CSCI 4470/6470 Algorithms, Fall 2019

Lecture Note II
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Part II Sorting and Order Statistics
Part II Sorting and Order Statistics

- Chapter 7. Quicksort, probabilistic analysis, randomized algorithms
- Chapter 8. Sorting in linear time, lower bounds
- Chapter 9. Medians and order statistics
Chapter 6. Heapsort

Chapter 6. Heapsort and the use of priority queue
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Chapter 6. Heapsort

Chapter 6. Heapsort and the use of priority queue

- key(parent) ≥ key(leftChild), key(rightChild);

![Diagram of a heap tree and the relationship between nodes and their positions in an array.](image-url)

- Index relationships are modeled with a complete binary tree:
  - index(leftChild) = 2 × index(parent) + 1
  - index(rightChild) = 2 × index(parent) + 2
Chapter 6. Heapsort

Chapter 6. Heapsort and the use of priority queue

• key(parent) ≥ key(leftChild), key(rightChild);
• relationships are modeled with a complete binary tree
Chapter 6. Heapsort

Chapter 6. Heapsort and the use of priority queue

- key(parent) \geq key(leftChild), key(rightChild);
- relationships are modeled with a complete binary tree
- can be stored in arrays (indexes begin with 0),
Chapter 6. Heapsort

Chapter 6. Heapsort and the use of priority queue

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Chapter 6. Heapsort

Chapter 6. Heapsort and the use of priority queue

- key(parent) ≥ key(leftChild), key(rightChild);
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- can be stored in arrays (indexes begin with 0),
  \[
  \text{index(leftChild)} = 2 \times \text{index(parent)} + 1 \\
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  \]
Chapter 6. Heapsort

The heap sort algorithm consists of subroutines:
Chapter 6. Heapsort

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- \text{Build-Max-Heap}(A)
Chapter 6. Heapsort

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- \texttt{Max-Heapify}(A, i)
Chapter 6. Heapsort

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- **Build-Max-Heap**\( (A) \)
- **Max-Heapify**\( (A, i) \)
- **HeapSort**\( (A) \)
Chapter 6. Heapsort

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heaps as priority queues
Chapter 6. Heapsort

The heap sort algorithm consists of subroutines:

- **Build-Max-Heap**($A$)
- **Max-Heapify**($A, i$)
- **HeapSort**($A$)

Heaps as priority queues

- **Heap-Maximum**($A$)
- **Heap-Extract-Max**($A$)
- **Heap-Increase-Key**($A, I, key$)
- **Max-Heap-Insert**($A, key$)
Chapter 6. Heapsort

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Chapter 6. Heapsort

Algorithm \textsc{HeapSort}(A)
Chapter 6. Heapsort

Algorithm HeapSort(A)

1. Build-Max-Heap(A)
Chapter 6. Heapsort

Algorithm HeapSort(A)
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2. for $i = \text{length}[A] - 1$ downto 1 { indexes begin from 0}
Algorithm $\text{HEAPSORT}(A)$

1. $\text{BUILD-MAX-HEAP}(A)$
2. for $i = \text{length}[A] - 1$ downto 1 { indexes begin from 0}
3. exchange $A[0] \leftrightarrow A[i]$

$T_{\text{BMH}}(n) = \lfloor n/2 \rfloor - 1 \sum_{i=0}^{n-1} c_2 T_{\text{MH}}(n,i)$, where $n = |A|$
Chapter 6. Heapsort

Algorithm HeapSort($A$)

1. Build-Max-Heap($A$)
2. for $i = \text{length}[A] - 1$ downto 1 \{ indexes begin from 0\}
3. exchange $A[0] \leftrightarrow A[i]$
4. $\text{heapsize}[A] = \text{heapsize}[A] - 1$
Chapter 6. Heapsort

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5. Max-Heapify(A, 0)
Chapter 6. Heapsort

Algorithm \texttt{HeapSort}(A)

1. \texttt{Build-Max-Heap}(A)
2. \texttt{for} \( i = \text{length}[A] - 1 \) \texttt{downto} 1 \quad \{ \text{indexes begin from 0} \}
3. exchange \( A[0] \leftrightarrow A[i] \)
4. \( \text{heapsize}[A] = \text{heapsize}[A] - 1 \)
5. \texttt{Max-Heapify}(A, 0)

\[ T_{HS}(n) = c_1 + T_{BMH}(n) + (n - 1)T_{MH}(n, 0), \text{ where } n = |A| \]
Chapter 6. Heapsort

Algorithm $\text{Heapsort}(A)$

1. $\text{Build-Max-Heap}(A)$
2. for $i = \text{length}[A] - 1$ downto 1 { indexes begin from 0 }
3. exchange $A[0] \leftrightarrow A[i]$
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Subroutine $\text{Build-Max-Heap}(A)$
Algorithm **HeapSort**\( (A) \)

1. **Build-Max-Heap**\( (A) \)
2. for \( i = \text{length}[A] - 1 \) downto 1 \{ indexes begin from 0\}
3. exchange \( A[0] \leftrightarrow A[i] \)
4. \( \text{heapsize}[A] = \text{heapsize}[A] - 1 \)
5. **Max-Heapify**\( (A, 0) \)

\[
T_{HS}(n) = c_1 + T_{BMH}(n) + (n - 1)T_{MH}(n, 0) , \text{ where } n = |A|
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Subroutine **Build-Max-Heap**\( (A) \)

1. \( \text{heapsize}[A] = \text{length}[A] \)
Chapter 6. Heapsort

Algorithm HEAPSORT($A$)

1. BUILD-MAX-HEAP($A$)
2. for $i = \text{length}[A] - 1$ downto 1  { indexes begin from 0}
3. exchange $A[0] \leftrightarrow A[i]$
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5. MAX-HEAPIFY($A, 0$)

\[ T_{HS}(n) = c_1 + T_{BMH}(n) + (n - 1)T_{MH}(n, 0), \text{ where } n = |A| \]

Subroutine BUILD-MAX-HEAP($A$)

1. $\text{heapsize}[A] = \text{length}[A]$
2. for $i = \lfloor \frac{1}{2} \text{length}[A] \rfloor - 1$ downto 0  { indexes begin from 0}
Algorithm **HEAPSORT**(\(A\))

1. **BUILD-MAX-HEAP**\((A)\)
2. **for** \(i = \text{length}[A] - 1 \text{ downto } 1\) \{ indexes begin from 0\}
3. exchange \(A[0] \leftrightarrow A[i]\)
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Subroutine **BUILD-MAX-HEAP**\((A)\)

1. \(\text{heapsize}[A] = \text{length}[A]\)
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3. **MAX-HEAPIFY**(\(A, i\))
Chapter 6. Heapsort

Algorithm HeapSort(A)

1. **Build-Max-Heap**(A)
2. for \(i = \text{length}[A] - 1\) **downto** 1 \{ indexes begin from 0\}
3. exchange \(A[0] \leftarrow A[i]\)
4. \(\text{heapsize}[A] = \text{heapsize}[A] - 1\)
5. **Max-Heapify**(A, 0)

\[T_{HS}(n) = c_1 + T_{BMH}(n) + (n - 1)T_{MH}(n, 0),\text{ where } n = |A|\]

Subroutine **Build-Max-Heap**(A)

1. \(\text{heapsize}[A] = \text{length}[A]\)
2. for \(i = \lfloor \frac{1}{2}\text{length}[A] \rfloor - 1\) **downto** 0 \{ indexes begin from 0\}
3. **Max-Heapify**(A, i)

\[T_{BMH}(n) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor - 1} c_2 T_{MH}(n, i)\]
Chapter 6. Heapsort

Subroutine $\text{MAX-HEAPIFY}(A, i)$

1. $l = 2 \times i + 1$
2. $r = 2 \times i + 2$
3. if $(l \leq \text{heapsize}[A])$ and $(A[l] > A[i])$
4. then $\text{largest} = l$
5. else $\text{largest} = i$
6. if $(r \leq \text{heapsize}[A])$ and $(A[r] > A[\text{largest}])$
7. then $\text{largest} = r$
8. if $\text{largest} \neq i$
9. then exchange $A[i] \leftrightarrow A[\text{largest}]
10. \text{Max-Heapify}(A, \text{largest})$

$T_{\text{MH}}(n, i) \leq c_3 + T_{\text{MH}}(n, 2i + 1)$

Because $T_{\text{MH}}(n, i) = c_3 + T_{\text{MH}}(n, 2i + 1)$, or $= c_3 + T_{\text{MH}}(n, 2i + 2)$

$T_{\text{MH}}(n, i) \leq c_4 \log_2 n$, for all $i = 0, 1, \ldots, n - 1$. (Prove it!)

$T_{\text{BMH}}(n) = \lfloor \frac{n}{2} \rfloor - 1 \sum_{i=0}^{n} c_2 T_{\text{MH}}(n, i)$

$\leq c_4 n^2 \log_2 n$

$T_{\text{HS}}(n) = c_1 + T_{\text{BMH}}(n) + (n - 1)c_2 T_{\text{MH}}(n, 0)$

$\leq c_4 n^2 \log_2 n + (n - 1)c_2 c_4 \log_2 n = O(n \log n)$
Chapter 6. Heapsort

Subroutine \texttt{MAX-HEAPIFY}(A, i)

1. \( l = 2 \times i + 1 \)
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Subroutine MAX-HEAPIFY(A, i)

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Subroutine MAX-HEAPIFY(A, i)

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Chapter 6. Heapsort

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4. then $\text{largest} = l$
Chapter 6. Heapsort

Subroutine MAX-HEAPIFY($A, i$)

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Chapter 6. Heapsort

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4. then \( \text{largest} = l \)
5. else \( \text{largest} = i \)
6. if \((r \leq \text{heapsize}[A]) \text{ and } (A[r] > A[\text{largest}])\)
Chapter 6. Heapsort

Subroutine MAX-HEAPIFY(A, i)

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3. \( \text{if } (l \leq \text{heapsize}[A]) \text{ and } (A[l] > A[i]) \) then largest = l
4. \( \quad \text{else largest} = i \)
5. \( \text{if } (r \leq \text{heapsize}[A]) \text{ and } (A[r] > A[\text{largest}]) \) then largest = r
Chapter 6. Heapsort

Subroutine MAX-HEAPIFY(A, i)

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3. if $(l \leq \text{heapsize}[A])$ and $(A[l] > A[i])$
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5. if $(r \leq \text{heapsize}[A])$ and $(A[r] > A[\text{largest}])$
   then largest = $r$
6. if largest $\neq i$

$T_{BH}(n) = \lfloor \frac{n}{2} \rfloor - 1 \sum_{i=0}^{n-1} c_{2} T_{MH}(n, i)$

$\leq c_{4} n \log_{2} n$

$T_{HS}(n) = c_{1} + T_{BH}(n) + (n - 1) c_{2} T_{MH}(n, 0)$

$\leq c_{4} n \log_{2} n + (n - 1) c_{2} c_{4} \log_{2} n = O(n \log n)$
Chapter 6. Heapsort

Subroutine \textsc{Max-Heapify}(A, i)

1. \( l = 2 \times i + 1 \)
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3. \textbf{if} \ (l \leq \text{heapsize}[A]) \textbf{and} \ (A[l] > A[i])
4. \textbf{then} \ largest = l
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6. \textbf{if} \ (r \leq \text{heapsize}[A]) \textbf{and} \ (A[r] > A[largest])
7. \textbf{then} \ largest = r
8. \textbf{if} \ largest \neq i
9. \textbf{then} \ exchange \ A[i] \leftrightarrow A[largest]
Chapter 6. Heapsort

Subroutine MAX-HEAPIFY($A, i$)

1. $l = 2 \times i + 1$
2. $r = 2 \times i + 2$
3. if ($l \leq \text{heapsize}[A]$ and ($A[l] > A[i]$))
   then largest = $l$
4. else largest = $i$
5. if ($r \leq \text{heapsize}[A]$ and ($A[r] > A[\text{largest}$])
   then largest = $r$
6. if largest $\neq i$
   then exchange $A[i] \leftrightarrow A[\text{largest}]
10. MAX-HEAPIFY($A, \text{largest}$)
Chapter 6. Heapsort

Subroutine **MAX-HEAPIFY**(*A*, *i*)

1. \( l = 2 \times i + 1 \)
2. \( r = 2 \times i + 2 \)
3. \( \text{if } (l \leq \text{heapsize}[A]) \text{ and } (A[l] > A[i]) \)
   \( \text{then } \text{largest} = l \)
4. \( \text{else } \text{largest} = i \)
5. \( \text{if } (r \leq \text{heapsize}[A]) \text{ and } (A[r] > A[\text{largest}]) \)
6. \( \text{then } \text{largest} = r \)
7. \( \text{if } \text{largest} \neq i \)
8. \( \text{then exchange } A[i] \leftrightarrow A[\text{largest}] \)
9. \( \text{MAX-HEAPIFY}(A, \text{largest}) \)

\[ T_{MH}(n, i) \leq c_3 + T_{MH}(n, 2i + 1) \]

Because \( T_{MH}(n, i) = c_3 + T_{MH}(n, 2i + 1) \), or
\[ = c_3 + T_{MH}(n, 2i + 2) \]
Chapter 6. Heapsort

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7. then \( \text{largest} = r \)
8. if \( \text{largest} \neq i \)
9. then exchange \( A[i] \leftrightarrow A[\text{largest}] \)
10. MAX-HEAPIFY(A, largest)

\[ T_{MH}(n, i) \leq c_3 + T_{MH}(n, 2i + 1) \quad \text{Because } T_{MH}(n, i) = c_3 + T_{MH}(n, 2i + 1), \text{ or} \]
\[ = c_3 + T_{MH}(n, 2i + 2) \]

\[ T_{MH}(n, i) \leq c_4 \log_2 n, \text{ for all } i = 0, 1, \ldots, n - 1. \ (\text{Prove it!}) \]
Chapter 6. Heapsort

Subroutine MAX-HEAPIFY(A, i)

1. \( l = 2 \times i + 1 \)
2. \( r = 2 \times i + 2 \)
3. if \( (l \leq \text{heapsize}[A]) \text{ and } (A[l] > A[i]) \)
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10. MAX-HEAPIFY(A, largest)

\( T_{MH}(n, i) \leq c_3 + T_{MH}(n, 2i + 1) \)
Because \( T_{MH}(n, i) = c_3 + T_{MH}(n, 2i + 1), \) or
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\( T_{MH}(n, i) \leq c_4 \log_2 n, \) for all \( i = 0, 1, \ldots, n - 1. \) (Prove it!)

\( T_{BMH}(n) = \sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor - 1} c_2 T_{MH}(n, i) \)
Chapter 6. Heapsort

Subroutine **MAX-HEAPIFY**\((A, i)\)

1. \(l = 2 \times i + 1\)
2. \(r = 2 \times i + 2\)
3. if \((l \leq \text{heapsize}[A])\) and \((A[l] > A[i])\)
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8. if \(\text{largest} \neq i\)
9. then exchange \(A[i] \leftrightarrow A[\text{largest}]\)
10. MAX-HEAPIFY\((A, \text{largest})\)

\[T_{MH}(n, i) \leq c_3 + T_{MH}(n, 2i + 1)\]  
Because \(T_{MH}(n, i) = c_3 + T_{MH}(n, 2i + 1)\), or \(= c_3 + T_{MH}(n, 2i + 2)\)

\[T_{MH}(n, i) \leq c_4 \log_2 n, \text{ for all } i = 0, 1, \ldots, n - 1. \] (Prove it!)

\[T_{BMH}(n) = \sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor - 1} c_2 T_{MH}(n, i) \leq c_4 \frac{n}{2} \log_2 n\]
Chapter 6. Heapsort

Subroutine MAX-HEAPIFY($A, i$)

1. $l = 2 \times i + 1$
2. $r = 2 \times i + 2$
3. if $(l \leq \text{heapsize}[A])$ and $(A[l] > A[i])$
   4. then largest $= l$
   5. else largest $= i$
6. if $(r \leq \text{heapsize}[A])$ and $(A[r] > A[\text{largest}])$
   7. then largest $= r$
8. if largest $\neq i$
   9. then exchange $A[i] \leftrightarrow A[\text{largest}]
10. \quad \text{MAX-HEAPIFY}(A, \text{largest})$

$T_{MH}(n, i) \leq c_3 + T_{MH}(n, 2i + 1)$  Because $T_{MH}(n, i) = c_3 + T_{MH}(n, 2i + 1)$, or
   $= c_3 + T_{MH}(n, 2i + 2)$

$T_{MH}(n, i) \leq c_4 \log_2 n$, for all $i = 0, 1, \ldots, n - 1$. (Prove it!)

$T_{BMH}(n) = \sum_{i=0}^{\lfloor n/2 \rfloor - 1} c_2 T_{MH}(n, i) \leq c_4 \frac{n}{2} \log_2 n$

$T_{HS}(n) = c_1 + T_{BMH}(n) + (n - 1)c_2 T_{MH}(n, 0)$
Chapter 6. Heapsort

Subroutine MAX-HEAPIFY(\(A, i\))

1. \(l = 2 \times i + 1\)
2. \(r = 2 \times i + 2\)
3. if \((l \leq \text{heapsize}[A]) \text{ and } (A[l] > A[i])\)
4. then \(\text{largest} = l\)
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10. MAX-HEAPIFY(\(A, \text{largest}\))

\(T_{MH}(n, i) \leq c_3 + T_{MH}(n, 2i + 1)\) Because \(T_{MH}(n, i) = c_3 + T_{MH}(n, 2i + 1)\), or \(= c_3 + T_{MH}(n, 2i + 2)\)

\(T_{MH}(n, i) \leq c_4 \log_2 n\), for all \(i = 0, 1, \ldots, n - 1\). (Prove it!)

\(T_{BMH}(n) = \sum_{i=0}^{\lfloor n/2 \rfloor - 1} c_2 T_{MH}(n, i) \leq c_4 \frac{n}{2} \log_2 n\)

\(T_{HS}(n) = c_1 + T_{BMH}(n) + (n - 1)c_2 T_{MH}(n, 0) \leq c_4 \frac{n}{2} \log_2 n + (n - 1)c_2 c_4 \log_2 n\)
Chapter 6. Heapsort

Subroutine MAX-HEAPIFY(A, i)

1. \( l = 2 \times i + 1 \)
2. \( r = 2 \times i + 2 \)
3. \( \text{if } (l \leq \text{heapsize}[A]) \text{ and } (A[l] > A[i]) \) then \( \text{largest} = l \)
4. \( \text{else largest} = i \)
5. \( \text{if } (r \leq \text{heapsize}[A]) \text{ and } (A[r] > A[\text{largest}]) \) then \( \text{largest} = r \)
6. \( \text{if largest} \neq i \) then exchange \( A[i] \leftrightarrow A[\text{largest}] \)
7. \( \text{MAX-HEAPIFY}(A, \text{largest}) \)

\[ T_{MH}(n, i) \leq c_3 + T_{MH}(n, 2i + 1) \quad \text{Because } T_{MH}(n, i) = c_3 + T_{MH}(n, 2i + 1), \text{ or } c_3 + T_{MH}(n, 2i + 2) \]

\[ T_{MH}(n, i) \leq c_4 \log_2 n, \text{ for all } i = 0, 1, \ldots, n - 1. \text{ (Prove it!)} \]

\[ T_{BMH}(n) = \sum_{i=0}^{[n/2]-1} c_2 T_{MH}(n, i) \leq c_4 \frac{n}{2} \log_2 n \]

\[ T_{HS}(n) = c_1 + T_{BMH}(n) + (n - 1)c_2 T_{MH}(n, 0) \leq c_4 \frac{n}{2} \log_2 n + (n - 1)c_2 c_4 \log_2 n = O(n \log n) \]
Chapter 6. Heapsort

Operations on heaps:
Chapter 6. Heapsort

Operations on heaps:

Function **Heap-Maximum**\((A)\)
1. **return** \((A[1])\)

obtain the maximum
Chapter 6. Heapsort

Operations on heaps:

Function **Heap-Maximum*(A)*
1. **return** *(A[1])* obtain the maximum

Function **Heap-Extract-Max*(A)*
1. **if** *heapsize[A] < 1*
2. **then return** *"heap underflow"
3. *max = A[1]*
6. **Max-Heapify**(A, 1)
7. **return** *(max)* obtain and remove the maximum
Chapter 6. Heapsort

Operations on heaps:

Function **Heap-Maximum**(A)
1. return (A[1])

Function **Heap-Extract-Max**(A)
1. if heapsize[A] < 1
2. then return ("heap underflow")
3. max = A[1]
5. heapsize[A] = heapsize[A] - 1
6. **Max-Heapify**(A, 1)
7. return (max)

Function **Heap-Increase-Key**(A, i, key)
1. if key < A[i]
2. then return ("new key is smaller than current key")
3. A[i] = key
4. while i > 1 and A[PARENT[i]] < A[i]
6. i = PARENT[i]
Chapter 6. Heapsort

Operations on heaps:

Function **Heap-Maximum(A)**
1. return \( A[1] \)

Function **Heap-Extract-Max(A)**
1. if heapsize\[A\] < 1
2. then return ("heap underflow")
3. max = \( A[1] \)
4. \( A[1] = A[\text{heapsize}[A]] \)
5. heapsize\[A\] = heapsize\[A\] − 1
6. **Max-Heapify(A, 1)**
7. return \( (max) \)

Function **Heap-Increase-Key(A, i, key)**
1. if key < \( A[i] \)
2. then return ("new key is smaller than current key")
3. \( A[i] = key \)
4. while \( i > 1 \) and \( A[PARENT[i]] < A[i] \)
5. exchange \( A[i] \leftrightarrow A[PARENT[i]] \)
6. \( i = PARENT[i] \)

Function **Max-Heap-Insert(A, key)**
1. heapsize\[A\] = heapsize\[A\] + 1
2. \( A[\text{heapsize}[A]] = -\infty \)
3. **Heap-Increase-Key(A, heapsize\[A\], key)**
Chapter 7. Quicksort

Chapter 7. Quicksort and randomized algorithms
Chapter 7. Quicksort

Chapter 7. Quicksort and randomized algorithms

Idea of the Quicksort: divide-and-conquer
Chapter 7. Quicksort and randomized algorithms

Idea of the Quicksort: divide-and-conquer

- divide: re-organize list $A[p, r]$ into two sublists $A[p, q - 1]$ and $A[q + 1, r]$ based on pivot $A[q]$, such that

  (a) $A[i] \leq A[q]$ for all $i = p, \ldots, q - 1$

  (b) $A[i] \geq A[q]$ for all $i = q + 1, \ldots, r$
Chapter 7. Quicksort

Chapter 7. Quicksort and randomized algorithms

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Chapter 7. Quicksort

Algorithm QuickSort \((A,p,r)\)

1. if \(p < r\)
2. then \(q = \text{Partition} (A,p,r)\)
3. QuickSort \((A,p,q-1)\)
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How the pivot \(A[q]\) is identified is crucial to the performance of Quicksort.

- Assume \(A[q]\) partitions list \(A,p,r\) evenly, then 
  \[ T(n) \leq 2T(n/2) + cn = O(n \log_2 n) \]

- Assume \(A[q]\) partitions the list 20% vs 80%, then
  \[ T(n) \leq T(5n) + T(4n5) + cn = O(n \log_2 n) \]

- Assume \(A[q]\) partitions the list 1% vs 99%, then
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Chapter 7. Quicksort

Chapter 7. Quicksort and Randomized algorithms

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Chapter 7. Quicksort

Chapter 7. Quicksort and Randomized algorithms

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Chapter 7. Quicksort

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Chapter 7. Quicksort

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Chapter 7. Quicksort

**Chapter 7. Quicksort and Randomized algorithms**

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Chapter 7. Quicksort

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How can we identify such a pivot?
Chapter 7. Quicksort
Chapter 7. Quicksort

\[
\begin{array}{cccccc}
\text{i} & \text{p} & \text{j} & \text{r} \\
2 & 8 & 7 & 1 & 3 & 5 & 6 & 4 \\
p & i & j & r \\
2 & 8 & 7 & 1 & 3 & 5 & 6 & 4 \\
p & i & j \\
2 & 8 & 7 & 1 & 3 & 5 & 6 & 4 \\
p & i & j \\
2 & 1 & 7 & 8 & 3 & 5 & 6 & 4 \\
p & i & j \\
2 & 1 & 3 & 8 & 7 & 5 & 6 & 4 \\
p & i & j \\
2 & 1 & 3 & 8 & 7 & 5 & 6 & 4 \\
p & i & j \\
2 & 1 & 3 & 4 & 7 & 5 & 6 & 8 \\
\end{array}
\]

\text{PARTITION}(A, p, r)

1. \( x \leftarrow A[r] \)
2. \( i \leftarrow p - 1 \)
3. \( \text{for } j \leftarrow p \text{ to } r - 1 \)
4. \( \text{do if } A[j] \leq x \)
5. \( \text{then } i \leftarrow i + 1 \)
6. \( \text{exchange } A[i] \leftrightarrow A[j] \)
7. \( \text{exchange } A[i + 1] \leftrightarrow A[r] \)
8. \( \text{return } i + 1 \)

\text{quicksort}(A, p, q-1) \quad \text{quicksort}(A, q+1, r)
Partition may not guarantee to partition the list to two fractions of sizes \( \epsilon n : (1 - \epsilon)n \), for a constant \( \epsilon > 0 \).
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Chapter 7. Quicksort

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- however, chances for skewed cases like above are very small.
- that is, the cases other than the skewed ones occur much more often.

So the idea of Quicksort may work well on a majority of data.
Chapter 7. Quicksort

Assume that the equal likely chance for every number to be in the last position, what is the chance to partition the list into

\[ x\% \quad \text{vs} \quad (100 - x)\% \]

fragments, for \(10 \leq x \leq 90\)?
Chapter 7. Quicksort

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fragments, for \( 10 \leq x \leq 90 \)?

The chance is \( = 80\% \)
Chapter 7. Quicksort

What running time would it be if 10:90 partition is always guaranteed?
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\[ T(n) \leq T(n/10) + T(9n/10) + cn \]
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\[ l_0: \quad cn \]

\[ l_1: \quad cn/10 \]

\[ l_2: \quad cn/10^2 \]

\[ \vdots \]

\[ l_h: \quad cn/10^h \]

\[ l_k: \quad c \]

where \((10/9)n = 1\), i.e., \(h = \log_{10} n\)

\[ \text{where } c' = c/\log_{10} 9 \]

\[ T(n) \leq cn \log_{10} n = O(n \log n) \]
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  l_1: & \quad cn/10 \quad 9cn/10
\end{align*}
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Chapter 7. Quicksort

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\[ l_2: \quad \text{cn/10^2} \quad \text{9cn/10^2} \quad \text{9^2cn/10^2} \]

\[ \cdots \cdots \]

\[ \text{where} \quad c' = c/\log_{10} 9 \]
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    l_h: & cn/10^h & \cdots & \cdots & c9^hn/10^h & cn \\
\end{array} \]
Chapter 7. Quicksort

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| \( l_2 \): | \( cn/10^2 \) | \( 9cn/10^2 \) | \( 9^2 cn/10^2 \) | \( cn \) |
| \( \ldots \ldots \): | \( \ldots \ldots \) | \( \ldots \ldots \) | \( \ldots \ldots \) | \( \ldots \ldots \) |
| \( l_h \): | \( cn/10^h \) | \( \ldots \ldots \) | \( c9^h n/10^h \) | \( cn \) |

where \( c' \) is a constant.
Chapter 7. Quicksort

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\[ \vdots \]
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l_h: & \quad cn/10^h, \ldots, c9^h n/10^h, \ldots, cn \\
l_k: & \quad \ldots, c9^k n/10^k \leq cn
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\[ cn \log_{10} n \leq T(n) \leq cn \log_{\frac{10}{9}} n \]

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\begin{align*}
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  \cdots & \\
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where \((\frac{1}{10})^hn = 1\), i.e., \(h = \log_{10} n\)

\((\frac{9}{10})^kn = 1\), i.e., \(k = \log_{\frac{9}{10}} n\)

\[ cn \log_{10} n \leq T(n) \leq cn \log_{\frac{9}{10}} n \]

\[ T(n) \leq cn \log_{\frac{9}{10}} n = cn \frac{\log_2 n}{\log_2 \frac{10}{9}} = c'n \log_2 n = O(n \log_2 n) \]

where \(c' = c/\log_2 \frac{10}{9}\)
Chapter 7. Quicksort

Instead of analyzing QuickSort (with uniformly distributed for input) we design a randomized version of the algorithm and analyze it.
Instead of analyzing \texttt{QUICKSORT} (with uniformly distributed for input) we design a randomized version of the algorithm and analyze it.

Algorithm \texttt{RANDOMIZED-PARTITION}(A, p, r)
1. \( i = random(p, r) \)
2. exchange \( A[r] \leftrightarrow A[i] \)
3. return \((\texttt{PARTITION}(A, p, r))\)
Chapter 7. Quicksort

Instead of analyzing QuickSort (with uniformly distributed for input) we design a randomized version of the algorithm and analyze it.

Algorithm Randomized-Partition($A, p, r$)
1. $i = \text{random}(p, r)$
2. exchange $A[r] \leftarrow A[i]$
3. return ($\text{Partiton}(A, p, r)$)

Algorithm Randomized QuickSort ($A, p, r$)
Chapter 7. Quicksort

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3. return (Partition($A, p, r$))

Algorithm Randomized QuickSort ($A, p, r$)
1. \( \text{if } p < r \)
Instead of analyzing QuickSort (with uniformly distributed for input) we design a randomized version of the algorithm and analyze it.

Algorithm **RANDOMIZED-PARTITION** \((A, p, r)\)
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Algorithm **RANDOMIZED QUICKSORT** \((A, p, r)\)
1. if \(p < r\)
2. then \(q = \text{RANDOMIZED-PARTITION}(A, p, r)\)
Chapter 7. Quicksort

Instead of analyzing \texttt{QuickSort} (with uniformly distributed for input) we design a randomized version of the algorithm and analyze it.

Algorithm \texttt{Randomized-Partition}(A, p, r)
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2. exchange \( A[r] \leftrightarrow A[i] \)
3. \textbf{return} (\texttt{Partition}(A, p, r))

Algorithm \texttt{Randomized QuickSort} (A, p, r)
1. \textbf{if} \( p < r \)
2. \textbf{then} \( q = \text{Randomized-Partition}(A, p, r) \)
3. \textbf{Randomized QuickSort} (A, p, q − 1)
Instead of analyzing \texttt{QUICKSORT} (with uniformly distributed for input) we design a randomized version of the algorithm and analyze it.

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Algorithm \texttt{RANDOMIZED QUICKSORT} (\(A, p, r\))
1. \textbf{if} \(p < r\)
2. \textbf{then} \(q = \text{RANDOMIZED-PARTITION}(A, p, r)\)
3. \texttt{RANDOMIZED QUICKSORT} \((A, p, q - 1)\)
4. \texttt{RANDOMIZED QUICKSORT} \((A, q + 1, r)\)
Chapter 7. Quicksort

Up to this point, you should have known:

1. the details of QuickSort algorithm, especially Partition;
2. Why it runs $O(n \log n)$ on uniformly distributed data, intuitively;
3. the connection between
   (a) requiring prob distribution in the input data;
   (b) randomized algorithms;
Chapter 7. Quicksort

Analysis of RANDOMIZED-QUICKSORT
Analysis of Randomized-QuickSort

• count the expected number of comparisons between $x_i$ and $x_j$;
Chapter 7. Quicksort

Analysis of \textsc{Randomized-QuickSort}

\begin{itemize}
  \item count the expected number of comparisons between $x_i$ and $x_j$;
\end{itemize}

\textbf{Observation 1}: $x_i$ is compared with $x_j$ \textbf{only when either is a pivot};
Chapter 7. Quicksort

Analysis of **RANDOMIZED-QUICKSORT**

- count the expected number of comparisons between $x_i$ and $x_j$;

**Observation 1:** $x_i$ is compared with $x_j$ only when either is a pivot;

**Observation 2:** $x_i$ is compared with $x_j$ at most once;
Chapter 7. Quicksort

Analysis of Randomized-QuickSort

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Analysis of Randomized-QuickSort

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$$X_{i,j} = 1 \text{ iff a comparison between } x_i \text{ and } x_j \text{ occurs}$$
Chapter 7. Quicksort

Analysis of Randomized-QuickSort

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- let \( X = \sum_{i<j} X_{i,j} \), total number of comparisons
Chapter 7. Quicksort

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\[
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Chapter 7. Quicksort

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Chapter 7. Quicksort

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\]

by linearity of expectations.
Analysis of \textsc{Randomized-Quicksort} (cont.)

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Analysis of RANDOMIZED-QUICKSORT (cont.)

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Chapter 7. Quicksort

Analysis of **RANDOMIZED-QUICKSORT** (cont.)

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Chapter 7. Quicksort

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Analysis of \texttt{RANDOMIZED-QUICKSORT} (cont.)

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but we do not know the size of the sublist \( L \)!
Analysis of **RANDOMIZED-QUICKSORT** (cont.)

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Analysis of **RANDOMIZED-QUICKSORT** (cont.)

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Chapter 7. Quicksort

Analysis of RANDOMIZED-QUICKSORT (cont.)

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\[ |L| \geq (j - i + 1) \]
Chapter 7. Quicksort

Analysis of \textsc{Randomized-Quicksort} (cont.)

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So
\[ P(X_{i,j} = 1) \leq 2 \frac{1}{|L|} \leq 2 \frac{1}{j - i + 1} \]
Chapter 7. Quicksort

original unsorted list

5 23 10

sublist \( L \) containing elements 5 and 10
10 is a pivot

\[
\text{L has to contain elements between 5 and 10}
\]
i.e., \( L \) has to contain elements 6, 7, 8, 9
\[
|L| \geq j - i + 1 = 10 - 5 + 1 = 6
\]

final sorted list

1 2 3 4 5 6 7 8 9 10

\( x_5 \) \hspace{1cm} \( x_{10} \)
Analysis of \textsc{Randomized-QuickSort} (cont.)

\[ E(X) = E(\sum_{i<j} X_{i,j}) = \sum_{i<j} E(X_{i,j}) = \sum_{i<j} \text{Prob}(X_{i,j} = 1) \]

\[ \leq \sum_{i<j} 2 \frac{1}{j - i + 1} \leq \sum_{i=1}^{n-1} \frac{1}{2} \frac{1}{k + 1} \leq cn \log_2 n \]

for some constant \( c > 0 \).

So \( E(X) = O(n \log_2 n) \).
Chapter 7. Quicksort

Analysis of \textsc{Randomized-QuickSort} (cont.)

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Chapter 7. Quicksort

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Chapter 7. Quicksort

Analysis of Randomized-QuickSort (cont.)

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So \( E(X) = O(n \log_2 n) \).
Chapter 7. Quicksort

O(n log n) Sorting Algorithms

![Graph showing O(n log n) Sorting Algorithms with curves for Heap, Merge, and Quick sorts.](image-url)
• We have used Big-$O$ for upper bounds.
• We need another notation for lower bounds.

Define $\Omega(g(n))$ be the set of functions that have growth rates not slower than $cg(n)$ for any given constant $c > 0$.

$\Omega(g(n)) = \{ f(n) : \exists c > 0, k > 0 \text{ such that } f(n) \geq cg(n) \text{ for all } n \geq k \}$

• e.g., $\Omega(n \log n)$ includes the following functions:
  14 \ n \log n, \frac{1}{100} \ n \log n, n^2, n^3 \log n, 3^{7002} \ n, n!,$ ...

• Proof techniques for Big-$\Omega$ are similar to those for Big-$O$. 

Chapter 8. Lower Bounds and Sorting in Linear Time
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- We have used Big-$O$ for upper bounds.
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• Proof techniques for Big-$\Omega$ are similar to those for Big-$O$. 
Upper bound of an algorithm

A time sufficient (i.e., enough) for the algorithm to solve all instances.

We make sure an upper bound should covers all instances; e.g., MergeSort has upper bound $O(n \log n)$.

Is it correct to say MergeSort has upper bound $O(n^2)$?

It is correct for two reasons:

1. since $cn \log n$ is sufficient, so is $cn^2$.
2. $O(n^2)$ contains all functions that $O(n \log n)$ contains.

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2. $O(n^2)$ contains all functions that $O(n \log n)$ contains.
Upper bound of an algorithm

A time sufficient (i.e., enough) for the algorithm to solve all instances.

We make sure an upper bound should covers all instances;

e.g., \textsc{MergeSort} has upper bound $O(n \log n)$.

Is it correct to say \textsc{MergeSort} has upper bound $O(n^2)$?

It is correct for two reasons:

(1) since $cn \log n$ is sufficient, so is $cn^2$.

(2) $O(n^2)$ contains all functions that $O(n \log n)$ contains.

Is it correct to say \textsc{MergeSort} has upper bound $O(n)$?
Chapter 8. Lower Bounds and Sorting in Linear Time

Lower bound of an algorithm: A time necessary (i.e., needed) for the algorithm to solve all instances. \( l(n) \) is a lower bound – if some (generic) instance requires time \( l(n) \) or more to be solved by the algorithm.

For example, MergeSort has lower bound \( \Omega(n \log n) \).

Is it correct to say MergeSort has lower bound \( \Omega(n) \)?

It is correct for two reasons:

1. Since \( cn \log n \) is necessary, so is \( cn \).
2. \( \Omega(n) \) contains all functions that \( \Omega(n \log n) \) contains.

Is it correct to say MergeSort has lower bound \( \Omega(n^2) \)?
Lower bound of an algorithm

It is correct to say \( \text{MergeSort} \) has lower bound \( \Omega(n \log n) \) because:

1. Since \( cn \log n \) is necessary, so is \( cn \).
2. \( \Omega(n \log n) \) contains all functions that \( \Omega(n \log^2 n) \) contains.

Is it correct to say \( \text{MergeSort} \) has lower bound \( \Omega(n^2) \)?
Lower bound of an algorithm

A time necessary (i.e., needed) for the algorithm to solve all instances.
Chapter 8. Lower Bounds and Sorting in Linear Time

**Lower bound of an algorithm**

A time necessary (i.e., needed) for the algorithm to solve all instances.

$l(n)$ is a lower bound – if some (generic) instance requires time $l(n)$ or more to be solved by the algorithm.
Chapter 8. Lower Bounds and Sorting in Linear Time

**Lower bound of an algorithm**

A time necessary (i.e., needed) for the algorithm to solve all instances.

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e.g., MergeSort has lower bound $\Omega(n \log n)$
Lower bound of an algorithm

A time necessary (i.e., needed) for the algorithm to solve all instances.

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Is it correct to say MergeSort has lower bound $\Omega(n)$?
Chapter 8. Lower Bounds and Sorting in Linear Time

**Lower bound of an algorithm**

A time necessary (i.e., needed) for the algorithm to solve all instances.

$l(n)$ is a lower bound – if some (generic) instance requires time $l(n)$ or more to be solved by the algorithm.

e.g., **mergeSort** has lower bound $\Omega(n \log n)$

Is it correct to say **mergeSort** has lower bound $\Omega(n)$?

It is correct for two reasons:
Chapter 8. Lower Bounds and Sorting in Linear Time

Lower bound of an algorithm
A time necessary (i.e., needed) for the algorithm to solve all instances.

\( l(n) \) is a lower bound – if some (generic) instance requires time \( l(n) \) or more to be solved by the algorithm.

e.g., \texttt{MergeSort} has lower bound \( \Omega(n \log n) \)

Is it correct to say \texttt{MergeSort} has lower bound \( \Omega(n) \)?

It is correct for two reasons:

(1) since \( cn \log n \) is necessary, so is \( cn \).
Chapter 8. Lower Bounds and Sorting in Linear Time

Lower bound of an algorithm

A time necessary (i.e., needed) for the algorithm to solve all instances.

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Is it correct to say `MergeSort` has lower bound $\Omega(n)$?

It is correct for two reasons:

1. since $cn \log n$ is necessary, so is $cn$.
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Chapter 8. Lower Bounds and Sorting in Linear Time

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Is it correct to say MergeSort has lower bound \( \Omega(n) \)?

It is correct for two reasons:

1. since \( cn \log n \) is necessary, so is \( cn \).
2. \( \Omega(n) \) contains all functions that \( \Omega(n \log n) \) contains.

Is it correct to say MergeSort has lower bound \( \Omega(n^2) \)?
Chapter 8. Lower Bounds and Sorting in Linear Time

- The best known upper bound for MergeSort is $O(n \log n)$, coinciding with the best known lower bound $\Omega(n \log n)$;
- Both bounds are tight (i.e., optimal). Thus complexity is denoted with $\Theta(n \log n)$, meaning both $O(n \log n)$ and $\Omega(n \log n)$.
The best known upper bound for `MERGESORT` is $O(n \log n)$,
The best known upper bound for \textsc{MergeSort} is $O(n \log n)$, coinciding with the best known lower bound $\Omega(n \log n)$;
The best known upper bound for \textsc{MergeSort} is $O(n \log n)$, coinciding with the best known lower bound $\Omega(n \log n)$;

Both bounds are tight (i.e., optimal). Thus complexity is denoted with $\theta(n \log n)$, meaning both $O(n \log n)$ and $\Omega(n \log n)$. 

The best known upper bound for \textsc{MergeSort} is $O(n \log n)$, coinciding with the best known lower bound $\Omega(n \log n)$; both bounds are tight (i.e., optimal). Thus complexity is denoted with $\theta(n \log n)$, meaning both $O(n \log n)$ and $\Omega(n \log n)$.

We may not be so lucky for some other algorithms.
Chapter 8. Lower Bounds and Sorting in Linear Time

\textbf{Rec-Fibonacci}(n)

\begin{verbatim}
if \( n = 1 \) or \( n = 2 \), \textbf{return} (1);
else
    \( T_1 = \textbf{Rec-Fibonacci}(n - 1) \);
    \( T_2 = \textbf{Rec-Fibonacci}(n - 2) \);
    \textbf{return} (\( T_1 + T_2 \));
\end{verbatim}
Chapter 8. Lower Bounds and Sorting in Linear Time

**Rec-Fibonacci**$(n)$

if $n = 1$ or $n = 2$, return $(1)$;
else
    $T_1 = \text{Rec-Fibonacci}(n - 1)$;
    $T_2 = \text{Rec-Fibonacci}(n - 2)$;
return $(T_1 + T_2)$;

Derive an upper bound:
Chapter 8. Lower Bounds and Sorting in Linear Time

**Rec-Fibonacci**\( (n)\)

if \( n = 1 \) or \( n = 2 \), return (1);
else
\[ T_1 = \text{Rec-Fibonacci}(n - 1); \]
\[ T_2 = \text{Rec-Fibonacci}(n - 2); \]
return \( T_1 + T_2 \);

Derive an upper bound:
\[ T(n) = c + T(n - 1) + T(n - 2), \text{ with } T(1) = T(2) = c \]
Chapter 8. Lower Bounds and Sorting in Linear Time

**Rec-Fibonacci**

if $n = 1$ or $n = 2$, return (1);
else

$T_1 = \text{Rec-Fibonacci}(n - 1);
T_2 = \text{Rec-Fibonacci}(n - 2);
return (T_1 + T_2);

Derive an upper bound:
$T(n) = c + T(n - 1) + T(n - 2)$, with $T(1) = T(2) = c$
$\leq c + 2T(n - 1)$
Chapter 8. Lower Bounds and Sorting in Linear Time

\texttt{Rec-Fibonacci}(n)

\begin{verbatim}
if \( n = 1 \) or \( n = 2 \), \texttt{return} (1);
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Derive an upper bound:
\[ T(n) = c + T(n - 1) + T(n - 2), \text{ with } T(1) = T(2) = c \]
\[ \leq c + 2T(n - 1) \]
\[ \leq c + 2c + 2^2 T(n - 2) \]
Chapter 8. Lower Bounds and Sorting in Linear Time

Rec-Fibonacci(n)

if $n = 1$ or $n = 2$, return (1);
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    $T_1 =$ Rec-Fibonacci($n - 1$);
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Derive an upper bound:
$T(n) = c + T(n - 1) + T(n - 2)$, with $T(1) = T(2) = c$
$\leq c + 2T(n - 1)$
$\leq c + 2c + 2^2 T(n - 2)$
\ldots
**Chapter 8. Lower Bounds and Sorting in Linear Time**

**Rec-Fibonacci**

\[
\text{if } n = 1 \text{ or } n = 2, \text{ return } (1); \\
\text{else} \\
\quad T_1 = \text{Rec-Fibonacci}(n - 1); \\
\quad T_2 = \text{Rec-Fibonacci}(n - 2); \\
\quad \text{return } (T_1 + T_2); \\
\]

Derive an upper bound:

\[
T(n) = c + T(n - 1) + T(n - 2), \text{ with } T(1) = T(2) = c \\
\leq c + 2T(n - 1) \\
\leq c + 2c + 2^2T(n - 2) \\
\ldots \\
\leq c + 2c + \ldots 2^{n-3}c + 2^{n-2}T(2)
\]

\[= O(2^n) \]
Chapter 8. Lower Bounds and Sorting in Linear Time

Rec-Fibonacci \( (n) \)

\[
\begin{align*}
\text{if } n = 1 \text{ or } n = 2, & \quad \text{return } (1); \\
\text{else} & \\
T_1 &= \text{Rec-Fibonacci} (n - 1); \\
T_2 &= \text{Rec-Fibonacci} (n - 2); \\
\text{return } (T_1 + T_2); \\
\end{align*}
\]

Derive an upper bound:
\[
T(n) = c + T(n - 1) + T(n - 2), \quad \text{with } T(1) = T(2) = c \\
\leq c + 2T(n - 1) \\
\leq c + 2c + 2^2T(n - 2) \\
\ldots \\
\leq c + 2c + \ldots 2^{n-3}c + 2^{n-2}T(2) \\
= \frac{2^{n-2}-1}{2-1}c + 2^{n-2}c
\]
Rec-Fibonacci(n)

if $n = 1$ or $n = 2$, return (1);
else
    $T_1 = \text{Rec-Fibonacci}(n - 1)$;
    $T_2 = \text{Rec-Fibonacci}(n - 2)$;
    return ($T_1 + T_2$);

Derive an upper bound:
$T(n) = c + T(n - 1) + T(n - 2)$, with $T(1) = T(2) = c$
    $\leq c + 2T(n - 1)$
    $\leq c + 2c + 2^2T(n - 2)$
    ...
    $\leq c + 2c + \ldots 2^{n-3}c + 2^{n-2}T(2)$
    $= \frac{2^{n-2} - 1}{2-1}c + 2^{n-2}c$
    $= (2^{n-2} - 1)c + 2^{n-2}c$
Chapter 8. Lower Bounds and Sorting in Linear Time

**Rec-Fibonacci**($n$)

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\text{if } n = 1 \text{ or } n = 2, \text{ return } (1);
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T_1 = \text{Rec-Fibonacci}(n - 1);
\]

\[
T_2 = \text{Rec-Fibonacci}(n - 2);
\]

\text{return } (T_1 + T_2);

Derive an upper bound:

\[
T(n) = c + T(n - 1) + T(n - 2), \text{ with } T(1) = T(2) = c
\]

\[
\leq c + 2T(n - 1)
\]

\[
\leq c + 2c + 2^2T(n - 2)
\]

\[
\ldots
\]

\[
\leq c + 2c + \ldots 2^{n-3}c + 2^{n-2}T(2)
\]

\[
= \frac{2^{n-2} - 1}{2-1}c + 2^{n-2}c
\]

\[
= (2^{n-2} - 1)c + 2^{n-2}c
\]

\[
= (2^{n-1} - 1)c
\]

\[O(2^n).\]
Rec-Fibonacci($n$)

if $n = 1$ or $n = 2$, return (1);
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$\leq c + 2T(n - 1)$
$\leq c + 2c + 2^2T(n - 2)$

$\ldots$

$\leq c + 2c + \ldots 2^{n-3}c + 2^{n-2}T(2)$

$= \frac{2^{n-2} - 1}{2 - 1}c + 2^{n-2}c$

$= (2^{n-2} - 1)c + 2^{n-2}c$

$= (2^{n-1} - 1)c$

$= O(2^n)$. 

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Rec-Fibonacci(n)

if $n = 1$ or $n = 2$, return (1);
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Chapter 8. Lower Bounds and Sorting in Linear Time

Rec-Fibonacci\( (n) \)

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T_2 &= \text{Rec-Fibonacci}(n - 2); \\
\text{return } (T_1 + T_2);
\end{align*}
\]

Derive a lower bound:

\[
T(n) = c + T(n - 1) + T(n - 2), \text{ with } T(1) = T(2) = c \geq 2T(n - 2) \geq 2 \cdot 2 \cdot T(n - 4) \ldots \geq 2 \cdot n - 2 \cdot 2 \cdot c = 2n^2 - 2c \geq c \cdot 2n \Rightarrow n = \Omega(n^{1.41}) \]

while the derived upper bound is \( O(2^n) \), not tight!
Chapter 8. Lower Bounds and Sorting in Linear Time

Rec-Fibonacci\( (n) \)

if \( n = 1 \) or \( n = 2 \), return \((1)\);
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Chapter 8. Lower Bounds and Sorting in Linear Time

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  return ($T_1 + T_2$);

Derive a lower bound:
$T(n) = c + T(n - 1) + T(n - 2)$, with $T(1) = T(2) = c$
$\geq 2T(n - 2)$
Chapter 8. Lower Bounds and Sorting in Linear Time

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Derive a lower bound:
\[ T(n) = c + T(n - 1) + T(n - 2), \text{ with } T(1) = T(2) = c \]
\[ \geq 2T(n - 2) \]
\[ \geq 2^2T(n - 4) \]
Chapter 8. Lower Bounds and Sorting in Linear Time

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\text{if } n &= 1 \text{ or } n = 2, \text{ return } (1); \\
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\end{align*}
\]

Derive a lower bound:
\[
T(n) = c + T(n - 1) + T(n - 2), \text{ with } T(1) = T(2) = c \\
\geq 2T(n - 2) \\
\geq 2^2T(n - 4) \\
\ldots
\]

while the derived upper bound is \(O(2^n)\), not tight!
Chapter 8. Lower Bounds and Sorting in Linear Time

**Rec-Fibonacci(n)**

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    return $(T_1 + T_2)$;

Derive a lower bound:

$T(n) = c + T(n - 1) + T(n - 2)$, with $T(1) = T(2) = c$

$\geq 2T(n - 2)$

$\geq 2^2 T(n - 4)$

$\ldots$

$\geq 2^{\frac{n-2}{2}} T(2)$
Chapter 8. Lower Bounds and Sorting in Linear Time

Rec-Fibonacci($n$)

if $n = 1$ or $n = 2$, return (1);
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    $T_1 = \text{Rec-Fibonacci}(n - 1)$;
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    return ($T_1 + T_2$);

Derive a lower bound:
$T(n) = c + T(n - 1) + T(n - 2)$, with $T(1) = T(2) = c$
\[ \geq 2T(n - 2) \]
\[ \geq 2^2 T(n - 4) \]
\[ \vdots \]
\[ \geq 2 \frac{n-2}{2} T(2) \]
\[ = 2 \frac{n-2}{2} c \]
Chapter 8. Lower Bounds and Sorting in Linear Time

Rec-Fibonacci\((n)\)

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\text{if } n = 1 \text{ or } n = 2, \text{ return } (1); \\
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T_2 = \text{Rec-Fibonacci}(n - 2); \\
\text{return } (T_1 + T_2);
\]

Derive a lower bound:
\(T(n) = c + T(n - 1) + T(n - 2), \text{ with } T(1) = T(2) = c\)
\[
\geq 2T(n - 2) \\
\geq 2^2 T(n - 4) \\
\ldots \\
\geq 2^{\frac{n-2}{2}} T(2) \\
= 2^{\frac{n-2}{2}} c \\
= 2^{\frac{n}{2}} 2^{-\frac{2}{2}} c
\]
Chapter 8. Lower Bounds and Sorting in Linear Time

**Rec-Fibonacci**

\[
\text{if } n = 1 \text{ or } n = 2, \text{ return } (1); \\
\text{else} \\
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T(n) = c + T(n - 1) + T(n - 2), \text{ with } T(1) = T(2) = c \\
\geq 2T(n - 2) \\
\geq 2^2T(n - 4) \\
\ldots \\
\geq 2^{\frac{n-2}{2}} T(2) \\
= 2^{\frac{n-2}{2}} c \\
= 2^{\frac{n}{2}} 2^{\frac{n}{2}} c \\
= \frac{c}{2} \left(2^{\frac{1}{2}}\right)^n
\]

while the derived upper bound is \(O(2^n)\), not tight!
Chapter 8. Lower Bounds and Sorting in Linear Time

Rec-Fibonacci(n)

if \( n = 1 \) or \( n = 2 \), return (1);
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\( T_2 = \text{Rec-Fibonacci}(n - 2); \)

return \((T_1 + T_2); \)

Derive a lower bound:
\[ T(n) = c + T(n - 1) + T(n - 2), \text{ with } T(1) = T(2) = c \]
\[ \geq 2T(n - 2) \]
\[ \geq 2^2 T(n - 4) \]
\[ \ldots \]
\[ \geq 2 \frac{n-2}{2} T(2) \]
\[ = 2 \frac{n-2}{2} c \]
\[ = 2^\frac{n}{2} 2^{-2} c \]
\[ = \frac{c}{2} (2^{\frac{1}{2}})^n \]
\[ = \frac{c}{2} \sqrt{2}^n \]
Chapter 8. Lower Bounds and Sorting in Linear Time

**Rec-Fibonacci**

if \( n = 1 \) or \( n = 2 \), return \( (1) \);
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T_1 = \text{Rec-Fibonacci}(n - 1); \\
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Derive a lower bound:

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T(n) = c + T(n - 1) + T(n - 2), \text{ with } T(1) = T(2) = c
\]

\[
\geq 2T(n - 2) \\
\geq 2^2 T(n - 4) \\
\ldots \\
\geq 2^{\frac{n-2}{2}} T(2) \\
= 2^{\frac{n-2}{2}} c \\
= 2^{\frac{n}{2}} 2^{-\frac{2}{2}} c \\
= \frac{c}{2} (2^{\frac{1}{2}})^n \\
= \frac{c}{2} \sqrt{2}^n \\
\geq \frac{c}{2} 1.41^n
\]
Chapter 8. Lower Bounds and Sorting in Linear Time

Rec-Fibonacci\( (n) \)

\[
\text{if } n = 1 \text{ or } n = 2, \text{ return } (1); \\
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\text{return } (T_1 + T_2);
\]

Derive a lower bound:
\[
T(n) = c + T(n - 1) + T(n - 2), \text{ with } T(1) = T(2) = c
\]
\[
\geq 2T(n - 2) \\
\geq 2^2 T(n - 4) \\
\ldots \\
\geq 2 \frac{n-2}{2} T(2) \\
= 2 \frac{n-2}{2} c \\
= 2 \frac{n}{2} 2^{\frac{1}{2}} c \\
= \frac{c}{2} (2^{\frac{1}{2}})^n \\
= \frac{c}{2} \sqrt{2}^n \\
\geq \frac{c}{2} 1.41^n \\
= \Omega(1.41^n).
\]
Chapter 8. Lower Bounds and Sorting in Linear Time

**Rec-Fibonacci**

\[
\text{if } n = 1 \text{ or } n = 2, \text{ return } (1);
\]

\[
\text{else}
\]

\[
T_1 = \text{Rec-Fibonacci}(n - 1);
\]

\[
T_2 = \text{Rec-Fibonacci}(n - 2);
\]

\[
\text{return } (T_1 + T_2);
\]

Derive a lower bound:

\[
T(n) = c + T(n - 1) + T(n - 2), \text{ with } T(1) = T(2) = c
\]

\[
\geq 2T(n - 2)
\]

\[
\geq 2^2T(n - 4)
\]

\[
\ldots
\]

\[
\geq 2^{\frac{n-2}{2}} T(2)
\]

\[
= 2^{\frac{n-2}{2}} c
\]

\[
= 2^{\frac{n}{2}} 2^{-\frac{1}{2}} c
\]

\[
= \frac{c}{2} (2^{\frac{1}{2}})^n
\]

\[
= \frac{c}{2} \sqrt{2}^n
\]

\[
\geq \frac{c}{2} 1.41^n
\]

\[
= \Omega(1.41^n). \text{ while the derived upper bound is } O(2^n),
\]
Chapter 8. Lower Bounds and Sorting in Linear Time

**Rec-Fibonacci** $(n)$

- If $n = 1$ or $n = 2$, return $(1)$;
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  - $T_1 = \text{Rec-Fibonacci}(n - 1)$;
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  - return $(T_1 + T_2)$;

Derive a lower bound:

$T(n) = c + T(n - 1) + T(n - 2)$, with $T(1) = T(2) = c$

\[
\begin{align*}
&\geq 2T(n - 2) \\
&\geq 2^2T(n - 4) \\
&\ldots \\
&\geq 2^{\frac{n-2}{2}}T(2) \\
&= 2^{\frac{n-2}{2}}c \\
&= 2^{\frac{n}{2}}2^{\frac{1}{2}}c \\
&= \frac{c}{2}(2^{\frac{1}{2}})^n \\
&= \frac{c}{2}\sqrt{2}^n \\
&\geq \frac{c}{2}1.41^n \\
&= \Omega(1.41^n). \text{ While the derived upper bound is } O(2^n), \text{ not tight!}
\end{align*}
\]
Chapter 8. Lower Bounds and Sorting in Linear Time

Upper bound of a problem

A time sufficient (i.e., enough) to solve all instances of the problem.

To derive an upper bound, we can resort to algorithms solving the problem; an upper bound is of such an algorithm is also an upper bound for the problem.

\( O(n^2) \) is an upper bound for \textit{Sorting} (why?)

\( O(n \log n) \) is also an upper bound for \textit{Sorting} (why?)

One important task in algorithm research: to design algorithms achieving better upper bounds (smaller time complexity)
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Chapter 8. Lower Bounds and Sorting in Linear Time

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Chapter 8. Lower Bounds and Sorting in Linear Time

Lower bound of a problem
A time necessary (i.e., needed) for all instances in the problem to be solved.

Can we use an algorithm lower bound for the problem lower bound?
For example, consider the Sorting problem, Insertion Sort has lower bound $\Omega(n^2)$ (why?),
Can we say the Sorting problem has lower bound $\Omega(n^2)$?
No! because...
MergeSort has upper bound $O(n \log n)$.
Likewise, we cannot say the Sorting problem has lower bound $\Omega(n \log n)$.

Statement “problem Sorting has lower bound $\Omega(n \log n)$” $\iff$ statement “there is no algorithm running faster than time $cn \log n$”.
Chapter 8. Lower Bounds and Sorting in Linear Time

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Chapter 8. Lower Bounds and Sorting in Linear Time

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• To derive a lower bound for a problem, we cannot examine an infinite number of algorithms!
Chapter 8. Lower Bounds and Sorting in Linear Time

Statement “problem **Sorting** has lower bound \( \Omega(n \log n) \)”

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statement “there is no algorithm running faster than time \( cn \log n \)”.

• To derive a lower bound for a **problem**, we **cannot** examine an infinite number of algorithms!

• Lower bounds can only be derived mathematically, but not from existing algorithms.
Chapter 8. Lower Bounds and Sorting in Linear Time

Deriving a lower bound for sorting

with decision tree as algorithm/computation model

Claim 1: total number of leaves is $\geq n!$.

Claim 2: the height of the tree at least $\geq \log n!$. (The minimum of heights of all such trees!)
Chapter 8. Lower Bounds and Sorting in Linear Time

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Chapter 8. Lower Bounds and Sorting in Linear Time

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Chapter 8. Lower Bounds and Sorting in Linear Time

**Theorem:** Sorting needs $\Omega(n \log n)$ comparisons on comparison-based computation models.
Chapter 8. Lower Bounds and Sorting in Linear Time

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**Prove.**
The longest path from the root to a leave is $\Omega(\log n!)$. I.e., the number of comparisons needed in the worst case is $\Omega(\log n!)$. 
Theorem: Sorting needs $\Omega(n \log n)$ comparisons on comparison-based computation models.

Prove.
The longest path from the root to a leaf is $\Omega(\log n!)$. I.e., the number of comparisons needed in the worst case is $\Omega(\log n!)$. 

$$n! = n(n - 1)(n - 2) \cdots (n - \frac{n}{2})(n - \frac{n}{2} - 1) \cdots 2 \times 1$$

$$\geq \left(\frac{n}{2}\right)^{\frac{n}{2}} \times 2^{\frac{n}{2} - 1} \geq \frac{1}{2} n^{\frac{n}{2}}$$

or by Stirling’s formula:

$$n! = \sqrt{2\pi n}(n/e)^n(1 + O(1/n))$$

$$\Omega(\log(n!)) = \Omega(n \log n)$$
Chapter 8. Lower Bounds and Sorting in Linear Time

Sorting algorithms with worst case linear time
(To be covered after the next chapter)
Sorting algorithms with worst case linear time
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- count sort
- radix sort
- bucket sort
Count sort

**Algorithm Counting-Sort** $(A, B, k)$

1. for $i = 0$ to $k$
2. $C[i] = 0$
3. for $j = 1$ to $\text{length}[A]$
5. $\{C[i] \text{ contains the number of elements whose values } = i\}$
6. for $i = 1$ to $k$
7. $C[i] = C[i] + C[i-1]$
8. $\{C[i] \text{ contains the number of elements whose values } \leq i\}$
9. for $j = \text{length}[A]$ downto $1$

Example: $A: 2 5 3 0 2 3 0 3$, $k = 5$, $C: 2 0 2 3 0 1$

**Analysis:** $T(n) = O(k + n)$
Count sort

Algorithm COUNTING-SORT \((A, B, k)\) \{\(A\) contains \(n\) integers; \(k\) is the max\}
Chapter 8. Lower Bounds and Sorting in Linear Time

Count sort

Algorithm COUNTING-SORT \((A, B, k)\) \{ \(A\) contains \(n\) integers; \(k\) is the max\}
1. \textbf{for} \(i = 0\) \textbf{to} \(k\)
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Chapter 8. Lower Bounds and Sorting in Linear Time

Count sort

Algorithm \textsc{Counting-Sort} (A, B, k) \hspace{1cm} \{A contains n integers; k is the max\}
1. \textbf{for} \ i = 0 \textbf{to} k
2. \hspace{1cm} C[i] = 0
3. \textbf{for} \ j = 1 \textbf{to} \text{length}[A]
Chapter 8. Lower Bounds and Sorting in Linear Time

Count sort

Algorithm Counting-Sort \((A, B, k)\) \{A contains \(n\) integers; \(k\) is the max\}
1. for \(i = 0\) to \(k\)
2. \(C[i] = 0\)
3. for \(j = 1\) to \(\text{length}[A]\)
4. \(C[A[j]] = C[A[j]] + 1\)
Count sort

Algorithm COUNTING-SORT \((A, B, k)\) \(\{A\ \text{contains}\ n\ \text{integers};\ k\ \text{is the max}\}\)

1. \textbf{for} \(i = 0\ \text{to} \ k\)
2. \hspace{0.5cm} \(C[i] = 0\)
3. \textbf{for} \(j = 1\ \text{to} \ \text{length}[A]\)
4. \hspace{1cm} \(C[A[j]] = C[A[j]] + 1\)
5. \hspace{0.5cm} \(\{C[i]\ \text{contains the number of elements whose values} = i\}\)
Chapter 8. Lower Bounds and Sorting in Linear Time

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Algorithm COUNTING-SORT \((A, B, k)\)  
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Chapter 8. Lower Bounds and Sorting in Linear Time

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Chapter 8. Lower Bounds and Sorting in Linear Time

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Example: \(A: 2 5 3 0 2 3 0 3, k = 5, C: 2 0 2 3 0 1\)

Analysis: \(T(n) = O(k + n)\)
Count sort

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11. \(C[A[j]] = C[A[j]] - 1\)

Example: \(A: 2 5 3 0 2 3 0 3\), \(k = 5\), \(C: 2 0 2 3 0 1\)

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Example: \(A: 2 \ 5 \ 3 \ 0 \ 2 \ 3 \ 0 \ 3, \ k = 5,\)
Chapter 8. Lower Bounds and Sorting in Linear Time

Count sort

Algorithm COUNTING-SORT \((A, B, k)\) \{\(A\) contains \(n\) integers; \(k\) is the max\}

1. \textbf{for} \(i = 0\) \textbf{to} \(k\)
2. \hspace{1em} \(C[i] = 0\)
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Example: \(A: 2 \ 5 \ 3 \ 0 \ 2 \ 3 \ 0 \ 3,\) \(k = 5,\) \(C: 2 \ 0 \ 2 \ 3 \ 0 \ 1\)

analysis:
Count sort

Algorithm **COUNTING-SORT** \((A, B, k)\) \{ \(A\) contains \(n\) integers; \(k\) is the max\}

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Example: \(A: 2 \ 5 \ 3 \ 0 \ 2 \ 3 \ 0 \ 3\), \(k = 5\), \(C: 2 \ 0 \ 2 \ 3 \ 0 \ 1\)  

analysis: \(T(n) = O(k + n)\)
Radix Sort:

Algorithm
Radix-Sort \((A, d)\)

1. for \(i = 1\) to \(d\)
2. sort \(A\) on the \(i\)th digit

Lemma. Given \(n\) \(b\)-bit binary numbers and any positive \(r \leq b\).

Radix-Sort uses \(\Theta(\lceil b/r \rceil (n + 2r))\) time.
Chapter 8. Lower Bounds and Sorting in Linear Time

Radix Sort:

```
329   720   720   329
457   355   329   355
657   436   436   436
839   457   839   457
436   657   355   657
720   329   457   720
355   839   657   839
```
Chapter 8. Lower Bounds and Sorting in Linear Time

Radix Sort:

<table>
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<tr>
<th>329</th>
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</tr>
</thead>
<tbody>
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Algorithm \textsc{Radix-Sort}(A, d)
Chapter 8. Lower Bounds and Sorting in Linear Time

Radix Sort:

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Algorithm \textsc{Radix-Sort}(A, d)

1. \textbf{for} \( i = 1 \) \textbf{to} \( d \)
Radix Sort:

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Algorithm \textbf{Radix-Sort}(A, d)

1. \textbf{for} $i = 1$ \textbf{to} $d$
2. \textbf{sort} $A$ on the $i$th digit
Chapter 8. Lower Bounds and Sorting in Linear Time

Radix Sort:

329  720  720  329
457  355  329  355
657  436  436  436
839  457  839  457
436  657  355  657
720  329  457  720
355  839  657  839

Algorithm Radix-Sort($A, d$)
1. for $i = 1$ to $d$
2. sort $A$ on the $i$th digit

Lemma. Given $n$ $b$-bit binary numbers and any positive $r \leq b$. Radix-Sort uses $\Theta([b/r](n + 2^r))$ time.
**Lemma.** Given $n$ $b$-bit binary numbers and any positive $r \leq b$. 
*Radix-Sort* uses $\Theta\left(\left\lceil \frac{b}{r} \right\rceil (n + 2^r)\right)$ time.
Lemma. Given $n$ $b$-bit binary numbers and any positive $r \leq b$. Radix-Sort uses $\Theta([b/r](n + 2^r))$ time.

Proof. Each $b$-digit binary number can be regarded as $[b/r]$ $r$-digit binary numbers. These $r$-digit binary numbers are of integer values in the range of $\{0, 1, \ldots, 2^r - 1\}$. 
Lemma. Given $n$ $b$-bit binary numbers and any positive $r \leq b$. Radix-Sort uses $\Theta(\lceil b/r \rceil (n + 2^r))$ time.

**Proof.** Each $b$-digit binary number can be regarded as $\lceil b/r \rceil$ $r$-digit binary numbers. These $r$-digit binary numbers are of integer values in the range of $\{0, 1, \ldots, 2^r - 1\}$.

Run Radix-Sort on the original binary numbers assumed to be $\lceil b/r \rceil$ columns.
Lemma. Given \( n \) \( b \)-bit binary numbers and any positive \( r \leq b \). 
Radix-Sort uses \( \Theta(\lceil b/r \rceil(n + 2^r)) \) time.

Proof. Each \( b \)-digit binary number can be regarded as \( \lfloor b/r \rfloor \) \( r \)-digit binary numbers. These \( r \)-digit binary numbers are of integer values in the range of \( \{0, 1, \ldots, 2^r - 1\} \).

Run Radix-Sort on the original binary numbers assumed to be \( \lceil b/r \rceil \) columns.

For every column, sorting by Counting-Sort with \( 2^r - 1 \) being the maximum.
Lemma. Given $n$ $b$-bit binary numbers and any positive $r \leq b$. Radix-Sort uses $\Theta([b/r](n + 2^r))$ time.

Proof. Each $b$-digit binary number can be regarded as $[b/r]$ $r$-digit binary numbers. These $r$-digit binary numbers are of integer values in the range of $\{0, 1, \ldots, 2^r - 1\}$.

Run Radix-Sort on the original binary numbers assumed to be $[b/r]$ columns.

For every column, sorting by Counting-Sort with $2^r - 1$ being the maximum.

The total time is $O([b/r](n + 2^r))$, where $(n + 2^r)$ is time for Counting-Sort.
Lemma. Given \( n \) \( b \)-bit binary numbers and any positive \( r \leq b \). Radix-Sort uses \( \Theta([b/r](n + 2^r)) \) time.

Proof. Each \( b \)-digit binary number can be regarded as \([b/r] \) \( r \)-digit binary numbers. These \( r \)-digit binary numbers are of integer values in the range of \( \{0, 1, \ldots, 2^r - 1\} \).

Run Radix-Sort on the original binary numbers assumed to be \([b/r]\) columns.

For every column, sorting by Counting-Sort with \( 2^r - 1 \) being the maximum.

The total time is \( O([b/r](n + 2^r)) \), where \( (n + 2^r) \) is time for Counting-Sort.

Since all steps in the two algorithms are mandatory, the total time is also \( \Omega([b/r](n + 2^r)) \), thus \( \Theta([b/r](n + 2^r)) \).
Lemma. Given $n$ $b$-bit binary numbers and any positive $r \leq b$. \textsc{Radix-Sort} uses $\Theta(\lceil b/r \rceil (n + 2r))$ time.

**Proof.** Each $b$-digit binary number can be regarded as $\lceil b/r \rceil$ $r$-digit binary numbers. These $r$-digit binary numbers are of integer values in the range of $\{0, 1, \ldots, 2^r - 1\}$.

Run \textsc{Radix-Sort} on the original binary numbers assumed to be $\lceil b/r \rceil$ columns.

For every column, sorting by \textsc{Counting-Sort} with $2^r - 1$ being the maximum.

The total time is $O(\lceil b/r \rceil (n + 2^r))$, where $(n + 2^r)$ is time for \textsc{Counting-Sort}.

Since all steps in the two algorithms are mandatory, the total time is also $\Omega(\lceil b/r \rceil (n + 2^r))$, thus $\Theta(\lceil b/r \rceil (n + 2^r))$.

Once $b$ and $n$ are given, we can choose $r$ to minimize the quantity $\lceil b/r \rceil (n + 2^r)$. 
Bucket Sort (assuming uniform distribution of inputs)
Bucket Sort (assuming uniform distribution of inputs)

Algorithm Bucket-Sort(A)
Bucket Sort (assuming uniform distribution of inputs)

Algorithm Bucket-Sort(A)
1. \( n = \text{length}[A] \)
Chapter 8. Lower Bounds and Sorting in Linear Time

**Bucket Sort** (assuming uniform distribution of inputs)

Algorithm `BUCKET-SORT(A)`
1. \( n = \text{length}[A] \)
2. \( \textbf{for} \ i = 1 \ \textbf{to} \ n \)
Chapter 8. Lower Bounds and Sorting in Linear Time

**Bucket Sort** (assuming uniform distribution of inputs)

Algorithm **Bucket-Sort**($A$)

1. $n = length[A]$
2. **for** $i = 1$ **to** $n$
3. insert $A[i]$ into list $B[\lfloor nA[i] \rfloor]$

A: .78 .17 .39 .26 .72 .94 .21 .12 .23 .68
B: 0 / 1 / 2 / 3 / 4 / 5 / 6 / 7 / 8 / 9
Chapter 8. Lower Bounds and Sorting in Linear Time

**Bucket Sort** (assuming uniform distribution of inputs)

Algorithm **Bucket-Sort**\((A)\)

1. \(n = \text{length}[A]\)
2. \(\text{for } i = 1 \text{ to } n\)
3. \(\text{insert } A[i] \text{ into list } B[\lfloor nA[i] \rfloor]\)
4. \(\text{for } i = 0 \text{ to } n - 1\)

\(A: .78 .17 .39 .26 .72 .94 .21 .12 .23 .68\)
\(B: 0 / 1 \rightarrow .12 \rightarrow .17 \rightarrow .21 \rightarrow .23 \rightarrow .26 \rightarrow .39 \rightarrow 4 / 5 / 6 \rightarrow .68 \rightarrow .72 \rightarrow .78 \rightarrow .94\)
Chapter 8. Lower Bounds and Sorting in Linear Time

**Bucket Sort** (assuming uniform distribution of inputs)

Algorithm **Bucket-Sort**(*A*)
1. \( n = \text{length}[A] \)
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5. sort list \( B[i] \) with **Insertion Sort**
Chapter 8. Lower Bounds and Sorting in Linear Time

**Bucket Sort** (assuming uniform distribution of inputs)

Algorithm **Bucket-Sort**($A$)

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4. for $i = 0$ to $n - 1$
5. sort list $B[i]$ with **Insertion Sort**
6. concatenate the lists $B[0], B[1], ..., B[n - 1]$

A: .78 .17 .39 .26 .72 .94 .21 .12 .23 .68

B: 0 / 1 → .12 / 2 → .17 / 3 → .21 / 2 → .23 / 3 → .26 / 4 / 5 / 6 → .39 / 7 → .68 / 8 / 9 → .72 / 7 → .78 / 7 / 9 → .94
Bucket Sort (assuming uniform distribution of inputs)

Algorithm Bucket-Sort(A)
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A: .78 .17 .39 .26 .72 .94 .21 .12 .23 .68
B: 0 /
Chapter 8. Lower Bounds and Sorting in Linear Time

**Bucket Sort** (assuming uniform distribution of inputs)

Algorithm $\text{BUCKET-SORT}(A)$

1. $n = \text{length}[A]$  
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4. for $i = 0$ to $n - 1$  
5. sort list $B[i]$ with $\text{INSERTION SORT}$  
6. concatenate the lists $B[0], B[1], ..., B[n - 1]$

A: 0.78 .17 .39 .26 .72 .21 .12 .23 .68  
B: 0 /  
   1 $\to$ .12 $\to$ .17
Chapter 8. Lower Bounds and Sorting in Linear Time

**Bucket Sort** (assuming uniform distribution of inputs)

Algorithm `Bucket-Sort(A)`

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A: .78 .17 .39 .26 .72 .94 .21 .12 .23 .68

B: 0 /
   1 → .12 → .17
   2 → .21 → .23 → .26
Chapter 8. Lower Bounds and Sorting in Linear Time

**Bucket Sort** (assuming uniform distribution of inputs)

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A: \( .78 \ .17 \ .39 \ .26 \ .72 \ .94 \ .21 \ .12 \ .23 \ .68 \)

B: \( 0 / \)
   \( 1 \rightarrow .12 \rightarrow .17 \)
   \( 2 \rightarrow .21 \rightarrow .23 \rightarrow .26 \)
   \( 3 \rightarrow .39 \)
Chapter 8. Lower Bounds and Sorting in Linear Time

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A: .78 .17 .39 .26 .72 .94 .21 .12 .23 .68

B: 0 /
   1 \( \rightarrow .12 \rightarrow .17 \)
   2 \( \rightarrow .21 \rightarrow .23 \rightarrow .26 \)
   3 \( \rightarrow .39 \)
   4 /
Chapter 8. Lower Bounds and Sorting in Linear Time

Bucket Sort (assuming uniform distribution of inputs)

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A: .78 .17 .39 .26 .72 .94 .21 .12 .23 .68
B: 0 /
   1 → .12 → .17
   2 → .21 → .23 → .26
   3 → .39
   4 /
   5 /

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Chapter 8. Lower Bounds and Sorting in Linear Time

**Bucket Sort** (assuming uniform distribution of inputs)

**Algorithm** `_BUCKET-SORT`(A)

1. \( n = \text{length}[A] \)
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6. \( \text{concatenate the lists } B[0], B[1], ..., B[n - 1] \)

\( A: \) .78 .17 .39 .26 .72 .94 .21 .12 .23 .68

\( B: \) 0 /
   1 → .12 → .17
   2 → .21 → .23 → .26
   3 → .39
   4 /
   5 /
   6 → .68
Chapter 8. Lower Bounds and Sorting in Linear Time

Bucket Sort (assuming uniform distribution of inputs)

Algorithm $\text{Bucket-Sort}(A)$
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6. concatenate the lists $B[0], B[1], ..., B[n - 1]$

A: \[ .78 \ .17 \ .39 \ .26 \ .72 \ .94 \ .21 \ .12 \ .23 \ .68 \]

B: \[
\begin{align*}
0 & / \\
1 & \rightarrow .12 \rightarrow .17 \\
2 & \rightarrow .21 \rightarrow .23 \rightarrow .26 \\
3 & \rightarrow .39 \\
4 & / \\
5 & / \\
6 & \rightarrow .68 \\
7 & \rightarrow .72 \rightarrow .78
\end{align*}
\]
Bucket Sort (assuming uniform distribution of inputs)

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A: .78 .17 .39 .26 .72 .94 .21 .12 .23 .68

B: 0 /
1 → .12 → .17
2 → .21 → .23 → .26
3 → .39
4 /
5 /
6 → .68
7 → .72 → .78
8 /
Chapter 8. Lower Bounds and Sorting in Linear Time

**Bucket Sort** (assuming uniform distribution of inputs)

Algorithm **Bucket-Sort**($A$)
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A: .78 .17 .39 .26 .72 .94 .21 .12 .23 .68

B: 0 /
1 → .12 → .17
2 → .21 → .23 → .26
3 → .39
4 /
5 /
6 → .68
7 → .72 → .78
8 /
9 → .94
Chapter 8. Lower Bounds and Sorting in Linear Time

**Bucket Sort** (assuming uniform distribution of inputs)

Algorithm **Bucket-Sort**\( (A) \)

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5. \( \text{sort list } B[i] \text{ with Insertion Sort} \)
6. \( \text{concatenate the lists } B[0], B[1], \ldots, B[n - 1] \)

\( A: \) \( .78 \ .17 \ .39 \ .26 \ .72 \ .94 \ .21 \ .12 \ .23 \ .68 \)

\( B: \) \( 0 / \)
\( 1 \to .12 \to .17 \)
\( 2 \to .21 \to .23 \to .26 \)
\( 3 \to .39 \)
\( 4 / \)
\( 5 / \)
\( 6 \to .68 \)
\( 7 \to .72 \to .78 \)
\( 8 / \)
\( 9 \to .94 \)
• find the maximum: linear time
• find the minimum: linear time
• find the median (i.e., the $n/2$th smallest element)?
  the problem has upper bound $O(n \log_2 n)$.

why?
Can we do better?
Chapter 9. Medians and Order Statistics

- find the maximum: linear time
- find the minimum: linear time
- find the median (i.e., the $n/2$th smallest element)?
  - the problem has upper bound $O(n \log_2 n)$.
  - why?
  - Can we do better?
Chapter 9. Medians and order statistics

- find the maximum: linear time
- find the minimum: linear time

The problem has upper bound $O(n \log_2 n)$. Why?

Can we do better?
Chapter 9. Medians and order statistics

• find the maximum: linear time
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Chapter 9. Medians and order statistics

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Can we do better?
Chapter 9. Medians and Order Statistics

Chapter 9. Medians and order statistics

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Chapter 9. Medians and Order Statistics

Chapter 9. Medians and order statistics

- find the maximum: linear time
- find the minimum: linear time
- find the median (i.e., the $\frac{n}{2}$th smallest element)?

the problem has upper bound $O(n \log_2 n)$. why?

Can we do better?
Chapter 9. Medians and Order Statistics

Selection problem

Input: a list \( A \) of elements, an integer \( i \); Output: the \( i \)th smallest element in \( A \); There are algorithms solving it in linear time.

Two types of algorithms:

• Selection in worst case linear time
• Selection in expected linear time (but worst case \( \Theta(n^2) \))
Selection problem
Selection problem

**Input:** a list $A$ of elements, an integer $i$;
Selection problem

**Input**: a list $A$ of elements, an integer $i$;

**Output**: the $i$th smallest element in $A$;
Selection problem

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There are algorithms solving it in linear time.
Chapter 9. Medians and Order Statistics

Selection problem

**Input:** a list $A$ of elements, an integer $i$;

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Two types of algorithms:
Selection problem

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Two types of algorithms:

- Selection in worst case linear time
Selection problem

**Input:** a list $A$ of elements, an integer $i$;

**Output:** the $i$th smallest element in $A$;

There are algorithms solving it in linear time.

Two types of algorithms:

- Selection in worst case linear time
- Selection in *expected* linear time (but worst case $\Theta(n^2)$)
Chapter 9. Medians and Order Statistics

Selection in worst case linear time

**Input**: set $S$ of $n$ elements and $i$;

**Output**: the $i$th smallest element in $S$;

**Main idea:**
- find a pivot $x$ to partition the list $S$ into two sublists $S_1$ and $S_2$, such that $\forall y \in S_1 y < x$ and $\forall z \in S_2 z > x$
- both $S_1$ and $S_2$ are guaranteed only a fraction of $S$;
- the $i$th smallest element is either $x$, or in $S_1$ or in $S_2$ (but not both);
- in either of the latter two cases, the algorithm is applied recursively.
Selection in worst case linear time
Selection in worst case linear time

**Input:** set $S$ of $n$ elements and $i$;
**Output:** the $i$th smallest element in $S$;
Chapter 9. Medians and Order Statistics

Selection in worst case linear time

**Input:** set \( S \) of \( n \) elements and \( i \);

**Output:** the \( i \)th smallest element in \( S \);

Main idea:
Chapter 9. Medians and Order Statistics

Selection in worst case linear time

**Input**: set $S$ of $n$ elements and $i$;
**Output**: the $ith$ smallest element in $S$;

Main idea:

- find a pivot $x$ to partition the list $S$ into two sublists $S_1$ and $S_2$, such that $\forall y \in S_1, y < x$ and $\forall z \in S_2, z > x$; both $S_1$ and $S_2$ are guaranteed only a fraction of $S$; the $ith$ smallest element is either $x$, or in $S_1$ or in $S_2$ (but not both); in either of the latter two cases, the algorithm is applied recursively.
Chapter 9. Medians and Order Statistics

Selection in worst case linear time

**INPUT:** set $S$ of $n$ elements and $i$;

**OUTPUT:** the $i$th smallest element in $S$;

Main idea:

- find a pivot $x$ to partition the list $S$ into two sublists $S_1$ and $S_2$, such that $\forall y \in S_1 \ y < x$ and $\forall z \in S_2 \ z > x$
Selection in worst case linear time

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Selection in worst case linear time

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Chapter 9. Medians and Order Statistics

Selection in worst case linear time

**INPUT:** set \( S \) of \( n \) elements and \( i \);
**OUTPUT:** the \( i \)th smallest element in \( S \);

**Main idea:**

- find a pivot \( x \) to partition the list \( S \) into two sublists \( S_1 \) and \( S_2 \), such that \( \forall y \in S_1 \ y < x \) and \( \forall z \in S_2 \ z > x \)
- both \( S_1 \) and \( S_2 \) are guaranteed only a fraction of \( S \);
- the \( i \)th smallest element is either \( x \), or in \( S_1 \) or in \( S_2 \) (but not both);
- in either of the latter two cases, the algorithm is applied recursively.
Chapter 9. Medians and Order Statistics

Assume total time complexity $T(n)$ if time for finding pivot:
$n c$ and time for the recursive step: $T(\beta n)$, for some $0 < \beta < 1$.

Then $T(n) \leq T(\beta n) + cn \leq cn + c\beta n + T(\beta^2 n) \leq cn + c\beta n + c\beta^2 n + \cdots + c\beta^{m-1} n + T(\beta^m n)$

where $\beta^m n = 1$.

$\leq c n (1 - \beta^m) + c' \leq c n (1 - 1) + c' = O(n)$
Chapter 9. Medians and Order Statistics

Assume total time complexity $T(n)$

\[
\textbf{if} \text{ time for finding pivot: } cn
\]
Chapter 9. Medians and Order Statistics

Assume total time complexity $T(n)$

\[
\text{if time for finding pivot: } cn \\
\text{and time for the recursive step: } T(\beta n), \text{ for some } 0 < \beta < 1
\]
Chapter 9. Medians and Order Statistics

Assume total time complexity $T(n)$

- if time for finding pivot: $cn$
- and time for the recursive step: $T(\beta n)$, for some $0 < \beta < 1$

Then $T(n) \leq T(\beta n) + cn$
Chapter 9. Medians and Order Statistics

Assume total time complexity $T(n)$

\[ \text{if time for finding pivot: } cn \]
\[ \text{and time for the recursive step: } T(\beta n), \text{ for some } 0 < \beta < 1 \]

Then $T(n) \leq T(\beta n) + cn$

\[ T(n) \leq cn + \]
Assume total time complexity $T(n)$

- if time for finding pivot: $cn$
- and time for the recursive step: $T(\beta n)$, for some $0 < \beta < 1$

Then $T(n) \leq T(\beta n) + cn$

$$T(n) \leq cn + c\beta n + T(\beta^2 n)$$
Chapter 9. Medians and Order Statistics

Assume total time complexity $T(n)$

**if** time for finding pivot: $cn$
and time for the recursive step: $T(\beta n)$, for some $0 < \beta < 1$

Then $T(n) \leq T(\beta n) + cn$

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Assume total time complexity $T(n)$

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- and time for the recursive step: $T(\beta n)$, for some $0 < \beta < 1$

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$$T(n) \leq cn + c\beta n + T(\beta^2 n)$$
Chapter 9. Medians and Order Statistics

Assume total time complexity $T(n)$

**if** time for finding pivot: $cn$
and time for the recursive step: $T(\beta n)$, for some $0 < \beta < 1$

Then $T(n) \leq T(\beta n) + cn$

$$T(n) \leq cn + c\beta n + T(\beta^2 n)$$

$T(n) \leq cn + c\beta n + c\beta^2 n + \cdots$

$$+ c\beta^{m-1} n + T(\beta^m n) \leq cn + c\beta n + c\beta^2 n + \cdots + c\beta^{m-1} n + T(\beta^m n)$$

$$\leq cn + c\beta n + c\beta^2 n + \cdots + c\beta^{m-1} n + T(\beta^m n)$$

$$\leq cn + c\beta n + c\beta^2 n + \cdots + c\beta^{m-1} n + T(\beta^m n) \leq c(1 - \beta^m) + c' \leq c(1 - \beta n) + c' = O(n)$$
Chapter 9. Medians and Order Statistics

Assume total time complexity $T(n)$

**if** time for finding pivot: $cn$
and time for the recursive step: $T(\beta n)$, for some $0 < \beta < 1$

Then $T(n) \leq T(\beta n) + cn$

$$T(n) \leq cn + c\beta n + T(\beta^2 n)$$

$$T(n) \leq cn + c\beta n + c\beta^2 n + \cdots + c\beta^{m-1} n + T(\beta^m n)$$
Chapter 9. Medians and Order Statistics

Assume total time complexity $T(n)$

- if time for finding pivot: $cn$
  and time for the recursive step: $T(\beta n)$, for some $0 < \beta < 1$

Then $T(n) \leq T(\beta n) + cn$

$$T(n) \leq cn + c\beta n + T(\beta^2 n)$$

$$T(n) \leq cn + c\beta n + c\beta^2 n + \cdots + c\beta^{m-1} n + T(\beta^m n) \quad \text{(where } \beta^m n = 1\text{)}$$
Chapter 9. Medians and Order Statistics

Assume total time complexity $T(n)$

if time for finding pivot: $cn$
and time for the recursive step: $T(\beta n)$, for some $0 < \beta < 1$

Then $T(n) \leq T(\beta n) + cn$

$$T(n) \leq cn + c\beta n + T(\beta^2 n)$$

$$T(n) \leq cn + c\beta n + c\beta^2 n + \cdots + c\beta^{m-1} n + T(\beta^m n) \quad (\text{where } \beta^m n = 1)$$

$$\leq cn \left( \frac{1 - \beta^m}{1 - \beta} \right) + c'$$
Chapter 9. Medians and Order Statistics

Assume total time complexity \( T(n) \)

\[ \text{if time for finding pivot: } cn \]
and time for the recursive step: \( T(\beta n) \), for some \( 0 < \beta < 1 \)

Then \( T(n) \leq T(\beta n) + cn \)

\[ T(n) \leq cn + c\beta n + T(\beta^2 n) \]

\[ T(n) \leq cn + c\beta n + c\beta^2 n + \cdots + c\beta^{m-1} n + T(\beta^m n) \text{ (where } \beta^m n = 1) \]

\[ \leq cn \left( \frac{1 - \beta^m}{1 - \beta} \right) + c' \leq c \frac{1}{1 - \beta} n + c' \]
Assume total time complexity $T(n)$

- if time for finding pivot: $cn$
- and time for the recursive step: $T(\beta n)$, for some $0 < \beta < 1$

Then $T(n) \leq T(\beta n) + cn$

$$T(n) \leq cn + c\beta n + T(\beta^2 n)$$

$$T(n) \leq cn + c\beta n + c\beta^2 n + \cdots + c\beta^{m-1} n + T(\beta^m n) \quad (\text{where } \beta^m n = 1)$$

$$\leq cn \left( \frac{1 - \beta^m}{1 - \beta} \right) + c' \leq c \frac{1}{1 - \beta} n + c' = O(n)$$
Chapter 9. Medians and Order Statistics

Assume total time complexity $T(n)$ if time for finding pivot: $cn + T(\alpha n)$, for some $0 < \alpha < 1$ and time for the recursive step: $T(\beta n)$, for some $0 < \beta < 1$. Assume $\alpha + \beta < 1$, then $T(n) \leq cn + T(\alpha n) + T(\beta n)$.

With the recursive tree method (you draw a picture): $T(n) \leq cn + c\alpha n + T(\alpha^2 n) + T(\alpha \beta n) + c\beta n + T(\beta^2 n)$.

\[
\leq cn + c(\alpha + \beta)n + c\alpha^2 n + 2c\alpha \beta n + c\beta^2 n + T(\alpha^3 n) + 3T(\alpha^2 \beta n) + 3T(\alpha \beta^2 n) + T(\beta^3 n) =
\]

$\leq cn + c(\alpha + \beta)n + c(\alpha + \beta)^2 n + c(\alpha + \beta)^3 n + \cdots + c(\alpha + \beta)^m n + c\prime T(1)$, the base case.

Therefore, $T(n) \leq c_1 - (\alpha + \beta)^n \leq c_1 - (\alpha + \beta)m = O(n)$. 
Chapter 9. Medians and Order Statistics

Assume total time complexity $T(n)$

\[ \text{if time for finding pivot: } cn + T(\alpha n), \text{ for some } 0 < \alpha < 1 \]
Chapter 9. Medians and Order Statistics

Assume total time complexity $T(n)$

if time for finding pivot: $cn + T(\alpha n)$, for some $0 < \alpha < 1$
and time for the recursive step: $T(\beta n)$, for some $0 < \beta < 1$
assume $\alpha + \beta < 1$,
Chapter 9. Medians and Order Statistics

Assume total time complexity \( T(n) \)

**if** time for finding pivot: \( cn + T(\alpha n) \), for some \( 0 < \alpha < 1 \)
and time for the recursive step: \( T(\beta n) \), for some \( 0 < \beta < 1 \)

**assume** \( \alpha + \beta < 1 \), then

\[
T(n) \leq cn + T(\alpha n) + T(\beta n)
\]
Chapter 9. Medians and Order Statistics

Assume total time complexity $T(n)$

- if time for finding pivot: $cn + T(\alpha n)$, for some $0 < \alpha < 1$
- and time for the recursive step: $T(\beta n)$, for some $0 < \beta < 1$

Assume $\alpha + \beta < 1$, then

$$T(n) \leq cn + T(\alpha n) + T(\beta n)$$

With the recursive tree method (you draw a picture):
Chapter 9. Medians and Order Statistics

Assume total time complexity $T(n)$

if time for finding pivot: $cn + T(\alpha n)$, for some $0 < \alpha < 1$
and time for the recursive step: $T(\beta n)$, for some $0 < \beta < 1$
assume $\alpha + \beta < 1$, then

$$T(n) \leq cn + T(\alpha n) + T(\beta n)$$

With the recursive tree method (you draw a picture):

$$T(n) \leq cn + c\alpha n + T(\alpha^2 n) + T(\alpha \beta n)$$
Chapter 9. Medians and Order Statistics

Assume total time complexity $T(n)$

if time for finding pivot: $cn + T(\alpha n)$, for some $0 < \alpha < 1$
and time for the recursive step: $T(\beta n)$, for some $0 < \beta < 1$
assume $\alpha + \beta < 1$, then

$T(n) \leq cn + T(\alpha n) + T(\beta n)$

With the recursive tree method (you draw a picture):

$T(n) \leq cn + c\alpha n + T(\alpha^2 n) + T(\alpha \beta n) + c\beta n + T(\beta \alpha n) + T(\beta^2 n)$
Chapter 9. Medians and Order Statistics

Assume total time complexity $T(n)$

if time for finding pivot: $cn + T(\alpha n)$, for some $0 < \alpha < 1$
and time for the recursive step: $T(\beta n)$, for some $0 < \beta < 1$
assume $\alpha + \beta < 1$, then

$$T(n) \leq cn + T(\alpha n) + T(\beta n)$$

With the recursive tree method (you draw a picture):

$$T(n) \leq cn + c\alpha n + T(\alpha^2 n) + T(\alpha \beta n) + c\beta n + T(\beta\alpha n) + T(\beta^2 n)$$

$$= cn + c\alpha n + c\beta n + T(\alpha^2 n) + 2T(\alpha \beta n) + T(\beta^2 n)$$
Chapter 9. Medians and Order Statistics

Assume total time complexity \( T(n) \)

**if** time for finding pivot: \( cn + T(\alpha n) \), for some \( 0 < \alpha < 1 \)
and time for the recursive step: \( T(\beta n) \), for some \( 0 < \beta < 1 \)
**assume** \( \alpha + \beta < 1 \), then

\[
T(n) \leq cn + T(\alpha n) + T(\beta n)
\]

With the recursive tree method (you draw a picture):

\[
T(n) \leq cn + c\alpha n + T(\alpha^2 n) + T(\alpha \beta n) + c\beta n + T(\beta \alpha n) + T(\beta^2 n)
\]

\[
= cn + c\alpha n + c\beta n + T(\alpha^2 n) + 2T(\alpha \beta n) + T(\beta^2 n)
\]

\[
\leq cn + c(\alpha + \beta)n + c\alpha^2 n + 2c\alpha \beta n + c\beta^2 n
\]
Chapter 9. Medians and Order Statistics

Assume total time complexity $T(n)$

**if** time for finding pivot: $cn + T(\alpha n)$, for some $0 < \alpha < 1$
and time for the recursive step: $T(\beta n)$, for some $0 < \beta < 1$
**assume** $\alpha + \beta < 1$, then

$$T(n) \leq cn + T(\alpha n) + T(\beta n)$$

With the recursive tree method (you draw a picture):

$$T(n) \leq cn + c\alpha n + T(\alpha^2 n) + T(\alpha \beta n) + c\beta n + T(\beta \alpha n) + T(\beta^2 n)$$
$$= cn + c\alpha n + c\beta n + T(\alpha^2 n) + 2T(\alpha \beta n) + T(\beta^2 n)$$
$$\leq cn + c(\alpha + \beta)n + c\alpha^2 n + 2c\alpha \beta n + c\beta^2 n + T(\alpha^3 n) + 3T(\alpha^2 \beta n) + 3T(\alpha \beta^2 n) + T(\beta^3 n)$$
Chapter 9. Medians and Order Statistics

Assume total time complexity $T(n)$

\[ \text{if time for finding pivot: } cn + T(\alpha n), \text{ for some } 0 < \alpha < 1 \]
and time for the recursive step: $T(\beta n)$, for some $0 < \beta < 1$

**assume** $\alpha + \beta < 1$, then

\[ T(n) \leq cn + T(\alpha n) + T(\beta n) \]

With the recursive tree method (you draw a picture):

\[ T(n) \leq cn + c\alpha n + T(\alpha^2 n) + T(\alpha \beta n) + c\beta n + T(\beta \alpha n) + T(\beta^2 n) \]

\[ = cn + c\alpha n + c\beta n + T(\alpha^2 n) + 2T(\alpha \beta n) + T(\beta^2 n) \]

\[ \leq cn + c(\alpha + \beta)n + c\alpha^2 n + 2c\alpha \beta n + c\beta^2 n + T(\alpha^3 n) + 3T(\alpha^2 \beta n) + 3T(\alpha \beta^2 n) + T(\beta^3 n) \]

\[ = cn + c(\alpha + \beta)n + c(\alpha + \beta)^2 n + T(\alpha^3 n) + 3T(\alpha^2 \beta n) + 3T(\alpha \beta^2 n) + T(\beta^3 n) \]
Chapter 9. Medians and Order Statistics

Assume total time complexity $T(n)$

\[
\text{if time for finding pivot: } cn + T(\alpha n), \text{ for some } 0 < \alpha < 1 \\
\text{and time for the recursive step: } T(\beta n), \text{ for some } 0 < \beta < 1
\]

Assume $\alpha + \beta < 1$, then

\[
T(n) \leq cn + T(\alpha n) + T(\beta n)
\]

With the recursive tree method (you draw a picture):

\[
T(n) \leq cn + c\alpha n + T(\alpha^2 n) + T(\alpha\beta n) + c\beta n + T(\beta\alpha n) + T(\beta^2 n)
\]

\[
= cn + c\alpha n + c\beta n + T(\alpha^2 n) + 2T(\alpha\beta n) + T(\beta^2 n)
\]

\[
\leq cn + c(\alpha + \beta)n + c\alpha^2 n + 2c\alpha\beta n + c\beta^2 n + T(\alpha^3 n) + 3T(\alpha^2 \beta n) + 3T(\alpha \beta^2 n) + T(\beta^3 n)
\]

\[
= cn + c(\alpha + \beta)n + c(\alpha + \beta)^2 n + T(\alpha^3 n) + 3T(\alpha^2 \beta n) + 3T(\alpha \beta^2 n) + T(\beta^3 n)
\]

\[
\leq cn + c(\alpha + \beta)n + c(\alpha + \beta)^2 n + c(\alpha + \beta)^3 n +
\]

\[
\]
Chapter 9. Medians and Order Statistics

Assume total time complexity $T(n)$

**if** time for finding pivot: $cn + T(\alpha n)$, for some $0 < \alpha < 1$
and time for the recursive step: $T(\beta n)$, for some $0 < \beta < 1$

**assume** $\alpha + \beta < 1$, then

$$T(n) \leq cn + T(\alpha n) + T(\beta n)$$

With the recursive tree method (you draw a picture):

$$T(n) \leq cn + c\alpha n + T(\alpha^2 n) + T(\alpha \beta n) + c\beta n + T(\beta \alpha n) + T(\beta^2 n)$$

$$= cn + c\alpha n + c\beta n + T(\alpha^2 n) + 2T(\alpha \beta n) + T(\beta^2 n)$$

$$\leq cn + c(\alpha + \beta)n + c\alpha^2 n + 2c\alpha \beta n + c\beta^2 n + T(\alpha^3 n) + 3T(\alpha^2 \beta n) + 3T(\alpha \beta^2 n) + T(\beta^3 n)$$

$$= cn + c(\alpha + \beta)n + c(\alpha + \beta)^2 n + T(\alpha^3 n) + 3T(\alpha^2 \beta n) + 3T(\alpha \beta^2 n) + T(\beta^3 n)$$

$$\leq cn + c(\alpha + \beta)n + c(\alpha + \beta)^2 n + c(\alpha + \beta)^3 n + \cdots + c(\alpha + \beta)^{m-1} n + c'$$
Chapter 9. Medians and Order Statistics

Assume total time complexity \( T(n) \)

\[ \text{if time for finding pivot: } cn + T(\alpha n), \text{ for some } 0 < \alpha < 1 \]
and time for the recursive step: \( T(\beta n) \), for some \( 0 < \beta < 1 \)
assume \( \alpha + \beta < 1 \), then

\[ T(n) \leq cn + T(\alpha n) + T(\beta n) \]

With the recursive tree method (you draw a picture):

\[ T(n) \leq cn + c\alpha n + T(\alpha^2 n) + T(\alpha \beta n) + c\beta n + T(\beta \alpha n) + T(\beta^2 n) \]

\[ = cn + c\alpha n + c\beta n + T(\alpha^2 n) + 2T(\alpha \beta n) + T(\beta^2 n) \]

\[ \leq cn + c(\alpha + \beta)n + c\alpha^2 n + 2c\alpha \beta n + c\beta^2 n + T(\alpha^3 n) + 3T(\alpha^2 \beta n) + 3T(\alpha \beta^2 n) + T(\beta^3 n) \]

\[ = cn + c(\alpha + \beta)n + c(\alpha + \beta)^2 n + T(\alpha^3 n) + 3T(\alpha^2 \beta n) + 3T(\alpha \beta^2 n) + T(\beta^3 n) \]

\[ \leq cn + c(\alpha + \beta)n + c(\alpha + \beta)^2 n + c(\alpha + \beta)^3 n + \cdots + c(\alpha + \beta)^{m-1} n + c' \]

where \( m = \max\{i, j\} \), for such \( i, j \) that \( \alpha^i n = 1 \) and \( \beta^j n = 1 \).
and \( c' = 2T(1) \), the base case.
Chapter 9. Medians and Order Statistics

Assume total time complexity \( T(n) \)

**if** time for finding pivot: \( cn + T(\alpha n) \), for some \( 0 < \alpha < 1 \)
and time for the recursive step: \( T(\beta n) \), for some \( 0 < \beta < 1 \)

**assume** \( \alpha + \beta < 1 \), then

\[
T(n) \leq cn + T(\alpha n) + T(\beta n)
\]

With the recursive tree method (you draw a picture):

\[
T(n) \leq cn + c\alpha n + T(\alpha^2 n) + T(\alpha \beta n) + c\beta n + T(\beta \alpha n) + T(\beta^2 n)
\]

\[
= cn + c\alpha n + c\beta n + T(\alpha^2 n) + 2T(\alpha \beta n) + T(\beta^2 n)
\]

\[
\leq cn + c(\alpha + \beta)n + c\alpha^2 n + 2c\alpha \beta n + c\beta^2 n + T(\alpha^3 n) + 3T(\alpha^2 \beta n) + 3T(\alpha \beta^2 n) + T(\beta^3 n)
\]

\[
= cn + c(\alpha + \beta)n + c(\alpha + \beta)^2 n + T(\alpha^3 n) + 3T(\alpha^2 \beta n) + 3T(\alpha \beta^2 n) + T(\beta^3 n)
\]

\[
\leq cn + c(\alpha + \beta)n + c(\alpha + \beta)^2 n + c(\alpha + \beta)^3 n + \cdots + c(\alpha + \beta)^{m-1} n + c'
\]

where \( m = \max\{i, j\} \), for such \( i, j \) that \( \alpha^i n = 1 \) and \( \beta^j n = 1 \).

and \( c' = 2T(1) \), the base case.

Therefore,

\[
T(n) \leq c \frac{1-(\alpha+\beta)^m}{1-(\alpha+\beta)} n
\]
Chapter 9. Medians and Order Statistics

Assume total time complexity $T(n)$

\[
\text{if} \ \text{time for finding pivot: } cn + T(\alpha n), \text{ for some } 0 < \alpha < 1
\]

and time for the recursive step: $T(\beta n)$, for some $0 < \beta < 1$

\text{assume } \alpha + \beta < 1, \text{ then }

\[T(n) \leq cn + T(\alpha n) + T(\beta n)\]

With the recursive tree method (you draw a picture):

\[T(n) \leq cn + c\alpha n + T(\alpha^2 n) + T(\alpha \beta n) + c\beta n + T(\beta \alpha n) + T(\beta^2 n)\]

\[= cn + c\alpha n + c\beta n + T(\alpha^2 n) + 2T(\alpha \beta n) + T(\beta^2 n)\]

\[\leq cn + c(\alpha + \beta)n + c\alpha^2 n + 2c\alpha \beta n + c\beta^2 n + T(\alpha^3 n) + 3T(\alpha^2 \beta n) + 3T(\alpha \beta^2 n) + T(\beta^3 n)\]

\[= cn + c(\alpha + \beta)n + c(\alpha + \beta)^2 n + T(\alpha^3 n) + 3T(\alpha^2 \beta n) + 3T(\alpha \beta^2 n) + T(\beta^3 n)\]

\[\leq cn + c(\alpha + \beta)n + c(\alpha + \beta)^2 n + c(\alpha + \beta)^3 n + \cdots + c(\alpha + \beta)^{m-1} n + c'\]

where $m = \max\{i, j\}$, for such $i, j$ that $\alpha^i n = 1$ and $\beta^j n = 1$.

and $c' = 2T(1)$, the base case.

Therefore, \[T(n) \leq c \frac{1-(\alpha+\beta)^m}{1-(\alpha+\beta)} n \leq c \frac{1}{1-(\alpha+\beta)} n\]
Chapter 9. Medians and Order Statistics

Assume total time complexity $T(n)$

\[ T(n) \leq cn + T(\alpha n) + T(\beta n) \]

if time for finding pivot: $cn + T(\alpha n)$, for some $0 < \alpha < 1$

and time for the recursive step: $T(\beta n)$, for some $0 < \beta < 1$

assume $\alpha + \beta < 1$, then

\[ T(n) \leq cn + T(\alpha n) + T(\beta n) \]

With the recursive tree method (you draw a picture):

\[
T(n) \leq cn + c\alpha n + T(\alpha^2 n) + T(\alpha \beta n) + c\beta n + T(\beta \alpha n) + T(\beta^2 n)
\]

\[
= cn + c\alpha n + c\beta n + T(\alpha^2 n) + 2T(\alpha \beta n) + T(\beta^2 n)
\]

\[
\leq cn + c(\alpha + \beta)n + c\alpha^2 n + 2c\alpha \beta n + c\beta^2 n + T(\alpha^3 n) + 3T(\alpha^2 \beta n) + 3T(\alpha \beta^2 n) + T(\beta^3 n)
\]

\[
= cn + c(\alpha + \beta)n + c(\alpha + \beta)^2 n + T(\alpha^3 n) + 3T(\alpha^2 \beta n) + 3T(\alpha \beta^2 n) + T(\beta^3 n)
\]

\[
\leq cn + c(\alpha + \beta)n + c(\alpha + \beta)^2 n + c(\alpha + \beta)^3 n + \cdots + c(\alpha + \beta)^{m-1} n + c'
\]

where $m = \max\{i, j\}$, for such $i, j$ that $\alpha^i n = 1$ and $\beta^j n = 1$.

and $c' = 2T(1)$, the base case.

Therefore,

\[
T(n) \leq c \frac{1-(\alpha+\beta)^m}{1-(\alpha+\beta)} n \leq c \frac{1}{1-(\alpha+\beta)} n = O(n).
\]
• the very selection algorithm is recursively called for finding the pivot
• the size of the sublist to find the pivot is also a fraction \( \frac{\alpha}{n} \) of the original list \( S \), |\( S \)| = \( n \);
• the total time actually is \( T(n) \leq T(\alpha n) + T(\beta n) + cn \) where \( \alpha + \beta < 1 \).
Chapter 9. Medians and Order Statistics

How to find such a pivot?
Chapter 9. Medians and Order Statistics

How to find such a pivot?

- the very selection algorithm is recursively called for finding the pivot
Chapter 9. Medians and Order Statistics

How to find such a pivot?

- the very selection algorithm is recursively called for finding the pivot
- the size of the sublist to find the pivot is also a fraction $\alpha n$
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How to find such a pivot?

- the very selection algorithm is recursively called for finding the pivot
- the size of the sublist to find the pivot is also a fraction $\alpha n$ of the original list $S$, $|S| = n$;
How to find such a pivot?

- the very selection algorithm is recursively called for finding the pivot
- the size of the sublist to find the pivot is also a fraction $\alpha n$ of the original list $S$, $|S| = n$;
- the total time actually is

$$T(n) \leq T(\alpha n) + T(\beta n) + cn$$

where $\alpha + \beta < 1$
Chapter 9. Medians and Order Statistics

Algorithm \texttt{Select} \((S, i)\); \{ where \(S\) contains \(n\) distinct elements\}
Algorithm \texttt{SELECT} \((S, i); \text{ where } S \text{ contains } n \text{ distinct elements}\)

(1) divide \(S\) into \(\lceil n/5 \rceil\) groups of 5 elements
Algorithm \textsc{Select} \((S, i)\); \{ where \(S\) contains \(n\) distinct elements\}

(1) divide \(S\) into \(\lceil n/5 \rceil\) groups of 5 elements

(2) sort each group (of 5) and find the median of each group;
Chapter 9. Medians and Order Statistics

Algorithm `SELECT (S, i);`  { where $S$ contains $n$ distinct elements }
(1) divide $S$ into $\lceil n/5 \rceil$ groups of 5 elements
(2) sort each group (of 5) and find the median of each group;
    let $M$ contain all these medians; where $|M| = \lceil n/5 \rceil$
Algorithm \texttt{Select} \((S, i); \{ \text{ where } S \text{ contains } n \text{ distinct elements} \}
(1) divide \(S\) into \(\lceil n/5 \rceil\) groups of 5 elements
(2) sort each group (of 5) and find the median of each group;
    let \(M\) contain all these medians; where \(|M| = \lceil n/5 \rceil\)
(3) \textbf{recursively call} \texttt{Select}(\(M, \lceil n/10 \rceil\));
Algorithm `Select` \((S, i)\); \{ where \(S\) contains \(n\) distinct elements\}

1. divide \(S\) into \(\lceil n/5 \rceil\) groups of 5 elements
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3. **recursively call** `Select` \((M, \lceil n/10 \rceil)\);
   - let the result be \(x\) and let the rank of \(x\) be \(k\) in \(S\)

if \(i = k\) return \((x)\)

else use \(x\) as the pivot to partition \(S\) resulting in \(S_1\) and \(S_2\), such that:
   - \(\forall y \in S_1\) \(y < x\)
   - \(\forall z \in S_2\) \(z > x\)
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6. if \(i < k\) recursively call \texttt{SELECT}(\(S_1, i\))
   else recursively call \texttt{SELECT}(\(S_2, i - k\))
Note: the number of elements $\leq x$ is at least:
\[ |S_1| \geq 3\left(\lceil \frac{n}{5} \rceil^2 \right) \geq 3\frac{n}{10} \]

\[ \Rightarrow |S_2| < n - 3\frac{n}{10} = 7\frac{n}{10} \]

Similarly, the number of elements $\geq x$ is at least:
\[ |S_2| \geq 3\left(\lceil \frac{n}{5} \rceil^2 - 2 \right) \geq 3\frac{n}{10} - 6 \geq 3\frac{n}{10} \]

\[ \Rightarrow |S_1| < n - 3\frac{n}{10} + 6 = 7\frac{n}{10} + 6 \]

So a time upper bound for Select is
\[ T(n) \leq T_{\text{mom}} + T_{\text{sub}} + cn T(n) \leq T\left(\lceil \frac{n}{5} \rceil\right) + T\left(\lceil 7\frac{n}{10} + 6 \rceil\right) + cn \]

when $n \geq 140$ (why?)
Note: the number of elements \( \leq x \) is at least:

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|S_1| \geq 3\left(\left\lceil \frac{n}{5} \right\rceil \right) \geq \frac{3n}{10}
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So a time upper bound for \texttt{SELECT} is \( T(n) \leq T_{mom} + T_{sub} + cn \)

\[ T(n) \leq T(\lceil n/5 \rceil) + T(\lfloor 7n/10 + 6 \rfloor) + cn \]

when \( n \geq 140 \) (why?)
Selection in *expected* linear time
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Selection in *expected* linear time

**Input:** a list $A$ of elements, an integer $i$;
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**Input:** a list $A$ of elements, an integer $i$;  
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Idea of the algorithm:

- randomly identify a pivot $x$ and partition the list $A$ into two sublists $A_l$ and $A_u$; assume the *rank* of $x$ is $k$;
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\begin{itemize}
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  \item if $i = k$, done, return $(x)$;
  \item else if $k > i$, recursively do for $A_l$ with $i$;
    \hspace{1cm} else recursively do for $A_u$ with $i - k$;
\end{itemize}
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Algorithm RANDOMIZED-SELECT \((A, p, r, i)\)
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1. \textbf{if} \(p = r\)
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Algorithm RANDOMIZED-SELECT \( (A, p, r, i) \)
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3. \( q = \) RANDOMIZED PARTITION \( (A, p, r) \)
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2. \hspace{1em} \textbf{return} \((A[p])\)
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4. \(k = q - p + 1\)
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If pivots always partition lists into \(\frac{n}{r} : \frac{r-1}{r} n\), for some \(r > 1\),
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6. \hspace{2em} \textbf{return} \(A[q]\)
7. \hspace{1em} \textbf{else if} \(i < k\)
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If pivots always partition lists into \(\frac{n}{r} : \frac{r - 1}{r}n\), for some \(r > 1\),
time \(T(n)\) would have the recurrence

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T(n) \leq \max\{T\left(\frac{n}{r}\right), T\left(\frac{(r - 1)n}{r}\right)\} + nc
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\]

assuming \(r \geq 2\),

\[
T(n) \leq cn\frac{(r-1)}{r} + cn\left(\frac{(r-1)}{r}\right)^2 + cn\left(\frac{(r-1)}{r}\right)^3 + \ldots + cn\left(\frac{(r-1)}{r}\right)^m = O(n)
\]

where \((\frac{(r-1)}{r})^m n = 1\), \(m = \log_{\frac{r-1}{r}} n\)
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Performance analysis
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Performance analysis

The worst case: running time $\Theta(n^2)$. 
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• on sublist $A[p..r]$, assume $n = r - p + 1$;
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Performance analysis

The worst case: running time $\Theta(n^2)$.

Average case: $E[T(n)]$

• on sublist $A[p..r]$, assume $n = r - p + 1$;

• the algorithm identifies a pivot and recursively computes on sublist $A[p..q]$ (or $A[q+1..r]$);
Performance analysis

The worst case: running time $\Theta(n^2)$.

Average case: $E[T(n)]$

• on sublist $A[p..r]$, assume $n = r - p + 1$;
• the algorithm identifies a pivot and recursively computes on sublist $A[p..q]$ (or $A[q+1..r]$);
• the pivot is chosen with probability $\frac{1}{n}$;
Average case: $E[T(n)]$ (cont’)

- so the expected time $E[T(n)]$ needs to include the average time of recursion on the case when sublist $A[p..q]$ possibly has lengths $k = 0, 1, 2, \ldots, n - 1$
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- thus the expected time $E[T(n)]$ is computed as

$$E[T(n)] \leq \sum_{k=1}^{n} \frac{1}{n} \cdot E[T(\max\{k-1, n-k\})] + an, \text{ for some constant } a > 0$$
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Average case: $E[T(n)]$ (cont’)

- so the expected time $E[T(n)]$ needs to include the average time of recursion on the case when sublist $A[p..q]$ possibly has lengths $k = 0, 1, 2, \ldots, n - 1$

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because $\max\{k-1, n-k\} = k - 1$ if $k > n/2$ and $\max\{k-1, n-k\} = n - k$ if $k \leq n/2$

$$E[T(n)] \leq \frac{2}{n} \sum_{k=n/2}^{n-1} E[T(k)] + an$$
We conclude that
\[ E[T(n)] \leq \frac{2}{n} \sum_{k=1}^{n-1} k = \frac{n}{2} E[T(k)] + an \]

Theorem.
\[ E[T(n)] = O(n). \]

Proof (by substitution method).
We will prove that
\[ E[T(n)] \leq cn \]
for some \( c > 0 \).

• Base case: \( n = \) ?, we will decide later;
• Assumption: for all \( k \leq n-1 \),
\[ E[T(k)] \leq ck; \]
• Induction:
\[ E[T(k)] \leq \frac{2}{n} \sum_{k=1}^{n-1} k = \frac{n}{2} E[T(k)] + an \]
\[ \leq 2c n \left( \frac{n-1}{2} \right) + an \]
\[ = 2c n \left( \frac{n^2}{4} - \frac{n}{2} + \frac{n}{2} \right) + an \]
\[ = \cdots = 3cn/4 + c/2 + an \]
\[ \leq cn \]
when \( \left( \frac{cn}{4} - \frac{c}{2} - an \right) \geq 0 \).

• Base case: \( T(n) \leq cn \), for \( n < \frac{2c}{c-4a} \),

How to prove?
We conclude that $E[T(n)] \leq 2/n \sum_{k=n/2}^{n-1} E[T(k)] + an$
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$$E[T(k)] \leq 2/n\sum_{k=n/2}^{n-1} E[T(k)] + an$$
We conclude that $E[T(n)] \leq 2/n\sum_{k=n/2}^{n-1} E[T(k)] + an$

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- **Base case:** $T(n) \leq cn$, for $n < 2c/(c - 4a)$, How to prove?
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Summary of Algorithm Analysis Scenarios

For example, given Insertion Sort:

• we first analyzed the algorithm and obtained
  \[ T(n) = c_1 n + c_2 (n-1) + c_4 (n-1) + c_5 \sum_{j=2}^{n} t_j + c_6 \sum_{j=2}^{n} (t_j - 1) + c_7 \sum_{j=2}^{n} (t_j - 1) + c_8 (n-1) \]

• we guessed upper bound \( T(n) = O(n^2) \), i.e., \( T(n) \leq cn^2 \);

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Summary of Algorithm Analysis Scenarios

Given an algorithm, carry out the following in order:
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For example, given Insertion Sort:

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Summary of Algorithm Analysis Scenarios

For recursive algorithms, for example, given the Binary Search algorithm,
• we first analyze the time $T(n)$ of the algorithm and obtained a recurrence for $T(n)$:
  $$T(n) \leq T(\lfloor n/2 \rfloor) + c$$
• we guess upper bound $T(n) = O(\log_2 n)$, i.e.,
  $$T(n) \leq c \log_2 n$$
• we prove the guessed bound.
  (1) we can use the recursive tree method by unfolding the time function,
  or (2) we can use the substitution method by the principle of induction.
  But we need the recurrence to apply induction.
  using the recurrence:
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see previous lecture notes
Summary of Algorithm Analysis Scenarios

For recursive algorithms

For example, given Binary Search algorithm,
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Chapter 9. Medians and Order Statistics

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