CSCI 4470/6470 Algorithms, Fall 2019

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Department of Computer Science, UGA

Syllabus: http://cobweb.cs.uga.edu/~cai/courses/algo/2019Fall/

August 14, 2019
An Introduction to the Introduction

There are two ways of constructing a software design: one way is to make it so simple that there are obviously no deficiencies, and the other way is to make it so complicated that there are no obvious deficiencies.

C.A.R. Hoare (1980 Turing Award recipient)

Which way do we take in algorithm design?

Both correctness and efficiency are desired.
An Introduction to the Introduction

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Which way do we take in algorithm design?

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An Introduction to the Introduction

Example 1

Sequence Homology Reveals Functions

- Homology reveals evolution of structure/function

<table>
<thead>
<tr>
<th>Species</th>
<th>Sequence</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>FOS_RAT</td>
<td>MMPSGFNADEYASSRSSASSPAGDLSSYYHSPADSFSMGPVNTQDFCADLSSSANNF 60</td>
<td></td>
</tr>
<tr>
<td>FOS_MOUSE</td>
<td>MMPSGFNADEYASSRSSASSPAGDLSSYYHSPADSFSMGPVNTQDFCADLSSSANNF 60</td>
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<tr>
<td>FOS_CHICK</td>
<td>MMPSGFNADEYASSRSSASSPAGDLSSYYHSPADSFSMGPVNTQDFCADLSSSANNF 60</td>
<td></td>
</tr>
<tr>
<td>FOSB_MOUSE</td>
<td>-MPQAPFGDYDS-GRCS-SPSAESQ--YLSVDFSPGTPTAAASQE-CAGLGEMPGS 54</td>
<td></td>
</tr>
<tr>
<td>FOSB_HUMAN</td>
<td>-MPQAPFGDYDS-GRCS-SPSAESQ--YLSVDFSPGTPTAAASQE-CAGLGEMPGS 54</td>
<td></td>
</tr>
</tbody>
</table>

Consensus

- Homology reveals regulatory structure (E. Coli promoters)

<table>
<thead>
<tr>
<th>Promoter</th>
<th>Sequence</th>
</tr>
</thead>
<tbody>
<tr>
<td>TATCAGCTATCCAATTCTTCTGAGTGCTGGGAGGTCATGCGACACCTGAGAAGAAG</td>
<td></td>
</tr>
</tbody>
</table>
An Introduction to the Introduction

Example 1

called **Multiple Sequence Alignment**.
An Introduction to the Introduction

Example 1

Problem **Multiple Sequence Alignment:**

Input: \( k \) sequences, each of length \( \approx n \);
Output: a biologically most "plausible" alignment
- e.g., for \( k = 2 \) editing through substitutions, insertions, deletions

Possible alignments \( \geq 2^n \), naive methods may not be efficient

Algorithm design goal 1: To craft algorithms as efficient as possible (to develop skills more than naive ideas!)
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Example 1

Problem **Multiple Sequence Alignment:**

Input: \( k \) sequences, each of length \( \approx n \);
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An Introduction to the Introduction

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+----------------+-----------------+-----------------+-----------------+-----------------+-----------------+-----------------+-----------------+-----------------+-----------------+-----------------+-----------------+-----------------+-----------------+
| FOS_CHICK      | MMYQGFAGEYEAPSSRCSSASPAGDSLTYYPSPADSFSSMGSPVNSQDFCTDLAVSSANF 60 |
| FOSB_MOUSE     | -MFQAFPGDYDS-GSRCSS-SPSAESQ--YLSSVDFGSPPATAASQE-CAGLGEMPGSF 54 |
+----------------+-----------------+-----------------+-----------------+-----------------+-----------------+-----------------+-----------------+-----------------+-----------------+-----------------+-----------------+-----------------+-----------------+
```
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FOSB_MOUSE   -MFQAFPGDYDS-GSRCSS-SPAESQ--YLSSVDSFGSPPTAAASE-CAGLEMPGEF 54
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editing through substitutions, insertions, deletions
An Introduction to the Introduction

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FOS_CHICK  MMYQGFAGEYEAPSSRCSSASPAGDSLTTYYPSPADSFSMSGSPVNSQDFCTDIAVSSANF 60
FOSB_MOUSE  -MFQAFPGDYS-GSRCSS-SPSAESQ--YLSSVDSFGSPPTAAASQE-CAGLGMPSGF 54
```

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Algorithm design goal 1:
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To craft algorithms as efficient as possible
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Problem **Multiple Sequence Alignment:**

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- e.g., for \( k = 2 \)

\[
\begin{align*}
\text{FOS}_\text{CHICK} & \quad \text{MMYQFAGEYEAPSSRCSSASPADSFSSMGSPVNSODFTCDLAVSSANF} & \quad 60 \\
\text{FOSB}_\text{MOUSE} & \quad \text{-MFQAPFGDYDS-GSRCSS-SPSAEQ--YLSSVDSFGSPPTAAASOE-CAGLGEMP} & \quad 54
\end{align*}
\]

editing through **substitutions, insertions, deletions**

possible alignments \( \geq 2^n \), naive methods may not be efficient

Algorithm design goal 1:

To craft algorithms as efficient as possible
(to develop skills more than naive ideas!)
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Example 2
An Introduction to the Introduction

Example 2

---

US Open 2016
Men's Singles

<table>
<thead>
<tr>
<th>Round 1</th>
<th>Round 2</th>
<th>Round 3</th>
<th>Round 4</th>
<th>Quarterfinals</th>
<th>Semifinals</th>
</tr>
</thead>
<tbody>
<tr>
<td>3. VESELY, Jan CZE</td>
<td>4. MYNENI, Saiuth IND (Q)</td>
<td>G. PELLA</td>
<td>M. YOUNGNY</td>
<td>6-16</td>
<td>6-1</td>
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<tr>
<td>5. FRATANGELO, Bem USA (W)</td>
<td>6. PELLA, Guido ARG</td>
<td>M. YOUNGNY</td>
<td>WALKOVER</td>
<td>6-16</td>
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<tr>
<td>7. YOUNGNY, Mikhail RUS</td>
<td>8. KUZAN, Martin SVK (Q)</td>
<td>J. INSEIER [20]</td>
<td>9. TSONGA, Jo-Wifted FRA (Q)</td>
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<tr>
<td>11. THOMPSON, Jordan AUS</td>
<td>12. DARCIS, Steve BEL (Q)</td>
<td>F. ESCOBEDO</td>
<td>K. EDMUND</td>
<td>6-16</td>
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<tr>
<td>13. JACO, Lukas SVK</td>
<td>14. ESBO, Ernesto USA (W)</td>
<td>T. BIANCHI</td>
<td>X. EDMUND</td>
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<td>15. EDMUND, Kyle GB</td>
<td>16. GHESQUIER, Richard FRA (T)</td>
<td>17. TSONGA, Jo-Wifted FRA</td>
<td>18. ANDREISI, Guido ARG (Q)</td>
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<td>6-16</td>
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<tr>
<td>19. DUCKWORTH, James AUS</td>
<td>20. HAAE, Robin NED</td>
<td>21. POSPISIL, Vasek CAN</td>
<td>22. KOVALIK, Josef SVK (Q)</td>
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<tr>
<td>23. NISHIOKA, Yshimoto JPN (L)</td>
<td>24. ANDERSON, Kevin RSA (Q)</td>
<td>J. SOCK [26]</td>
<td>V. POSPISIL</td>
<td>6-16</td>
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<td>25. SCK, Jack USA</td>
<td>26. FRITZ, Taylor USA</td>
<td>27. ZVEREV, Mischa GER (Q)</td>
<td>28. HERBERT, Pierre-Huques FRA</td>
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<td>29. STACTOVI, Semyi UKR</td>
<td>30. DJOKOVIC, Novak SRB</td>
<td>31. TSONGA, Jo-Wifted FRA</td>
<td>32. CILIC, Marin CRO</td>
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<td>33. NADAL, Rafael ESP</td>
<td>34. ISTOMIN, Denis UZ</td>
<td>35. ROBERT, Stephane FRA</td>
<td>36. NADAL [4]</td>
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<td>37. BELZI, Thomas BRA</td>
<td>38. ZVEREV, Mischa GER</td>
<td>39. BENNETTE, Julien FRA</td>
<td>40. RAMOS-VINIALAS, Albert ESP</td>
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<td>41. POULIE, Lucas FRA</td>
<td>42. KUKUSHKIN, Mikhail KAZ</td>
<td>43. CLEAR, Guilherme BRA</td>
<td>44. CHIUVINELLI, Marc SUL (Q)</td>
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<td>45. BAKER, Brian USA</td>
<td>46. DELBONIS, Federico ARG</td>
<td>47. GARCIA-LOPEZ, Guillermo ESP</td>
<td>48. BAUSTIA AGUT, Roberto ESP (Q)</td>
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<td>49. MONFILS, Gael FRA</td>
<td>50. MULLER, Gilles LUX</td>
<td>51. SATRAW, Jan CZE (Q)</td>
<td>52. MONCORN, Mackenzie USA (W)</td>
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<td>53. FUCSIOVIC, Marin HUN</td>
<td>54. ALMARGO, Nicolas ESP (P)</td>
<td>55. SBA, Dudi ISR</td>
<td>56. CUEVAS, Pablo URO (Q)</td>
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<tr>
<td>57. PAIRE, Benoit FRA [26]</td>
<td>58. LAJOVIC, Dusan SRB</td>
<td>59. BAGHDATIS, Marcos CYP (Q)</td>
<td>60. BAGNIS, Facundo ARG</td>
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<tr>
<td>61. HARRISON, Ryan USA (Q)</td>
<td>62. HARRISON, Ryan USA</td>
<td>63. BROWN, Dustin GBR</td>
<td>64. RAONIC, Milos CAN (Q)</td>
<td>6-16</td>
<td>6-16</td>
</tr>
</tbody>
</table>

Champion: WAHRIN, Stan SUI
8-7(1) 6-4 1-6 7-6(1)
An Introduction to the Introduction
An Introduction to the Introduction

128 players in total, single elimination

How many matches are played to determine a champion?

Is it possible to just play fewer matches?
• 128 players in total, single elimination
● 128 players in total, single elimination
● How many matches are played to determine a champion?
• 128 players in total, single elimination
• How many matches are played to determine a champion? 127
An Introduction to the Introduction

- 128 players in total, single elimination
- How many matches are played to determine a champion? 127
- Is it possible to just play fewer matches?
An Introduction to the Introduction

• 7 matches is sufficient to determine the champion. What does it mean? The specific arrangement of 7 matches finds the champion.

• Are 7 matches necessary to determine the champion? What does that mean? \( \leq 6 \) matches, regardless of arrangement, cannot find a champion.

![Tournament Bracket]

<table>
<thead>
<tr>
<th>First Round</th>
<th>Semi-finals</th>
<th>Final</th>
<th>Winner</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 Abbott</td>
<td>Abbott</td>
<td></td>
<td>Abbott</td>
</tr>
<tr>
<td>2 Bolton</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3 Cross</td>
<td>Cross</td>
<td></td>
<td>Abbott</td>
</tr>
<tr>
<td>4 Dent</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5 Elphick</td>
<td>Flower</td>
<td></td>
<td>Flower</td>
</tr>
<tr>
<td>6 Flower</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7 Gough</td>
<td>Gough</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8 Handy</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
• 7 matches is *sufficient* to determine the champion.
• 7 matches is **sufficient** to determine the champion.

**What does it mean?**
• 7 matches is sufficient to determine the champion.

What does it mean?
The specific arrangement of 7 matches finds the champion.
An Introduction to the Introduction

- 7 matches is **sufficient** to determine the champion.

**What does it mean?**

The specific arrangement of 7 matches finds the champion.

- Are 7 matches **necessary** to determine the champion?
An Introduction to the Introduction

• 7 matches is **sufficient** to determine the champion.

  **What does it mean?**
  The specific arrangement of 7 matches finds the champion.

• Are 7 matches **necessary** to determine the champion?

  **What does that mean?**
An Introduction to the Introduction

- 7 matches is **sufficient** to determine the champion.

  **What does it mean?**
  The specific arrangement of 7 matches finds the champion.

- Are 7 matches **necessary** to determine the champion?

  **What does that mean?**
  $\leq 6$ matches, regardless arrangement, cannot find a champion.
Problem \textbf{FIND MAX:} 

Input: a set of \( n \) numbers \( x_1, \ldots, x_n \); 
Output: \( x_i \), for some \( i \) (\( 1 \leq i \leq n \)), such that for \( j = 1, 2, \ldots, n \), \( x_i \geq x_j \). 

The problem can be solved with \( n - 1 \) comparisons. There is a way to find the maximum using \( n - 1 \) comparisons. 

Regardless how to compare these numbers.
Problem **FIND MAX:**

Input: a set of \( n \) numbers \( x_1, \ldots, x_n \);
Problem \textsc{Find Max}:

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- The problem can be solved with \( n - 1 \) comparisons. There is a way to find the maximum using \( n - 1 \) comparisons.
- The problem requires at least \( n - 1 \) comparisons.
An Introduction to the Introduction

Problem **Find Max:**

- **Input:** a set of \(n\) numbers \(x_1, \ldots, x_n\);
- **Output:** \(x_i\), for some \(i\) \((1 \leq i \leq n)\), such that for \(j = 1, 2, \ldots, n\),

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- The problem can be solved with \(n - 1\) comparisons.
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- $n - 1$ comparisons are **sufficient** to find the maximum called an upper bound;
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Problem **Find Max**:

- **Input:** a set of $n$ numbers $x_1, \ldots, x_n$;
- **Output:** $x_i$, for some $i$ (1 ≤ $i$ ≤ $n$), such that for $j = 1, 2, \ldots, n$,
  \[ x_i \geq x_j \]

- **$n - 1$ comparisons are sufficient** to find the maximum called an upper bound;

You only need to show an algorithm to show an upper bound;

- **$n - 1$ comparisons are necessary** to find the maximum called a lower bound;
Problem \textbf{FIND MAX}:

Input: a set of \( n \) numbers \( x_1, \ldots, x_n \);
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- \( n - 1 \) comparisons are \textbf{sufficient} to find the maximum called an \textbf{upper bound};

You only need to show an \textbf{algorithm} to show an upper bound;

- \( n - 1 \) comparisons are \textbf{necessary} to find the maximum called a \textbf{lower bound};
Problem \textbf{Find Max}:

Input: a set of \( n \) numbers \( x_1, \ldots, x_n \);
Output: \( x_i \), for some \( i \) (\( 1 \leq i \leq n \)), such that for \( j = 1, 2, \ldots, n \),

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\begin{itemize}
  \item \( n - 1 \) comparisons are \textit{sufficient} to find the maximum called an upper bound;
  
  You only need to show an \textit{algorithm} to show an upper bound;
  
  \item \( n - 1 \) comparisons are \textit{necessary} to find the maximum called a lower bound;
  
  you need a \textit{(mathematical) proof} to show a lower bound.
\end{itemize}
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Problem **Find Max**:

- Input: a set of \( n \) numbers \( x_1, \ldots, x_n \);
- Output: \( x_i \), for some \( i \) (\( 1 \leq i \leq n \)), such that for \( j = 1, 2, \ldots, n \),

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• Is \( n + 3 \) an upper bound for problem **Find Max**?
Problem \texttt{Find Max}:

Input: a set of \( n \) numbers \( x_1, \ldots, x_n \);
Output: \( x_i \), for some \( i \) (\( 1 \leq i \leq n \)), such that for \( j = 1, 2, \ldots, n \),
\[ x_i \geq x_j \]

- Is \( n + 3 \) an upper bound for problem \texttt{Find Max}?
- Is \( n + 3 \) a lower bound for the problem?
Problem \texttt{Find Max}:

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- Is \( n + 3 \) an upper bound for problem \texttt{Find Max}?
- Is \( n + 3 \) a lower bound for the problem?
- Is \( n - 3 \) an upper bound? How about a lower bound?
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Problem Find Max:

Input: a set of $n$ numbers $x_1, \ldots, x_n$;
Output: $x_i$, for some $i$ ($1 \leq i \leq n$), such that for $j = 1, 2, \ldots, n$,

$$x_i \geq x_j$$

• Is $n + 3$ an upper bound for problem Find Max?
• Is $n + 3$ a lower bound for the problem?
• Is $n - 3$ an upper bound? How about a lower bound?

Algorithm design goal 2:
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Problem **Find Max**:

Input: a set of $n$ numbers $x_1, \ldots, x_n$;
Output: $x_i$, for some $i$ ($1 \leq i \leq n$), such that for $j = 1, 2, \ldots, n$,

$$x_i \geq x_j$$

- Is $n + 3$ an upper bound for problem **Find Max**?
- Is $n + 3$ a lower bound for the problem?
- Is $n - 3$ an upper bound? How about a lower bound?

**Algorithm design goal 2:**

To understand challenges posed by a problem to be solved
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Problem \texttt{Find Max}:

Input: a set of \( n \) numbers \( x_1, \ldots, x_n \);
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- Is \( n + 3 \) an upper bound for problem \texttt{Find Max}?
- Is \( n + 3 \) a lower bound for the problem?
- Is \( n - 3 \) an upper bound? How about a lower bound?

Algorithm design goal 2:

To understand challenges posed by a problem to be solved
(need to develop capability to observe and analyze!)
Example 3 Toss coin over the phone
An Introduction to the Introduction

Example 3 Toss coin over the phone

- Two distant cities toss coin to decide who to a football match;
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Example 3 Toss coin over the phone

- Two distant cities toss coin to decide who to a football match;
- Lack of visual communication tools, coin toss is done via phone calls;
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Example 3 Toss coin over the phone

- Two distant cities toss coin to decide who to a football match;
- Lack of visual communication tools, coin toss is done via phone calls;
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Example 3 Toss coin over the phone

- Two distant cities toss coin to decide who to a football match;
- Lack of visual communication tools, coin toss is done via phone calls;

- How would it work?
They decided the following protocol:

A told B a huge Boolean circuit through the phone; along with a binary string $Y$, the output produced from some binary string input $X$ (but the input string $X$ is not given to B); then B guesses if the number of 0's in $X$ is odd or even.
They decided the following protocol:

- A told B a huge Boolean circuit through the phone;
They decided the following protocol:

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They decided the following protocol:

- A told B a huge Boolean circuit through the phone; along with a binary string $Y$, the output produced from some binary string input $X$
  (but the input string $X$ is not given to $B$);
- Then B guesses if the number of 0’s in $X$ is odd or even;
An Introduction to the Introduction

- From $Y$, it would be very difficult for $B$ to figure out which $X$ is used to produce $Y$ (one-way function, an intractable problem);
- So the best $B$ can do is to randomly guess (odd/even), which has the same effect as tossing a coin.

Algorithm design goal 3: To understand inherent difficulties of some prominent problems (need to study math underlying the intractability)
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An Introduction to the Introduction

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An Introduction to the Introduction

To design good algorithms for computational problems,
An Introduction to the Introduction

To design good algorithms for computational problems,
• need to be familiar with techniques for complexity analysis;
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To design good algorithms for computational problems,
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• need to master skills for efficient algorithm design;
An Introduction to the Introduction

To design good algorithms for computational problems,
• need to be familiar with techniques for complexity analysis;
• need to master skills for efficient algorithm design;
• need to be able to know if an algorithm is optimal (how?).
The Introduction

What is this course about (and why is it needed)?

• about basic yet indispensable skills for problem solving
• algorithm design leads to writing code, i.e., creative thinking without programming languages

How different is this course from other algorithm courses?

• design technique-oriented, not application-oriented
• emphasis on guaranteed performance (typically in efficiency)

Goals to achieve

• to learn to measure performance of algorithms
• to master some fundamental algorithmic techniques
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Part I. Foundations
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- Chapter 1. The role of algorithms in computing
- Chapter 2. Getting started
- Chapter 3. Growth of functions
- Chapter 4. Solving recurrences
- Chapter 5. Probabilistic analysis and randomized algorithms
Part I. Foundations

The theme of the course
Part I. Foundations

The theme of the course

- Goal: learning techniques to design efficient algorithms
Part I. Foundations

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• Goal: learning techniques to design efficient algorithms
• mean: through developing skills to analyze algorithms
Part I. Foundations

The theme of the course

• Goal: learning techniques to design efficient algorithms
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Design and analysis of algorithms are closely related.
Part I. Foundations

Example: the Fibonacci sequence.

\[ f(n) = \begin{cases} 
  f(n - 1) + f(n - 2) & \text{if } n \geq 3 \\
  1, & \text{otherwise}
\end{cases} \]
Example: the Fibonacci sequence.

\[
 f(n) = \begin{cases} 
 f(n - 1) + f(n - 2) & \text{if } n \geq 3 \\
 1, & \text{otherwise}
\end{cases}
\]

That is:

\[
\begin{array}{cccccccccc}
 n & : & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \ldots \\
 f(n) & : & 1 & 1 & 2 & 3 & 5 & 8 & 13 & 21 & 34 & \ldots 
\end{array}
\]
Problem 1: Computing the $n$th Fibonacci number:
Part I. Foundations

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**Input:** $n \geq 1$;

**Output:** the $n$th number in the Fibonacci sequence.
Part I. Foundations

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- *recursive*: task decomposition, top-down, recursive calls;
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Two different types of algorithms: *recursive* and *iterative*

- **recursive:** task decomposition, top-down, recursive calls;
- **iterative:** more tightly coupled tasks, bottom-up approaches;
Rec-Fibonacci(n)

if n = 1 or n = 2, return 1;
else $T_1 = \text{Rec-Fibonacci}(n-1)$;
$T_2 = \text{Rec-Fibonacci}(n-2)$;
return ($T_1 + T_2$);

But how efficient is it? Or how slow is it?
Its execution is via a run-time stack:
- suitable for execution of subroutines
- but oblivious, cannot remember any completed subroutine.
Part I. Foundations

Rec-Fibonacci$(n)$

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Part I. Foundations

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Part I. Foundations

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Part I. Foundations

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**Part I. Foundations**

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Part I. Foundations

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Repeated computations everywhere!
Part I. Foundations

The size of tree is the number of recursive calls;
The size of tree is the number of recursive calls; How big is it?
Part I. Foundations

\[ \text{small triangle} \leq \text{size of tree} \leq \text{large triangle} \]

\[ 2^n \leq \text{size of tree} \leq 2^{n+1} \]
Part I. Foundations

small triangle \(\leq\) size of tree \(\leq\) large triangle
Part I. Foundations

\[
\text{small triangle} \leq \text{size of tree} \leq \text{large triangle}
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roughly: \( 2^{\frac{n}{2}} \leq \text{size of tree} \leq 2^n \)
Iterative-Fibonacci\((n)\)
Iterative-Fibonacci\( (n) \)

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\text{if } n = 1 \text{ or } n = 2 \text{ return } (1);
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Part I. Foundations

Iterative-Fibonacci($n$)

if $n = 1$ or $n = 2$ return (1);
else
    $M[1] = 1$, $M[2] = 1$

How fast is it?

$T_{total} = \max\{T_{if}, T_{else}\}$

where

$T_{if} = c_1$,

$T_{else} = c_2 + T_{for}$

$T_{total} \leq c_1 + c_2 + d(n - 2)$

Iterative-Fibonacci($n$) is a simple dynamic programming algorithm.
Iterative-Fibonacci\( (n) \)

if \( n = 1 \) or \( n = 2 \) return \( (1) \);
else

\[ M[1] = 1, \ M[2] = 1 \]

for \( i = 3 \) to \( n \) do

\[ M[i] = M[i - 1] + M[i - 2] \]
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Part I. Foundations

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$$T_{total} = \max\{T_{if}, T_{else}\}$$

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$$T_{total} \leq c_1$$
Part I. Foundations

**Iterative-Fibonacci** \(n\)

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\[
\text{else }
\]

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M[1] = 1, \ M[2] = 1
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\[
\text{for } i = 3 \text{ to } n \text{ do}
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M[i] = M[i - 1] + M[i - 2]
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\[
\text{return } (M[n])
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How fast is it?

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T_{total} = \max\{T_{if}, T_{else}\}
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where \(T_{if} = c_1, \ T_{else} = c_2 + T_{for} = c_2 + d \times (n - 2)\)

\[
T_{total} \leq c_1 + c_2 + d(n - 2),
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Iterative-Fibonacci(n)

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$$T_{total} \leq c_1 + c_2 + d(n - 2), \text{ a linear function in } n$$
Part I. Foundations

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**Iterative-Fibonacci**$(n)$ is a simple **dynamic programming** algorithm.
Part I. Foundations

Actually, all the algorithm \texttt{ITERATIVE-FIBONACCI}(n) does is:
Part I. Foundations

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To fill out a table of size $n$, with
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To fill out a table of size $n$, with

- each entry being filled out exactly once, and
Part I. Foundations

Actually, all the algorithm \texttt{Iterative-Fibonacci}(n) does is:

To fill out a table of size $n$, with

\begin{itemize}
  \item each entry being filled out exactly once, and
  \item filling out an entry takes a constant, say $c$ steps.
\end{itemize}
Part 1. Foundations

Actually, all the algorithm \textsc{Iterative-Fibonacci}(n) does is:

To fill out a table of size \( n \), with

- each entry being filled out exactly once, and
- filling out an entry takes a constant, say \( c \) steps.

So the total time \textsc{Iterative-Fibonacci}(n) uses is
Part I. Foundations

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To fill out a table of size \( n \), with

- each entry being filled out exactly once, and
- filling out an entry takes a constant, say \( c \) steps.

So the total time \textsc{Iterative-Fibonacci}(n) uses is

\[
T(n) = c \times n
\]
Chapter 1. The role of algorithms in computing
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What is an Algorithm: a well-defined, finite procedure that takes an input and produces an output.
Chapter 1. The role of algorithms in computing

What is an Algorithm: a well-defined, finite procedure that takes an input and produces an output.

Example 2: An algorithm skeleton;

Algorithm Maximum;

INPUT: list $X = \{a_1, \cdots, a_n\}$;

Body that is a series of instructions;

OUTPUT: $y$, the maximum of $a_1, \cdots, a_n$. 
Alternatively, an algorithm specifies a finite process to compute a function or a relation.
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e.g., algorithm \textsc{Maximum} computes the following function:

\[ f_{\text{max}}(X) = y, \text{ where } \forall a \in X, y \geq a, \]
Alternatively, an algorithm specifies a finite process to compute a function or a relation.

e.g., algorithm \textsc{Maximum} computes the following function:

$$f_{\text{max}}(X) = y, \text{ where } \forall a \in X, y \geq a,$$

For some problems, the functions computed are predicates, i.e., output $y \in \{\text{TRUE, FALSE}\}$.
Chapter 1. The Role of Algorithms in Computing

Algorithms as a technology to resolve efficiency issues
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Algorithms as a technology to resolve efficiency issues

Efficient use of computer resources such as time and space is necessary.
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Two typical situations:
Chapter 1. The Role of Algorithms in Computing

Algorithms as a technology to resolve efficiency issues

Efficient use of computer resources such as time and space is necessary.

Two typical situations:

- very large input data for “easy” problems;
Chapter 1. The Role of Algorithms in Computing

Algorithms as a technology to resolve efficiency issues

Efficient use of computer resources such as time and space is necessary.

Two typical situations:

- very large input data for “easy” problems;
- moderately large input data for “hard” problems.
Chapter 2. Getting Started

The Sorting Problem

Input: \( n \) numbers \( \langle a_1, \cdots, a_n \rangle \);
Output: a reordering \( \langle a'_1, \cdots, a'_n \rangle \) of the input such that \( a'_1 \leq a'_2 \leq \cdots \leq a'_n \).

Insertion Sort

idea: an iterative process to produce a new list such that at each iteration, the new list consists of two sublists,
• a sorted sublist followed by an unsorted sublist, and
• the leftmost number of the unsorted is being inserted into the sorted.
As the process goes, the sorted sublist gets longer, the unsorted sublist gets shorter, until the unsorted becomes empty.
Chapter 2. Getting Started

Chapter 2. Getting started

The Sorting Problem

**INPUT:** $n$ numbers $\langle a_1, \cdots, a_n \rangle$;
Chapter 2. Getting Started

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**Input:** \( n \) numbers \( \langle a_1, \cdots, a_n \rangle \);

**Output:** a reordering \( \langle a'_1, \cdots, a'_n \rangle \) of the input such that
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Chapter 2. Getting Started

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Insertion Sort

**Idea:** an iterative process to produce a new list such that
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Chapter 2. Getting Started

Algorithm INSERTION-SORT(A)

1. for $j = 2$ to length[A]
do
2. key = A[j]
4. $i = j - 1$
5. while $i > 0$ and $A[i] > key$
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Analysis of the algorithm:
• (correctness proof): to show that the algorithm is as desired;
• (efficiency proof): to show a guaranteed efficiency of the algorithm
Chapter 2. Getting Started

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Algorithm INSERTION-SORT($A$)

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Chapter 2. Getting Started

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2 \hspace{1em} key \hspace{0.5em} = \hspace{0.5em} A[j]
3 \hspace{1em} \{\text{Insert} \hspace{0.5em} A[j] \hspace{0.5em} \text{into sorted} \hspace{0.5em} A[1..j-1]\}
Chapter 2. Getting Started

Algorithm **INSERTION-SORT**\( (A) \)

1. **for** \( j = 2 \) **to** length\( [A] \) **do**
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Algorithm Insertion-Sort($A$)

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Chapter 2. Getting Started

Algorithm **INSERTION-SORT**(\(A\))

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6\hspace{1em} \hspace{1em} \textbf{do} \hspace{0.5em} A[i + 1] \hspace{0.5em} = \hspace{0.5em} A[i] \\
7\hspace{1em} \hspace{1em} i \hspace{0.5em} = \hspace{0.5em} i - 1 \\
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Chapter 2. Getting Started

Correctness proof: this is to prove
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Chapter 2. Getting Started

**Correctness proof:** this is to prove

the pre-condition (condition for the input)

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If the algorithm consists of sequential blocks of instructions,

the task is to prove the correct transformation by each block.

This means we need to prove that every sequential statement
in the algorithm transforms the given pre-condition to
the given post-condition.
Chapter 2. Getting Started

The most difficult task is to do this for a loop statement.
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The most difficult task is to do this for a loop statement. Finding loop invariant becomes necessary and sufficient. In Insertion-Sort, the loop invariant is

at each iteration, the sublist $A[1..j - 1]$ consists of the elements originally in the positions $[1..j-1]$ but in sorted order.

However, finding loop invariants is difficult!
Chapter 2. Getting Started

Efficiency analysis: This is to show that
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- For all cases of input, the needed computation resources for the algorithm.
- resources can be CPU time and memory space used in the computation.
- however, the unit measured is not real time or memory unit.
Chapter 2. Getting Started

Time of an algorithm $A(x)$

Input Instances

$x$
Chapter 2. Getting Started

Time of an algorithm $A(x)$

Input Instances   worst case
Chapter 2. Getting Started

Time of an algorithm $A(x)$

**Upper bound** for algorithm $A$, bounding all cases of instances

Input Instances  worst case
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Resource measurement based on

• random-access machine (RAM)
• counting primitive operations: addition, subtraction, floor, ceiling, multiplication, jump, memory movement, these operations differ in time by a constant multiplicative factor.
• speed between different machines: a constant multiplicative factor.
Chapter 2. Getting Started

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Chapter 2. Getting Started

Analysis of

Algorithm \texttt{INSERTION-SORT}(A)
Chapter 2. Getting Started

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Algorithm `INSERTION-SORT(A)`

1. `for j = 2 to length[A] do`
2. `key = A[j]`
3. `{Insert A[j] into sorted A[1..j − 1]}`
4. `i = j − 1`
5. `while i > 0 and A[i] > key`
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8. `A[i + 1] = key`

Assume $t_j$ to be the number of times while is executed for every $j$.

$$T(n) = c_1 n + c_2 (n − 1) + c_4 (n − 1) + c_5 n \sum_{j=2}^{n} t_j + c_6 n \sum_{j=2}^{n} (t_j − 1) + c_7 n \sum_{j=2}^{n} (t_j − 1) + c_8 (n − 1)$$
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Chapter 2. Getting Started

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Chapter 2. Getting Started

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Chapter 2. Getting Started

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\[ T(n) \leq a \frac{n}{2} (n + 1) + bn + c - a \leq xn^2 + yn + z \]

for some constants \( x, y, z \).
Chapter 2. Getting Started

So we have proved:

\[ T(n) \leq x_n^2 + y_n + z \]

for some constants \( x, y, z \leq \) means in all cases for which \( \leq \) holds; \( x_n^2 + y_n + z \) is a complexity upper bound for \( T(n) \).
Chapter 2. Getting Started

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Chapter 2. Getting Started

Important complexity issues:

1. size of input $n$: the number of bits encoding input $x$, i.e., $n = \|x\|$. It is inaccurate for $n$ to represent the number of items in the input.

Consider to sort 4 items $\langle x_1, x_2, x_3, x_4 \rangle$ of values in the scale of $2^N$, for some very large $N$.

• If $n$ is the number of items, $n = 4$, then any sorting algorithm would run in constant time.

• However, since $x_1, x_2, x_3, x_4$ are of very large values, a single comparison $x_1 \leq x_2$ would need a time proportional to $N$.

Hence, if $n = \|\langle x_1, x_2, x_3, x_4 \rangle\|$, then $n \approx N$.

To sort the 4 items, a constant number of comparisons is needed, each taking a time linear in $N$ (i.e., total time is linear in $n$).
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Chapter 2. Getting Started

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Chapter 2. Getting Started

2. Running time

\[ T(n) \]: the number of primitive operations executed,

worst-case running time: the running time upper bound for all inputs.

order of growth: \[ T(n) = an^2 + bn + c \] grows the same rate as \[ an^2 \] (if \( a > 0 \)).
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Chapter 3. Growth of Functions

Big-O: \(O(n^2)\) contains all functions of growth rate \(\leq cn^2\).

So for function \(T(n) = an^2 + bn + c\), \(T(n) \in O(n^2)\), but written as \(T(n) = O(n^2)\).

In general, \(O(g(n)) = \{f(n): \exists c > 0, k > 0 \text{ such that } 0 \leq f(n) \leq cg(n), \text{ for all } n \geq k\}\).
Chapter 3. Growth of Functions

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Chapter 3. Growth of Functions

For example: the following functions are all of the order of $O(n^2)$:

1. $3n^2$
2. $5.3n^2 + 6n \log_2 n + 90$
3. $0.001n^2 - 200n - 5000$
4. $3n \log_2 n^2 + 6n$
5. $\sqrt{n} - 20 \log_2 n + 56$
6. $345$

But the following are not:

8. $3n^2 \log_2 n - 400n$
9. $n^2$.001
Chapter 3. Growth of Functions

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Chapter 3. Growth of Functions

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Chapter 3. Growth of Functions

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But the following are not:

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Chapter 4. Solving Recurrences

Example: binary search algorithm

Algorithm Binary Search \((A, p, r, \text{key})\)

1. if \(p > r\) return \((\text{null})\)
2. else
3. \(q = \lfloor \frac{p + r}{2} \rfloor\)
4. if \(A[q] = \text{key}\) return \((k)\)
5. else
6. if \(A[q] > \text{key}\) Binary Search \((A, p, q - 1, \text{key})\)
7. else Binary Search \((A, q + 1, r, \text{key})\)

Let \(T(n)\) be the worst case time complexity for Binary Search, where parameter \(n = r - p + 1\).

The time \(T(n)\) can be upper-bounded with the recurrence, 

\[
T(n) \leq T\left(\frac{n}{2}\right) + c
\]

and \(T(0) \leq c\) where \(c > 0\) is a constant.

Note: we can also set base case \(T(1) \leq c\).
Example: binary search algorithm

Algorithm Binary Search \((A, p, r, \text{key})\)

1. if \(p > r\) return (NULL)
2. else
3. \(q = \left\lceil \frac{p + r}{2} \right\rceil\)
4. if \(A[q] = \text{key}\) return \((k)\)
5. else
6. if \(A[q] > \text{key}\) Binary Search \((A, p, q - 1, \text{key})\)
7. else
8. Binary Search \((A, q + 1, r, \text{key})\)
Chapter 4. Solving Recurrences

Example: binary search algorithm

Algorithm \textbf{Binary Search}(A, p, r, key)

1. \textbf{if} \ p > r \ \textbf{return} \ (\text{NULL})
2. \textbf{else}
3. \hspace{1em} q = \left\lfloor \frac{p + r}{2} \right\rfloor
4. \hspace{1em} \textbf{if} \ A[q] = key \ \textbf{return} \ (k)
5. \hspace{1em} \textbf{else}
6. \hspace{2em} \textbf{if} \ A[q] > key \ \textbf{Binary Search} \ (A, p, q - 1, key)
7. \hspace{2em} \textbf{else}
8. \hspace{3em} \textbf{Binary Search} \ (A, q + 1, r, key)

Let $T(n)$ be the worst case time complexity for \textbf{Binary Search}, where parameter $n = r - p + 1$. 
Chapter 4. Solving Recurrences

Example: binary search algorithm

Algorithm Binary Search($A, p, r, key$)

1. if $p > r$ return (NULL)
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Let $T(n)$ be the worst case time complexity for Binary Search, where parameter $n = r - p + 1$.

The time $T(n)$ can be upper-bounded with the recurrence,

$$T(n) \leq T\left(\frac{n}{2}\right) + c \text{ and } T(0) \leq c$$

where $c > 0$ is a constant.
Chapter 4. Solving Recurrences

Example: binary search algorithm

Algorithm \textsc{Binary Search}(A, p, r, key)

1. \hspace{1em} \textbf{if} \; p > r \; \textbf{return} \; (\text{NULL})
2. \hspace{1em} \textbf{else}
3. \hspace{2em} \textbf{q} = \left\lfloor \frac{p+r}{2} \right\rfloor
4. \hspace{2em} \textbf{if} \; A[q] = key \; \textbf{return} \; (k)
5. \hspace{2em} \textbf{else}
6. \hspace{3em} \textbf{if} \; A[q] > key \; \textsc{Binary Search} \; (A, p, q - 1, key)
7. \hspace{3em} \textbf{else}
8. \hspace{4em} \textsc{Binary Search} \; (A, q + 1, r, key)

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where $c > 0$ is a constant.

\textbf{Note}: we can also set base case $T(1) \leq c$. 
Chapter 4. Solving Recurrences

We have recurrence $T(n) \leq T(n/2) + c$ and $T(1) \leq c$ for the Binary Search algorithm. The recurrence can be resolved by unfolding the recursive terms. The process to unfold the recurrence $T(n)$ can be:

1. simple, straightforward
2. induction based (called substitution method)
3. graph based (called recursive tree method)
Chapter 4. Solving Recurrences

We have recurrence

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The recurrence can be resolved by unfolding the recursive terms. The process to unfold the recurrence \( T(n) \) can be:

1. simple, straightforward
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3. graph based (called recursive tree method).
Chapter 4. Solving Recurrences

1. Simple, straightforward unfolding:

\[ T(n) \leq T(n^2) + c \]

\[ T(n) \leq T(n^2) + cT(n^2) \leq T(n^2^2) + cT(n^2^2) \leq T(n^2^3) + cT(n^2^3) \]

\[ \quad \vdots \]

\[ T(n^{2^h}) \leq T(n^{2^h+1}) + c \]

where

\[ n^{2^{h+1}} = 1, \quad 2^{h+1} = n, \quad h + 1 = \log_2 n \]

proved this last equation!
Chapter 4. Solving Recurrences

1. Simple, straightforward unfolding:
Chapter 4. Solving Recurrences

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\[ T(n) \leq T\left(\frac{n}{2}\right) + c \] as a template;
Chapter 4. Solving Recurrences

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......
Chapter 4. Solving Recurrences

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   \begin{align*}
   T(n) &\leq T\left(\frac{n}{2}\right) + c \\
   T\left(\frac{n}{2}\right) &\leq T\left(\frac{n}{2^2}\right) + c \\
   T\left(\frac{n}{2^2}\right) &\leq T\left(\frac{n}{2^3}\right) + c \\
   \cdots \\
   T\left(\frac{n}{2^h}\right) &\leq T\left(\frac{n}{2^{h+1}}\right) + c 
   \end{align*}
   \]

   where \( \frac{n}{2^{h+1}} = 1 \), \( 2^{h+1} = n \), \( h + 1 = \log_2 n \)
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T\left(\frac{n}{2^2}\right) \leq T\left(\frac{n}{2^3}\right) + c \\
\vdots \\
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$$

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Chapter 4. Solving Recurrences

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\ldots \\
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\]

where $\frac{n}{2^{h+1}} = 1$, $2^{h+1} = n$, $h + 1 = \log_2 n$

\[
T(n) \leq T\left(\frac{n}{2^{h+1}}\right) + (h + 1)c
\]

there are $h + 1$ inequalities
Chapter 4. Solving Recurrences

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& \quad \cdots \\
T\left(\frac{n}{2^h}\right) & \leq T\left(\frac{n}{2^{h+1}}\right) + c \\
& \quad \text{where } \frac{n}{2^{h+1}} = 1, \ 2^{h+1} = n, \ h + 1 = \log_2 n
\end{align*}
\]

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T(n) \leq T\left(\frac{n}{2^{h+1}}\right) + (h + 1)c
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\[
\leq c + c \log_2 n
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Chapter 4. Solving Recurrences

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\ldots \ldots \\
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\]

\[
T(n) \leq T\left(\frac{n}{2^{h+1}}\right) + (h + 1)c \
\leq c + c \log_2 n \\
= O(\log_2 n)
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1. Simple, straightforward unfolding:

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\[\ldots\]

$$T\left(\frac{n}{2^h}\right) \leq T\left(\frac{n}{2^{h+1}}\right) + c \quad \text{where} \quad \frac{n}{2^{h+1}} = 1, \quad 2^{h+1} = n, \quad h + 1 = \log_2 n$$

\[\text{proved this last equation!}\]
Chapter 4. Solving Recurrences

Another example:

Algorithm Merge Sort \((A, p, r)\)

1. if \(p < r\) then
2. \(q = \lfloor \frac{p + r}{2} \rfloor\)
3. Merge Sort \((A, p, q)\)
4. Merge Sort \((A, q + 1, r)\)
5. Merging2Lists \((A, p, q, r)\)

Analysis of the algorithm.

- Assume \(n = r - p + 1\), a power of 2; also assume \(T(n)\) is time for Merge Sort \((A, p, r)\).

- \(t_1, t_2 = c\)
- \(t_3 = t_4 = T(n/2)\)
- \(t_5 \leq n\) (why?)

\[ T(n) = t_1 + t_2 + t_3 + t_4 + t_5 \leq 2T(n/2) + n + c \]

Base case: \(T(1) = c\).
Another example:

Algorithm Merge Sort \((A, p, r)\)

1. if \(p < r\)
2. then \(q = \lfloor p + r \rfloor 2\)
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4. Merge Sort \((A, q + 1, r)\)
5. Merging Lists \((A, p, q, r)\)

Analysis of the algorithm.

• Assume \(n = r - p + 1\) is a power of 2; also assume \(T(n)\) is time for Merge Sort \((A, p, r)\). Then

\[ t_1, t_2 = c \]
\[ t_3 = t_4 = T(n/2) \]
\[ t_5 \leq n \text{ (why?)} \]

\[ T(n) = t_1 + t_2 + t_3 + t_4 + t_5 \leq 2T(n/2) + n + c \]

Base case: \(T(1) = c\).
Chapter 4. Solving Recurrences

Another example:

Algorithm \textsc{Merge Sort}(A, p, r)

1. \textbf{if} \ p < r
2. \textbf{then} \ q = \lfloor \frac{p+r}{2} \rfloor
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Analysis of the algorithm.

- Assume \( n = r - p + 1 \), a power of 2;
  also assume \( T(n) \) is time for \textsc{Merge Sort}(A, p, r). Then
Chapter 4. Solving Recurrences

Another example:

Algorithm Merge Sort($A, p, r$)

1. if $p < r$
2. then $q = \lfloor \frac{p+r}{2} \rfloor$
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4. Merge Sort($A, q+1, r$)
5. Merging2Lists($A, p, q, r$)

Analysis of the algorithm.

- Assume $n = r - p + 1$, a power of 2;
  also assume $T(n)$ is time for Merge Sort($A, p, r$). Then

- $t_{1,2} = c$
Another example:

Algorithm **Merge Sort**\((A, p, r)\)

1. **if** \(p < r\)
2. **then** \(q = \lfloor \frac{p + r}{2} \rfloor\)
3. **Merge Sort**\((A, p, q)\)
4. **Merge Sort**\((A, q + 1, r)\)
5. **Merging2Lists**\((A.p, q, r)\)

Analysis of the algorithm.

- Assume \(n = r - p + 1\), a power of 2; also assume \(T(n)\) is time for **Merge Sort**\((A, p, r)\). Then

- \(t_{1,2} = c\)
- \(t_3 = t_4 = T(\frac{n}{2})\)
Chapter 4. Solving Recurrences

Another example:

Algorithm $\text{Merge Sort}(A, p, r)$

1. \textbf{if} $p < r$
2. \hspace{10pt} \textbf{then} $q = \left\lfloor \frac{p+r}{2} \right\rfloor$
3. $\text{Merge Sort}(A, p, q)$
4. $\text{Merge Sort}(A, q+1, r)$
5. $\text{Merging2Lists}(A, p, q, r)$

Analysis of the algorithm.

- Assume $n = r - p + 1$, a power of 2;
  also assume $T(n)$ is time for $\text{Merge Sort}(A, p, r)$. Then

- $t_{1,2} = c$
- $t_3 = t_4 = T\left(\frac{n}{2}\right)$
- $t_5 \leq n$ (why?)
Chapter 4. Solving Recurrences

Another example:

Algorithm MERGE SORT(A, p, r)

1. if \( p < r \)
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3. MERGE SORT(A, p, q)
4. MERGE SORT(A, q + 1, r)
5. MERGING2LISTS(A, p, q, r)

Analysis of the algorithm.

- Assume \( n = r - p + 1 \), a power of 2; also assume \( T(n) \) is time for MERGE SORT(A, p, r). Then

- \( t_{1,2} = c \)
- \( t_3 = t_4 = T(\frac{n}{2}) \)
- \( t_5 \leq n \) (why?)

\[
T(n) = t_{1,2} + t_3 + t_4 + t_5 \leq 2T(\frac{n}{2}) + n + c
\]
Chapter 4. Solving Recurrences

Another example:

Algorithm **MERGE SORT**(*A*, *p*, *r*)

1. if *p < r*
2. then *q* = \(\lfloor \frac{p+r}{2} \rfloor\)
3. **MERGE SORT**(*A*, *p*, *q*)
4. **MERGE SORT**(*A*, *q + 1*, *r*)
5. **MERGING2LISTS**(*A*.*p*, *q*, *r*)

Analysis of the algorithm.

- Assume *n = r − p + 1*, a power of 2;
  also assume *T*(n) is time for **MERGE SORT**(*A*, *p*, *r*). Then

- *t*\(_{1,2} = c*
- *t*\(_{3} = t*_{4} = T(\frac{n}{2})*
- *t*\(_{5} \leq n* (why?)

\[
T(n) = t_{1,2} + t_{3} + t_{4} + t_{5} \leq 2T(\frac{n}{2}) + n + c
\]

base case: *T*(1) = *c.*
Chapter 4. Solving Recurrences

Solve recurrence

$$T(n) \leq 2T(n^2) + n + c$$

with base case

$$T(1) = c$$

with a simple method:

$$T(n) \leq 2T(n^2) + n + c$$

$$\leq 2T(n^2) + n^2 + c$$

$$\leq 2T(n^2) + n^2 + c$$

$$\leq 2T(n^2) + n^2 + c$$

$$\leq 2T(n^2) + n^2 + c$$

$$\cdots$$

$$T(n^h) \leq 2T(n^h) + n^h + c$$

where

$$n^{2h+1} = 1$$

Multiplying

$$2, 2^2, 2^3, \ldots$$

to the second, third, \ldots inequalities, respectively,

$$T(n) \leq 2T(n^2) + n + c$$

$$\leq 2T(n^2) + n^2 + c$$

$$\leq 2T(n^2) + n^2 + c$$

$$\leq 2T(n^2) + n^2 + c$$

$$\cdots$$

$$2^h T(n^h) \leq n^h + c$$

where

$$n^{2h+1} = 1$$
Chapter 4. Solving Recurrences

Solve recurrence

\[ T(n) \leq 2T\left(\frac{n}{2}\right) + n + c \quad \text{with base case } T(1) = c \]

with a simple method:
Chapter 4. Solving Recurrences

Solve recurrence

\[ T(n) \leq 2T\left(\frac{n}{2}\right) + n + c \quad \text{with base case } T(1) = c \]

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Chapter 4. Solving Recurrences

Solve recurrence

\[ T(n) \leq 2T\left(\frac{n}{2}\right) + n + c \quad \text{with base case } T(1) = c \]

with a simple method:

\[
\begin{align*}
T(n) & \leq 2T\left(\frac{n}{2}\right) + n + c \\
T\left(\frac{n}{2}\right) & \leq 2T\left(\frac{n}{2^2}\right) + \frac{n}{2} + c
\end{align*}
\]
Chapter 4. Solving Recurrences

Solve recurrence

\[ T(n) \leq 2T\left(\frac{n}{2}\right) + n + c \] with base case \( T(1) = c \)

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\begin{align*}
T(n) &\leq 2T\left(\frac{n}{2}\right) + n + c \\
T\left(\frac{n}{2}\right) &\leq 2T\left(\frac{n}{2^2}\right) + \frac{n}{2} + c \\
T\left(\frac{n}{2^2}\right) &\leq 2T\left(\frac{n}{2^3}\right) + \frac{n}{2^2} + c
\end{align*}
\]
Chapter 4. Solving Recurrences

Solve recurrence

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T\left(\frac{n}{2^2}\right) & \leq 2T\left(\frac{n}{2^3}\right) + \frac{n}{2^2} + c \\
\vdots
\end{align*}
\]
Chapter 4. Solving Recurrences

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T\left(\frac{n}{2^2}\right) & \leq 2T\left(\frac{n}{2^3}\right) + \frac{n}{2^2} + c \\
& \vdots \\
T\left(\frac{n}{2^h}\right) & \leq 2T\left(\frac{n}{2^{h+1}}\right) + \frac{n}{2^h} + c \quad \text{where} \quad \frac{n}{2^{h+1}} = 1
\end{align*}
\]
Chapter 4. Solving Recurrences

Solve recurrence

\[ T(n) \leq 2T\left(\frac{n}{2}\right) + n + c \quad \text{with base case } T(1) = c \]

with a simple method:

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\begin{align*}
T(n) & \leq 2T\left(\frac{n}{2}\right) + n + c \\
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multiplying \(2, 2^2, \ldots\) to the second, third, \(\ldots\) inequalities, respectively,
Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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T(n) & \leq 2T\left(\frac{n}{2}\right) + n + c \\
2T\left(\frac{n}{2}\right) & \leq 2^2T\left(\frac{n}{2^2}\right) + 2 \times \frac{n}{2} + 2c
\end{align*}
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Chapter 4. Solving Recurrences

Solve recurrence

\[ T(n) \leq 2T\left(\frac{n}{2}\right) + n + c \] with base case \( T(1) = c \)

with a simple method:

\[
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Chapter 4. Solving Recurrences

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\vdots \\
2^h T\left(\frac{n}{2^h}\right) &\leq 2^{h+1} T\left(\frac{n}{2^{h+1}}\right) + 2^h \times \frac{n}{2^h} + 2^h c \quad \text{where } \frac{n}{2^{h+1}} = 1
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Chapter 4. Solving Recurrences

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2^2T\left(\frac{n}{2^2}\right) & \leq 2^3T\left(\frac{n}{2^3}\right) + 2^2 \times \frac{n}{2^2} + 2^2c \\
\vdots & \\
2^hT\left(\frac{n}{2^h}\right) & \leq 2^{h+1}T\left(\frac{n}{2^{h+1}}\right) + 2^h \times \frac{n}{2^h} + 2^hc \quad \text{where } \frac{n}{2^{h+1}} = 1
\end{align*}
\]

\[
T(n) \leq 2^{h+1}T\left(\frac{n}{2^{h+1}}\right) + (h + 1)n + c \sum_{i=0}^{h} 2^i
\]
Chapter 4. Solving Recurrences

\[ T(n) \leq 2h + 1 \quad T(n/2) + (h + 1)n + c h \sum_{i=0}^{2^h} \]

With \( n/2^h + 1 = 1 \), we have \( n = 2^h + 1 \) or \( h + 1 = \log_2 n \).

\[ T(n) \leq 2h + 1 + n \log_2 n + c(2^h + 1 - 1) = cn + n \log_2 n + c(n - 1) = O(n \log_2 n) \]

We need to prove the last equality, i.e., find constants \( a \) and \( k \) such that

\[ n \log_2 n + 2cn - c \leq an \log_2 n (1) \]

when \( n > k \).

Choose \( a = 2 \). Then to make (1) holds, we need \( \log_2 n > 2c \). So \( k = 2^{2c} \) suffices.

That is, \( n \log_2 n + 2cn - c \leq 2n \log_2 n \) when \( n > k = 2^{2c} \).

So \( T(n) = O(n \log_2 n) \).
Chapter 4. Solving Recurrences

\[ T(n) \leq 2^{h+1} T\left(\frac{n}{2^{h+1}}\right) + (h + 1)n + c \sum_{i=0}^{h} 2^i \]
Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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\[ n \log_2 n + 2cn - c \leq an \log_2 n \quad (1) \]

when \( n > k \).
Chapter 4. Solving Recurrences

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= cn + n \log_2 n + c(n - 1) 
= n \log_2 n + 2cn - c 
= O(n \log_2 n) \]

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Choose \( a = 2 \). Then to make (1) holds, we need \( \log_2 n > 2c \). So \( k = 2^{2c} \) suffices.
Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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= cn + n \log_2 n + c(n - 1) \\
= n \log_2 n + 2cn - c \\
= O(n \log_2 n) \]

We need to prove the last equality, i.e., find constants \( a \) and \( k \) such that

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So

\[ T(n) = O(n \log_2 n) \]
Chapter 4. Solving Recurrences

When $n$ is not a power of 2

- choose $m \cdot n$ such that $m \cdot n$ is a power of 2 and the smallest such that $n \leq m \cdot n$;
- $T(n) \leq T(m \cdot n)$, why? assume $T$ to be monotonic;
- use the analysis we just did, $T(m \cdot n) = O(m \cdot n \log_2 m \cdot n)$; that is, $\exists c, k, T(m \cdot n) \leq cm \cdot n \log_2 m \cdot n$ when $m \cdot n \geq k$;
- but $m \cdot n < 2^n$, why? because $m \cdot n < n$;
- So $T(n) \leq T(m \cdot n) \leq cm \cdot n \log_2 m \cdot n \leq 2^{cn} \log_2 (2^n) \leq 2^{cn} \log_2 n = c'n \log_2 n$, here $c' = 4c$, when $m \cdot n \geq k (\geq 4)$, i.e., when $n \geq \lceil k/2 \rceil (\geq 2)$;
- therefore, $T(n) = O(n \log_2 n)$.
When \( n \) is not a power of 2
When $n$ is not a power of 2

- choose $m_n$ such that $m_n$ is a power of 2
Chapter 4. Solving Recurrences

When $n$ is not a power of 2

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When $n$ is not a power of 2

- choose $m_n$ such that $m_n$ is a power of 2 and the smallest such that $n \leq m_n$;
- $T(n) \leq T(m_n)$, why?
Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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- but $m_n < 2n$, why?
Chapter 4. Solving Recurrences

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When $n$ is not a power of 2

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Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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- but $m_n < 2n$, why? because $\frac{m_n}{2} < n$;

- So $T(n) \leq T(m_n) \leq cm_n \log_2 m_n \leq 2cn \log_2(2n)$
When $n$ is not a power of 2

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Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

Methods for solving recurrences
Chapter 4. Solving Recurrences

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1. Substitution method (based on math induction)
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First we recall the principle of the math induction:

To prove a property $P(n)$ for every natural number $n \geq 1$, it suffices to prove

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"The principle for dominos to fall".
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Chapter 4. Solving Recurrences

**Theorem:** Arithmetic sequence of first \( n \) terms

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1 + 2 + 3 + \cdots + n = \frac{n}{2}(n + 1)
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Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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Algorithm Merge Sort($A, p, r$)

1. if $p < r$
2. then $q = \left\lfloor \frac{p+r}{2} \right\rfloor$
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T(n) \leq 2T\left(\frac{n}{2}\right) + n + c \quad \text{with base case } T(1) = c
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Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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$$= an \log_2 n - c3n - 2n + n + c = an \log_2 n + c(1 - 3n) - n$$
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step 2. assumption: for \( n ≥ 2 \), the claim is true, i.e.,

\[
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\[
T(n) ≤ 2T\left( \frac{n}{2} \right) + n + c = 2T\left( \frac{n}{2} \right) + n + c ≤ 2a \frac{n}{2} \log_2 \frac{n}{2} + n + c
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because \( c(1 - 3n) - n < 0 \)
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$$= an \log_2 n - c3n - 2n + n + c = an \log_2 n + c(1 - 3n) - n \leq an \log_2 n$$

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Chapter 4. Solving Recurrences

A little review on logarithm functions:

- \( \log_a n + \log_a m = \log_a nm \);
- \( \log_a n^b = b \log_a n \), especially \( \log_a 1^n = -\log_a n \);
- \( a^{\log_a n} = n \);
- \( \log_a n = \log_b n \log_b a = \frac{1}{\log_b a} \log_b n \);
- \( \log_m a^n = (\log_a n)^m \neq \log_a n^m \).
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Chapter 4. Solving Recurrences

A little review on logarithm functions:

- \( \log_a n + \log_a m = \log_a nm; \)
- \( \log_a n^b = b \log_a n, \) especially \( \log_a \frac{1}{n} = -\log_a n; \)
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Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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\[ \log_a^m n = (\log_a n)^m \neq \log_a n^m. \]
Chapter 4. Solving Recurrences

Solving recurrence with the substitution method (guess then verify)

\[ T(n) = \frac{3}{2} T(\lfloor \frac{2n}{3} \rfloor) + n, \text{ where } T(1) = 2 \]
Chapter 4. Solving Recurrences

Solving recurrence with the **substitution method** (guess then verify)

\[ T(n) = \frac{3}{2} T\left(\left\lfloor \frac{2n}{3} \right\rfloor \right) + n, \quad \text{where } T(1) = 2 \]

**Guess** \( T(n) \leq cn \log_2 n \), for some constant \( c \) to be determined later.
Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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$$T(2) = \frac{3}{2}T(\lceil \frac{2 \cdot 2}{3} \rceil) + 2 = \frac{3}{2}T(\lceil \frac{4}{3} \rceil) + 2 = \frac{3}{2}T(1) + 2 = 5$$

We make $T(2) = 5 \leq cn \log_2 n$ holds for $c = 3$, and when $n \geq k = 2$. 

Chapter 4. Solving Recurrences

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Step 2, assumption: assume the guessed upper bound holds for \( \lfloor \frac{2n}{3} \rfloor \), i.e.,

\[ T(\lfloor \frac{2n}{3} \rfloor) \leq c \lfloor \frac{2n}{3} \rfloor \log_2 \lfloor \frac{2n}{3} \rfloor \]
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Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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\[ \leq cn \left( \log_2 n - \log_2 \frac{3}{2} \right) + n \]
Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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\[ T(n) \leq cn \log_2 n - cn \log_2 \frac{3}{2} + n \leq cn \log_2 n \]
Chapter 4. Solving Recurrences

2. Changing variables

Example:

\[ T(n) = 2T(\sqrt{n}) + \log_2 n \]

Define \( m = \log_2 n \), i.e., \( n = 2^m \)

\[ T(2^m) = 2T(2^{m/2}) + m \]

rename the function:

\[ S(m) = T(2^m) \]

solve it, we have

\[ S(m) = O(m \log m) \]

so

\[ T(n) = O(m \log m) = O(\log n \log \log n) \].
Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

3. Recursive tree method
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(1) $T(n)$ is a tree with non-recursive terms as the root and recursive terms as its children.
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(1) $T(n)$ is a tree with non-recursive terms as the root and recursive terms as its children.

(2) for each child, replace it with then non-recursive terms and produce children that are then recursive terms
3. Recursive tree method

By unfolding the recurrence to make a recursive-tree.

(1) \( T(n) \) is a tree with non-recursive terms as the root and recursive terms as its children.

(2) for each child, replace it with then non-recursive terms and produce children that are then recursive terms

(3) repeat (2), expand the tree until all children are the base case.
Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$
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\begin{align*}
  l_0: & \quad T(n) \\
  \end{align*}
Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

\[
\begin{array}{ccc}
l_0: & T(n/4) & T(n) \\
l_1: & T(n/4) & T(n/4) & T(n/4) \\
\end{array}
\]
Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

$l_0$:

$l_1$:

$T(n/4)$

$T(n/4)$

$T(n/4)$

$n^2$
Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

<table>
<thead>
<tr>
<th>$l_0$:</th>
<th>$T(n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$l_1$:</td>
<td>$T(n/4)$</td>
</tr>
</tbody>
</table>

$T(n) = n^2 + \sum_{i=0}^{m-1} 3^i \cdot T(n/4)^{m-1}$

where $n/4^m = 1$, i.e., $m = \log_4 n$.

Then $T(n)$ is the sum $T(n) = n^2 + \sum_{i=0}^{m-1} 3^i \cdot T(n/4)^{m-1}$
Chapter 4. Solving Recurrences

Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

$l_0$: $T(n)$

$l_1$: $T(n/4)$ $T(n/4)$ $T(n/4)$

$l_2$: $T(n/4^2)$ $T(n/4^2)$ $T(n/4^2)$ $T(n/4^2)$ $T(n/4^2)$ $T(n/4^2)$ $T(n/4^2)$ $T(n/4^2)$ $n^2$
Chapter 4. Solving Recurrences

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$l_0$: $\begin{array}{cccc}
T(n)
\end{array}$

$l_1$: $\begin{array}{ccc}
T(n/4) & T(n/4)
\end{array}$

$l_2$: $\begin{array}{cccc}
T(n/4) & T(n/4) & T(n/4) & n^2
\end{array}$
Chapter 4. Solving Recurrences

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\[
\begin{align*}
l_0: & & T(n) \\
l_1: & & T(n/4) \\
l_2: & & T(n/4) T(n/4) T(n/4) \\
l_3: & & \ldots
\end{align*}
\]
Chapter 4. Solving Recurrences

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$l_0$: 

$l_1$: $T(n/4)$ $T(n/4)$ $T(n/4)$

$l_2$: $T(n/4^2)$ $T(n/4^2)$ $T(n/4^2)$ $T(n/4^2)$ $T(n/4^2)$ $T(n/4^2)$ $T(n/4^2)$ $T(n/4^2)$ $T(n/4^2)$ $T(n/4^2)$ $T(n/4^2)$ $T(n/4^2)$ $T(n/4^2)$ $T(n/4^2)$

$l_3$: ......

$n^2$ $3(n/4)^2$ $3^2(n/4^2)^2$
Chapter 4. Solving Recurrences

Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

$l_0$: 

$l_1$: 

$l_2$: $T(n/4) \quad T(n/4) \quad T(n/4) \quad n^2$

$l_3$: ......

$l_4$: ......

$l_0$: $T(n)$

$l_1$: $T(n/4)$

$l_2$: $T(n/4) \quad T(n/4) \quad T(n/4) \quad T(n/4) \quad T(n/4) \quad 3(n/4)^2$

$l_3$: ......

$l_4$: ......

$n^2$

$3(n/4)^2$
Chapter 4. Solving Recurrences

Example \( T(n) = 3T(n/4) + n^2 \), with base case \( T(1) = 1 \)

\[
\begin{align*}
  l_0: & \quad T(n) \\
  l_1: & \quad T(n/4) \\
  l_2: & \quad T(n/4) T(n/4) T(n/4) \\
  l_3: & \quad \ldots \\
  l_4: & \quad \ldots \\
\end{align*}
\]
Chapter 4. Solving Recurrences

Example \( T(n) = 3T(n/4) + n^2 \), with base case \( T(1) = 1 \)

\[
\begin{align*}
  l_0: & \quad T(n) \\
  l_1: & \quad T(n/4) \\
  l_2: & \quad T(n/4) T(n/4) T(n/4) \\
  l_3: & \quad \ldots \\
  l_4: & \quad \ldots \\
  l_5: & \quad \ldots
\end{align*}
\]
Chapter 4. Solving Recurrences

Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

$l_0$: 

$l_1$: 

$l_2$: $T(n/4)^3$ $T(n/4)^2$ $T(n/4)$ $n^2$

$l_3$: 

$l_4$: 

$l_5$: 

$l_{m-1}$: 

$3(n/4)^2$ 

$3^2(n/4)^2$ 

$3^3(n/4)^2$ 

$n^2$
Chapter 4. Solving Recurrences

Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

$l_0: T(n)$
$l_1: T(n/4)$
$l_2: T(n/4^2) T(n/4^2) T(n/4^2)$
$l_3: \ldots$
$l_4: \ldots$
$l_5: \ldots$
$l_{m-1}: \ldots$

$T(n)$
$T(n/4)$
$T(n/4^2)$
$T(n/4^2)$
$T(n/4^2)$
$T(n/4^2)$
$T(n/4^2)$
$3(n/4)^2$
$3^2(n/4^2)^2$
$3^3(n/4^3)^2$

$3^{m-2}(n/4^{m-2})^2$

$n^2$

$n^2$
Chapter 4. Solving Recurrences

Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

\begin{align*}
  l_0: & \quad T(n) \\
  l_1: & \quad T(n/4) \quad T(n/4) \\
  l_2: & \quad T(n/4^2) \quad T(n/4^2) \quad T(n/4^2) \quad T(n/4^2) \quad T(n/4^2) \\
  l_3: & \quad \ldots \\
  l_4: & \quad \ldots \\
  l_5: & \quad \ldots \\
  l_{m-1}: & \quad \ldots \\
  l_m: & \quad T(1), T(1), T(1), T(1), \ldots, T(1)
\end{align*}

\[3^{m-2} \left( \frac{n}{4^{m-2}} \right)^2 \quad 3^{m-1} \left( \frac{n}{4^{m-1}} \right)^2\]
Chapter 4. Solving Recurrences

Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

\[l_0: \ \ \ \ \ \ \ \ \ \ \ \ T(n) \ \ \ \ \ \ \ \ \ \ \ \ T(n)
\]
\[l_1: \ T(n/4) \ \ \ \ \ \ \ \ \ \ T(n/4) \ \ \ \ \ \ \ \ \ \ T(n/4) \ \ \ \ \ \ \ \ \ \ T(n/4) \ \ \ \ \ \ \ \ \ \ T(n/4) \ \ \ \ \ n^2
\]
\[l_2: \ T(\frac{n}{4^2}) \ T(\frac{n}{4^2}) \ T(\frac{n}{4^2}) \ T(\frac{n}{4^2}) \ T(\frac{n}{4^2}) \ \ \ \ \ \ \ \ \ \ \ \ 3(\frac{n}{4})^2
\]
\[l_3: \ \ \ldots
\]
\[l_4: \ \ \ldots
\]
\[l_5: \ \ \ldots
\]
\[l_{m-1}: \ \ \ldots
\]
\[l_m: \ T(1), T(1), T(1), T(1), T(1), \ldots, T(1)
\]

where $\frac{n}{4^m} = 1$, i.e., $m = \log_4 n$. 
Chapter 4. Solving Recurrences

Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

\begin{align*}
l_0: & \quad T(n) \\
l_1: & \quad T(n/4) \\
l_2: & \quad T(n/4) T(n/4) T(n/4) \\
l_3: & \quad \ldots \\
l_4: & \quad \ldots \\
l_5: & \quad \ldots \\
l_{m-1}: & \quad \ldots \\
l_m: & \quad T(1), T(1), T(1), T(1), T(1), \ldots, T(1)
\end{align*}

where $\frac{n}{4^m} = 1$, i.e., $m = \log_4 n$.

Then $T(n)$ is the sum

$$T(n) = n^2[1 + 3\left(\frac{1}{4}\right)^2 + 3^2\left(\frac{1}{4^2}\right)^2 + 3^3\left(\frac{1}{4^3}\right)^2 + \cdots + 3^{m-1}\left(\frac{1}{4^{m-1}}\right)^2] + 3^m T(1)$$
Chapter 4. Solving Recurrences

Example \( T(n) = 3T(n/4) + n^2 \), with base case \( T(1) = 1 \)

\[
\begin{align*}
&l_0: & T(n) \\
&l_1: & T(n/4) & T(n/4) \\
&l_2: & T(n/4^2) & T(n/4^2) & T(n/4^2) \\
&l_3: & \\
&l_4: & \\
&l_5: & \\
&l_{m-1}: & \\
&l_m: & T(1), T(1), T(1), T(1), T(1), \ldots, T(1) \\
\end{align*}
\]

where \( \frac{n}{4^m} = 1 \), i.e., \( m = \log_4 n \).

Then \( T(n) \) is the sum

\[
T(n) = n^2[1 + 3\left(\frac{1}{4}\right)^2 + 3^2\left(\frac{1}{4^2}\right)^2 + 3^3\left(\frac{1}{4^3}\right)^2 + \cdots + 3^{m-1}\left(\frac{1}{4^{m-1}}\right)^2] + 3^m T(1)
\]

\[
T(n) = n^2[1 + 3\left(\frac{1}{4}\right)^2 + 3^2\left(\frac{1}{4^2}\right)^2 + 3^3\left(\frac{1}{4^3}\right)^2 + \cdots + 3^{m-1}\left(\frac{1}{4^{m-1}}\right)^2] + 3^m \times 1
\]
Chapter 4. Solving Recurrences

Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

$l_0: \quad T(n)$

$l_1: \quad T(n/4) \quad T(n/4) \quad T(n/4) \quad n^2$

$l_2: \quad T(n/4^2) \quad T(n/4^2) \quad T(n/4^2) \quad T(n/4^2) \quad T(n/4^2) \quad T(n/4^2) \quad T(n/4^2) \quad 3(\frac{n}{4})^2$

$l_3: \quad \ldots$

$l_4: \quad \ldots$

$l_5: \quad \ldots$

$l_{m-1}: \quad \ldots$

$l_m: \quad T(1), T(1), T(1), T(1), T(1), \ldots, T(1)$

where $\frac{n}{4^m} = 1$, i.e., $m = \log_4 n$.

Then $T(n)$ is the sum

$T(n) = n^2[1 + 3(\frac{1}{4})^2 + 3^2(\frac{1}{4^2})^2 + 3^3(\frac{1}{4^3})^2 + \cdots + 3^{m-1}(\frac{1}{4^{m-1}})^2] + 3^m T(1)$

$T(n) = n^2[1 + 3(\frac{1}{4})^2 + 3^2(\frac{1}{4^2})^2 + 3^3(\frac{1}{4^3})^2 + \cdots + 3^{m-1}(\frac{1}{4^{m-1}})^2] + 3^m \times 1$

$= n^2[1 + 3(\frac{1}{4})^2 + 3^2(\frac{1}{4^2})^2 + 3^3(\frac{1}{4^3})^2 + \cdots + 3^{m-1}(\frac{1}{4^{m-1}})^2] + 3^m \left(\frac{n}{4^m}\right)^2$
Chapter 4. Solving Recurrences

Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

\[ l_0: \quad T(n) \]
\[ l_1: \quad T(n/4) \]
\[ l_2: \quad T(n/4^2) T(n/4^2) T(n/4^2) \]
\[ l_3: \quad \ldots \]
\[ l_4: \quad \ldots \]
\[ l_5: \quad \ldots \]
\[ l_{m-1}: \quad \ldots \]
\[ l_m: \quad T(1), T(1), T(1), T(1), T(1), \ldots, T(1) \]

where $\frac{n}{4^m} = 1$, i.e., $m = \log_4 n$.

Then $T(n)$ is the sum

\[
T(n) = n^2 \left[ 1 + 3 \left( \frac{1}{4} \right)^2 + 3^2 \left( \frac{1}{4^2} \right)^2 + 3^3 \left( \frac{1}{4^3} \right)^2 + \cdots + 3^{m-1} \left( \frac{1}{4^{m-1}} \right)^2 \right] + 3^m T(1) \\
= n^2 \left[ 1 + 3 \left( \frac{1}{4^2} \right)^2 + 3^2 \left( \frac{1}{4^3} \right)^2 + \cdots + 3^{m-1} \left( \frac{1}{4^{m-1}} \right)^2 \right] + 3^m \times 1 \\
= \frac{n^2}{1 - \frac{3}{16}} + \frac{3}{16} + \frac{3^2}{16^2} + \frac{3^3}{16^3} + \cdots + \frac{3^{m-1}}{16^{m-1}} + 3^{m-1} \left( \frac{n}{4^m} \right)^2 \\
= \frac{n^2}{1 - \frac{3}{16}} + \left( \frac{3}{16} \right)^2 + \left( \frac{3}{16} \right)^3 + \cdots + \left( \frac{3}{16} \right)^m \\]
Chapter 4. Solving Recurrences

Example \( T(n) = 3T(n/4) + n^2 \), with base case \( T(1) = 1 \)

\begin{align*}
\text{l}_0: & \quad T(n) \\
\text{l}_1: & \quad T(n/4) \\
\text{l}_2: & \quad T(n/4^2) T(n/4^2) T(n/4^2) \\
\text{l}_3: & \quad \ldots \\
\text{l}_4: & \quad \ldots \\
\text{l}_5: & \quad \ldots \\
\text{l}_{m-1}: & \quad \ldots \\
\text{l}_m: & \quad T(1), T(1), T(1), T(1), T(1), \ldots, T(1) \\
\end{align*}

where \( \frac{n}{4^m} = 1 \), i.e., \( m = \log_4 n \).

Then \( T(n) \) is the sum

\[
T(n) = n^2 \left[ 1 + 3\left(\frac{1}{4}\right)^2 + 3^2\left(\frac{1}{4^2}\right)^2 + 3^3\left(\frac{1}{4^3}\right)^2 + \cdots + 3^{m-1}\left(\frac{1}{4^{m-1}}\right)^2 \right] + 3^m T(1) \\
T(n) = n^2 \left[ 1 + 3\left(\frac{1}{4}\right)^2 + 3^2\left(\frac{1}{4^2}\right)^2 + 3^3\left(\frac{1}{4^3}\right)^2 + \cdots + 3^{m-1}\left(\frac{1}{4^{m-1}}\right)^2 \right] + 3^m \times 1 \\
= n^2 \left[ 1 + 3\left(\frac{1}{4}\right)^2 + 3^2\left(\frac{1}{4^2}\right)^2 + 3^3\left(\frac{1}{4^3}\right)^2 + \cdots + 3^{m-1}\left(\frac{1}{4^{m-1}}\right)^2 \right] + 3^m \left(\frac{n}{4^m}\right)^2 \\
= n^2 \left[ 1 + \frac{3}{16} + \left(\frac{3}{16}\right)^2 + \left(\frac{3}{16}\right)^3 + \cdots + \left(\frac{3}{16}\right)^m \right] \\
= n^2 \left(\frac{1 - \left(\frac{3}{16}\right)^{m+1}}{1 - \frac{3}{16}}\right)
\]
Chapter 4. Solving Recurrences

Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

\[
\begin{align*}
&l_0: \\
&l_1: \\
&l_2: T(n/4) T(n/4) T(n/4) T(n/4) T(n/4) T(n/4) T(n/4) T(n/4) T(n/4) T(n/4) T(n/4) T(n/4) T(n/4)
&= \left(1 + 3\left(\frac{1}{4}\right)^2 + 3^2\left(\frac{1}{4}\right)^2 + 3^3\left(\frac{1}{4}\right)^2 + \cdots + 3^{m-1}\left(\frac{1}{4}\right)^2\right) + 3^m T(1)
&= n^2[1 + \frac{3}{16} + \left(\frac{3}{16}\right)^2 + \left(\frac{3}{16}\right)^3 + \cdots + \left(\frac{3}{16}\right)^m]
&= n^2\left(\frac{1 - \left(\frac{3}{16}\right)^{m+1}}{1 - \frac{3}{16}}\right)
&\leq n^2\left(\frac{1}{1 - \frac{3}{16}}\right)
\end{align*}
\]
Chapter 4. Solving Recurrences

Example \( T(n) = 3T(n/4) + n^2 \), with base case \( T(1) = 1 \)

\[
\begin{align*}
l_0: \quad & T(n) \\
l_1: \quad & T(n/4) \\
l_2: \quad & T(n/4^2) T(n/4^2) T(n/4^2) \\
l_3: \quad & \ldots \ldots \\
l_4: \quad & \ldots \ldots \\
l_5: \quad & \ldots \ldots \\
l_{m-1}: \quad & \ldots \ldots \\
l_m: \quad & T(1), T(1), T(1), T(1), T(1), \ldots, T(1)
\end{align*}
\]

where \( \frac{n}{4^m} = 1 \), i.e., \( m = \log_4 n \).

Then \( T(n) \) is the sum

\[
T(n) = n^2 \left[ 1 + 3 \left( \frac{1}{4} \right)^2 + 3^2 \left( \frac{1}{4^2} \right)^2 + 3^3 \left( \frac{1}{4^3} \right)^2 + \cdots + 3^{m-1} \left( \frac{1}{4^{m-1}} \right)^2 \right] + 3^m T(1)
\]

\[
T(n) = n^2 \left[ 1 + 3 \left( \frac{1}{4} \right)^2 + 3^2 \left( \frac{1}{4^2} \right)^2 + 3^3 \left( \frac{1}{4^3} \right)^2 + \cdots + 3^{m-1} \left( \frac{1}{4^{m-1}} \right)^2 \right] + 3^m \times 1
\]

\[
= n^2 \left[ 1 + 3 \left( \frac{1}{4} \right)^2 + 3^2 \left( \frac{1}{4^2} \right)^2 + 3^3 \left( \frac{1}{4^3} \right)^2 + \cdots + 3^{m-1} \left( \frac{1}{4^{m-1}} \right)^2 \right] + 3^m \left( \frac{n}{4^m} \right)^2
\]

\[
= n^2 \left[ 1 + \frac{3}{16} + \left( \frac{3}{16} \right)^2 + \left( \frac{3}{16} \right)^3 + \cdots + \left( \frac{3}{16} \right)^m \right]
\]

\[
= n^2 \left( \frac{1 - \left( \frac{3}{16} \right)^{m+1}}{1 - \frac{3}{16}} \right)
\]

\[
\leq n^2 \left( \frac{1}{1 - \frac{3}{16}} \right)
\]

\[
= \frac{16}{13} n^2
\]

for all \( n > 0 \).
Chapter 4. Solving Recurrences

Recursive tree notation in the textbook:

\[ T(n) = 3T(n/4) + n^2, \]  with base case \( T(1) = 1 \)

\( l_0: \quad n^2 \)
Chapter 4. Solving Recurrences

Recursive tree notation in the textbook:

\[ T(n) = 3T(n/4) + n^2, \text{ with base case } T(1) = 1 \]

- \( l_0: \) 
  \( (n/4)^2 \) 
  \( n^2 \) 

- \( l_1: \) 
  \( (n/4)^2 \) 
  \( (n/4)^2 \) 
  \( (n/4)^2 \)
Chapter 4. Solving Recurrences

Recursive tree notation in the textbook:

\[ T(n) = 3T(n/4) + n^2, \text{ with base case } T(1) = 1 \]

\[ l_0: \]

\[ l_1: \]

\[ l_2: \]

\[ \left( \frac{n}{4^2} \right)^2 \left( \frac{n}{4^2} \right)^2 \left( \frac{n}{4^2} \right)^2 \left( \frac{n}{4^2} \right)^2 \left( \frac{n}{4^2} \right)^2 \left( \frac{n}{4^2} \right)^2 \left( \frac{n}{4^2} \right)^2 \left( \frac{n}{4^2} \right)^2 \left( \frac{n}{4^2} \right)^2 \left( \frac{n}{4^2} \right)^2 \]
Chapter 4. Solving Recurrences

Recursive tree notation in the textbook:

\[ T(n) = 3T(n/4) + n^2, \text{ with base case } T(1) = 1 \]

\[ \begin{align*}
  l_0: & \quad \text{ } & & \quad n^2 \\
  l_1: & \quad \left(\frac{n}{4}\right)^2 & \left(\frac{n}{4}\right)^2 & \left(\frac{n}{4}\right)^2 \\
  l_2: & \quad \left(\frac{n}{4^2}\right)^2 & \left(\frac{n}{4^2}\right)^2 & \left(\frac{n}{4^2}\right)^2 & \left(\frac{n}{4^2}\right)^2 & \left(\frac{n}{4^2}\right)^2 & \left(\frac{n}{4^2}\right)^2 \\
  l_3: & \quad \ldots \ldots \\
\end{align*} \]
Recursive tree notation in the textbook:

\[ T(n) = 3T(n/4) + n^2, \text{ with base case } T(1) = 1 \]

\( l_0: \)
\[ n^2 \]

\( l_1: \)
\[ \left(\frac{n}{4^2}\right)^2 \left(\frac{n}{4^2}\right)^2 \left(\frac{n}{4^2}\right)^2 \left(\frac{n}{4^2}\right)^2 \left(\frac{n}{4^2}\right)^2 \left(\frac{n}{4^2}\right)^2 \left(\frac{n}{4^2}\right)^2 \left(\frac{n}{4^2}\right)^2 \left(\frac{n}{4^2}\right)^2 \]

\( l_2: \)
\[ \left(\frac{n}{4^2}\right)^2 \left(\frac{n}{4^2}\right)^2 \left(\frac{n}{4^2}\right)^2 \left(\frac{n}{4^2}\right)^2 \left(\frac{n}{4^2}\right)^2 \left(\frac{n}{4^2}\right)^2 \left(\frac{n}{4^2}\right)^2 \left(\frac{n}{4^2}\right)^2 \left(\frac{n}{4^2}\right)^2 \]

\( l_3: \ldots \)

\( l_4: \ldots \)

3^3 \text{ nodes of } \left(\frac{n}{4^3}\right)^2
Chapter 4. Solving Recurrences

Recursive tree notation in the textbook:

\[ T(n) = 3T(n/4) + n^2, \text{ with base case } T(1) = 1 \]

\[ l_0: \]
\[ l_1: \]
\[ l_2: \quad \left(\frac{n}{4^2}\right)^2 \left(\frac{n}{4^2}\right)^2 \left(\frac{n}{4^2}\right)^2 \]
\[ l_3: \quad \ldots \ldots \]
\[ l_4: \quad \ldots \ldots \]
\[ l_{m-1}: \quad \ldots \ldots \]

\[ 3^3 \text{ nodes of } \left(\frac{n}{4^3}\right)^2 \]
Chapter 4. Solving Recurrences

Recursive tree notation in the textbook:

\[ T(n) = 3T(n/4) + n^2, \text{ with base case } T(1) = 1 \]

\[ l_0: \]
\[ l_1: \]
\[ l_2: \]
\[ l_3: \ldots \]
\[ l_4: \ldots \]
\[ l_{m-1}: \ldots \]

\[ 3^3 \text{ nodes of } \left(\frac{n}{4^3}\right)^2 \]
\[ 3^{m-1} \text{ of } \left(\frac{n}{4^{m-1}}\right)^2 \]
Recursive tree notation in the textbook:

\[ T(n) = 3T(n/4) + n^2, \text{ with base case } T(1) = 1 \]

- \( l_0: \) \( n^2 \)
- \( l_1: \) \( (n/4)^2 (n/4)^2 (n/4)^2 \)
- \( l_2: \) \( (n/4)^2 (n/4)^2 (n/4)^2 (n/4)^2 (n/4)^2 \)
- \( l_3: \) \( \ldots \)
- \( l_4: \) \( \ldots \)
- \( l_{m-1}: \) \( \ldots \)
- \( l_m: \) \( T(1), T(1), T(1), T(1), T(1), \ldots \)

3\(^3\) nodes of \((n/4^3)^2\)

3\(^{m-1}\) of \((n/4^{m-1})^2\)

3\(^m\) nodes of \(T(1)\)
Recursive tree notation in the textbook:

\[ T(n) = 3T(n/4) + n^2, \text{ with base case } T(1) = 1 \]

\[ \begin{align*}
  l_0: & & n^2 \\
  l_1: & & (n/4)^2 \\
  l_2: & & (n/4)^2 (n/4)^2 (n/4)^2 \\
  l_3: & & \ldots \ldots \\
  l_4: & & \ldots \ldots \\
  l_{m-1}: & & \ldots \ldots \\
  l_m: & & T(1), T(1), T(1), T(1), T(1), \ldots \\
\end{align*} \]

3^{m-1} of \((\frac{n}{4^{m-1}})^2\)

3^m nodes of \(T(1)\)
Another example (page 91 textbook):

Assume time function $T(n)$ of some algorithm has the recurrence

$$T(n) = T\left(\frac{n}{3}\right) + T\left(\frac{2n}{3}\right) + n$$

with base case $T(1) = T(2) = T(3) = c > 0$, a constant. We assume $n$ is a power of 3.
Another example (page 91 textbook):

Assume time function $T(n)$ of some algorithm has the recurrence

$$T(n) = T\left(\frac{n}{3}\right) + T\left(\frac{2n}{3}\right) + n$$

with base case $T(1) = T(2) = T(3) = c > 0$, a constant. We assume $n$ is a power of 3.

(1) using recursive tree method to derive an upper bound
Another example (page 91 textbook):

Assume time function $T(n)$ of some algorithm has the recurrence

$$T(n) = T\left(\frac{n}{3}\right) + T\left(\frac{2n}{3}\right) + n$$

with base case $T(1) = T(2) = T(3) = c > 0$, a constant.

We assume $n$ is a power of 3.

(1) using recursive tree method to derive an upper bound

(2) using substitution method to verify the upper bound
Chapter 4. Solving Recurrences

(1) using recursive tree method to derive an upper bound
Chapter 4. Solving Recurrences

(1) using recursive tree method to derive an upper bound

\[ l_0: \quad T(n) \]
\[ l_1: \quad T\left(\frac{n}{3}\right) \quad T\left(\frac{2n}{3}\right) \quad n \]
(1) using recursive tree method to derive an upper bound

\begin{align*}
l_0 &: T(n) \\
l_1 &: T\left(\frac{n}{3}\right) \quad T\left(\frac{2n}{3}\right) \\
l_2 &: T\left(\frac{n}{3^2}\right) \quad T\left(\frac{2n}{3^2}\right) \quad T\left(\frac{2n}{3^2}\right) \quad T\left(\frac{2^2n}{3^2}\right) \\
\end{align*}

\[
\frac{n}{3} + \frac{2n}{3} = n
\]
Chapter 4. Solving Recurrences

(1) using recursive tree method to derive an upper bound

\[ l_0: \begin{array}{c} T(n) \end{array} \]
\[ l_1: \begin{array}{ccc} T\left(\frac{n}{3}\right) & T\left(\frac{2n}{3}\right) \end{array} \]
\[ l_2: \begin{array}{cccc} T\left(\frac{n}{3^2}\right) & T\left(\frac{2n}{3^2}\right) & T\left(\frac{2n}{3^2}\right) & T\left(\frac{2^2n}{3^2}\right) \end{array} \]
\[ \ldots \]
\[ l_{m-1}: \begin{array}{c} T\left(\frac{n}{3^{m-1}}\right) \ldots \end{array} \]

\[ n = \frac{n}{3} + \frac{2n}{3} = n \]

\[ l_m: \begin{array}{c} T\left(\frac{2^{m-1}n}{3^{m-1}}\right) \end{array} \]

\[ n \]
Chapter 4. Solving Recurrences

(1) using recursive tree method to derive an upper bound

\[ l_0: \quad T(n) \]
\[ l_1: \quad T(\frac{n}{3}) \quad T(\frac{2n}{3}) \]
\[ l_2: \quad T(\frac{n}{3^2}) \quad T(\frac{2n}{3^2}) \quad T(\frac{2n}{3^2}) \quad T(\frac{2^2n}{3^2}) \]
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\[ \ldots \]
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\[ l_m: \quad T\left(\frac{n}{3^m}\right), \quad T\left(\frac{2n}{3^m}\right), \ldots \]
\[ l_{m+1}: \quad \ldots \]
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\[ l_{m+1}: \quad \ldots \]
\[ \ldots \]
\[ l_r: \quad \ldots \]
\[ \ldots \]
\[ l_{m+1}: \quad \ldots \]
\[ \ldots \]
\[ l_r: \quad \ldots \]
Chapter 4. Solving Recurrences

(1) using recursive tree method to derive an upper bound

\[\begin{align*}
  l_0: & & T(n) \\
  l_1: & & T\left(\frac{n}{3}\right) & & T\left(\frac{2n}{3}\right) \\
  l_2: & & T\left(\frac{n}{3^2}\right) & & T\left(\frac{2n}{3^2}\right) & & T\left(\frac{2^2n}{3^2}\right) \\
  \vdots & & & & & & \frac{n}{3} + \frac{2n}{3} = n \\
  l_{m-1}: & & T\left(\frac{n}{3^{m-1}}\right) \cdots & & T\left(\frac{2^{m-1}n}{3^{m-1}}\right) & & T\left(\frac{2^m n}{3^m}\right) \cdots \\
  l_m: & & T\left(\frac{n}{3^m}\right) & & T\left(\frac{2n}{3^m}\right) \cdots & & T\left(\frac{2^{m+1}n}{3^{m+1}}\right) < n \\
  l_{m+1}: & & \vdots & & \vdots & & \vdots \\
  \vdots & & \vdots & & \vdots & & \vdots \\
  l_r: & & \vdots & & T\left(\frac{2^r n}{3^r}\right) < n
\end{align*}\]
Chapter 4. Solving Recurrences

(1) using recursive tree method to derive an upper bound
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(1) using recursive tree method to derive an upper bound

- the leftmost branch is the shortest; the rightmost is the longest;
- \( \frac{n}{3m} = 1, \ m = \log_3 n; \ \frac{2^r n}{3^m} = 1, \ r = \log_{\frac{3}{2}} n. \)
- beginning from level \( m + 1 \), some nodes will gradually disappear, so the number of leaves is NOT \( 2^{\log_{\frac{3}{2}} n} \neq O(n \log_2 n) \) (why?),
- we do not need to give an accurate account for the sum of all quantities in blue color and those at leaves, but only an estimated upper bound.
- we estimate an upper bound to be \( O(n \log_2 n). \)
Chapter 4. Solving Recurrences

(2) using the substitution method to verify the upper bound

We prove that

\[ T(n) = O(n \log_2 n) \]

That is to prove:

\[ \exists a, k > 0 \text{ such that } T(n) \leq an \log_2 n \text{ when } n \geq k \]

Base case: for \( n = 3 \), \( T(3) = c \leq a \cdot 3 \log_2 3 \) will be true if we choose \( a \geq c \) because \( \log_2 3 > 1 \).

Assume that \( T(n^3) \leq a \cdot n^3 \log_2 n^3 \); and \( T(2n^3) \leq a \cdot 2n^3 \log_2 2n^3 \);

Then

\[ T(n) = T(n^3) + T(2n^3) + n \leq an^3 \log_2 n^3 + a \cdot 2n^3 \log_2 2n^3 + n = an \log_2 n - an^3 (\log_2 3 + 2 \log_2 3) + n \]
Chapter 4. Solving Recurrences

(2) using the substitution method to verify the upper bound
We prove that \( T(n) = O(n \log_2 n) \).

That is to prove: \( \exists a, k > 0 \) such that

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T(n) \leq an \log_2 n \quad \text{when } n \geq k
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Chapter 4. Solving Recurrences

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\[
T(n) = T\left(\frac{n}{3}\right) + T\left(\frac{2n}{3}\right) + n
\]

\[
\leq a \frac{n}{3} \log_2 \frac{n}{3} + a \frac{2n}{3} \log_2 \frac{2n}{3} + n
\]

\[
= \frac{an}{3} \log_2 n + \frac{an}{3} \log_2 \frac{1}{3} + \frac{2an}{3} \log_2 n + \frac{2an}{3} \log_2 \frac{2}{3} + n
\]

\[
= an \log_2 n - \frac{an}{3} \left(\log_2 3 + 2 \log_2 \frac{3}{2}\right) + n
\]
Chapter 4. Solving Recurrences

Therefore,

\[ T(n) \leq an \log_2 n - \frac{an}{3} \left( \log_2 3 + \log_2 \frac{9}{4} \right) + n \]

\[ = an \log_2 n - \frac{an}{3} \log_2 \frac{27}{4} + n \]

\[ \leq an \log_2 n \]

when \( a \) is chosen such that \( a \geq \max\{3, c\} \), since it makes

\[ -\frac{an}{3} \log_2 \frac{27}{4} + n \leq 0 \]
Chapter 5. Probabilistic Analysis of Algorithms

Chapter 5. Probabilistic analysis and randomized algorithms
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• Close relationship with randomized algorithms
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Alternatively, we can enforce the desired distribution by using randomness (tossing coins) in the algorithms. 

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  - on ‘NO’ instances, 100% accuracy; \( \text{Prob}(\text{to answer 'NO' on 'NO' instance}) = 1 \)
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Accuracy 75% can be improved to 99.99% with multiple trials.

Las Vegas algorithms is as powerful as Monte Carlo algorithms, if not more.
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