapter 2. Getting Started

Algorithm INSERTION-SORT($A$)

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5. while $i > 0$ and $A[i] > key$

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Chapter 2. Getting Started

Algorithm INSERTION-SORT($A$)
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4. \hspace{1em} $i = j - 1$
5. while $i > 0$ and $A[i] > key$
6. \hspace{2em} do $A[i + 1] = A[i]$
7. \hspace{1em} $i = i - 1$
Chapter 2. Getting Started

Algorithm \textsc{Insertion-Sort}(A)

1   \textbf{for} $j = 2$ \textbf{to} length[A] \textbf{do}
2       $key = A[j]$
3       \{Insert $A[j]$ into sorted $A[1..j - 1]$\}
4       $i = j - 1$
5       \textbf{while} $i > 0$ and $A[i] > key$
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Algorithm **INSERTION-SORT**(*A*)

1. **for** *j* = 2 **to** length[*A*] **do**
2.   
3.   
4.   
5. **while** *i* > 0 **and** *A*[i] > key **do** *A*[i + 1] = *A*[i]
6.   
7.   
8. **end while**
9. **end for**
10. *A*[i + 1] = key

Analysis of the algorithm:
Chapter 2. Getting Started

Algorithm Insertion-Sort($A$)

1. for $j = 2$ to length[$A$] do
2.     key = $A[j]$
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5.     while $i > 0$ and $A[i] > key$
7.             $i = i - 1$
8.     $A[i+1] = key$

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Chapter 2. Getting Started

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If the algorithm consists of sequential blocks of instructions, the task is to prove the correct transformation by each block.
Chapter 2. Getting Started

Correctness proof: this is to prove

the pre-condition (condition for the input)

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If the algorithm consists of sequential blocks of instructions, the task is to prove the correct transformation by each block.

This means we need to prove that every sequential statement in the algorithm transforms the given pre-condition to the given post-condition.
The most difficult task is to do this for a loop statement.
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In Insertion-Sort, the loop invariant is

at each iteration, the sublist $A[1..j - 1]$ consists of the elements originally in the positions $[1..j-1]$ but in sorted order.

However, finding loop invariants is difficult!
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Efficiency analysis: This is to show that...
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- For all cases of input, the needed computation resources for the algorithm.
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Efficiency analysis: This is to show that

- For all cases of input, the needed computation resources for the algorithm.
- resources can be CPU time and memory space used in the computation.
- however, the unit measured is not real time or memory unit.
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Chapter 2. Getting Started

Time of an algorithm $A(x)$

**Upper bound** for algorithm $A$, bounding all cases of instances

- **Input Instances**
- **worst case**

$x$
Chapter 2. Getting Started

Resource measurement based on

- random-access machine (RAM)
- counting primitive operations: addition, subtraction, floor, ceiling, multiplication, jump, memory movement, these operations differ in time by a constant multiplicative factor.
- speed between different machines: a constant multiplicative factor.
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Analysis of Algorithm \textsc{Insertion-Sort}(A)
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Algorithm \textsc{Insertion-Sort}(A)

1. \textbf{for} \( j = 2 \) \textbf{to} \( \text{length}[A] \) \textbf{do}
2. \hspace{1em} \textit{key} = \( A[j] \)
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4. \hspace{1em} \textit{i} = \( j - 1 \)
5. \hspace{1em} \textbf{while} \( i > 0 \) \textit{and} \( A[i] > \textit{key} \)
6. \hspace{2em} \textbf{do} \( A[i + 1] = A[i] \)
7. \hspace{2em} \hspace{1em} \textit{i} = \( i - 1 \)
8. \hspace{1em} \hspace{2em} \textit{A[i + 1]} = \textit{key}
Chapter 2. Getting Started

Analysis of

Algorithm INSERTrnION-SORT\((A)\)

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6 \hspace{1em} \hspace{1em} \hspace{1em} \textbf{do} \hspace{0.5em} A[i+1] = A[i] \\
7 \hspace{1em} \hspace{1em} \hspace{1em} \hspace{1em} i = i - 1 \\
8 \hspace{1em} \hspace{1em} A[i+1] = \text{key}

Assume \(t_j\) to be the number of times \textbf{while} is executed for every \(j\).

\[
T(n) = c_1 n + c_2 (n-1) + c_4 (n-1) + c_5 \sum_{j=2}^{n} t_j + c_6 \sum_{j=2}^{n} (t_j - 1) + c_7 \sum_{j=2}^{n} (t_j - 1) + c_8 (n-1)
\]
Chapter 2. Getting Started

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for some constants \(a, b, c,\)
Chapter 2. Getting Started

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for some constants \( a, b, c \), for example, \( a \geq c_5 + c_6 + c_7 \), \( b \geq c_1 + c_2 + c_4 + c_8 \).
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\[
T(n) \leq a \frac{n}{2} (n + 1) + bn + c - a
\]
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\[ T(n) \leq an^2 \sum_{j=2}^{n} t_j + bn + c \]

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\[ T(n) \leq a \frac{n}{2} (n + 1) + bn + c - a \leq xn^2 + yn + z \]

for some constants \( x, y, z \).
Chapter 2. Getting Started

So we have proved:

\[ T(n) \leq x n^2 + y n + z \]

for some constants \( x, y, z \) \leq \text{means in all cases for which} \leq \text{holds;} \]

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Chapter 2. Getting Started

Important complexity issues:

1. size of input $n$: the number of bits encoding input $x$, i.e., $n = |x|$. It is inaccurate for $n$ to represent the number of items in the input.

Consider to sort 4 items $\langle x_1, x_2, x_3, x_4 \rangle$ of values in the scale of $2^N$, for some very large $N$.

- If $n$ is the number of items, $n = 4$, then any sorting algorithm would run in constant time.
- However, since $x_1, x_2, x_3, x_4$ are of very large values, a single comparison $x_1 \leq x_2$ would need a time proportional to $N$.

Hence, if $n = |\langle x_1, x_2, x_3, x_4 \rangle|$, then $n \approx N$.

To sort the 4 items, a constant number of comparisons is needed, each taking a time linear in $N$ (i.e., total time is linear in $n$).
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Chapter 2. Getting Started

2. Running time

\( T(n) \): the number of primitive operations executed, a function in worst-case running time: the running time upper bound for all inputs.

order of growth: \( T(n) = an^2 + bn + c \) grows the same rate as \( an^2 \) (if \( a > 0 \)).
2. Running time $T(n)$: the number of primitive operations executed, a function in $n$
Chapter 2. Getting Started

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Chapter 3. Growth of Functions

Big-O: set \( O(n^2) \) contains all functions of growth rate \( \leq cn^2 \).

So for function \( T(n) = an^2 + bn + c \), \( T(n) \in O(n^2) \), but written as \( T(n) = O(n^2) \).

In general, \( O(g(n)) = \{ f(n) : \exists c > 0, k > 0 \text{ such that } 0 \leq f(n) \leq cg(n), \text{ for all } n \geq k \} \).
Chapter 3. Growth of Functions

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Chapter 3. Growth of Functions

For example: the following functions are all of the order of $O(n^2)$:

(1) $3n^2$
(2) $5.3n^2 + 6n \log_2 n + 90$
(3) $0.001n^2 - 200n - 5000$
(4) $3n \log_2 n^2 + 6n$
(5) $\sqrt{n} - 20 \log_2 n$
(6) $\log_2 n + 56$
(7) $345$

But the following are not:

(8) $3n^2 \log_2 n - 400n$
(9) $n^2$.001
Chapter 3. Growth of Functions

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But the following are not:

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Let us do proofs for some of these examples.
For example: the following functions are all of the order of $O(n^2)$:

1. $3n^2$
2. $5.3n^2 + 6n \log_2 n + 90$
3. $0.001n^2 - 200n - 5000$
4. $3n \log_2 n + 6n$
5. $\sqrt{n} - 20 \log_2 n$
6. $\log_2 n + 56$
7. $345$
Chapter 3. Growth of Functions

For example: the following functions are all of the order of $O(n^2)$:

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(8) $3n^2 \log_2 n - 400n$
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Chapter 3. Growth of Functions

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8. $3n^2 \log_2 n - 400n$
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Let us do proofs for some of these examples.
Chapter 4. Solving Recurrences

Example: binary search algorithm

Algorithm

Binary Search (A, p, r, key)

1. if p > r return (NULL)
2. else
3. q = ⌊p + r / 2⌋
4. if A[q] = key return (q)
5. else
6. if A[q] > key Binary Search (A, p, q − 1, key)
7. else Binary Search (A, q + 1, r, key)

Let T(n) be the worst case time complexity for Binary Search, where parameter n = r − p + 1.
The time T(n) can be upper-bounded with the recurrence,

\[ T(n) \leq T(n/2) + c \]

and T(0) ≤ c where c > 0 is a constant.

Note: we can also set base case T(1) ≤ c.
Example: binary search algorithm

Algorithm BINARY SEARCH\((A, p, r, key)\)

1. if \(p > r\) return (NULL)
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6. if \(A[q] > key\) BINARY SEARCH \((A, p, q - 1, key)\)
7. else
8. BINARY SEARCH \((A, q + 1, r, key)\)
Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

We have recurrence
\[ T(n) \leq T(n^2) + c \]
and
\[ T(1) \leq c \]
for the Binary Search algorithm.

The recurrence can be resolved by unfolding the recursive terms. The process to unfold the recurrence \( T(n) \) can be:

1. straightforward unfolding (for simpler recurrences)
2. graph based (called recursive tree method)
3. guess and induction based (called substitution method)
Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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We have recurrence

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Chapter 4. Solving Recurrences

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2. graph based (called recursive tree method).
3. guess and induction based (called substitution method)
Chapter 4. Solving Recurrences

1. Simple, straightforward unfolding:

\[ T(n) \leq T(n^2) + c \]

\[ T(n) \leq T(n^2) + cT(n^2) \leq T(n^{2^2}) + cT(n^{2^2}) \leq T(n^{2^3}) + cT(n^{2^3}) \leq \ldots \leq T(n^{2^h}) \leq T(n^{2^h+1}) + c \]

where \( n^{2^h+1} = 1 \), \( 2^{h+1} = n \), \( h+1 = \log_2 n \)

\[ T(n) \leq T(n^{2^h+1}) + c(h+1) \]

there are \( h+1 \) inequalities

\[ = T(1) + c(h+1) = c + c \log_2 n = O(\log_2 n) \]

\( \Rightarrow \) proved this equation using definition of big-O
1. Simple, straightforward unfolding:
Chapter 4. Solving Recurrences

1. Simple, straightforward unfolding:

   \[ T(n) \leq T\left(\frac{n}{2}\right) + c \]

   use \( T(n) \leq T\left(\frac{n}{2}\right) + c \) as a template;
Chapter 4. Solving Recurrences

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use $T(n) \leq T\left(\frac{n}{2}\right) + c$ as a template;

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Chapter 4. Solving Recurrences

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1. Simple, straightforward unfolding:

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$$
T(n) \leq T\left(\frac{n}{2}\right) + c \\
T\left(\frac{n}{2}\right) \leq T\left(\frac{n}{2^2}\right) + c \\
T\left(\frac{n}{2^2}\right) \leq T\left(\frac{n}{2^3}\right) + c
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Chapter 4. Solving Recurrences

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T(n) & \leq T\left(\frac{n}{2}\right) + c \\
T\left(\frac{n}{2}\right) & \leq T\left(\frac{n}{2^2}\right) + c \\
T\left(\frac{n}{2^2}\right) & \leq T\left(\frac{n}{2^3}\right) + c \\
& \ldots.
\end{align*}

......
Chapter 4. Solving Recurrences

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\[
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T(n) & \leq T\left(\frac{n}{2}\right) + c \\
T\left(\frac{n}{2}\right) & \leq T\left(\frac{n}{2^2}\right) + c \\
T\left(\frac{n}{2^2}\right) & \leq T\left(\frac{n}{2^3}\right) + c \\
\cdots & \\
T\left(\frac{n}{2^h}\right) & \leq T\left(\frac{n}{2^{h+1}}\right) + c &= T(1) + c(h+1) \leq O(\log_2 n)
\end{align*}
\]

where $\frac{n}{2^{h+1}} = 1$, $2^{h+1} = n$, $h + 1 = \log_2 n$
Chapter 4. Solving Recurrences

1. Simple, straightforward unfolding:

   use \( T(n) \leq T(\frac{n}{2}) + c \) as a template;

   \[
   \begin{align*}
   T(n) & \leq T(\frac{n}{2}) + c \\
   T(\frac{n}{2}) & \leq T(\frac{n}{2^2}) + c \\
   T(\frac{n}{2^2}) & \leq T(\frac{n}{2^3}) + c \\
   \cdots & \\
   T(\frac{n}{2^h}) & \leq T(\frac{n}{2^{h+1}}) + c
   \end{align*}
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Chapter 4. Solving Recurrences

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   $T(n) \leq T\left(\frac{n}{2}\right) + c$
   $T\left(\frac{n}{2}\right) \leq T\left(\frac{n}{2^2}\right) + c$
   $T\left(\frac{n}{2^2}\right) \leq T\left(\frac{n}{2^3}\right) + c$

   $\ldots$

   $T\left(\frac{n}{2^h}\right) \leq T\left(\frac{n}{2^{h+1}}\right) + c$ \hspace{1cm} \text{where } \frac{n}{2^{h+1}} = 1, \ 2^{h+1} = n, \ h+1 = \log_2 n$

   $+$

   $T(n) \leq T\left(\frac{n}{2^{h+1}}\right) + c(h+1)$ \hspace{1cm} \text{there are } h+1 \text{ inequalities}
1. Simple, straightforward unfolding:

use $T(n) \leq T\left(\frac{n}{2}\right) + c$ as a template;

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T(n) \leq T\left(\frac{n}{2}\right) + c \\
T\left(\frac{n}{2}\right) \leq T\left(\frac{n}{2^2}\right) + c \\
T\left(\frac{n}{2^2}\right) \leq T\left(\frac{n}{2^3}\right) + c \\
\vdots \\
T\left(\frac{n}{2^h}\right) \leq T\left(\frac{n}{2^{h+1}}\right) + c \quad \text{where } \frac{n}{2^{h+1}} = 1, \ 2^{h+1} = n, \ h + 1 = \log_2 n
\]

\[
T(n) \leq T\left(\frac{n}{2^{h+1}}\right) + c(h + 1) \quad \text{there are } h + 1 \text{ inequalities} \\
= T(1) + c(h + 1)
\]
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T(n) \leq T\left(\frac{n}{2}\right) + c
\]
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\]
\[
T\left(\frac{n}{2^2}\right) \leq T\left(\frac{n}{2^3}\right) + c
\]

\[\cdots\]
\[
T\left(\frac{n}{2^h}\right) \leq T\left(\frac{n}{2^{h+1}}\right) + c
\]

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T(n) \leq T\left(\frac{n}{2^{h+1}}\right) + c(h + 1)
\]

there are $h + 1$ inequalities

\[
= T(1) + c(h + 1)
\]
\[
= c + c \log_2 n
\]
Chapter 4. Solving Recurrences

1. Simple, straightforward unfolding:

use \( T(n) \leq T\left(\frac{n}{2}\right) + c \) as a template;

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T(n) & \leq T\left(\frac{n}{2}\right) + c \\
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& +
\end{align*}
\]

\[
T(n) \leq T\left(\frac{n}{2^{h+1}}\right) + c(h + 1) \quad \text{there are } h + 1 \text{ inequalities}
\]

\[
= T(1) + c(h + 1) \\
= c + c \log_2 n \\
= O(\log_2 n)
\]
1. Simple, straightforward unfolding:

use $T(n) \leq T\left(\frac{n}{2}\right) + c$ as a template;

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T(n) \leq T\left(\frac{n}{2^{h+1}}\right) + c(h + 1) \quad \text{there are } h + 1 \text{ inequalities}
\]

\[
= T(1) + c(h + 1) \\
= c + c \log_2 n \\
= O(\log_2 n) \quad \leftarrow \text{proved this equation using definition of big-}O
\]
Chapter 4. Solving Recurrences

Another example:

Algorithm Merge Sort (A, p, r)
1. if p < r
2. then q = ⌊p + r/2⌋
3. Merge Sort (A, p, q)
4. Merge Sort (A, q + 1, r)
5. Merging 2 Lists (A, p, q, r)

Analysis of the algorithm.

• Assume n = r − p + 1, a power of 2; also assume T(n) is time for Merge Sort (A, p, r).

• t_1, 2 = c • t_3 = t_4 = T(n/2)
• t_5 ≤ n (why?)

T(n) = t_1, 2 + t_3 + t_4 + t_5 ≤ 2T(n/2) + n + c

base case: T(1) = d, constant
Chapter 4. Solving Recurrences

Another example:

Algorithm

\begin{align*}
\text{Merge Sort} & (A, p, r) \\
1 & \text{if } p < r \\
2 & \text{then } q = \lfloor \frac{p + r}{2} \rfloor \\
3 & \text{Merge Sort} (A, p, q) \\
4 & \text{Merge Sort} (A, q + 1, r) \\
5 & \text{Merging2Lists} (A, p, q, r)
\end{align*}

Analysis of the algorithm.

- Assume \( n = r - p + 1 \), a power of 2; also assume \( T(n) \) is the time for \( \text{Merge Sort} (A, p, r) \).
- \( t_1, t_2 = c \cdot t_3 = t_4 = T \left( \frac{n}{2} \right) \)
- \( t_5 \leq n \) (why?)

\[ T(n) = t_1, t_2 + t_3 + t_4 + t_5 \leq 2T \left( \frac{n}{2} \right) + n + c \]

Base case: \( T(1) = d \), constant
Chapter 4. Solving Recurrences

Another example:

Algorithm `MERGE SORT(A, p, r)`

1. `if p < r` 
2. `then q = ⌊(p+r)/2⌋` 
3. `MERGE SORT(A, p, q)` 
4. `MERGE SORT(A, q + 1, r)` 
5. `MERGING2LISTS(A.p, q, r)`

Analysis of the algorithm.

• Assume $n = r - p + 1$, a power of 2; also assume $T(n)$ is time for `MERGE SORT(A, p, r)`.

• $t_1 = t_2 = c$ 
• $t_3 = t_4 = T(n/2)$ 
• $t_5 \leq n$ (why?)

$T(n) = t_1 + t_2 + t_3 + t_4 + t_5 \leq 2T(n/2) + n + c$

base case: $T(1) = d$, constant
Chapter 4. Solving Recurrences

Another example:

Algorithm MERGE SORT\((A, p, r)\)

1. \textbf{if} \(p < r\)
2. \textbf{then} \(q = \left\lfloor \frac{p+r}{2} \right\rfloor\)
3. MERGE SORT\((A, p, q)\)
4. MERGE SORT\((A, q + 1, r)\)
5. MERGING2LISTS\((A.p, q, r)\)

Analysis of the algorithm.

- Assume \(n = r - p + 1\), a power of 2;
  also assume \(T(n)\) is time for \textbf{MERGE SORT\((A, p, r)\). Then}
Chapter 4. Solving Recurrences

Another example:

Algorithm \textsc{Merge Sort}(A, p, r)

1. \textbf{if} \ p < r
2. \quad \textbf{then} \ q = \left\lfloor \frac{p+r}{2} \right\rfloor
3. \quad \textsc{Merge Sort}(A, p, q)
4. \quad \textsc{Merge Sort}(A, q + 1, r)
5. \quad \textsc{Merging2Lists}(A,p, q, r)

Analysis of the algorithm.

- Assume \(n = r - p + 1\), a power of 2;
  also assume \(T(n)\) is time for \textsc{Merge Sort}(A, p, r). Then

- \(t_{1,2} = c\)
Chapter 4. Solving Recurrences

Another example:

Algorithm \textsc{Merge Sort}(A, p, r)

1. \textbf{if} \ p < r
2. \textbf{then} \ q = \left\lfloor \frac{p+r}{2} \right\rfloor
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4. \textsc{Merge Sort}(A, q + 1, r)
5. \textsc{Merging2Lists}(A, p, q, r)

Analysis of the algorithm.

- Assume \( n = r - p + 1 \), a power of 2;
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- \( t_{1,2} = c \)
- \( t_3 = t_4 = T\left(\frac{n}{2}\right) \)
Chapter 4. Solving Recurrences

Another example:

Algorithm MERGE SORT\((A, p, r)\)

1.  \textbf{if} \( p < r \)
2.     \textbf{then} \( q = \left\lfloor \frac{p+r}{2} \right\rfloor \)
3.     MERGE SORT\((A, p, q)\)
4.     MERGE SORT\((A, q + 1, r)\)
5.     Merging2Lists\((A.p, q, r)\)

Analysis of the algorithm.

- Assume \( n = r - p + 1 \), a power of 2;
- also assume \( T(n) \) is time for MERGE SORT\((A, p, r)\). Then

- \( t_{1,2} = c \)
- \( t_3 = t_4 = T\left(\frac{n}{2}\right) \)
- \( t_5 \leq n \) (why?)
Chapter 4. Solving Recurrences

Another example:

Algorithm \textsc{Merge Sort}(A, p, r)

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4. \ \textsc{Merge Sort}(A, q+1, r)
5. \ \textsc{Merging2Lists}(A,p, q, r)

Analysis of the algorithm.

- Assume \( n = r - p + 1 \), a power of 2; also assume \( T(n) \) is time for \textsc{Merge Sort}(A, p, r). Then

- \( t_{1,2} = c \)
- \( t_3 = t_4 = T\left(\frac{n}{2}\right) \)
- \( t_5 \leq n \) (\textbf{why?})

\[
T(n) = t_{1,2} + t_3 + t_4 + t_5 \leq 2T\left(\frac{n}{2}\right) + n + c
\]
Chapter 4. Solving Recurrences

Another example:

Algorithm \texttt{Merge Sort}(A, p, r)

1. \textbf{if} \( p < r \)
2. \textbf{then} \( q = \lfloor \frac{p+r}{2} \rfloor \)
3. \texttt{Merge Sort}(A, p, q)
4. \texttt{Merge Sort}(A, q + 1, r)
5. \texttt{Merging2Lists}(A.p, q, r)

Analysis of the algorithm.

- Assume \( n = r - p + 1 \), a power of 2;
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T(n) = t_{1,2} + t_3 + t_4 + t_5 \leq 2T\left(\frac{n}{2}\right) + n + c
\]

base case: \( T(1) = d \), constant
Chapter 4. Solving Recurrences

Solve recurrence

\[ T(n) \leq 2T(n^2) + n + c \]

with base case \( T(1) = d \)

with a simple method:

\[ T(n) \leq 2T(n^2) + n + c \]

\[ 2T(n^2) \leq 2T(n^2^2) + 2n + 2c \]

\[ 2T(n^2^2) \leq 2T(n^2^3) + 2^2n + 2^2c \]

\[ \vdots \]

\[ 2^h T(n^2^h) \leq 2^h T(n^2^{h+1}) + 2^h n + 2^h c \]

where \( n^{2^{h+1}} = 1 \)
Chapter 4. Solving Recurrences

Solve recurrence

\[ T(n) \leq 2T\left(\frac{n}{2}\right) + n + c \]  with base case \( T(1) = d \)

with a simple method:
Chapter 4. Solving Recurrences

Solve recurrence

\[ T(n) \leq 2T\left(\frac{n}{2}\right) + n + c \quad \text{with base case} \quad T(1) = d \]

with a simple method:

\[ T(n) \leq 2T\left(\frac{n}{2}\right) + n + c \]
Chapter 4. Solving Recurrences

Solve recurrence

\[ T(n) \leq 2T\left(\frac{n}{2}\right) + n + c \]  with base case \( T(1) = d \)

with a simple method:

\[ T(n) \leq 2T\left(\frac{n}{2}\right) + n + c \]
\[ T\left(\frac{n}{2}\right) \leq 2T\left(\frac{n}{2^2}\right) + \frac{n}{2} + c \]
Chapter 4. Solving Recurrences

Solve recurrence

\[ T(n) \leq 2T\left(\frac{n}{2}\right) + n + c \quad \text{with base case } T(1) = d \]

with a simple method:

\[
\begin{align*}
T(n) & \leq 2T\left(\frac{n}{2}\right) + n + c \\
T\left(\frac{n}{2}\right) & \leq 2T\left(\frac{n}{2^2}\right) + \frac{n}{2} + c \\
T\left(\frac{n}{2^2}\right) & \leq 2T\left(\frac{n}{2^3}\right) + \frac{n}{2^2} + c
\end{align*}
\]
Chapter 4. Solving Recurrences

Solve recurrence

\[ T(n) \leq 2T\left(\frac{n}{2}\right) + n + c \]

with base case \( T(1) = d \)

with a simple method:

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\begin{align*}
T(n) & \leq 2T\left(\frac{n}{2}\right) + n + c \\
T\left(\frac{n}{2}\right) & \leq 2T\left(\frac{n}{4}\right) + \frac{n}{2} + c \\
T\left(\frac{n}{4}\right) & \leq 2T\left(\frac{n}{8}\right) + \frac{n}{4} + c \\
\vdots & \\
T\left(\frac{n}{2^h}\right) & \leq 2T\left(\frac{n}{2^{h+1}}\right) + \frac{n}{2^h} + c \\
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\[ T(n) \leq 2^{h+1} T\left(\frac{n}{2^{h+1}}\right) + (h + 1)n + c \sum_{i=0}^{h} 2^i \]
Chapter 4. Solving Recurrences

\[ T(n) \leq 2h + 1 \]

\[ T(n) \leq (h + 1)n + c \sum_{i=0}^{2i} \]

With \( n^2h + 1 = 1 \), we have \( n = 2h + 1 \) or \( h + 1 = \log_2 n \).

\[ T(n) \leq 2h + 1 + n \log_2 n + c(n - 1) = \mathcal{O}(n \log_2 n) \]

We need to prove the last equality, i.e., find constants \( a \) and \( k \) such that

\[ n \log_2 n + (c + d)n - c \leq an \log_2 n \]

when \( n > k = 2c + d \).

Proof:

Choose \( a = 2 \). Then to make (1) holds, we need

\[ \log_2 n > c + d \].

So \( k = 2c + d \) suffices.

That is,

\[ n \log_2 n + (c + d)n - c \leq 2n \log_2 n \]

when \( n > k = 2c + d \).

So \( T(n) = \mathcal{O}(n \log_2 n) \).
Chapter 4. Solving Recurrences

\[ T(n) \leq 2^{h+1}T\left(\frac{n}{2^h+1}\right) + (h + 1)n + c \sum_{i=0}^{h} 2^i \]
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\[ = nd + n \log_2 n + c(n - 1) \]

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when \( n > k \).

Proof: Choose \( a = 2 \). Then to make (1) holds, we need \( \log_2 n > c + d \). So \( k = 2c + d \) suffices.

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So \( T(n) = O(n \log_2 n) \).
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Choose \( a = 2 \). Then to make (1) holds, we need \( \log_2 n > c + d \). So \( k = 2(c + d) \) suffices.
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Chapter 4. Solving Recurrences

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So

\[ T(n) = O(n \log_2 n) \]
Chapter 4. Solving Recurrences

When $n$ is not a power of 2

- choose $m_n$ such that $m_n n$ is a power of 2 and the smallest such that $n \leq m_n n$;
- $T(n) \leq T(m_n n)$, why? assume $T$ to be monotonic;
- use the analysis we just did, $T(m_n n) = O(m_n n \log_2 m_n n)$; that is, $\exists c, k$, $T(m_n n) \leq cm_n n \log_2 m_n n$ when $m_n n \geq k$;
- but $m_n n < 2^n$, why? because $m_n n 2^k < n$;
- so $T(n) \leq T(m_n n) \leq cm_n n \log_2 m_n n \leq 2^cn \log_2 (2^n) \leq 2^cn \log_2 n = c'n \log_2 n$, here $c' = 4c$, when $m_n n \geq k'$ ($\geq 4$), i.e., when $n \geq \lceil k/2 \rceil$ ($\geq 2$);
- therefore, $T(n) = O(n \log_2 n)$. 


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Chapter 4. Solving Recurrences

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that is, \exists c, k, $T(m_n) \leq cm_n \log_2 m_n$ when $m_n \geq k$;

- but $m_n < 2n$, why?

because $m_n \leq n$;

- So $T(n) \leq T(m_n) \leq cm_n \log_2 m_n \leq 2cn \log_2 (2n) \leq 2cn \log_2 n \leq c'n \log_2 n$, here $c' = 4c$, when $m_n \geq k$ ($\geq 4$), i.e., when $n \geq \lceil k/2 \rceil$ ($\geq 2$);

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- but $m_n < 2n$, why?
Chapter 4. Solving Recurrences

When \( n \) is not a power of 2

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Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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  when $m_n \geq k(\geq 4)$, i.e., when $n \geq \lceil \frac{k}{2} \rceil(\geq 2)$;
- therefore, $T(n) = O(n \log_2 n)$. 


Chapter 4. Solving Recurrences

Methods for solving recurrences
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1. Substitution method (based on math induction)
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First we recall the principle of the math induction:

To prove a property $\mathcal{P}(n)$ for every natural number $n \geq 1$, it suffices to prove

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"The principle for dominos to fall".
Math induction comes with different forms or variants
Chapter 4. Solving Recurrences

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$$P(k) \rightarrow P(2k) \land P(2k+1)$$
Chapter 4. Solving Recurrences

**Theorem**: Arithmetic sequence of first $n$ terms

$$1 + 2 + 3 + \cdots + n = \frac{n(n + 1)}{2}$$
Chapter 4. Solving Recurrences

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$$1 + 2 + 3 + \cdots + n = \frac{n}{2}(n + 1)$$

**Proof:** (use math induction)
Chapter 4. Solving Recurrences

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step 1: base case: $n = 1$, left = 1, right = $\frac{1}{2}(1 + 1) = 1$;
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Chapter 4. Solving Recurrences

Algorithm MERGE SORT(A, p, r)

1. if $p < r$
2. then $q = \lfloor \frac{p+r}{2} \rfloor$
3. MERGE SORT(A, p, q)
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Chapter 4. Solving Recurrences

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\[ T(n) \leq 2T\left(\frac{n}{2}\right) + n + c \text{ \ with base case } T(1) = c \]
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step 2. assumption: for $\frac{n}{2}$, the claim is true,
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Chapter 4. Solving Recurrences

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$$T(n) \leq 2T\left(\frac{n}{2}\right) + n + c = 2T\left(\frac{n}{2}\right) + n + c \leq 2a \frac{n}{2} \log_2 \frac{n}{2} + n + c$$
Chapter 4. Solving Recurrences

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\[
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Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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step 3. induction:

\[
T(n) \leq 2T\left(\frac{n}{2}\right) + n + c = 2T\left(\frac{n}{2}\right) + n + c \leq 2a \frac{n}{2} \log_2 \frac{n}{2} + n + c
\]

\[
= an(\log_2 n - \log_2 2) + n + c = an \log_2 n - an + n + c = an \log_2 n - (3c + 2)n + n + c
\]

\[
= an \log_2 n - c3n - 2n + n + c = an \log_2 n + c(1 - 3n) - n
\]
Chapter 4. Solving Recurrences

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because $c(1 - 3n) - n < 0$
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Chapter 4. Solving Recurrences

A little review on logarithm functions:

\[
\begin{align*}
\log_a n + \log_a m &= \log_a (nm); \\
\log_a n^b &= b \log_a n, \text{ especially } \log_a 1^n &= -\log_a n; \\
a^{\log_a n} &= n; \\
\log_a n &= \log_b n \cdot \frac{1}{\log_b a}; \\
\log_m a^n &= (\log_a n)^m \neq \log_a n^m.
\end{align*}
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A little review on logarithm functions:

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Chapter 4. Solving Recurrences

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A little review on logarithm functions:

- $\log a^n + \log a^m = \log a^{nm}$;
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Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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Solving recurrence with the substitution method (guess then verify)

\[ T(n) = \frac{3}{2} T\left(\left\lfloor \frac{2n}{3} \right\rfloor \right) + n, \text{ where } T(1) = 2 \]
Chapter 4. Solving Recurrences

Solving recurrence with the **substitution method** (guess then verify)

\[ T(n) = \frac{3}{2} T\left(\left\lfloor \frac{2n}{3} \right\rfloor \right) + n, \quad \text{where } T(1) = 2 \]

**Guess** \( T(n) \leq cn \log_2 n \), for some constant \( c \) to be determined later.
Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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We make \( T(2) = 5 \leq cn \log_2 n \) holds for \( c = 3 \), and when \( n \geq k = 2 \),
Chapter 4. Solving Recurrences

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\[ T(\lfloor \frac{2n}{3} \rfloor) \leq c \lfloor \frac{2n}{3} \rfloor \log_2 \lfloor \frac{2n}{3} \rfloor \]
Chapter 4. Solving Recurrences

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\[ \leq \frac{3}{2} \left( c \frac{2n}{3} \log_2 \frac{2n}{3} \right) + n \leq cn (\log_2 n + \log_2 \frac{2}{3}) + n \]
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\[
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Chapter 4. Solving Recurrences

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Since $\log_2 \frac{3}{2} > 0.5,$
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Since \( \log_2 \frac{3}{2} > 0.5 \), \( -cn \log_2 \frac{3}{2} + n < 0 \) when \( c \geq 2 \).

But we already know \( c = 3 \) from the base case proof,
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Step 3, substitute it in the recurrence, we get

\[ T(n) = \frac{3}{2} T(\lfloor \frac{2n}{3} \rfloor) + n \leq \frac{3}{2} \left( c \lfloor \frac{2n}{3} \rfloor \log_2 \lfloor \frac{2n}{3} \rfloor \right) + n \]

\[ \leq \frac{3}{2} \left( c \frac{2n}{3} \log_2 \frac{2n}{3} \right) + n \leq cn \log_2 n + \log_2 \frac{2}{3} + n \]

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\[ T(n) \leq cn \log_2 n - cn \log_2 \frac{3}{2} + n \leq cn \log_2 n \]
Chapter 4. Solving Recurrences

2. Changing variables

Example:

\[ T(n) = 2T(\sqrt{n}) + \log_2 n \]

Define \( m = \log_2 n \), i.e., \( n = 2^m \).

\[ T(2^m) = 2T(2^{m/2}) + m \]

rename the function:

\[ S(m) = T(2^m) \]

solve it, we have

\[ S(m) = O(m \log m) \]

so

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Chapter 4. Solving Recurrences

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3. Recursive tree method
Chapter 4. Solving Recurrences

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By unfolding the recurrence to make a recursive-tree.
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(1) \( T(n) \) is a tree with non-recursive terms as the root and recursive terms as its children.
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3. Recursive tree method

By **unfolding** the recurrence to make a recursive-tree.

1. $T(n)$ is a tree with non-recursive terms as the root and recursive terms as its children.

2. For each child, replace it with then non-recursive terms and produce children that are then recursive terms.
3. Recursive tree method

By unfolding the recurrence to make a recursive-tree.

(1) $T(n)$ is a tree with non-recursive terms as the root and recursive terms as its children.

(2) for each child, replace it with then non-recursive terms and produce children that are then recursive terms

(3) repeat (2), expand the tree until all children are the base case.
Chapter 4. Solving Recurrences

Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$
Chapter 4. Solving Recurrences

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$l_0$:

\[
T(n)
\]
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| \( l_0 \) | \( T(n/4) \) | \( T(n) \) |
| \( l_1 \) | \( T(n/4) \) | \( T(n/4) \) | \( T(n/4) \) |
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Example \( T(n) = 3T(n/4) + n^2 \), with base case \( T(1) = 1 \)

\[
\begin{array}{cccc}
l_0: & T(n/4) & T(n) & T(n/4) \\
l_1: & T(n/4) & T(n/4) & T(n/4) & n^2
\end{array}
\]
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Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

\[
\begin{array}{cccc}
\text{l}_0: & & & \\
\text{l}_1: & T(n/4) & T(n/4) & T(n/4) & n^2 \\
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\]
Chapter 4. Solving Recurrences

Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

\begin{align*}
l_0: & \\
l_1: & T(n/4) \\
l_2: & T(n/4) T(n/4) T(n/4) T(n/4) T(n/4) T(n/4) T(n/4) T(n/4) T(n/4) T(n/4) T(n/4) T(n/4)\end{align*}

\[n^2\]
Chapter 4. Solving Recurrences

Example \( T(n) = 3T(n/4) + n^2 \), with base case \( T(1) = 1 \)

\[
\begin{align*}
l_0: & & T(n) \\
l_1: & & T(n/4) \quad T(n/4) \\
l_2: & & T(n/4) \quad T(n/4) \quad T(n/4) \\
      & & \vdots \\
l_{m-1}: & & \vdots \\
l_m: & & T(1) = 1, T(1) = 1, T(1) = 1, \ldots, T(1) = 1 \\
\end{align*}
\]

where \( n/4^m = 1 \), i.e., \( m = \log_4 n \).

Then \( T(n) = n^2 \left[ 1 + 3 \left( \frac{1}{4} \right)^2 + 3 \left( \frac{1}{4} \right)^4 + 3 \left( \frac{1}{4} \right)^6 + \cdots + 3 \left( \frac{1}{4} \right)^{2m-2} \right] + 3 \left( \frac{n}{4} \right)^2 \leq n^2 \left( 1 - \frac{3}{16} \right)^{m-1} \leq n^2 \left( 1 - \frac{3}{4} \right) = \frac{13}{16} n^2 \) for all \( n > 0 \).
### Chapter 4. Solving Recurrences

Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

<table>
<thead>
<tr>
<th>$l_0$</th>
<th>$T(n)$</th>
<th>$T(n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$l_1$</td>
<td>$T(n/4)$</td>
<td>$T(n/4)$</td>
</tr>
<tr>
<td>$l_2$</td>
<td>$T(n/4^2)$</td>
<td>$T(n/4^2)$</td>
</tr>
<tr>
<td>$l_3$</td>
<td>......</td>
<td>......</td>
</tr>
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Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

\(l_0:\)  
\(T(n)\)

\(l_1:\)
\(T(n/4)\)  
\(T(n/4)\)

\(l_2:\)
\(T(n/4^2)\)  
\(T(n/4^2)\)

\(l_3:\)  
\(\ldots\)
Chapter 4. Solving Recurrences

Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

$l_0$: 

$l_1$: 

$l_2$: $T(n/4) T(n/4) T(n/4)$

$l_3$: \ldots

$l_4$: \ldots
Chapter 4. Solving Recurrences

Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

\[ l_0: \quad T(n) \quad T(n/4) \quad T(n/4) \quad T(n/4) \quad n^2 \]
\[ l_1: \quad T(n/4) \quad T(n/4) \quad T(n/4) \quad T(n/4) \quad 3(n/4)^2 \]
\[ l_2: \quad T(n/4^2) \quad T(n/4^2) \quad T(n/4^2) \quad T(n/4^2) \quad T(n/4^2) \quad T(n/4^2) \quad 3^2(n/4^2)^2 \]
\[ l_3: \quad \ldots \]
\[ l_4: \quad \ldots \]
Chapter 4. Solving Recurrences

Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

\begin{align*}
l_0: & & & & T(n) \\
l_1: & & T(n/4) & & T(n/4) \quad T(n/4) \quad T(n/4) \\
l_2: & & T(n/4) & & T(n/4) \quad T(n/4) \quad T(n/4) \\
l_3: & & \ldots \ldots \quad \ldots \ldots \quad \ldots \ldots \quad \ldots \ldots \\
l_4: & & \ldots \ldots \quad \ldots \ldots \quad \ldots \ldots \quad \ldots \ldots \\
l_5: & & \ldots \ldots \quad \ldots \ldots \quad \ldots \ldots \quad \ldots \ldots \\
\end{align*}

\[ n^2 \]

\[ 3(\frac{n}{4})^2 \]

\[ 3^2(\frac{n}{4^2})^2 \]

\[ 3^3(\frac{n}{4^3})^2 \]
Chapter 4. Solving Recurrences

Example \( T(n) = 3T(n/4) + n^2 \), with base case \( T(1) = 1 \)

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  l_0: & & & T(n) \\
  l_1: & T(n/4) & T(n/4) & T(n/4) \\
  l_2: & T(n/4^2) & T(n/4^2) & T(n/4^2) & T(n/4^2) \\
  l_3: & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
  l_4: & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
  l_5: & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
  l_{m-1}: & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{align*} \]
Chapter 4. Solving Recurrences

Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

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\begin{align*}
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  l_1: & & T(n/4) & & T(n/4) & & n^2 \\
  l_2: & & T(n/4) & & T(n/4) & & T(n/4) & & 3(n/4)^2 \\
  l_3: & & \ldots & & \ldots & & \ldots & & \ldots \\
  l_4: & & \ldots & & \ldots & & \ldots & & \ldots \\
  l_5: & & \ldots & & \ldots & & \ldots & & \ldots \\
  l_{m-1}: & & \ldots & & \ldots & & \ldots & & \ldots \\
\end{align*}
\]
Chapter 4. Solving Recurrences

Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

$l_0$: $T(n)$

$l_1$: $T(n/4)$ $T(n/4)$ $T(n/4)$ $n^2$

$l_2$: $T(n/4^2)$ $T(n/4^2)$ $T(n/4^2)$ $T(n/4^2)$ $T(n/4^2)$ $T(n/4^2)$ $3(n/4)^2$

$l_3$: $\ldots$ $3^2(n/4^2)^2$

$l_4$: $\ldots$ $3^3(n/4^3)^2$

$l_5$: $\ldots$ $3^{m-2}(n/4^{m-2})^2$

$l_m$: $T(1), T(1), T(1), T(1), T(1), \ldots, T(1)$ $3^{m-1}(n/4^{m-1})^2$
Chapter 4. Solving Recurrences

Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

\begin{align*}
l_0: & \quad T(n) \\
l_1: & \quad T(n/4) \\
l_2: & \quad T(n/4)^2 \\
l_3: & \quad \ldots \\
l_4: & \quad \ldots \\
l_5: & \quad \ldots \\
l_{m-1}: & \quad \ldots \\
l_m: & \quad T(1), T(1), T(1), T(1), T(1), \ldots, T(1)
\end{align*}

where $\frac{n}{4^m} = 1$, i.e., $m = \log_4 n$. 

$T(n) = 3^n T(1)$

Then $T(n)$ is the sum

$T(n) = n^2 \left[ 1 + 3 \left( \frac{1}{4} \right)^2 + 3 \left( \frac{1}{4} \right)^4 + 3 \left( \frac{1}{4} \right)^6 + \cdots + 3^{m-1} \left( \frac{1}{4} \right)^{2m-2} \right] + 3^m T(1)$

for all $n > 0$. 

$T(n) = n^2 \left[ 1 + 3 \left( \frac{1}{4} \right)^2 + 3 \left( \frac{1}{4} \right)^4 + 3 \left( \frac{1}{4} \right)^6 + \cdots \right] = n^2 \left[ 1 + \frac{3}{32} \left[ 1 - \left( \frac{1}{4} \right)^m \right] \right] = n^2 \left[ 1 + \frac{3}{32} \right] = \frac{35}{32} n^2$
Chapter 4. Solving Recurrences

Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

$l_0$: $T(n)$

$l_1$: $T(n/4)$ $T(n/4)$ $T(n/4)$

$l_2$: $T(n/4^2)$ $T(n/4^2)$ $T(n/4^2)$ $T(n/4^2)$ $T(n/4^2)$ $T(n/4^2)$

$l_3$: $3(n/4)^2$

$l_4$: $3^2(n/4^2)^2$

$l_5$: $3^3(n/4^3)^2$

$l_{m-1}$: $3^{m-2}(n/4^{m-2})^2$

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Then $T(n)$ is the sum

$$T(n) = n^2[1 + 3(1/4)^2 + 3^2(1/4^2)^2 + 3^3(1/4^3)^2 + \cdots + 3^{m-1}(1/4^{m-1})^2] + 3^mT(1)$$
Chapter 4. Solving Recurrences

Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

$l_0$: $T(n)$

$l_1$: $T(n/4)$

$l_2$: $T(n/4^2)$ $T(n/4^2)$ $T(n/4^2)$ $T(n/4^2)$ $T(n/4^2)$ $T(n/4^2)$ $T(n/4^2)$ $T(n/4^2)$ $T(n/4^2)$ $T(n/4^2)$ $n^2$

$l_3$: $3(n/4)^2$

$l_4$: $3^2(n/4^2)^2$

$l_5$: $3^3(n/4^3)^2$

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$l_m$: $T(1), T(1), T(1), T(1), T(1), \ldots, T(1)$ $3^{m-1}(n/4^{m-1})^2$

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Then $T(n)$ is the sum

$T(n) = n^2[1 + 3(\frac{1}{4})^2 + 3^2(\frac{1}{4^2})^2 + 3^3(\frac{1}{4^3})^2 + \cdots + 3^{m-1}(\frac{1}{4^{m-1}})^2] + 3^mT(1)$

$T(n) = n^2[1 + 3(\frac{1}{4})^2 + 3^2(\frac{1}{4^2})^2 + 3^3(\frac{1}{4^3})^2 + \cdots + 3^{m-1}(\frac{1}{4^{m-1}})^2] + 3^m \times 1$
Chapter 4. Solving Recurrences

Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

\[
\begin{align*}
l_0: & \quad T(n) \\
l_1: & \quad T(n/4) \\
l_2: & \quad T(n/4)T(n/4)T(n/4) \\
l_3: & \quad \ldots \\
l_4: & \quad \ldots \\
l_5: & \quad \ldots \\
l_{m-1}: & \quad \ldots \\
l_m: & \quad T(1), T(1), T(1), T(1), T(1), \ldots, T(1)
\end{align*}
\]

where $\frac{n}{4^m} = 1$, i.e., $m = \log_4 n$.

Then $T(n)$ is the sum

\[
\begin{align*}
T(n) &= n^2[1 + 3\left(\frac{1}{4}\right)^2 + 3^2\left(\frac{1}{4^2}\right)^2 + 3^3\left(\frac{1}{4^3}\right)^2 + \cdots + 3^{m-1}\left(\frac{1}{4^{m-1}}\right)^2] + 3^m T(1) \\
T(n) &= n^2[1 + 3\left(\frac{1}{4}\right)^2 + 3^2\left(\frac{1}{4^2}\right)^2 + 3^3\left(\frac{1}{4^3}\right)^2 + \cdots + 3^{m-1}\left(\frac{1}{4^{m-1}}\right)^2] + 3^m \times 1 \\
&= n^2\left[1 + 3\left(\frac{1}{4}\right)^2 + 3^2\left(\frac{1}{4^2}\right)^2 + 3^3\left(\frac{1}{4^3}\right)^2 + \cdots + 3^{m-1}\left(\frac{1}{4^{m-1}}\right)^2\right] + 3^m\left(\frac{n}{4^m}\right)^2
\end{align*}
\]
Chapter 4. Solving Recurrences

Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

$$l_0: \quad T(n)$$
$$l_1: \quad T(n/4) \quad T(n/4) \quad T(n/4)$$
$$l_2: \quad T(n/4^2) \quad T(n/4^2) \quad T(n/4^2) \quad T(n/4^2)$$
$$l_3: \quad \ldots$$
$$l_4: \quad \ldots$$
$$l_5: \quad \ldots$$
$$l_{m-1}: \quad \ldots$$
$$l_m: \quad T(1), T(1), T(1), T(1), T(1), \ldots, T(1)$$

where $\frac{n}{4^m} = 1$, i.e., $m = \log_4 n$.

Then $T(n)$ is the sum

$$T(n) = n^2[1 + 3\left(\frac{1}{4}\right)^2 + 3^2\left(\frac{1}{4^2}\right)^2 + 3^3\left(\frac{1}{4^3}\right)^2 + \cdots + 3^{m-1}\left(\frac{1}{4^{m-1}}\right)^2] + 3^m T(1)$$

$$T(n) = n^2[1 + 3\left(\frac{1}{4}\right)^2 + 3^2\left(\frac{1}{4^2}\right)^2 + 3^3\left(\frac{1}{4^3}\right)^2 + \cdots + 3^{m-1}\left(\frac{1}{4^{m-1}}\right)^2] + 3^m \times 1$$

$$= n^2[1 + 3\left(\frac{1}{4}\right)^2 + 3^2\left(\frac{1}{4^2}\right)^2 + 3^3\left(\frac{1}{4^3}\right)^2 + \cdots + 3^{m-1}\left(\frac{1}{4^{m-1}}\right)^2] + 3^m \left(\frac{n}{4m}\right)^2$$

$$= n^2[1 + \frac{3}{16} + \left(\frac{3}{16}\right)^2 + \left(\frac{3}{16}\right)^3 + \cdots + \left(\frac{3}{16}\right)^m]$$
Chapter 4. Solving Recurrences

Example \( T(n) = 3T(n/4) + n^2 \), with base case \( T(1) = 1 \)

\[
\begin{align*}
 l_0: & \quad T(n) \\
 l_1: & \quad T(n/4) \\
l_2: & \quad T(n/4) T(n/4) T(n/4) \\
l_3: & \quad \ldots \\
l_4: & \quad \ldots \\
l_5: & \quad \ldots \\
l_{m-1}: & \quad \ldots \\
l_m: & \quad T(1), T(1), T(1), T(1), T(1), \ldots, T(1)
\end{align*}
\]

where \( \frac{n}{4^m} = 1 \), i.e., \( m = \log_4 n \).

Then \( T(n) \) is the sum

\[
T(n) = n^2 \left[ 1 + 3 \left( \frac{1}{4} \right)^2 + 3^2 \left( \frac{1}{4^2} \right)^2 + 3^3 \left( \frac{1}{4^3} \right)^2 + \cdots + 3^{m-1} \left( \frac{1}{4^{m-1}} \right)^2 \right] + 3^m T(1)
\]

\[
T(n) = n^2 \left[ 1 + 3 \left( \frac{1}{4} \right)^2 + 3^2 \left( \frac{1}{4^2} \right)^2 + 3^3 \left( \frac{1}{4^3} \right)^2 + \cdots + 3^{m-1} \left( \frac{1}{4^{m-1}} \right)^2 \right] + 3^m \times 1
\]

\[
= n^2 \left[ 1 + 3 \left( \frac{1}{4} \right)^2 + 3^2 \left( \frac{1}{4^2} \right)^2 + 3^3 \left( \frac{1}{4^3} \right)^2 + \cdots + 3^{m-1} \left( \frac{1}{4^{m-1}} \right)^2 \right] + 3^m \left( \frac{n}{4^m} \right)^2
\]

\[
= n^2 \left[ 1 + \frac{3}{16} + \left( \frac{3}{16} \right)^2 + \left( \frac{3}{16} \right)^3 + \cdots + \left( \frac{3}{16} \right)^m \right]
\]

\[
= n^2 \left( 1 - \left( \frac{3}{16} \right)^{m+1} \right)
\]

\[
\leq n^2 \left( 1 - \left( \frac{3}{16} \right) \right)
\]

for all \( n > 0 \).

Chapter 4. Solving Recurrences

Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

$l_0$:

$l_1$:

$l_2$:

$l_3$: . . . .

$l_4$: . . . .

$l_5$: . . . .

$l_{m-1}$: . . . .

$l_m$: $T(1), T(1), T(1), T(1), T(1), \ldots, T(1)$

where $\frac{n}{4^m} = 1$, i.e., $m = \log_4 n$.

Then $T(n)$ is the sum

$T(n) = n^2[1 + 3(\frac{1}{4})^2 + 3^2(\frac{1}{4^2})^2 + 3^3(\frac{1}{4^3})^2 + \cdots + 3^{m-1}(\frac{1}{4^{m-1}})^2] + 3^m T(1)$

$T(n) = n^2[1 + 3(\frac{1}{4})^2 + 3^2(\frac{1}{4^2})^2 + 3^3(\frac{1}{4^3})^2 + \cdots + 3^{m-1}(\frac{1}{4^{m-1}})^2] + 3^m \times 1$

$= n^2[1 + 3(\frac{1}{4})^2 + 3^2(\frac{1}{4^2})^2 + 3^3(\frac{1}{4^3})^2 + \cdots + 3^{m-1}(\frac{1}{4^{m-1}})^2] + 3^m \times \frac{n}{4^m}$

$= n^2[1 + \frac{3}{16} + (\frac{3}{16})^2 + (\frac{3}{16})^3 + \cdots + (\frac{3}{16})^m]

= n^2(\frac{1 - (\frac{3}{16})^{m+1}}{1 - \frac{3}{16}})

\leq n^2(\frac{1}{1 - \frac{3}{16}})$
Chapter 4. Solving Recurrences

Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

\[
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  l_1: & & T(n/4) \\
  l_2: & & T(n/4^2) T(n/4^2) T(n/4^2) T(n/4^2) T(n/4^2) T(n/4^2) T(n/4^2) \quad n^2 \\
  l_3: & & \ldots \\
  l_4: & & \ldots \\
  l_5: & & \ldots \\
  l_{m-1}: & & \ldots \\
  l_m: & & T(1), T(1), T(1), T(1), T(1), \ldots, T(1)
\end{align*}
\]

where $\frac{n}{4^m} = 1$, i.e., $m = \log_4 n$.

Then $T(n)$ is the sum

\[
T(n) = n^2 [1 + 3(\frac{1}{4})^2 + 3^2(\frac{1}{4^2})^2 + 3^3(\frac{1}{4^3})^2 + \cdots + 3^{m-1}(\frac{1}{4^{m-1}})^2] + 3^m T(1)
\]

\[
T(n) = n^2 [1 + 3(\frac{1}{4})^2 + 3^2(\frac{1}{4^2})^2 + 3^3(\frac{1}{4^3})^2 + \cdots + 3^{m-1}(\frac{1}{4^{m-1}})^2] + 3^m \times 1
\]

\[
= n^2 [1 + 3(\frac{1}{4})^2 + 3^2(\frac{1}{4^2})^2 + 3^3(\frac{1}{4^3})^2 + \cdots + 3^{m-1}(\frac{1}{4^{m-1}})^2] + 3^m \left(\frac{n}{4^m}\right)^2
\]

\[
= n^2 [1 + \frac{3}{16} + \left(\frac{3}{16}\right)^2 + \left(\frac{3}{16}\right)^3 + \cdots + \left(\frac{3}{16}\right)^m]
\]

\[
= n^2 \left(\frac{1 - \left(\frac{3}{16}\right)^{m+1}}{1 - \frac{3}{16}}\right)
\]

\[
\leq n^2 \left(\frac{1}{1 - \frac{3}{16}}\right)
\]

\[
= \frac{16}{13} n^2
\]

for all $n > 0$. 
Chapter 4. Solving Recurrences

Recursive tree notation in the textbook:

\[ T(n) = 3T(n/4) + n^2, \text{ with base case } T(1) = 1 \]

\[ l_0: \quad n^2 \]
Chapter 4. Solving Recurrences

Recursive tree notation in the textbook:

\[ T(n) = 3T(n/4) + n^2, \text{ with base case } T(1) = 1 \]

- \( l_0: \)
  - \( n^2 \)
  - \( (n/4)^2 \)
  - \( (n/4)^2 \)

- \( l_1: \)
  - \( (n/4)^2 \)
  - \( (n/4)^2 \)
  - \( (n/4)^2 \)
Chapter 4. Solving Recurrences

Recursive tree notation in the textbook:

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- \( l_0: \) 
  - \( n^2 \)

- \( l_1: \) 
  - \( (n/4)^2 \)
  - \( (n/4)^2 \)
  - \( (n/4)^2 \)

- \( l_2: \) 
  - \( \left( \frac{n}{4^2} \right)^2 \)
  - \( \left( \frac{n}{4^2} \right)^2 \)
  - \( \left( \frac{n}{4^2} \right)^2 \)
  - \( \left( \frac{n}{4^2} \right)^2 \)
  - \( \left( \frac{n}{4^2} \right)^2 \)
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Recursive tree notation in the textbook:

\[ T(n) = 3T(n/4) + n^2, \text{ with base case } T(1) = 1 \]

- \( l_0: \) \( n^2 \)
- \( l_1: \) \( (n/4)^2 \) \( (n/4)^2 \) \( (n/4)^2 \) \( (n/4)^2 \) \( (n/4)^2 \)
- \( l_2: \) \( (n/4^2)^2 \) \( (n/4^2)^2 \) \( (n/4^2)^2 \) \( (n/4^2)^2 \) \( (n/4^2)^2 \) \( (n/4^2)^2 \) \( (n/4^2)^2 \) \( (n/4^2)^2 \)
- \( l_3: \) \( \ldots \)
Chapter 4. Solving Recurrences

Recursive tree notation in the textbook:

\[ T(n) = 3T(n/4) + n^2, \text{ with base case } T(1) = 1 \]

- \( l_0: \) \( n^2 \)
- \( l_1: \) \( (n/4)^2 \) \( (n/4)^2 \) \( (n/4)^2 \)
- \( l_2: \) \( (n/4^2)^2 \) \( (n/4^2)^2 \) \( (n/4^2)^2 \) \( (n/4^2)^2 \) \( (n/4^2)^2 \) \( (n/4^2)^2 \)
- \( l_3: \) .......
- \( l_4: \) .......

3^3 \) nodes of \( (n/4^3)^2 \)
Recursive tree notation in the textbook:

\[ T(n) = 3T(n/4) + n^2, \] with base case \[ T(1) = 1 \]

\[ l_0: \quad n^2 \]
\[ l_1: \quad (n/4)^2 \quad (n/4)^2 \quad (n/4)^2 \]
\[ l_2: \quad (n/4^2)^2 \quad (n/4^2)^2 \quad (n/4^2)^2 \quad (n/4^2)^2 \quad (n/4^2)^2 \quad (n/4^2)^2 \quad (n/4^2)^2 \quad (n/4^2)^2 \]
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\[ \text{3}^3 \text{ nodes of } (n/4^3)^2 \]
Chapter 4. Solving Recurrences

Recursive tree notation in the textbook:

\( T(n) = 3T(n/4) + n^2 \), with base case \( T(1) = 1 \)

\( l_0: \quad n^2 \)
\( l_1: \quad \frac{n}{4^2} \quad \left( \frac{n}{4^2} \right)^2 \quad \left( \frac{n}{4^2} \right)^2 \quad \left( \frac{n}{4^2} \right)^2 \quad \left( \frac{n}{4^2} \right)^2 \quad \left( \frac{n}{4^2} \right)^2 \quad \left( \frac{n}{4^2} \right)^2 \quad \left( \frac{n}{4^2} \right)^2 \\ 3^3 \) nodes of \( \left( \frac{n}{4^3} \right)^2 \)
\( l_2: \quad \frac{n}{4^2} \quad \left( \frac{n}{4^2} \right)^2 \quad \left( \frac{n}{4^2} \right)^2 \quad \left( \frac{n}{4^2} \right)^2 \quad \left( \frac{n}{4^2} \right)^2 \quad \left( \frac{n}{4^2} \right)^2 \quad \left( \frac{n}{4^2} \right)^2 \quad \left( \frac{n}{4^2} \right)^2 \\ 3^3 \) nodes of \( \left( \frac{n}{4^3} \right)^2 \)
\( l_3: \quad \ldots \ldots \)
\( l_4: \quad \ldots \ldots \)
\( l_{m-1}: \quad \ldots \ldots \)
\( 3^{m-1} \) of \( \left( \frac{n}{4^{m-1}} \right)^2 \)
Chapter 4. Solving Recurrences

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\[ l_1: \quad (n/4)^2 \quad (n/4)^2 \quad (n/4)^2 \quad \ldots \]
\[ l_2: \quad (n/4^2)^2 \quad (n/4^2)^2 \quad (n/4^2)^2 \quad \ldots \]
\[ l_3: \quad \ldots \ldots \]
\[ l_4: \quad \ldots \ldots \]
\[ l_{m-1}: \quad \ldots \ldots \]
\[ l_m: \quad T(1), T(1), T(1), T(1), T(1), \ldots \]

\[ 3^3 \text{ nodes of } (n/4^3)^2 \]
\[ 3^{m-1} \text{ of } (n/4^{m-1})^2 \]
\[ 3^m \text{ nodes of } T(1) \]
Chapter 4. Solving Recurrences

Recursive tree notation in the textbook:

\[ T(n) = 3T\left(\frac{n}{4}\right) + n^2, \text{ with base case } T(1) = 1 \]

\( l_0: \)  
\( l_1: \) \( n^2 \)  
\( l_2: \) \( \left(\frac{n}{4^2}\right)^2 \left(\frac{n}{4^2}\right)^2 \left(\frac{n}{4^2}\right)^2 \)  
\( l_3: \) \( \ldots \)  
\( l_4: \) \( \ldots \)  
\( l_{m-1}: \) \( \ldots \)  
\( l_m: \) \( T(1), T(1), T(1), T(1), T(1), \ldots \)  

3\(^3\) nodes of \( \left(\frac{n}{4^3}\right)^2 \)

3\(^{m-1}\) of \( \left(\frac{n}{4^{m-1}}\right)^2 \)

3\(^m\) nodes of \( T(1) \)
Another example (page 91 textbook):

Assume time function $T(n)$ of some algorithm has the recurrence

$$T(n) = T\left(\frac{n}{3}\right) + T\left(\frac{2n}{3}\right) + n$$

with base case $T(1) = T(2) = T(3) = c > 0$, a constant. We assume $n$ is a power of 3.
Chapter 4. Solving Recurrences

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(1) using recursive tree method to derive an upper bound

(2) using substitution method to verify the upper bound
(1) using recursive tree method to derive an upper bound
(1) using recursive tree method to derive an upper bound

\[l_0: \quad T(n)\]
\[l_1: \quad T\left(\frac{n}{3}\right) \quad T\left(\frac{2n}{3}\right) \quad n\]
Chapter 4. Solving Recurrences

(1) using recursive tree method to derive an upper bound

\[ l_0: \quad T(n) \]
\[ l_1: \quad T\left(\frac{n}{3}\right) \quad T\left(\frac{2n}{3}\right) \quad n \]
\[ l_2: \quad T\left(\frac{n}{3^2}\right) \quad T\left(\frac{2n}{3^2}\right) \quad T\left(\frac{2n}{3^2}\right) \quad T\left(\frac{2^2n}{3^2}\right) \quad \frac{n}{3} + \frac{2n}{3} = n \]
(1) using recursive tree method to derive an upper bound

\[
\begin{align*}
l_0: & & T(n) \\
l_1: & & T\left(\frac{n}{3}\right) & T\left(\frac{2n}{3}\right) \\
l_2: & & T\left(\frac{n}{3^2}\right) & T\left(\frac{2n}{3^2}\right) & T\left(\frac{2^2 n}{3^2}\right) \\
& & \frac{n}{3} + \frac{2n}{3} = n \\
\vdots \\
l_{m-1}: & & T\left(\frac{n}{3^{m-1}}\right) \ldots \\
& & T\left(\frac{2^{m-1} n}{3^{m-1}}\right) \quad n
\end{align*}
\]
(1) using recursive tree method to derive an upper bound

\[ l_0: \quad T(n) \]

\[ l_1: \quad T\left(\frac{n}{3}\right) \quad T\left(\frac{2n}{3}\right) \]

\[ l_2: \quad T\left(\frac{n}{3^2}\right) \quad T\left(\frac{2n}{3^2}\right) \quad T\left(\frac{2^2n}{3^2}\right) \quad \frac{n}{3} + \frac{2n}{3} = n \]

\[ \ldots \]

\[ l_{m-1}: \quad T\left(\frac{n}{3^{m-1}}\right) \quad \ldots \]

\[ l_m: \quad T\left(\frac{n}{3^m}\right) \quad T\left(\frac{2n}{3^m}\right) \quad \ldots \]

\[ \vdots \]

\[ l_{m-1}: \quad T\left(\frac{n}{3^{m-1}}\right) \quad \ldots \]

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(1) using recursive tree method to derive an upper bound

\[ l_0: \quad T(n) \]
\[ l_1: \quad T\left(\frac{n}{3}\right) \quad T\left(\frac{2n}{3}\right) \quad n = \frac{n}{3} + \frac{2n}{3} \]
\[ l_2: \quad T\left(\frac{n}{3^2}\right) \quad T\left(\frac{2n}{3^2}\right) \quad T\left(\frac{2^2n}{3^2}\right) \quad \frac{n}{3} + \frac{2n}{3} = n \]
\[ \cdots \]
\[ l_{m-1}: \quad T\left(\frac{n}{3^{m-1}}\right) \quad T\left(\frac{2^{m-1}n}{3^{m-1}}\right) \quad T\left(\frac{2^mn}{3^m}\right) \quad n \]
\[ l_m: \quad T\left(\frac{n}{3^m}\right) \quad T\left(\frac{2n}{3^m}\right) \quad T\left(\frac{2^mn}{3^m}\right) \quad n \]
\[ l_{m+1}: \quad \cdots \]
\[ l_{m+1}: \quad \cdots \quad T\left(\frac{2^{m+1}n}{3^{m+1}}\right) < n \]
Chapter 4. Solving Recurrences

(1) using recursive tree method to derive an upper bound

\[ l_0: \quad T(n) \]
\[ l_1: \quad T\left(\frac{n}{3}\right) \quad T\left(\frac{2n}{3}\right) \quad n \]
\[ l_2: \quad T\left(\frac{n}{3^2}\right) \quad T\left(\frac{2n}{3^2}\right) \quad T\left(\frac{2^2n}{3^2}\right) \quad \frac{n}{3} + \frac{2n}{3} = n \]
\[ \ldots \]
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\[ \ldots \]
\[ l_r: \quad \ldots \]
\[ \quad T\left(\frac{2^{m+1}n}{3^{m+1}}\right) < n \]
\[ \quad T\left(\frac{2^rn}{3^r}\right) < n \]
Chapter 4. Solving Recurrences

(1) using recursive tree method to derive an upper bound

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\[ l_m: \quad T\left(\frac{n}{3^m}\right) \quad T\left(\frac{2n}{3^m}\right) \quad T\left(\frac{2^2n}{3^m}\right) \]
\[ \vdots \]
\[ l_{m+1}: \quad \ldots \]
\[ \vdots \]
\[ l_r: \quad \ldots \]

\[ \frac{n}{3} + \frac{2n}{3} = n \]
\[ T\left(\frac{2^{m-1}n}{3^{m-1}}\right) \quad n \]
\[ T\left(\frac{2^m n}{3^m}\right) \quad n \]
\[ T\left(\frac{2^{m+1}n}{3^{m+1}}\right) < n \]
\[ T\left(\frac{2^r n}{3^r}\right) < n \]
Chapter 4. Solving Recurrences

(1) using recursive tree method to derive an upper bound
(1) using recursive tree method to derive an upper bound

- the leftmost branch is the shortest; the rightmost is the longest;
- \( \frac{n}{3m} = 1, \ m = \log_3 n; \ \frac{2^r n}{3^r} = 1, \ r = \log_3 \frac{n}{2} n. \)
- beginning from level \( m + 1 \), some nodes will gradually disappear, so the number of leaves is NOT \( 2^{\log_3 \frac{n}{2}} n \neq O(n \log_2 n) \) (why?),
- we do not need to give an accurate account for the sum of all quantities in blue color and those at leaves, but only an estimated upper bound.
- we estimate an upper bound to be \( O(n \log_2 n). \)
(2) using the substitution method to verify the upper bound

We prove that $T(n) = O(n \log_2 n)$.

That is to prove:

$\exists a, k > 0$ such that $T(n) \leq an \log_2 n$ when $n \geq k$

Base case: for $n = 3$, $T(3) = c \leq a \cdot 3 \log_2 3$ will be true if we choose $a \geq c$ because $\log_2 3 > 1$.

Assume that $T(n/3) \leq a(n/3) \log_2 (n/3)$; and $T(2n/3) \leq a(2n/3) \log_2 (2n/3)$;

Then

$T(n) = T(n/3) + T(2n/3) + n \leq an \log_2 n + an \log_2 n + n = an \log_2 n - an/3 (\log_2 3 + 2 \log_2 3) + n$
Chapter 4. Solving Recurrences

(2) using the substitution method to verify the upper bound
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That is to prove: \( \exists a, k > 0 \) such that

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Chapter 4. Solving Recurrences

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Base case: for \( n = 3 \), \( T(3) = c \leq a3 \log_2 3 \) will be true if we choose \( a \geq c \) because \( \log_2 3 > 1 \).
Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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Then

$$T(n) = T\left(\frac{n}{3}\right) + T\left(\frac{2n}{3}\right) + n$$

$$\leq a \frac{n}{3} \log_2 \frac{n}{3} + a \frac{2n}{3} \log_2 \frac{2n}{3} + n$$

$$= an \log_2 n - \frac{an}{3} \left(\log_2 3 + 2 \log_2 \frac{3}{2}\right) + n$$

(1)
Therefore,

\[ T(n) \leq an \log_2 n - \frac{an}{3} (\log_2 3 + \log_2 \frac{9}{4}) + n \]

\[ = an \log_2 n - \frac{an}{3} \log_2 \frac{27}{4} + n \]

\[ \leq an \log_2 n \]

when \( a \) is chosen such that \( a \geq \max\{3, c\} \), since it makes

\[ -\frac{an}{3} \log_2 \frac{27}{4} + n \leq 0 \]
Chapter 4. Solving Recurrences

How do we know if a guessed upper bound is incorrect?

\[ T(n) \leq 2T\left(\frac{n}{2}\right) + n, \quad T(1) = 1 \]

- Suppose we guess \( T(n) = O(n) \),

\[ T(n) \leq cn \quad (1) \]

for some \( c, k > 0 \) when \( n > k \).

- Then using the substitution method to prove (1),
  - based case is okay since \( T(1) = 1 \leq c \times 1 \) holds for any chosen \( c \geq 1 \);
  - assume \( T\left(\frac{n}{2}\right) \leq c\frac{n}{2} \), using this on the recurrence

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\[ T(n) \leq 2T\left(\frac{n}{2}\right) + n \leq 2c\frac{n}{2} + n = (1 + c)n \]
Chapter 4. Solving Recurrences

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\[ T(n) \leq 2T\left(\frac{n}{2}\right) + n \leq 2c\frac{n}{2} + n = (1 + c)n \not\leq cn \]
Chapter 5. Probabilistic Analysis of Algorithms

Chapter 5. Probabilistic analysis and randomized algorithms
Chapter 5. Probabilistic Analysis of Algorithms

Chapter 5. Probabilistic analysis and randomized algorithms

- Estimate efficiency of algorithms on a majority of inputs, not all inputs;
Chapter 5. Probabilistic Analysis of Algorithms

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Chapter 5. Probabilistic Analysis of Algorithms

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Chapter 5. Probabilistic Analysis of Algorithms

Chapter 5. Probabilistic analysis and randomized algorithms

• Estimate efficiency of algorithms on a majority of inputs, not all inputs;
• Performance is “average” cases, not the worst case;
• With assumption that input data are in a probabilistic distribution
• Close relationship with randomized algorithms
A good example: **Quick Sort** algorithm [Hoare’1959]

- It has the worst case time \( T_{wc}(n) \geq an^2 \) for some constant \( a > 0 \).
- It has the average case time \( T_{ac}(n) \leq bn \log_2 n \) for a constant \( b > 0 \).

In other words, it is efficient in most cases; assumption: the input data are of the uniform distribution.

But usually the input data do not satisfy uniform distribution. Alternatively, we can enforce the desired distribution by using randomness (tossing coins) in the algorithms. That is, we use randomized algorithms.
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Chapter 5. Probabilistic Analysis of Algorithms

In both probabilistic analysis of deterministic algorithms and analysis of randomized algorithms:

• actions in the algorithm are considered random events

random events
In both probabilistic analysis of deterministic algorithms and analysis of randomized algorithms

- actions in the algorithm are considered random events
- such random events are driven by random data

You can compute the expected time (i.e., averaged time)
Chapter 5. Probabilistic Analysis of Algorithms

In both probabilistic analysis of deterministic algorithms and analysis of randomized algorithms

- actions in the algorithm are considered random events
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Chapter 5. Probabilistic Analysis of Algorithms

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Chapter 5. Probabilistic Analysis of Algorithms

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Chapter 5. Probabilistic Analysis of Algorithms

There are two types of randomized algorithms:

- Las Vegas algorithms
  - always gives answer correctly;
  - running time comes with a probability distribution
    - e.g., for QuickSort, $\Pr(T(n) \leq cn \log n) \geq 0.75$.
- Monte Carlo algorithms
  - on 'NO' instances, 100% accuracy;
  - $\Pr(\text{to answer 'NO' on 'NO' instance}) = 1$
  - on 'YES' instances, $\geq 75\%$ accuracy;
  - $\Pr(\text{to answer 'YES' on 'YES' instance}) \geq 0.75$.

Accuracy 75% can be improved to 99.99% with multiple trials.

Las Vegas algorithms is as powerful as Monte Carlo algorithms, if not more.
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