Part VI. Graph Algorithms
Chapter 22 Elementary graph algorithms
Chapter 23. Minimum spanning trees
Chapter 24. Single-source shortest paths
Chapter 25. All-pairs shortest paths
Chapter 22. Elementary graph algorithms

Chapter 22. Elementary graph algorithms
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- Representations of graphs
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- Traverse graphs:
  1. breadth-first-search (BFS)
  2. depth-first-search (DFS)
- Applications:
  1. topological sort
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Chapter 22. Elementary graph algorithms

Graph: $G = (V, E)$
Chapter 22. Elementary graph algorithms

Terminologies and notations:

• graph \( G = (V,E) \), where \( V = \{v_1,\ldots,v_n\} \) and \( E \subseteq V \times V \)

\( V = \{1,2,3,4,5,6,7\} \)

\( E = \{(1,2), (1,3), (2,3), (2,5), (3,4), (3,5), (4,5), (4,6), (5,6), (5,7)\} \)

• weight \( w: E \rightarrow \mathbb{R} \), e.g., \( w(1,2) = 4 \), \( w(5,6) = 3 \), etc.

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• cycle: when \( v_1 = v_k \).

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- **digraphs**: directed graphs

- **complete graphs**: $K_n$, e.g., $K_6$

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- **planar graphs**: embedded in the plane without crossing edges: However, $K_5$ is not planar, neither is $K_{3,3}$
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![Graph Diagram](image)
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  1-tree is a tree;

  2-tree is a graph but with **tree width** $= 2$
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Representations of graphs

adjacency-matrix
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adjacency-matrix
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adjacency-matrix for a weighted graph
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Traverse graphs

basic ideas of depth-first-search (DFS) and breadth-first-search (BFS)

Both methods yield "search trees"
or "search forest" (if the graph is not connected)
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DFS on directed graphs, search tree
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DFS on non-directed graphs, search tree
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DFS and BFS are two fundamental algorithms for graph traversal!
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First recursive DFS algorithm, assuming $G$ is connected.
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**Recursive-DFS**($G, u$);

How does the algorithm start?
- Initially set $u.visit = false$ for every vertex $u \in G.V$.
- Set $s.\pi = NULL$ for some specific $s \in G.V$.
- Call **Recursive-DFS**($G, s$).

But if $G$ is not connected, what should we do?
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5. Recursive-DFS ($G, v$);
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To-Start-DFS ($G$)
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To-Start-DFS(\(G\))
1. \textbf{for each} \(s \in G.V\) \{ initialize visit values \}
2. \(s.visit = \text{false};\)
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5. \[\text{Recursive-DFS}(G, v); \]
6. \[\text{return } ()\; ;\]
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DFS (from the textbook) computes \textit{discover} and \textit{finish} time stamps \((u.d \text{ and } u.f)\) for every visited vertex \(u\).

\begin{verbatim}
DFS(G)
1  for each vertex \(u \in G.V\)
2      \(u.color = \text{WHITE}\)
3      \(u.\pi = \text{NIL}\)
4  \(time = 0\)
5  for each vertex \(u \in G.V\)
6      if \(u.color = \text{WHITE}\)
7          DFS-VISIT(G, u)

DFS-VISIT(G, u)
1  \(time = time + 1\)  // white vertex \(u\) has just been discovered
2  \(u.d = time\)
3  \(u.color = \text{GRAY}\)
4  for each \(v \in G.Adj[u]\)  // explore edge \((u, v)\)
5      if \(v.color = \text{WHITE}\)
6          \(v.\pi = u\)
7          DFS-VISIT(G, v)
8  \(u.color = \text{BLACK}\)  // blacken \(u\); it is finished
9  \(time = time + 1\)
10  \(u.f = time\)
\end{verbatim}
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DFS-VISIT(G, u)
1  time = time + 1           // white vertex u has just been discovered
2  u.d = time
3  u.color = GRAY
4  for each v ∈ G.Adj[u]      // explore edge (u, v)
5      if v.color == WHITE
6          v.π = u
7          DFS-VISIT(G, v)
8  u.color = BLACK           // blacken u; it is finished
9  time = time + 1
10 u.f = time

→: edge being explored;
→: edge path taken by DFS
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DFS-\textsc{Visit}(G, u)
1 \hspace{1cm} \text{time} = \text{time} + 1 \hspace{2cm} \text{\texttt{// white vertex }u\text{ has just been discovered}
2 \hspace{1cm} u.d = \text{time}
3 \hspace{1cm} u.color = \text{GRAY}
4 \hspace{1cm} \text{for each } v \in G.\text{Adj}[u] \hspace{2cm} \text{\texttt{// explore edge } (u, v)\texttt{)
5 \hspace{2cm} \text{if } v.color = \text{WHITE}
6 \hspace{2cm} \hspace{1cm} v.\pi = u
7 \hspace{2cm} \hspace{1cm} \text{DFS-\textsc{Visit}}(G, v)
8 \hspace{2cm} u.color = \text{BLACK} \hspace{2cm} \text{\texttt{// blacken }u; \texttt{it is finished
9 \hspace{1cm} \text{time} = \text{time} + 1
10 \hspace{1cm} u.f = \text{time}

→: edge being explored;
→: edge path taken by DFS
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DFS-VISIT\( (G, u) \)
\begin{align*}
1 & \text{time} = \text{time} + 1 \quad \text{// white vertex } u \text{ has just been discovered} \\
2 & \text{u.d} = \text{time} \\
3 & \text{u.color} = \text{GRAY} \\
4 & \text{for each } v \in G.\text{Adj}[u] \quad \text{// explore edge } (u, v) \\
5 & \quad \text{if } v.\text{color} == \text{WHITE} \\
6 & \quad \quad v.\pi = u \\
7 & \quad \quad \text{DFS-VISIT}(G, v) \\
8 & \quad \text{u.color} = \text{BLACK} \quad \text{// blacken } u; \text{ it is finished} \\
9 & \text{time} = \text{time} + 1 \\
10 & \text{u.f} = \text{time}
\end{align*}

→: edge being explored;
→: edge path taken by DFS
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DFS-VISIT(G, u)
1. \( time = time + 1 \) // white vertex \( u \) has just been discovered
2. \( u.d = time \)
3. \( u.color = GRAY \)
4. for each \( v \in G.Adj[u] \) // explore edge \((u, v)\)
5. \( \text{if } v.color == WHITE \)
6. \( v.pi = u \)
7. DFS-VISIT(G, v)
8. \( u.color = BLACK \) // blacken \( u \); it is finished
9. \( time = time + 1 \)
10. \( u.f = time \)

→: edge being explored;
→: edge path taken by DFS
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Another example of DFS execution (page 605)
Time complexity of DFS algorithm

\[ \Theta(|E| + |V|), \] where \(|E|\) is the number of edges in \(G\).
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Time complexity of DFS algorithm

**DFS(G)**

1. for each vertex \( u \in G.V \)
2. \( u.color = \text{WHITE} \)
3. \( u.\pi = \text{NIL} \)
4. \( time = 0 \)
5. for each vertex \( u \in G.V \)
6. if \( u.color == \text{WHITE} \)
7. DFS-VISIT\( (G, u) \)

**DFS-VISIT(G, u)**

1. \( time = time + 1 \) % white vertex \( u \) has just been discovered
2. \( u.d = time \)
3. \( u.color = \text{GRAY} \)
4. for each \( v \in G.Adj[u] \) % explore edge \((u, v)\)
5. if \( v.color == \text{WHITE} \)
6. \( v.\pi = u \)
7. DFS-VISIT\( (G, v) \)
8. \( u.color = \text{BLACK} \) % blacken \( u \); it is finished
9. \( time = time + 1 \)
10. \( u.f = time \)
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Time complexity of DFS algorithm

DFS(G)
1 for each vertex \( u \in G.V \)
2 \( u.color = \text{WHITE} \)
3 \( u.\pi = \text{NIL} \)
4 \( \text{time} = 0 \)
5 for each vertex \( u \in G.V \)
6 if \( u.color == \text{WHITE} \)
7 \( \text{DFS-VISIT}(G, u) \)

DFS-VISIT(G, u)
1 \( \text{time} = \text{time} + 1 \) \hspace{1cm} // white vertex \( u \) has just been discovered
2 \( u.d = \text{time} \)
3 \( u.color = \text{GRAY} \)
4 for each \( v \in G.Adj[u] \) \hspace{1cm} // explore edge \( (u, v) \)
5 if \( v.color == \text{WHITE} \)
6 \( v.\pi = u \)
7 \( \text{DFS-VISIT}(G, v) \)
8 \( u.color = \text{BLACK} \) \hspace{1cm} // blacken \( u \); it is finished
9 \( \text{time} = \text{time} + 1 \)
10 \( u.f = \text{time} \)

\( \Theta(|E| + |V|) \), where \( |E| \) is the number of edges in \( G \).
Properties of depth-first-search:

(1) $u = v.\pi \iff \text{DFS-Visit}(G,v)$ is called.

(2) Theorem 22.7 (Parenthesis Theorem): for any $u, v$, exactly one of the following three conditions holds:

- $[u.d, u.f]$ and $[v.d, v.f]$ are entirely disjoint, and neither $u$ nor $v$ is a descendant of the other in the search tree.
- $[u.d, u.f]$ is contained entirely within $[v.d, v.f]$ and $u$ is a descendant of $v$.
- $[v.d, v.f]$ is contained entirely within $[u.d, u.f]$ and $v$ is a descendant of $u$.

Corollary 22.8 (Nesting of descendants’ intervals): Vertex $v$ is a proper descendant of $u$ in the depth-first search forest if and only if $u.d < v.d < v.f < u.f$. 

Properties of depth-first-search:

(1) \( u = v.\pi \)
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Properties of depth-first-search:

1. \( u = v.\pi \) iff \( \text{DFS-Visit}(G, v) \) is called.
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- \([u.d, u.f]\) and \([v.d, v.f]\) are entirely disjoint, and neither \( u \) nor \( v \) is a descendant of the other in the search tree.
- \([u.d, u.f]\) is contained entirely within \([v.d, v.f]\) and \( u \) is a descendant of \( v \), or
- \([v.d, v.f]\) is contained entirely within \([u.d, u.f]\) and \( v \) is a descendant of \( u \).
Properties of depth-first-search:

1. $u = v.\pi$ iff $\text{DFS-Visit}(G, v)$ is called.

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   - $[u.d, u.f]$ and $[v.d, v.f]$ are entirely disjoint, and neither $u$ nor $v$ is a descendant of the other in the search tree.
Properties of depth-first-search:

1. $u = v.\pi$ iff DFS-Visit$(G, v)$ is called.

2. **Theorem 22.7 (Parenthesis Theorem):** for any $u, v$, exactly one of the following three conditions holds:
   
   - $[u.d, u.f]$ and $[v.d, v.f]$ are entirely disjoint, and neither $u$ nor $v$ is a descendant of the other in the search tree.
   - $[u.d, u.f]$ is contained entirely within $[v.d, v.f]$ and $u$ is a descendant of $v$, or
Properties of depth-first-search:

1. \( u = v.\pi \) iff DFS-Visit\((G, v)\) is called.

2. Theorem 22.7 (Parenthesis Theorem): for any \( u, v \), exactly one of the following three conditions holds:

- \([u.d, u.f]\) and \([v.d, v.f]\) are entirely disjoint, and neither \( u \) nor \( v \) is a descendant of the other in the search tree.
- \([u.d, u.f]\) is contained entirely within \([v.d, v.f]\) and \( u \) is a descendant of \( v \), or
- \([v.d, v.f]\) is contained entirely within \([u.d, u.f]\) and \( v \) is a descendant of \( u \).
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Properties of depth-first-search:

(1) \( u = v.\pi \) iff \( \text{DFS-Visit}(G, v) \) is called.

(2) **Theorem 22.7 (Parenthesis Theorem):** for any \( u, v \), exactly one of the following three conditions holds:

- \([u.d, u.f]\) and \([v.d, v.f]\) are entirely disjoint, and neither \( u \) nor \( v \) is a descendant of the other in the search tree.

- \([u.d, u.f]\) is contained entirely within \([v.d, v.f]\) and \( u \) is a descendant of \( v \), or

- \([v.d, v.f]\) is contained entirely within \([u.d, u.f]\) and \( v \) is a descendant of \( u \).

**Corollary 22.8 (Nesting of descendants’ intervals)** Vertex \( v \) is a proper descendant of \( u \) in the depth-first search forest if and only if \( u.d < v.d < v.f < u.f \).
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**Theorem 22.9 (White-path Theorem)** In a depth-first search forest of graph $G$, vertex $v$ is a descendant of $u$ if and only if at the time $u.d$ that the search discovers $u$, there is a path from $u$ to $v$ consisting of entirely of white vertices.
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Theorem 22.9 (White-path Theorem) In a depth-first search forest of graph $G$, vertex $v$ is a descendant of $u$ if and only if at the time $u.d$ that the search discovers $u$, there is a path from $u$ to $v$ consisting of entirely of white vertices.

Proof: \(\Rightarrow\)
Theorem 22.9 (White-path Theorem) In a depth-first search forest of graph $G$, vertex $v$ is a descendant of $u$ if and only if at the time $u.d$ that the search discovers $u$, there is a path from $u$ to $v$ consisting of entirely of white vertices.

Proof: $\Rightarrow$
   
   case 1: $u = v$, apparently the claim is true;
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**Theorem 22.9 (White-path Theorem)** In a depth-first search forest of graph $G$, vertex $v$ is a descendant of $u$ if and only if at the time $u.d$ that the search discovers $u$, there is a path from $u$ to $v$ consisting of entirely of white vertices.

**Proof:** $\Rightarrow$

- case 1: $u = v$, apparently the claim is true;
- case 2: $v$ is a proper descendant of $u$, use **Corollary 22.8** on every vertex on the path from $u$ to $v$; the claim is true;
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Theorem 22.9 (White-path Theorem) In a depth-first search forest of graph $G$, vertex $v$ is a descendant of $u$ if and only if at the time $u.d$ that the search discovers $u$, there is a path from $u$ to $v$ consisting of entirely of white vertices.

Proof: $\Rightarrow$

- case 1: $u = v$, apparently the claim is true;
- case 2: $v$ is a proper descendant of $u$, use Corollary 22.8 on every vertex on the path from $u$ to $v$; the claim is true;

$\Leftarrow$
Theorem 22.9 (White-path Theorem) In a depth-first search forest of graph $G$, vertex $v$ is a descendant of $u$ if and only if at the time $u.d$ that the search discovers $u$, there is a path from $u$ to $v$ consisting of entirely of white vertices.

Proof: $\Rightarrow$

- case 1: $u = v$, apparently the claim is true;
- case 2: $v$ is a proper descendant of $u$, use Corollary 22.8 on every vertex on the path from $u$ to $v$; the claim is true;

$\Leftarrow$

Assume that at the time $u.d$, there is a path from $u$ to $v$ as stated in the theorem but $v$ is not descendant of $u$.
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Theorem 22.9 (White-path Theorem) In a depth-first search forest of graph $G$, vertex $v$ is a descendant of $u$ if and only if at the time $u.d$ that the search discovers $u$, there is a path from $u$ to $v$ consisting of entirely of white vertices.

Proof: $\Rightarrow$

- case 1: $u = v$, apparently the claim is true;
- case 2: $v$ is a proper descendant of $u$, use Corollary 22.8 on every vertex on the path from $u$ to $v$; the claim is true;

$\Leftarrow$

Assume that at the time $u.d$, there is a path from $u$ to $v$ as stated in the theorem but $v$ is not descendant of $u$.

Assume $(w, x)$ be an edge on the path and $x$ is the first vertex on the path which is not descendant of $u$. 
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**Theorem 22.9 (White-path Theorem)** In a depth-first search forest of graph $G$, vertex $v$ is a descendant of $u$ if and only if at the time $u.d$ that the search discovers $u$, there is a path from $u$ to $v$ consisting of entirely of white vertices.

**Proof:** $\Rightarrow$
- case 1: $u = v$, apparently the claim is true;
- case 2: $v$ is a proper descendant of $u$, use Corollary 22.8 on every vertex on the path from $u$ to $v$; the claim is true;

$\Leftarrow$
- Assume that at the time $u.d$, there is a path from $u$ to $v$ as stated in the theorem but $v$ is not descendant of $u$.
- Assume $(w, x)$ be an edge on the path and $x$ is the first vertex on the path which is not descendant of $u$. Note that $w$ is a descendant of $u$ (or just $u$)
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**Theorem 22.9 (White-path Theorem)** In a depth-first search forest of graph $G$, vertex $v$ is a descendant of $u$ if and only if at the time $u.d$ that the search discovers $u$, there is a path from $u$ to $v$ consisting of entirely of white vertices.

**Proof:** $\Rightarrow$
- case 1: $u = v$, apparently the claim is true;
- case 2: $v$ is a proper descendant of $u$, use Corollary 22.8 on every vertex on the path from $u$ to $v$; the claim is true;

$\Leftarrow$
- Assume that at the time $u.d$, there is a path from $u$ to $v$ as stated in the theorem but $v$ is not descendant of $u$.

Assume $(w, x)$ be an edge on the path and $x$ is the first vertex on the path which is not descendant of $u$. Note that $w$ is a descendant of $u$ (or just $u$). Because at time $u.d$ $x$ is of WHITE color, $u.d < x.d$. 
Theorem 22.9 (White-path Theorem) In a depth-first search forest of graph $G$, vertex $v$ is a descendant of $u$ if and only if at the time $u.d$ that the search discovers $u$, there is a path from $u$ to $v$ consisting of entirely of white vertices.

Proof: $\Rightarrow$

- case 1: $u = v$, apparently the claim is true;
- case 2: $v$ is a proper descendant of $u$, use Corollary 22.8 on every vertex on the path from $u$ to $v$; the claim is true;

$\Leftarrow$

Assume that at the time $u.d$, there is a path from $u$ to $v$ as stated in the theorem but $v$ is not descendant of $u$.

Assume $(w, x)$ be an edge on the path and $x$ is the first vertex on the path which is not descendant of $u$. Note that $w$ is a descendant of $u$ (or just $u$). Because at time $u.d$ $x$ is of WHITE color, $u.d < x.d$. Because $(w, x)$ is an edge, there are two possible scenarios:

1. when $(w, x)$ is being explored, $x$ has already been discovered;
Theorem 22.9 (White-path Theorem) In a depth-first search forest of graph $G$, vertex $v$ is a descendant of $u$ if and only if at the time $u.d$ that the search discovers $u$, there is a path from $u$ to $v$ consisting of entirely of white vertices.

Proof: $\Rightarrow$
- case 1: $u = v$, apparently the claim is true;
- case 2: $v$ is a proper descendant of $u$, use Corollary 22.8 on every vertex on the path from $u$ to $v$; the claim is true;

$\Leftarrow$
- Assume that at the time $u.d$, there is a path from $u$ to $v$ as stated in the theorem but $v$ is not descendant of $u$.

Assume $(w, x)$ be an edge on the path and $x$ is the first vertex on the path which is not descendant of $u$. Note that $w$ is a descendant of $u$ (or just $u$)

Because at time $u.d$ $x$ is of WHITE color, $u.d < x.d$.

Because $(w, x)$ is an edge, there are two possible scenarios:
- (1) when $(w, x)$ is being explored, $x$ has already been discovered; we thus have $x.d < w.f$;
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Theorem 22.9 (White-path Theorem) In a depth-first search forest of graph $G$, vertex $v$ is a descendant of $u$ if and only if at the time $u.d$ that the search discovers $u$, there is a path from $u$ to $v$ consisting of entirely of white vertices.

Proof: $\Rightarrow$

- case 1: $u = v$, apparently the claim is true;
- case 2: $v$ is a proper descendant of $u$, use Corollary 22.8 on every vertex on the path from $u$ to $v$; the claim is true;

$\Leftarrow$

Assume that at the time $u.d$, there is a path from $u$ to $v$ as stated in the theorem but $v$ is not descendant of $u$.

Assume $(w, x)$ be an edge on the path and $x$ is the first vertex on the path which is not descendant of $u$. Note that $w$ is a descendant of $u$ (or just $u$).

Because at time $u.d$, $x$ is of WHITE color, $u.d < x.d$.

Because $(w, x)$ is an edge, there are two possible scenarios:

1. when $(w, x)$ is being explored, $x$ has already been discovered; we thus have $x.d < w.f$;
2. when $(w, x)$ is being explored, $x$ has WHITE color but will then be discovered
Theorem 22.9 (White-path Theorem) In a depth-first search forest of graph $G$, vertex $v$ is a descendant of $u$ if and only if at the time $u.d$ that the search discovers $u$, there is a path from $u$ to $v$ consisting of entirely of white vertices.

Proof: $\Rightarrow$

- case 1: $u = v$, apparently the claim is true;
- case 2: $v$ is a proper descendant of $u$, use Corollary 22.8 on every vertex on the path from $u$ to $v$; the claim is true;

$\Leftarrow$

Assume that at the time $u.d$, there is a path from $u$ to $v$ as stated in the theorem but $v$ is not descendant of $u$.

Assume $(w, x)$ be an edge on the path and $x$ is the first vertex on the path which is not descendant of $u$. Note that $w$ is a descendant of $u$ (or just $u$).

Because at time $u.d$ $x$ is of WHITE color, $u.d < x.d$.

Because $(w, x)$ is an edge, there are two possible scenarios:

1. when $(w, x)$ is being explored, $x$ has already been discovered; we thus have $x.d < w.f$;
2. when $(w, x)$ is being explored, $x$ has WHITE color but will then be discovered we thus also have $x.d < w.f$;
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Theorem 22.9 (White-path Theorem) In a depth-first search forest of graph $G$, vertex $v$ is a descendant of $u$ if and only if at the time $u.d$ that the search discovers $u$, there is a path from $u$ to $v$ consisting of entirely of white vertices.

Proof: $\Rightarrow$
\begin{itemize}
  \item case 1: $u = v$, apparently the claim is true;
  \item case 2: $v$ is a proper descendant of $u$, use Corollary 22.8 on every vertex on the path from $u$ to $v$; the claim is true;
\end{itemize}

$\Leftarrow$
Assume that at the time $u.d$, there is a path from $u$ to $v$ as stated in the theorem but $v$ is not descendant of $u$.

Assume $(w, x)$ be an edge on the path and $x$ is the first vertex on the path which is not descendant of $u$. Note that $w$ is a descendant of $u$ (or just $u$)

Because at time $u.d$ $x$ is of WHITE color, $u.d < x.d$.

Because $(w, x)$ is an edge, there are two possible scenarios:
\begin{itemize}
  \item (1) when $(w, x)$ is being explored, $x$ has already been discovered; we thus have $x.d < w.f$;
  \item (2) when $(w, x)$ is being explored, $x$ has WHITE color but will then be discovered we thus also have $x.d < w.f$;
\end{itemize}

According to Corollary 22.8, $u.d < x.d < w.f < u.f$. 

Theorem 22.9 (White-path Theorem) In a depth-first search forest of graph $G$, vertex $v$ is a descendant of $u$ if and only if at the time $u.d$ that the search discovers $u$, there is a path from $u$ to $v$ consisting of entirely of white vertices.

Proof: $\Rightarrow$

- case 1: $u = v$, apparently the claim is true;
- case 2: $v$ is a proper descendant of $u$, use Corollary 22.8 on every vertex on the path from $u$ to $v$; the claim is true;

$\Leftarrow$

Assume that at the time $u.d$, there is a path from $u$ to $v$ as stated in the theorem but $v$ is not descendant of $u$.

Assume $(w, x)$ be an edge on the path and $x$ is the first vertex on the path which is not descendant of $u$. Note that $w$ is a descendant of $u$ (or just $u$)

Because at time $u.d$ $x$ is of WHITE color, $u.d < x.d$.

Because $(w, x)$ is an edge, there are two possible scenarios:

1. when $(w, x)$ is being explored, $x$ has already been discovered; we thus have $x.d < w.f$;
2. when $(w, x)$ is being explored, $x$ has WHITE color but will then be discovered we thus also have $x.d < w.f$;

According to Corollary 22.8, $u.d < x.d < w.f < u.f$. Thus, $u.d < x.d < u.f$
Theorem 22.9 (White-path Theorem) In a depth-first search forest of graph $G$, vertex $v$ is a descendant of $u$ if and only if at the time $u.d$ that the search discovers $u$, there is a path from $u$ to $v$ consisting of entirely of white vertices.

Proof: $\Rightarrow$

case 1: $u = v$, apparently the claim is true;
case 2: $v$ is a proper descendant of $u$, use Corollary 22.8 on every vertex on the path from $u$ to $v$; the claim is true;

$\Leftarrow$

Assume that at the time $u.d$, there is a path from $u$ to $v$ as stated in the theorem but $v$ is not descendant of $u$.

Assume $(w, x)$ be an edge on the path and $x$ is the first vertex on the path which is not descendant of $u$. Note that $w$ is a descendant of $u$ (or just $u$).

Because at time $u.d$, $x$ is of WHITE color, $u.d < x.d$.

Because $(w, x)$ is an edge, there are two possible scenarios:

1. when $(w, x)$ is being explored, $x$ has already been discovered;
   we thus have $x.d < w.f$;
2. when $(w, x)$ is being explored, $x$ has WHITE color but will then be discovered
   we thus also have $x.d < w.f$;

According to Corollary 22.8, $u.d < x.d < w.f < u.f$. Thus, $u.d < x.d < u.f$
By Theorem 22.7, the interval $[x.d, x.f]$ is entirely contained within interval $[u.d, u.f]$. 
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Theorem 22.9 (White-path Theorem) In a depth-first search forest of graph $G$, vertex $v$ is a descendant of $u$ if and only if at the time $u.d$ that the search discovers $u$, there is a path from $u$ to $v$ consisting of entirely of white vertices.

Proof: $\Rightarrow$

case 1: $u = v$, apparently the claim is true;

case 2: $v$ is a proper descendant of $u$, use Corollary 22.8 on every vertex on the path from $u$ to $v$; the claim is true;

$\Leftarrow$

Assume that at the time $u.d$, there is a path from $u$ to $v$ as stated in the theorem but $v$ is not descendant of $u$.

Assume $(w, x)$ be an edge on the path and $x$ is the first vertex on the path which is not descendant of $u$. Note that $w$ is a descendant of $u$ (or just $u$)

Because at time $u.d$ $x$ is of WHITE color, $u.d < x.d$.

Because $(w, x)$ is an edge, there are two possible scenarios:

(1) when $(w, x)$ is being explored, $x$ has already been discovered;
   we thus have $x.d < w.f$;

(2) when $(w, x)$ is being explored, $x$ has WHITE color but will then be discovered
   we thus also have $x.d < w.f$;

According to Corollary 22.8, $u.d < x.d < w.f < u.f$. Thus, $u.d < x.d < u.f$

By Theorem 22.7, the interval $[x.d, x.f]$ is entirely contained within interval $[u.d, u.f]$. Therefore, $x$ is a descendant of $u$. Contradicts to the earlier assumption.
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**Theorem 22.9 (White-path Theorem)** In a depth-first search forest of graph $G$, vertex $v$ is a descendant of $u$ if and only if at the time $u.d$ that the search discovers $u$, there is a path from $u$ to $v$ consisting of entirely of white vertices.

**Proof:**  \[\Rightarrow\]  
- case 1: $u = v$, apparently the claim is true;
- case 2: $v$ is a proper descendant of $u$, use **Corollary 22.8** on every vertex on the path from $u$ to $v$; the claim is true;

\[\Leftarrow\]  
Assume that at the time $u.d$, there is a path from $u$ to $v$ as stated in the theorem but $v$ is not descendant of $u$.

Assume $(w, x)$ be an edge on the path and $x$ is the first vertex on the path which is not descendant of $u$. Note that $w$ is a descendant of $u$ (or just $u$)

Because at time $u.d$ $x$ is of WHITE color, $u.d < x.d$.

Because $(w, x)$ is an edge, there are two possible scenarios:
- (1) when $(w, x)$ is being explored, $x$ has already been discovered; we thus have $x.d < w.f$;
- (2) when $(w, x)$ is being explored, $x$ has WHITE color but will then be discovered we thus also have $x.d < w.f$;

According to **Corollary 22.8**, $u.d < x.d < w.f < u.f$. Thus, $u.d < x.d < u.f$

By **Theorem 22.7**, the interval $[x.d, x.f]$ is entirely contained within interval $[u.d, u.f]$. Therefore, $x$ is a descendant of $u$. Contradicts to the earlier assumption. $v$ should be a descendant of $u$. 
Classification of edges (for directed graphs)
Chapter 22. Elementary graph algorithms

Classification of edges (for directed graphs)

- **Tree edges**: those in the search tree (forest);
- **Back edges**: those connecting a vertex to an ancestor; a self-loop, in a directed graph, can be a back edge;
- **Forward edges**: those connecting a vertex to a descendant;
- **Cross edges**: all other edges.

DFS forest with back, forward, & cross edges.
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![Diagram showing classification of edges](image)
Chapter 22. Elementary graph algorithms

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Chapter 22. Elementary graph algorithms

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Chapter 22. Elementary graph algorithms

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First stages of a Directed DFS, showing **Edges**, the **DFS TREE**, a **Tree Edge**, a **Back Edge**, a **Forward Edge**, and a **Cross Edge**.
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To identify the type of edge \((u, v)\) with the color of \(v\):

First stages of a Directed DFS, showing Edges, the DFS TREE, a Tree Edge, a Back Edge, a Forward Edge, and a Cross Edge.
Chapter 22. Elementary graph algorithms

To identify the type of edge \((u, v)\) with the color of \(v\):

WHITE: tree edge;
To identify the type of edge \((u, v)\) with the color of \(v\):

- **WHITE**: tree edge;
- **GRAY**: back edge;
- **BLACK**: forward or cross edge;
Chapter 22. Elementary graph algorithms

To identify the type of edge \((u, v)\) with the color of \(v\):

- WHITE: tree edge;
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Theorem 22.10 In a depth-first search of undirected graph $G$, every edge of $G$ is either tree edge or back edge.
Chapter 22. Elementary graph algorithms

Theorem 22.10 In a depth-first search of undirected graph $G$, every edge of $G$ is either tree edge or back edge.

Proof. Let $(u, v)$ be an edge in $G$. Assume in the depth-first search of $G$, $u.d < v.d$. 
Theorem 22.10 In a depth-first search of undirected graph $G$, every edge of $G$ is either tree edge or back edge.

Proof. Let $(u, v)$ be an edge in $G$. Assume in the depth-first search of $G$, $u.d < v.d$. Then there are two scenarios:
Theorem 22.10 In a depth-first search of undirected graph $G$, every edge of $G$ is either tree edge or back edge.

Proof. Let $(u, v)$ be an edge in $G$. Assume in the depth-first search of $G$, $u.d < v.d$. Then there are two scenarios:

1. $v$ is discovered by exploring edge $(u, v)$, then $(u, v)$ is a tree edge;
2. $v$ is discovered not through exploring edge $(u, v)$. Because $(u, v)$ is an edge, $v$ is discovered when $u$ is in gray color. Since $u$ is in the adjacency list of $v$, $(v, u)$ will eventually be explored and thus a back edge.
**Theorem 22.10** In a depth-first search of undirected graph $G$, every edge of $G$ is either tree edge or back edge.

**Proof.** Let $(u, v)$ be an edge in $G$. Assume in the depth-first search of $G$, $u.d < v.d$. Then there are two scenarios:

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Breadth First Search (BFS)
Chapter 22. Elementary graph algorithms

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Chapter 22. Elementary graph algorithms

Breadth First Search Algorithm
(with a queue)

Time complexity of BFS: \( O(|V| + |E|) \)

Note: BFS can find a shortest path from \( s \) to all other nodes (non-weighted). (Why?)
Breadth First Search Algorithm (with a queue)

Time complexity of BFS: $O(|V| + |E|)$

Note: BFS can find a shortest path from $s$ to all other nodes (non-weighted).
Chapter 22. Elementary graph algorithms

Breadth First Search Algorithm (with a queue)

BFS($G, s$)
1. for each vertex $u \in G.V - \{s\}$
2. \hspace{1em} $u.color = \text{WHITE}$
3. \hspace{2em} $u.d = \infty$
4. \hspace{2em} $u.\pi = \text{NIL}$
5. $s.color = \text{GRAY}$
6. $s.d = 0$
7. $s.\pi = \text{NIL}$
8. $Q = \emptyset$
9. \hspace{1em} $\text{ENQUEUE}(Q, s)$
10. while $Q \neq \emptyset$
11. \hspace{2em} $u = \text{DEQUEUE}(Q)$
12. \hspace{3em} for each $v \in G.\text{Adj}[u]$
13. \hspace{4em} \text{if } v.color == \text{WHITE}
14. \hspace{5em} \hspace{1em} $v.color = \text{GRAY}$
15. \hspace{5em} \hspace{1em} $v.d = u.d + 1$
16. \hspace{5em} \hspace{1em} $v.\pi = u$
17. \hspace{5em} \hspace{1em} $\text{ENQUEUE}(Q, v)$
18. \hspace{2em} $u.color = \text{BLACK}$
Chapter 22. Elementary graph algorithms

Breadth First Search Algorithm (with a queue)

BFS(G, s)
1 for each vertex u ∈ G.V − {s}
2 u.color = WHITE
3 u.d = ∞
4 u.π = NIL
5 s.color = GRAY
6 s.d = 0
7 s.π = NIL
8 Q = ∅
9 ENQUEUE(Q, s)
10 while Q ≠ ∅
11 u = DEQUEUE(Q)
12 for each v ∈ G.Adj[u]
13 if v.color == WHITE
14 v.color = GRAY
15 v.d = u.d + 1
16 v.π = u
17 ENQUEUE(Q, v)
18 u.color = BLACK

Time complexity of BFS:
Chapter 22. Elementary graph algorithms

Breadth First Search Algorithm (with a queue)

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7. $s.\pi = \text{NIL}$
8. $Q = \emptyset$
9. ENQUEUE($Q, s$)
10. while $Q \neq \emptyset$
11. $u = \text{DEQUEUE}(Q)$
12. for each $v \in G.\text{Adj}[u]$
13. \quad if $v.color == \text{WHITE}$
14. \quad \quad $v.color = \text{GRAY}$
15. \quad \quad $v.d = u.d + 1$
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17. \quad \quad ENQUEUE($Q, v$)
18. $u.color = \text{BLACK}$

Time complexity of BFS: $O(|V| + |E|)$
Chapter 22. Elementary graph algorithms

Breadth First Search Algorithm (with a queue)

```plaintext
BFS(G, s)
1  for each vertex u ∈ G.V − {s}
2     u.color = WHITE
3     u.d = ∞
4     u.π = NIL
5  s.color = GRAY
6  s.d = 0
7  s.π = NIL
8  Q = Ø
9  ENQUEUE(Q, s)
10  while Q ≠ Ø
11     u = DEQUEUE(Q)
12        for each v ∈ G.Adj[u]
13           if v.color == WHITE
14              v.color = GRAY
15              v.d = u.d + 1
16              v.π = u
17              ENQUEUE(Q, v)
18     u.color = BLACK
```

Time complexity of BFS: $O(|V| + |E|)$

Note: BFS can find a shortest path from $s$ to all other nodes (non-weighted). (Why?)
Chapter 22. Elementary graph algorithms

Applications

Reachability Problem

Input: \( G = (V,E) \), and \( s,t \in V \)

Output: YES if and only there is a path \( s \Rightarrow t \) in \( G \).

• The problem can be solved with DFS and BFS by search on the graph from \( s \) until \( t \) shows up.
Applications

Reachability Problem

Input: \( G = (V, E) \), and \( s, t \in V \);

Output: YES if and only if there is a path \( s \rightleftharpoons t \) in \( G \).

\[ \text{The problem can be solved with DFS and BFS by search on the graph from } s \text{ until } t \text{ shows up.} \]
Applications

Reachability Problem
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Reachability Problem

INPUT: $G = (V, E)$, and $s, t \in V$,
Applications

Reachability Problem

**Input:** \( G = (V, E) \), and \( s, t \in V \);
**Output:** \textbf{YES} if and only there is a path \( s \sim t \) in \( G \).
Applications

Reachability Problem

**Input:** $G = (V, E)$, and $s, t \in V$;

**Output:** *YES* if and only there is a path $s \leadsto t$ in $G$.

- The problem can be solved with DFS and BFS
Applications

Reachability Problem

**Input:** $G = (V, E)$, and $s, t \in V$;

**Output:** YES if and only there is a path $s \leadsto t$ in $G$.

- The problem can be solved with DFS and BFS
  - by search on the graph from $s$ until $t$ shows up.
Applications

Reachability Problem

**Input:** $G = (V, E)$, and $s, t \in V$;

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- The problem can be solved with DFS and BFS by search on the graph from $s$ until $t$ shows up.
Chapter 22. Elementary graph algorithms

Reachability Problem

Reachability \((G, u, t)\);

1. \(u.\text{visit} = \text{true}\);
2. for each \(v \in \text{Adj}[u]\) and not \(v.\text{visit}\);
3. if \(v = t\) then reachable = Yes; exit;
4. else \(v.\pi = u\);
5. Reachability \((G, v, t)\);
6. return ()

Main()

reachable = No;
Reachability \((G, s, t)\);
print (reachable);
Reachability Problem

Reachability Problem
Reachability Problem

\textsc{Reachability}(G, u, t);
1. \hspace{1em} u.visit = \textbf{true};
2. \hspace{1em} for each \hspace{0.5em} v \in Adj[u] \hspace{1em} and \hspace{1em} \textbf{not} \hspace{1em} v.visit;
3. \hspace{1em} \textbf{if} \hspace{1em} v = t \hspace{1em} \textbf{then} \hspace{1em} \text{reachable} = \textbf{Yes}; \hspace{1em} \textbf{exit};
4. \hspace{1em} \textbf{else} \hspace{1em} v.\pi = u;
5. \hspace{1em} \textsc{Reachability}(G, v, t);
6. \hspace{1em} \textbf{return} ( );

Main()
\begin{itemize}
\item reachable = \textbf{No};
\item \textsc{Reachability}(G, s, t);
\item \textbf{print} (reachable);
\end{itemize}
Reachability Problem

**REACHABILITY**($G, u, t$);
1. $u.visit = true$;
2. **for** each $v \in Adj[u]$ and **not** $v.visit$;
3. **if** $v = t$ **then** reachable = Yes; **exit**;
4. **else** $v.\pi = u$;
5. **REACHABILITY**($G, v, t$);
6. **return** ( );

**MAIN()**

reachable = No;
**REACHABILITY**($G, s, t$);
**print**(reachable);
Chapter 22. Elementary graph algorithms

Path Counting Problem

Input:
\( G = (V, E) \), and 
\( s, t \in V \);

Output: the number of paths from 
\( s \mapsto t \) in 
\( G \).

- we modify Reachability to count paths.

\[
\text{PathCounting}(G, u, t);
\]

1. \( u.\text{visit} = \text{true} \);
2. for each \( v \in \text{Adj}[u] \);
3. if \( v.\text{visit} \) then 
   \( u.\text{c} = u.\text{c} + v.\text{c} \);
4. else 
   \( v.\pi = u \);
5. \( \text{PathCounting}(G, v, t) \);
6. \( u.\text{c} = u.\text{c} + v.\text{c} \);
7. return ( );

Main()

1. for each \( u \in G \) and \( u \neq t \);
2. \( u.\text{c} = 0 \);
3. \( t.\text{c} = 1 \);
4. \( \text{PathCounting}(G, s, t) \);
5. print ( \( s.\text{c} \) )
Path Counting Problem

Input: $G = (V,E)$, and $s, t \in V$; 
Output: the number of paths from $s \rightarrow t$ in $G$.

- we modify Reachability to count paths.

PathCounting($G,u,t$):
1. $u.visit = true$;
2. for each $v \in \text{Adj}[u]$:
3. if $v.visit$ then $u.c = u.c + v.c$;
4. else $v.\pi = u$;
5. PathCounting($G,v,t$);
6. $u.c = u.c + v.c$;
7. return ();

Main():
1. for each $u \in G$ and $u \neq t$:
2. $u.c = 0$;
3. $t.c = 1$;
4. PathCounting($G,s,t$);
5. print ($s.c$)
Chapter 22. Elementary graph algorithms

Path Counting Problem

Input: $G = (V, E)$, and $s, t \in V$;
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Path Counting Problem

Input: \( G = (V, E) \), and \( s, t \in V \) ;
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Path Counting Problem

INPUT: \( G = (V, E) \), and \( s, t \in V \);  
OUTPUT: the number of paths from \( s \rightsquigarrow t \) in \( G \).

• we modify Reachability to count paths.
Path Counting Problem

**Input:** $G = (V, E)$, and $s, t \in V$;

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PathCounting($G, u, t$);
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7. return ( );
```
Chapter 22. Elementary graph algorithms

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Chapter 22. Elementary graph algorithms

Topological sorting

- On directed acyclic graphs (DAGs)
- A sorted order: socks, shorts, pants, shoes, shirt, tie, belt, jacket, watch.
Chapter 22. Elementary graph algorithms

Topological sorting
Chapter 22. Elementary graph algorithms

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• apply DFS algorithm.

• reversed order of finish times: p, n, o, s, m, r, y, v, x, w, z, u, q, t

• Correctness proof?
Chapter 22. Elementary graph algorithms

• apply DFS algorithm.
Chapter 22. Elementary graph algorithms

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Chapter 22. Elementary graph algorithms

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- Correctness proof?
Chapter 22. Elementary graph algorithms

Strongly connected components (SCC)

Let $G = (V, E)$ be a digraph. A strongly connected component is a maximal subgraph $H = (V_H, E_H)$ of $G$ such that for every two nodes $v, u \in V_H$,

1. there is a directed path $v \Rightarrow u$ consisting of edges in $E_H$; and
2. there is a directed path $u \Rightarrow v$ consisting of edges in $E_H$.
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Strongly connected components (SCC)
Chapter 22. Elementary graph algorithms

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Strongly connected components (SCC)

Let $G = (V, E)$ be a digraph. A strongly connected component is a maximal subgraph $H = (V_H, E_H)$ of $G$ such that for every two nodes $v, u \in V_H$, there is a directed path $v \Rightarrow u$ consisting of edges in $E_H$ and a directed path $u \Rightarrow v$ consisting of edges in $E_H$. 
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Chapter 22. Elementary graph algorithms

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(1) there is a directed path $v \leadsto u$ consisting of edges in $E_H$; and

(2) there is a directed path $u \leadsto v$ consisting of edges in $E_H$. 
Chapter 22. Elementary graph algorithms

Strongly connected components (SCC)

Let $G = (V, E)$ be a digraph. A strongly connected component is a maximal subgraph $H = (V_H, E_H)$ of $G$ such that for every two nodes $v, u \in V_H$, (1) there is a directed path $v \rightarrow u$ consisting of edges in $E_H$; and (2) there is a directed path $u \rightarrow v$ consisting of edges in $E_H$. 
Idea of an algorithm to use DFS to solve SCC problem.

• use DFS to generate DFS forest; each search tree $T_u$ (rooted at $u$) consists of vertices $v$ such that $u \rightarrow v$;

• use DFS again on $T_u$; hope to search from every one $v$ within $T_u$ to make sure $v \rightarrow u$ as well.

• however, this may be difficult (proof is left as an exercise).
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- however, this may be difficult (proof is left as an exercise).
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Algorithm

1. call DFS(G) to compute u.f for each u ∈ G.V
2. compute G_{T} the transpose of G \{reverse all edges in G\}
3. call DFS(G_{T}) (vertices are considered in the decreasing order of finish times computed in step 1)
4. output each tree in the depth-first forest produced by step 3.
Chapter 22. Elementary graph algorithms

Algorithm **Strongly Connected Components**($G$)

1. call DFS($G$) to compute $u.f$ for each $u \in G.V$
2. compute $G^T$ the transpose of $G$ (reverse all edges in $G$)
3. call DFS($G^T$) (vertices are considered in the decreasing order of finish times computed in step 1)
4. output each tree in the depth-first forest produced by step 3.
Chapter 22. Elementary graph algorithms

Algorithm Strongly Connected Components(G)
1. call DFS(G) to compute u.f for each u ∈ G.V
Chapter 22. Elementary graph algorithms

Algorithm **Strongly Connected Components** \((G)\)
1. *call* \(\text{DFS}(G)\) to compute \(u.f\) for each \(u \in G.V\)
2. compute \(G^T\) the transpose of \(G\)
Chapter 22. Elementary graph algorithms

Algorithm **Strongly Connected Components** (*G*)

1. **call** DFS(*G*) to compute *u.f* for each *u ∈ G.V*
2. compute *G^T* the transpose of *G* { reverse all edges in *G* }
3. **call** DFS(*G^T*) (vertices are considered in the decreasing order of finish times computed in step 1)
4. output each tree in the depth-first forest produced by step 3.
Chapter 22. Elementary graph algorithms

Algorithm Strongly Connected Components($G$)
1. call DFS($G$) to compute $u.f$ for each $u \in G.V$
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Chapter 22. Elementary graph algorithms

Algorithm **Strongly Connected Components**$(G)$
1. **call** DFS$(G)$ to compute $u.f$ for each $u \in G.V$
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Chapter 22. Elementary graph algorithms

Algorithm **STRONGLY CONNECTED COMPONENTS** \((G)\)

1. **call** DFS \((G)\) to compute \(u.f\) for each \(u \in G.V\)
2. compute \(G^T\) the transpose of \(G\) \{ reverse all edges in \(G\) \}
3. **call** DFS \((G^T)\) (vertices are considered in the decreasing order of finish times computed in step 1)
4. output each tree in the depth-first forest produced by step 3.
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Ideas behind the algorithm:
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- the first pass DFS results in a DFS forest, let $T$ be a tree with root $r$;

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Properties from algorithm

Strongly Connected Components \((G)\)

(1) Component graph: \(G_{SCC} = (V_{SCC},E_{SCC})\) is defined as follow:

\[
\text{let } C_1, C_2, ..., C_k \text{ be } k \text{ distinct SCCs for } G. \text{ Then } V_{SCC} = \{v_1, v_2, v_k\}; E_{SCC} = \{ (v_i, v_j) : \exists u \in C_i, v \in C_j, (u, v) \in E \}\.
\]

Then \(G_{SCC}\) is a DAG (directed acyclic graph).

Proof. Assume the opposite to the claim that, for some \(v_i, v_j \in V_{SCC}\), there is a path \(v_i \Rightarrow v_j\) and another path \(v_j \Rightarrow v_i\), forming a cycle in \(V_{SCC}\).

By the definition of \(G_{SCC}\), there must be a path in \(G\), from some vertex in \(C_i\) to some vertex in \(C_j\); at the same time, there is a path in \(G\), from some vertex in \(C_j\) to some vertex in \(C_i\). Then \(C_i\) and \(C_j\) should form a single SCC, not two distinct SCCs. Contradicts.
Properties from algorithm \textsc{Strongly Connected Components}(G)

\begin{align*}
\text{Component graph:} \\
G_{\text{SCC}} = \left( V_{\text{SCC}}, E_{\text{SCC}} \right)
\end{align*}

Let $C_1, C_2, \ldots, C_k$ be $k$ distinct SCCs for $G$. Then $V_{\text{SCC}} = \{v_1, v_2, \ldots, v_k\}$; $E_{\text{SCC}} = \{ (v_i, v_j) : \exists u \in C_i, v \in C_j, (u, v) \in E \}$. Then $G_{\text{SCC}}$ is a DAG (directed acyclic graph).

\textbf{Proof.} Assume the opposite to the claim that, for some $v_i, v_j \in V_{\text{SCC}}$, there is a path $v_i \rightarrow v_j$ and another path $v_j \rightarrow v_i$, forming a cycle in $V_{\text{SCC}}$. By the definition of $G_{\text{SCC}}$, there must be a path in $G$, from some vertex in $C_i$ to some vertex in $C_j$; at the same time, there is a path in $G$, from some vertex in $C_j$ to some vertex in $C_i$. Then $C_i$ and $C_j$ should form a single SCC, not two distinct SCCs. Contradicts.
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Let $C$ be a SCC, define $f(C) = \max_{u \in C} \{ u.f \}$, (with the finish times from the first DFS call).

Lemma 22.14: Let $C$ and $C'$ be distinct strongly connected components for directed graph $G$. If $(u, v) \in E$, where $u \in C$ and $v \in C'$, then $f(C) > f(C')$.

Proof: Assume the opposite, i.e., $f(C) < f(C')$. Then there must be vertices $x \in C$ and $y \in C'$ such that $x.f < y.f$.

Now consider the first DFS call, there are two situations:

1. $y$ was searched before $x$: by property (1) there is no path from $y$ to $x$, $x.f > y.f$.

2. $y$ was searched after $x$: since there is a path from $x$ to $y$ because of $(u, v)$, $x.f > y.f$.

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The algorithm
Strongly Connected Components

correctly computes the strongly connected components for a directed graph \( G \).

We need to prove two statements:

1. If \( v \xrightarrow{} u \) and \( u \xrightarrow{} v \) in \( G \), then \( u \) and \( v \) belong to the same component \( C \) produced by the algorithm.

2. If \( u, v \in C \), then we have \( v \xrightarrow{} u \) and \( u \xrightarrow{} v \) in \( G \).
(3) The algorithm **Strongly Connected Components** \((G)\) correctly computes the strongly connected components for a directed graph \(G\).
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(3) The algorithm **Strongly Connected Components**($G$) correctly computes the strongly connected components for a directed graph $G$.

We need to prove two statements:

(1) If $v \rightsquigarrow u$ and $u \rightsquigarrow v$ in $G$, then $u$ and $v$ belong to the same component $C$ produced by the algorithm.

(2) If $u, v \in C$, then we have $v \rightsquigarrow u$ and $u \rightsquigarrow v$ in $G$. 
Proof:
(1) If $v \rightarrow u$ and $u \rightarrow v$ in $G$, then $u$ and $v$ belong to the same component $C$ produced by the algorithm.

Sketch of proof:
• assume in 1st DFS, $v$ was discovered before $u$ (or opposite);
• as $v \rightarrow u$ in $G$, $u$ and $v$ belong to the same search tree rooted at $r$ with $r.f \geq v.f > u.f$ (note: $r$ could be just $v$);
• as $u \rightarrow v$ in $G$, $v \rightarrow u$ in $G_T$;
• now consider the 2nd DFS; there are 2 situations:
  (1) searching from some $w$ with $w.f \geq v.f$ (note: $w$ could be $v$) finds $v$ first; then it finds $u$;
  (2) the search finds $u$ first; because $v \rightarrow u$ in $G$, $u \rightarrow v$ is in $G_T$, it finds also $v$.
In both situations, $u$ and $v$ belongs to the same search tree in the 2nd DFS search. Therefore, $u$ and $v$ belong to the same component.
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Proof:

• Assume in 1st DFS, v was discovered before u (or opposite);
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• assume in 1st DFS, \( v \) was discovered before \( u \) (or opposite);
• as \( v \leadsto u \) in \( G \), \( u \) and \( v \) belong to the same search tree rooted at \( r \) with \( r.f \geq v.f > u.f \) (note: \( r \) could be just \( v \))
• as \( u \leadsto v \) in \( G \), \( v \leadsto u \) in \( G^T \);
• now consider the 2nd DFS; there are 2 situations:

(1) searching from some \( w \) with \( w.f \geq v.f \) (note: \( w \) could be \( v \)) finds \( v \) first; then it finds \( u \);

(2) the search finds \( u \) first; because \( v \leadsto u \) in \( G \), \( u \leadsto v \) is in \( G^T \), it finds also \( v \).
Chapter 22. Elementary graph algorithms

Proof:

(1) If \( v \sim u \) and \( u \sim v \) in \( G \) in \( G \), then \( u \) and \( v \) belong to the same component \( C \) produced by the algorithm.

Sketch of proof:

• assume in 1st DFS, \( v \) was discovered before \( u \) (or opposite);
• as \( v \sim u \) in \( G \), \( u \) and \( v \) belong to the same search tree rooted at \( r \) with \( r.f \geq v.f > u.f \) (note: \( r \) could be just \( v \))
• as \( u \sim v \) in \( G \), \( v \sim u \) in \( G^T \);
• now consider the 2nd DFS; there are 2 situations:

  (1) searching from some \( w \) with \( w.f \geq v.f \) (note: \( w \) could be \( v \)) finds \( v \) first; then it finds \( u \);

  (2) the search finds \( u \) first; because \( v \sim u \) in \( G \), \( u \sim v \) is in \( G^T \), it finds also \( v \).

In both situations, \( u \) and \( v \) belongs to the same search tree in the 2nd DFS search. Therefore, \( u \) and \( u \) belong to the same component.
Chapter 22. Elementary graph algorithms

If \( u, v \in C \), then we have \( v \xrightarrow{} u \) and \( u \xrightarrow{} v \) in \( G \).

Sketch of proof:
1. Assume \( u, v \) belong to the same tree of root \( r \) in 2nd DFS.
2. Then \( r.f > u.f \) and \( r.f > v.f \) in 1st DFS.
3. The assumption in (1) also implies:
   - \( r \xrightarrow{} u \) and \( r \xrightarrow{} v \) in \( G_T \);
   - That is, \( u \xrightarrow{} r \) and \( v \xrightarrow{} r \) in \( G \);
   - Then \( u.f > r.f \) and \( v.f > r.f \) in 1st DFS, which conflict with conclusions in (2), UNLESS \( r \xrightarrow{} u \) and \( r \xrightarrow{} v \) in \( G \) also.
4. This means: through \( r \), \( v \xrightarrow{} u \) and \( u \xrightarrow{} v \) in \( G \).
(2) If $u, v \in C$, then we have $v \rightsquigarrow u$ and $u \rightsquigarrow v$ in $G$. 
(2) If \( u, v \in C \), then we have \( v \leadsto u \) and \( u \leadsto v \) in \( G \).

Sketch of proof:
Chapter 22. Elementary graph algorithms

(2) If \( u, v \in C \), then we have \( v \leadsto u \) and \( u \leadsto v \) in \( G \).

Sketch of proof:

(1) assume \( u, v \) belong to the same tree of root \( r \) in 2nd DFS;

(2) then \( r.f > u.f \) and \( r.f > v.f \) in 1st DFS;

(3) the assumption in (1) also implies:
   • \( r \leadsto u \) and \( r \leadsto v \) in \( G_T \);
   • that is, \( u \leadsto r \) and \( v \leadsto r \) in \( G \);
   • then \( u.f > r.f \) and \( v.f > r.f \) in 1st DFS,
     which conflict with conclusions in (2), UNLESS \( r \leadsto u \) and \( r \leadsto v \) in \( G \) also.

(4) This means: through \( r \), \( v \leadsto u \) and \( u \leadsto v \) in \( G \).
(2) If \( u, v \in C \), then we have \( v \xrightarrow{} u \) and \( u \xrightarrow{} v \) in \( G \).

Sketch of proof:

(1) assume \( u, v \) belong to the same tree of root \( r \) in 2nd DFS;
(2) then \( r.f > u.f \); and \( r.f > v.f \) in 1st DFS;
(2) If \( u, v \in C \), then we have \( v \rightsquigarrow u \) and \( u \rightsquigarrow v \) in \( G \).

Sketch of proof:

(1) assume \( u, v \) belong to the same tree of root \( r \) in 2nd DFS;
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(3) the assumption in (1) also implies:

• \( r \rightsquigarrow u \) and \( r \rightsquigarrow v \) in \( G \);
• that is, \( u \rightsquigarrow r \) and \( v \rightsquigarrow r \) in \( G \);
• then \( u.f > r.f \) and \( v.f > r.f \) in 1st DFS,
  which conflict with conclusions in (2), UNLESS
  \( r \rightsquigarrow u \) and \( r \rightsquigarrow v \) in \( G \) also.

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(2) If $u, v \in C$, then we have $v \leadsto u$ and $u \leadsto v$ in $G$.

Sketch of proof:

(1) assume $u, v$ belong to the same tree of root $r$ in 2nd DFS;
(2) then $r.f > u.f$; and $r.f > v.f$ in 1st DFS;
(3) the assumption in (1) also implies:
   • $r \leadsto u$ and $r \leadsto v$ in $G^{T}$;
(2) If $u, v \in C$, then we have $v \sim u$ and $u \sim v$ in $G$.

Sketch of proof:

(1) assume $u, v$ belong to the same tree of root $r$ in 2nd DFS;
(2) then $r.f > u.f$; and $r.f > v.f$ in 1st DFS;
(3) the assumption in (1) also implies:
   - $r \sim u$ and $r \sim v$ in $G^T$;
   - that is, $u \sim r$ and $v \sim r$ in $G$;
(2) If $u, v \in C$, then we have $v \leadsto u$ and $u \leadsto v$ in $G$.

Sketch of proof:

(1) assume $u, v$ belong to the same tree of root $r$ in 2nd DFS;
(2) then $r.f > u.f$; and $r.f > v.f$ in 1st DFS;
(3) the assumption in (1) also implies:

- $r \leadsto u$ and $r \leadsto v$ in $G^T$;
- that is, $u \leadsto r$ and $v \leadsto r$ in $G$;
- then $u.f > r.f$ and
(2) If $u, v \in C$, then we have $v \leadsto u$ and $u \leadsto v$ in $G$.

Sketch of proof:

(1) assume $u, v$ belong to the same tree of root $r$ in 2nd DFS;
(2) then $r.f > u.f$; and $r.f > v.f$ in 1st DFS;
(3) the assumption in (1) also implies:
   • $r \leadsto u$ and $r \leadsto v$ in $G^T$;
   • that is, $u \leadsto r$ and $v \leadsto r$ in $G$;
   • then $u.f > r.f$ and
     $v.f > r.f$ in 1st DFS,
(2) If \( u, v \in C \), then we have \( v \sim u \) and \( u \sim v \) in \( G \).

Sketch of proof:

(1) assume \( u, v \) belong to the same tree of root \( r \) in 2nd DFS;
(2) then \( r.f > u.f \); and \( r.f > v.f \) in 1st DFS;
(3) the assumption in (1) also implies:
   - \( r \sim u \) and \( r \sim v \) in \( G^T \);
   - that is, \( u \sim r \) and \( v \sim r \) in \( G' \);
   - then \( u.f > r.f \) and
     \( v.f > r.f \) in 1st DFS,
which conflict with conclusions in (2),
(2) If \( u, v \in C \), then we have \( v \leadsto u \) and \( u \leadsto v \) in \( G \).

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(2) then \( r.f > u.f \); and \( r.f > v.f \) in 1st DFS;
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   • that is, \( u \leadsto r \) and \( v \leadsto r \) in \( G \);
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     which conflict with conclusions in (2), **UNLESS** $r \leadsto u$ and $r \leadsto v$ in $G$ also.
(4) This means: through $r$, $v \leadsto u$ and $u \leadsto v$ in $G$. 
Chapter 22. Elementary graph algorithms

Reachability Problem

Input:

\[ \text{G} = (V, E), \text{s, t} \in V, \]

Output: YES if and only there is a path \( \text{s} \rightarrow \text{t} \) in \( \text{G} \).

- The problem can be solved with DFS and BFS by search on the graph from \( \text{s} \) until \( \text{t} \) shows up.
- Linear time \( O(|E| + |V|) \). Can we do better?
- But first answer the following question: Can you write an SQL program to solve Reachability?
- It appears that a loop is needed to solve Reachability.
  - Why?
    - Inherent difficulty in parallel computation.
    - P-complete, it cannot be solved in time \( O(\log n) \) even if \( \Theta(n) \) CPUs are used.
Reachability Problem

Input: \( G = (V, E) \), and \( s, t \in V \);
Output: YES if and only there is a path \( s \rightarrow t \) in \( G \).

• The problem can be solved with DFS and BFS by search on the graph from \( s \) until \( t \) shows up.

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Reachability Problem

**Input:** \( G = (V, E) \), and \( s, t \in V \);
Chapter 22. Elementary graph algorithms

Reachability Problem

**Input:** \( G = (V, E) \), and \( s, t \in V \);

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Chapter 22. Elementary graph algorithms

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**Input**: $G = (V, E)$, and $s, t \in V$;
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Chapter 22. Elementary graph algorithms

Reachability Problem

**Input:** $G = (V, E)$, and $s, t \in V$;

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- The problem can be solved with DFS and BFS by search on the graph from $s$ until $t$ shows up.
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Reachability Problem

**Input:** \( G = (V, E) \), and \( s, t \in V \);  
**Output:** YES if and only there is a path \( s \leadsto t \) in \( G \).

- The problem can be solved with DFS and BFS by search on the graph from \( s \) until \( t \) shows up. Linear time \( O(|E| + |V|) \). Can we do better?

- But first answer the following question:  
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Reachability Problem

**Input:** $G = (V, E)$, and $s, t \in V$;
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Chapter 22. Elementary graph algorithms

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  Inherent difficulty in parallel computation.
Reachability Problem

**Input:** \( G = (V, E) \), and \( s, t \in V \), ;

**Output:** YES if and only there is a path \( s \rightsquigarrow t \) in \( G \).

- The problem can be solved with DFS and BFS by search on the graph from \( s \) until \( t \) shows up. Linear time \( O(|E| + |V|) \). Can we do better?

- But first answer the following question:
  Can you write an SQL program to solve Reachability?

- It appears that a loop is needed to solve Reachability. **Why?**
  Inherent difficulty in parallel computation.
  
  **P-complete**, it cannot be solved in time \( O(\log n) \) even if \( \Theta(n) \) CPUs are used.
Chapter 23. Minimum Spanning Trees

A spanning tree of a graph $G = (V,E)$ is a tree as a subgraph in $G$ which contains all vertices in $V$.

A minimum spanning tree (MST) of an edge-weighted graph $G$ is a spanning tree with the least edge weight sum.
A spanning tree of a graph $G = (V, E)$ is a tree as a subgraph in $G$ which contains all vertices in $V$. A minimum spanning tree (MST) of an edge-weighted graph $G$ is a spanning tree with the least edge weight sum.
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Chapter 23. Minimum Spanning Trees

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Chapter 23. Minimum Spanning Trees

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Chapter 23. Minimum Spanning Trees

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Chapter 23. Minimum Spanning Trees

A spanning tree of a graph $G = (V, E)$ is a tree as a subgraph in $G$ which contains all vertices in $V$.

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Chapter 23. Minimum Spanning Trees

The MST problem
Chapter 23. Minimum Spanning Trees

The MST problem

**INPUT:** connected, undirected graph $G = (V, E)$ with weight $w : E \rightarrow R$,
Chapter 23. Minimum Spanning Trees

The MST problem

**Input:** connected, undirected graph \( G = (V, E) \) with weight \( w : E \rightarrow R \),

**Output:** a spanning tree \( T = (V, E') \) such that

\[
W(T) = \sum_{(u,v) \in E'} w(u,v) \text{ is the minimum}
\]
The MST problem

Input: connected, undirected graph $G = (V, E)$ with weight $w : E \to R$,
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is the minimum

We will introduce two greedy algorithms: (1) Kruskal’s and (2) Prim’s
The MST problem

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We will introduce two **greedy algorithms**: (1) Kruskal’s and (2) Prim’s

- They **have the same generic process** to grow a spanning tree;
Chapter 23. Minimum Spanning Trees

The MST problem

**Input:** connected, undirected graph $G = (V, E)$ with weight $w : E \to \mathbb{R}$,

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is the minimum

We will introduce two **greedy algorithms**: (1) Kruskal’s and (2) Prim’s

- They **have the same generic process** to grow a spanning tree;
- but **differ** in which edge to add the partially grown tree.
Chapter 23. Minimum Spanning Trees

Growing an MST

A generic process to grow an MST.

Generic MST \((G,w)\) {
  given graph \(G\) and weight function \(w\)

1. \(A = \emptyset\)
2. while \(A\) does not form a spanning tree
3. find an edge \((u,v)\) that is safe for \(A\)
4. \(A = A \cup \{(u,v)\}\)
5. return \(A\)

Loop invariant: \(A\) is always a subset of some MST;

Note: when the loop terminates, \(A\) is a MST.

safe edge: edge \((u,v)\) is safe for \(A\) if does not violate the loop invariant, i.e, \(A \cup \{(u,v)\}\) is a subset of some MST.
Growing an MST

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Growing an MST

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Growing an MST

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\textbf{Generic MST}(G, w) \quad \{ \text{ given graph } G \text{ and weight function } w \} 

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Chapter 23. Minimum Spanning Trees

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Chapter 23. Minimum Spanning Trees

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Growing an MST

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1. \( A = \emptyset \);
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4. \hspace{1em} \( A = A \cup \{(u, v)\} \)
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Growing an MST
A generic process to grow an MST.

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Chapter 23. Minimum Spanning Trees

Growing an MST

A generic process to grow an MST.

\textsc{Generic MST}(G, w) \quad \{ \text{ given graph } G \text{ and weight function } w \} 

1. \hspace{0.5cm} A = \emptyset; \\
2. \hspace{0.5cm} \textbf{while} A does not form a spanning tree \\
3. \hspace{0.5cm} \textbf{find an edge} (u, v) \text{ that is safe for } A \\
4. \hspace{0.5cm} A = A \cup \{ (u, v) \} \\
5. \hspace{0.5cm} \textbf{return} (A) \\

Loop invariant: \( A \) is always a subset of some MST;
Chapter 23. Minimum Spanning Trees

Growing an MST
A generic process to grow an MST.

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Growing an MST

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\text{Generic MST}(G, w) \quad \{ \text{given graph } G \text{ and weight function } w \} \\
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safe edge:
Growing an MST

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edge \((u, v)\) is safe for \( A \) if \textbf{does not violate the loop invariant},
Chapter 23. Minimum Spanning Trees

**Growing an MST**

A generic process to grow an MST.

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1. \( A = \emptyset; \)
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Note: when the loop terminates, \( A \) is a MST.

**safe edge:**

edge \((u, v)\) is safe for \( A \) if \textbf{does not violate the loop invariant, i.e, } \( A \cup \{(u, v)\} \) is a subset of some MST.
Chapter 23. Minimum Spanning Trees

We first need some terminologies
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- **cut**: \((S, V-S)\), a partition of \(V\)
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Chapter 23. Minimum Spanning Trees

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- **cut**: $(S, V-S)$, a partition of $V$

- **crossing**: $(u, v)$ crosses cut $(S, V - S)$ if $u$ and $v$ are in $S$ and $V - S$, respectively
Chapter 23. Minimum Spanning Trees

Some more terminologies
Some more terminologies

- **respect**: a cut respects a set $A$ of edges if no edge in $A$ crosses the cut.
Some more terminologies

- **respect**: a cut respects a set $A$ of edges if no edge in $A$ crosses the cut.

- **light edge**: an edge is a light edge crossing a cut if its weight is the minimum of any edge that crosses the cut.
Theorem 23.1 Let $G = (V, E)$. 

Sketch of proof:

(1) If $A \cup \{(u,v)\}$ forms a cycle there must have been another edge in $A$ that crosses cut $(S, V-S)$ (WHY?), implying the cut did not respect $A$. Contradicts.

(2) Assume that some MST $T$, $A \subset T$. First, $T \cup \{(u,v)\}$ forms a circle! Why? There must be another edge $(x,y)$ cross the cut $(S, V-S)$. Since $(u,v)$ is light edge, $T' = T - \{(x,y)\} \cup \{(u,v)\}$ is an MST. Now $A \cup \{(u,v)\} \subset T'$ because $(x,y) \not\in A$ (otherwise, the cut would not respect $A$).
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Chapter 23. Minimum Spanning Trees

Theorem 23.1 gives a sufficient information for how to identify a safe edge to make the **Generic MST** algorithm work.

- Specific algorithms can be produced from **Generic MST** based on how the set $A$ is grown.
- $A$ may always be a tree (Prim's algorithm) or could be a forest (Kruskal's algorithm).

**MST-Kruskal** $(G,w)$

1. $A = \emptyset$
2. for each vertex $v \in G.V$
3. Make-Set ($v$)
4. sort edges in $E$ into non-decreasing order by their weight $w$
5. for each edge $(u,v) \in E$, taken in the order
6. if Find Set ($u$) $\neq$ Find Set ($v$)
7. $A = A \cup \{(u,v)\}$
8. Union ($u,v$)
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Chapter 23. Minimum Spanning Trees

Execution of Kruskal's algorithm for MST

disjoint sets

[A] [B] [C] [D] [E] [F] [G] [H]

[A] [B] [C] [D] [E] [F] [G] [H]

[A] [B] [C] [D] [E] [F, G] [H]

[A] [B] [D] [E] [C, F, G] [H]

[A] [D] [E] [B, C, F, G] [H]

[D] [E] [A, B, C, F, G] [H]

[D, E] [A, B, C, F, G] [H]

[D, E, H] [A, B, C, F, G]

[A, B, C, D, E, F, G, H]
Chapter 23. Minimum Spanning Trees

At each iteration of the for loop, e.g., identify

\[
\begin{align*}
A &= \{(A, F), (B, F), (C, G), (F, G), (D, E)\} \\
S &= \{A, B, C, D, F, G\} \\
V - S &= \{E, H\} \\
\end{align*}
\]

\[
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\begin{itemize}
\item $A = \{(A,F), (B,F), (C,G), (F,G), (D,E)\}$,
\item $S = \{A,B,C,F,G\}$, $V - S = \{E,H\}$, light edge $(D,E)$ crosses the cut;
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[D] [E] [A, B, C, F, G] [H]

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\[ A = \{(A, F), (B, F), (C, G), (F, G)\}, \]

cut that respects \( A \): \( S = \{A, B, C, D, F, G\}, \ V-S = \{E, H\}, \]

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  cut that respect $\mathcal{A}$: $S = \{A, B, C, F, G, H\}$, $V - S = \{D, E\}$,
Chapter 23. Minimum Spanning Trees

At each iteration of the \texttt{for} loop, e.g., identify

\begin{itemize}
  \item \(\mathcal{A} = \{(A, F), (B, F), (C, G), (F, G)\}\),
    cut that respects \(\mathcal{A}\): \(S = \{A, B, C, D, F, G\}\), \(V-S = \{E, H\}\),
    light edge \((D, E)\) crosses the cut;
  
  \item \(\mathcal{A} = \{(A, F), (B, F), (C, G), (F, G), (D, E)\}\),
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\end{itemize}
Chapter 23. Minimum Spanning Trees

The Kruskal’s algorithm uses disjoint-set data structures (where elements are partitioned into disjoint sets)
Chapter 23. Minimum Spanning Trees

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- **Make Set**$(x)$: create a set of single element $x$;
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- **Make Set($x$)**: create a set of single element $x$;
- **Find Set($x$)**: identify the set that contains element $x$;

Implementations (left: linked lists, Right: disjoint-set forest)

**Time complexity:** $O(\log n)$ for Make Set($x$), Find Set($x$), Union($x,y$) with disjoint-set forest implementation.

**Time complexity of Kruskal's algorithm:** $O(|E| \log |V| + |V|)$. 
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![Diagram of disjoint-set forest implementation]

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Chapter 23. Minimum Spanning Trees

MST-Prim($G, w, r$)
Chapter 23. Minimum Spanning Trees

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Chapter 23. Minimum Spanning Trees

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2. $u.key = \infty$ \{ $u.key$ is the $u$'s shortest distance to set $A = V-Q$ \}
3. $u.\pi = NULL$
4. $r.key = 0$ \{ start from vertex $r$ \}
5. $Q = G.V$ \{ establish priority queue $Q$ with key values \}
6. while $Q \neq \emptyset$
7. $u = \text{Extract Min}(Q)$
8. for each $v \in \text{Adj}[u]$
9. if $v \in Q$ and $w(u,v) < v.key$ \{ for those not in $A$, update distances \}
10. then $v.\pi = u$
11. $v.key = w(u,v)$
12. return $\pi$ usage of Priority queue: $Q$, Extract Min takes $O(\log n)$ time.

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Chapter 23. Minimum Spanning Trees

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Chapter 23. Minimum Spanning Trees

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Chapter 23. Minimum Spanning Trees

\textbf{MST-Prim}(G, w, r)

1. \textbf{for} each \( u \in G.V \)
2. \hspace{1cm} \( u.key = \infty \) \quad \{ \text{u.key is the u's shortest distance to set } A = V-Q \}
3. \hspace{1cm} u.\pi = NULL
4. \hspace{1cm} r.key = 0 \quad \{ \text{start from vertex } r \} \}
5. \hspace{1cm} Q = G.V \quad \{ \text{establish priority queue } Q \text{ wit key values}\}
6. \hspace{1cm} \textbf{while } Q \neq \emptyset
7. \hspace{2cm} u = \text{Extract Min}(Q)
8. \hspace{2cm} \textbf{for} each \( v \in Adj[u] \)
9. \hspace{3cm} \textbf{if} \( v \in Q \text{ and } w(u, v) < v.key \) \quad \{ \text{for those not in } A, \text{ update distances}\}
10. \hspace{3cm} \textbf{then} \( v.\pi = u \)
11. \hspace{3cm} \hspace{1cm} v.key = w(u, v)
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\textbf{usage of Priority queue: } Q, \textbf{Extract Min} \textbf{ takes } O(\log n) \textbf{ time.}

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Chapter 23. Minimum Spanning Trees
Summary of Kruskal’s and Prim’s algorithms:
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Chapter 23. Minimum Spanning Trees

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• repeatedly choosing from the remaining edges;
  pick a light edge that respects a cut
  add it to $A$,
  ensure that $A$ is a subset of some MST
• until $A$ forms a spanning tree.
Some questions about MST
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• What are the "cuts" implied in Kruskal’s algorithm and in Prim’s algorithm, respectively?
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- What are the "cuts" implied in Kruskal's algorithm and in Prim's algorithm, respectively?

- Can we develop a DP algorithm for the MST problem?
  
  the main issue: how solutions to subproblems help build solution for the problem

  what are subproblems, or what do subsolutions look like?
Chapter 24. Single Source Shortest Paths

Chapter 24. Single-source shortest paths
Chapter 24. Single-source shortest paths

Given a graph $G = (V, E)$, with weight $w : E \rightarrow R$; and single vertex $s \in V$;
Chapter 24. Single Source Shortest Paths

Chapter 24. Single-source shortest paths

Given a graph $G = (V,E)$, with weight $w : E \rightarrow \mathbb{R}$; and single vertex $s \in V$;
for each vertex $v \in V$, find a shortest path $s \leadsto v$. 

- Shortest path is a simple path.
- "Distance" is measured by the total edge weight on the path.
  - If the path $v_0 p \leadsto v_k$ is $p = (v_0, v_1, ..., v_k)$ then the path weight is $w(p) = \sum_{i=1}^{k} w(v_{i-1}, v_i)$.
- Shortest distance between $u$ and $v$ is $\delta(u,v) = \min_{u p \leadsto v} \{w(p)\}$. 


Chapter 24. Single Source Shortest Paths

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Chapter 24. Single Source Shortest Paths

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Chapter 24. Single-source shortest paths

Given a graph $G = (V, E)$, with weight $w : E \to \mathbb{R}$; and single vertex $s \in V$; for each vertex $v \in V$, find a shortest path $s \rightsquigarrow v$.

- **Shortest path is a simple path.**
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  i.e., if the path $v_0 \rightsquigarrow v_k$ is $p = (v_0, v_1, \ldots, v_k)$

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Chapter 24. Single-source shortest paths

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- “distance” is measured by the total edge weight on the path
  i.e., if the path $v_0 \sim^p v_k$ is $p = (v_0, v_1, \ldots, v_k)$
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• **Single-source shortest paths**: from $s$ to each vertex $v \in V$

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Chapter 24. Single Source Shortest Paths

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- A special case: **Single-pair shortest path**: from $s$ to $t$
- **All-pairs shortest paths**: from $s$ to $t$ for all pairs $s, t \in V$. 
Lemma 24.1 (a subpath of a shortest path is a shortest path)
Chapter 24. Single Source Shortest Paths

**Lemma 24.1** (a subpath of a shortest path is a shortest path)

Given a weighted directed graph $G = (V, E)$ with edge weight function $w$. Let $p = (v_0, v_1, \ldots, v_k)$ be a shortest path $v_0 \xrightarrow{p} v_k$. Then $p_{i,j} = (v_i, \ldots, v_j)$ is a shortest path $v_i \xrightarrow{p_{i,j}} v_j$. 

Proof idea: (proof by contradiction)

Assume that $p_{i,j}$ is not the shortest path from $v_i$ to $v_j$. Then there is a shorter path $q_{i,j}$ from $v_i$ to $v_j$. Define path $q = (v_0, \ldots, v_i, q_{i,j}, v_j, \ldots, v_k)$ has weight $w(q) = \sum_{t=1}^{i} w(v_{t-1}, v_t) + w(q_{i,j}) + \sum_{t=j}^{k-1} w(v_t, v_{t+1}) < \sum_{t=1}^{i} w(v_{t-1}, v_t) + w(p_{i,j}) + \sum_{t=j}^{k-1} w(v_t, v_{t+1}) = w(p)$ contradicts to the assumption that $p$ is the shortest path from $v_0$ to $v_k$. 
Lemma 24.1 (a subpath of a shortest path is a shortest path)

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Define path

$$q = (v_0, \ldots, v_i, q_{i,j}, v_j, \ldots, v_k)$$

has weight

$$w(q) =$$
Chapter 24. Single Source Shortest Paths

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Chapter 24. Single Source Shortest Paths

**Lemma 24.1** (a subpath of a shortest path is a shortest path)

Given a weighted directed graph $G = (V, E)$ with edge weight function $w$. Let $p = (v_0, v_1, \ldots, v_k)$ be a shortest path $v_0 \overset{p}{\leadsto} v_k$. Then $p_{i,j} = (v_i, \ldots, v_j)$ is a shortest path $v_i \overset{p_{i,j}}{\leadsto} v_j$.

**Proof idea:** (proof by contradiction) Assume that $p_{i,j}$ is not the shortest path from $v_i$ to $v_j$. Then there is a shorter path $q_{i,j}$ from $v_i$ to $v_j$.

Define path

$$q = (v_0, \ldots, v_i, q_{i,j}, v_j, \ldots, v_k)$$

has weight

$$w(q) = \sum_{t=1}^{i} w(v_{t-1}, v_t) + w(q_{i,j}) + \sum_{t=j}^{k-1} w(v_t, v_{t+1})$$

<
Chapter 24. Single Source Shortest Paths

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$$< \sum_{t=1}^{i} w(v_{t-1}, v_t)$$
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$$q = (v_0, \ldots, v_i, q_{i,j}, v_j, \ldots, v_k)$$

has weight

$$w(q) = \sum_{t=1}^{i} w(v_{t-1}, v_t) + w(q_{i,j}) + \sum_{t=j}^{k-1} w(v_t, v_{t+1})$$

$$< \sum_{t=1}^{i} w(v_{t-1}, v_t) + w(p_{i,j}) + \sum_{t=j}^{k-1} w(v_t, v_{t+1}) = w(p)$$

contradicts to the assumption that $p$ is the shortest path from $v_0$ to $v_k$. 
Chapter 24. Single Source Shortest Paths

**Lemma 24.1** (a subpath of a shortest path is a shortest path)

Given a weighted directed graph $G = (V, E)$ with edge weight function $w$. Let $p = (v_0, v_1, \ldots, v_k)$ be a shortest path $v_0 \xrightarrow{p} v_k$. Then $p_{i,j} = (v_i, \ldots, v_j)$ is a shortest path $v_i \xrightarrow{p_{i,j}} v_j$.

**Proof idea:** (proof by contradiction) Assume that $p_{i,j}$ is not the shortest path from $v_i$ to $v_j$. Then there is a **shorter path** $q_{i,j}$ from $v_i$ to $v_j$.

Define path

$$q = (v_0, \ldots, v_i, q_{i,j}, v_j, \ldots, v_k)$$

has weight

$$w(q) = \sum_{t=1}^{i} w(v_{t-1}, v_t) + w(q_{i,j}) + \sum_{t=j}^{k-1} w(v_t, v_{t+1})$$

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Chapter 24. Single Source Shortest Paths

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Proof idea: (proof by contradiction) Assume that $p_{i,j}$ is not the shortest path from $v_i$ to $v_j$. Then there is a shorter path $q_{i,j}$ from $v_i$ to $v_j$. Define path

$$q = (v_0, \ldots, v_i, q_{i,j}, v_j, \ldots, v_k)$$

has weight

$$w(q) = \sum_{t=1}^{i} w(v_{t-1}, v_t) + w(q_{i,j}) + \sum_{t=j}^{k-1} w(v_t, v_{t+1})$$

$$< \sum_{t=1}^{i} w(v_{t-1}, v_t) + w(p_{i,j}) + \sum_{t=j}^{k-1} w(v_t, v_{t+1}) = w(p)$$
Chapter 24. Single Source Shortest Paths

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Define path

$q = (v_0, \ldots, v_i, q_{i,j}, v_j, \ldots, v_k)$

has weight

\[
w(q) = \sum_{t=1}^{i} w(v_{t-1}, v_t) + w(q_{i,j}) + \sum_{t=j}^{k-1} w(v_t, v_{t+1})
\]

\[
< \sum_{t=1}^{i} w(v_{t-1}, v_t) + w(p_{i,j}) + \sum_{t=j}^{k-1} w(v_t, v_{t+1}) = w(p)
\]

contradicts to the assumption that $p$ is the shortest path from $v_0$ to $v_k$. 
Some terminologies:
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- **negative weights** are allowed;
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- cycles on a path: not a simple path;
Chapter 24. Single Source Shortest Paths

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- **negative weights** are allowed;
- cycles on a path: not a simple path;
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Chapter 24. Single Source Shortest Paths

Some terminologies:

- **negative weights** are allowed;
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Chapter 24. Single Source Shortest Paths

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Some terminologies:

- **negative weights** are allowed;
- cycles on a path: not a simple path;
- **negative weight cycles**, 0 weight cycles
- representing shortest paths: predecessor $\pi$
  shortest path tree:

  http://graphserver.sourceforge.net/gallery.html

  (width $\propto 1$/distance)
Technique: relaxation

- Intuition:
  
  if \( s \leadsto v \) has distance \( v.d \) (computed so far),
Technique: relaxation

- Intuition:
  - If $s \xrightarrow{p} v$ has distance $v.d$ (computed so far),
  - $s \xrightarrow{q} u$ is newly discovered.
Chapter 24. Single Source Shortest Paths

Technique: relaxation

- Intuition:

  if $s \xrightarrow{p} v$ has distance $v.d$ (computed so far),
  $s \xrightarrow{q} u$ is newly discovered. Then

\[
v.d = \min\{v.d, u.d + w(u, v)\}
\]
Technique: relaxation

Intuition:

if $s \xrightarrow{p} v$ has distance $v.d$ (computed so far),
$s \xrightarrow{q} u$ is newly discovered. Then

$$v.d = \min\{v.d, u.d + w(u,v)\}$$
Chapter 24. Single Source Shortest Paths

**Technique: relaxation**

- **Intuition:**
  
  If $s \xrightarrow{p} v$ has distance $v.d$ (computed so far),
  
  $s \xrightarrow{q} u$ is newly discovered. Then

  $$v.d = \min\{v.d, u.d + w(u, v)\}$$

- **In other words:**
Chapter 24. Single Source Shortest Paths

**Technique: relaxation**

- **Intuition:**
  
  if $s \xrightarrow{p} v$ has distance $v.d$ (computed so far), $s \xrightarrow{q} u$ is **newly discovered**. Then

  $$v.d = \min\{v.d, u.d + w(u,v)\}$$

- **In other words:**
  
  Let $v.d$ be an weight upper bound of a shortest path from $s$ to $v$, 

Chapter 24. Single Source Shortest Paths

Technique: relaxation

• Intuition:
  
  if \( s \xrightarrow{p} v \) has distance \( v.d \) (computed so far),
  
  \( s \xrightarrow{q} u \) is newly discovered. Then

  \[
  v.d = \min\{v.d, u.d + w(u,v)\}
  \]

• In other words:
  
  Let \( v.d \) be an weight upper bound of a shortest path from \( s \) to \( v \),
  
  initialized \( \infty \).
Chapter 24. Single Source Shortest Paths

Technique: relaxation

- Intuition:
  
  if \( s \xrightarrow{p} v \) has distance \( v.d \) (computed so far),
  
  \( s \xrightarrow{q} u \) is newly discovered. Then

  \[
  v.d = \min\{v.d, u.d + w(u, v)\}
  \]

- In other words:

  Let \( v.d \) be an weight upper bound of a shortest path from \( s \) to \( v \), initialized \( \infty \).

  The process of relaxing edge \((u, v)\): improves \( v.d \) by taking the path through \( u \), and update \( v.d \) and \( v.\pi \).
Bellman-Ford algorithm

1. for each vertex \( v \in G.V \) initialization
2. \( v.d = \infty \)
3. \( v.\pi = \text{NULL} \)
4. \( s.d = 0 \)
5. for \( i = 1 \) to \(|V| - 1\) relaxation
6. for each edge \((u,v)\in G.E\)
7. if \( v.d > u.d + w(u,v) \)
8. \( v.d = u.d + w(u,v) \)
9. \( v.\pi = u \)
10. for each edge \((u,v)\in G.E\) checking negative weight cycle
11. if \( v.d > u.d + w(u,v) \)
12. return (FALSE)
13. return (TRUE)

Running time: \( O(|V||E|) \)
Chapter 24. Single Source Shortest Paths

Bellman-Ford algorithm

\texttt{Bellman-Ford}(G, w, s)
Bellman-Ford algorithm

\textsc{Bellman-Ford}(G, w, s)

1. \textbf{for} each vertex $v \in G.V$ \hspace{1cm} \textbf{initialization}

\begin{itemize}
  \item[2.] $v.d = \infty$
  \item[3.] $v.\pi = \text{NULL}$
  \item[4.] $s.d = 0$
  \item[5.] \textbf{for} $i = 1$ \textbf{to} $|V| - 1$ \textbf{relaxation}
  \item[6.] \textbf{for} each edge $(u, v) \in G.E$
  \item[7.] \textbf{if} $v.d > u.d + w(u, v)$
  \item[8.] $v.d = u.d + w(u, v)$
  \item[9.] $v.\pi = u$
  \item[10.] \textbf{for} each edge $(u, v) \in G.E$ \textbf{checking negative weight cycle}
  \item[11.] \textbf{if} $v.d > u.d + w(u, v)$
  \item[12.] \textbf{return} \text{FALSE}
  \item[13.] \textbf{return} \text{TRUE}
\end{itemize}

\textbf{Running time:} $O(|V| |E|)$
Chapter 24. Single Source Shortest Paths

Bellman-Ford algorithm

\textsc{Bellman-Ford}(G, w, s)

1. \textbf{for} each vertex \( v \in G.V \) \textbf{ initialization}
2. \( v.d = \infty \)

Running time: \( O(|V||E|) \)
Chapter 24. Single Source Shortest Paths

Bellman-Ford algorithm

\textsc{Bellman-Ford}\((G, w, s)\)

1. \textbf{for} each vertex \(v \in G.V\) \textbf{ initialization}\n
2. \(v.d = \infty\) \n
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Chapter 24. Single Source Shortest Paths

Bellman-Ford algorithm

\textsc{Bellman-Ford}(G, w, s)

\begin{enumerate}
\item \textbf{for} each vertex \( v \in G.V \) \quad \text{initialization}
\item \( v.d = \infty \)
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Running time: \( O(|V||E|) \)
Chapter 24. Single Source Shortest Paths

Bellman-Ford algorithm

\textsc{Bellman-Ford}(G, w, s)

1. \textbf{for} each vertex \( v \in G.V \) \textit{ initialization}
   \[ v.d = \infty \]
   \[ v.\pi = NULL \]
2. \[ s.d = 0 \]
3. \textbf{for} \( i = 1 \) to \( |V| - 1 \) \textit{ relaxation}
   \[ \text{ if } v.d > u.d + w(u, v) \]
   \[ v.d = u.d + w(u, v) \]
   \[ v.\pi = u \]
4. \textbf{for} each edge \((u, v)\) \textit{ checking negative weight cycle}
   \[ \text{ if } v.d > u.d + w(u, v) \]
   \[ \text{ return } \text{FALSE} \]
5. \text{ return } \text{TRUE} \]

Running time: \( \mathcal{O}(|V| \times |E|) \)
Chapter 24. Single Source Shortest Paths

Bellman-Ford algorithm

**Bellman-Ford**(G, w, s)

1. **for** each vertex \( v \in G.V \) \hspace{1cm} \text{initialization}
2. \( v.d = \infty \)
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5. **for** \( i = 1 \) **to** \(|V| - 1\) \hspace{1cm} \text{relaxation}
6. **for** each edge \((u, v) \in G.E\)
Bellman-Ford algorithm

\textsc{Bellman-Ford}(G, w, s)
1. \textbf{for} each vertex \( v \in G.V \) \textbf{ initialization}
2. \( v.d = \infty \)
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5. \textbf{for} \( i = 1 \) \textbf{ to } \mid V \mid - 1 \textbf{ relaxation}
6. \textbf{for} each edge \((u, v) \in G.E\)
7. \textbf{if} \( v.d > u.d + w(u, v) \)
Chapter 24. Single Source Shortest Paths

Bellman-Ford algorithm

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\item \textbf{for} each edge \( (u, v) \in G.E \)
\item \textbf{if} \( v.d > u.d + w(u, v) \)
\item \( v.d = u.d + w(u, v) \)
\end{enumerate}
Bellman-Ford algorithm

\textsc{Bellman-Ford}(G, w, s)

1. \textbf{for} each vertex \( v \in G.V \) \hspace{1cm} \text{initialization}
2. \hspace{1cm} \( v.d = \infty \)
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5. \hspace{1cm} \textbf{for} \( i = 1 \) \textbf{to} \( |V| - 1 \) \hspace{1cm} \text{relaxation}
6. \hspace{1cm} \textbf{for} each edge \( (u, v) \in G.E \)
7. \hspace{1cm} \hspace{1cm} \textbf{if} \( v.d > u.d + w(u, v) \)
8. \hspace{1cm} \hspace{1cm} \hspace{1cm} \( v.d = u.d + w(u, v) \)
9. \hspace{1cm} \hspace{1cm} \hspace{1cm} \( v.\pi = u \)

10. \hspace{1cm} \textbf{for} each edge \( (u, v) \in G.E \) \hspace{1cm} \text{checking negative weight cycle}
11. \hspace{1cm} \hspace{1cm} \textbf{if} \( v.d > u.d + w(u, v) \)
12. \hspace{1cm} \hspace{1cm} \hspace{1cm} \textbf{return} \( \text{FALSE} \)
13. \hspace{1cm} \hspace{1cm} \hspace{1cm} \textbf{return} \( \text{TRUE} \)

Running time: \( O(|V||E|) \)
Bellman-Ford algorithm

**Bellman-Ford**$(G, w, s)$
1. **for** each vertex $v \in G.V$ \hspace{1cm} initialization
2. \hspace{1cm} $v.d = \infty$
3. \hspace{1cm} $v.\pi = NULL$
4. \hspace{1cm} $s.d = 0$
5. **for** $i = 1$ **to** $|V| - 1$ \hspace{1cm} relaxation
6. \hspace{1cm} **for** each edge $(u, v) \in G.E$
7. \hspace{1cm} **if** $v.d > u.d + w(u, v)$
8. \hspace{1cm} \hspace{1cm} $v.d = u.d + w(u, v)$
9. \hspace{1cm} \hspace{1cm} $v.\pi = u$
10. **for** each edge $(u, v) \in G.E$ \hspace{1cm} checking negative weight cycle

Running time: $O(|V||E|)$
Chapter 24. Single Source Shortest Paths

Bellman-Ford algorithm

\textbf{Bellman-Ford}(G, w, s)
1. for each vertex \( v \in G.V \) initialization
2. \( v.d = \infty \)
3. \( v.\pi = NULL \)
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5. \textbf{for} \( i = 1 \) to \(|V| - 1\) relaxation
6. \textbf{for} each edge \((u, v) \in G.E\)
7. \textbf{if} \( v.d > u.d + w(u, v) \)
8. \( v.d = u.d + w(u, v) \)
9. \( v.\pi = u \)
10. \textbf{for} each edge \((u, v) \in G.E\) checking negative weight cycle
11. \textbf{if} \( v.d > u.d + w(u, v) \)

Running time: \( O(|V|\cdot|E|) \)
Chapter 24. Single Source Shortest Paths

Bellman-Ford algorithm

\texttt{BELLMAN-FORD}(G, w, s)

\begin{itemize}
  \item 1. \textbf{for} each vertex \( v \in G.V \) \textit{ initialization}
  \item 2. \( v.d = \infty \)
  \item 3. \( v.\pi = NULL \)
  \item 4. \( s.d = 0 \)
  \item 5. \textbf{for} \( i = 1 \) to \( |V| - 1 \) \textit{ relaxation}
  \item 6. \textbf{for} each edge \( (u, v) \in G.E \)
  \item 7. \textbf{if} \( v.d > u.d + w(u, v) \)
  \item 8. \( v.d = u.d + w(u, v) \)
  \item 9. \( v.\pi = u \)
  \item 10. \textbf{for} each edge \( (u, v) \in G.E \) \textit{ checking negative weight cycle}
  \item 11. \textbf{if} \( v.d > u.d + w(u, v) \)
  \item 12. \textbf{return} (FALSE)
\end{itemize}

Running time: \( O(|V| |E|) \)
Bellman-Ford algorithm

\textsc{Bellman-Ford}(G, w, s)

1. \textbf{for} each vertex \( v \in G.V \) \hspace{1cm} \text{initialization}
2. \hspace{1cm} \( v.d = \infty \)
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4. \hspace{1cm} \( s.d = 0 \)
5. \textbf{for} \( i = 1 \) \textbf{to} \( |V| - 1 \) \hspace{1cm} \text{relaxation}
6. \hspace{1cm} \textbf{for} each edge \( (u, v) \in G.E \)
7. \hspace{1cm} \hspace{1cm} \textbf{if} \( v.d > u.d + w(u, v) \)
8. \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \( v.d = u.d + w(u, v) \)
9. \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \( v.\pi = u \)
10. \textbf{for} each edge \( (u, v) \in G.E \) \hspace{1cm} \text{checking negative weight cycle}
11. \hspace{1cm} \hspace{1cm} \textbf{if} \( v.d > u.d + w(u, v) \)
12. \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \textbf{return} (FALSE)
13. \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \textbf{return} (TRUE)

\textbf{Running time}: \( O(|V| \cdot |E|) \)
Chapter 24. Single Source Shortest Paths

Bellman-Ford algorithm

\textsc{Bellman-Ford}(G, w, s)

1. \textbf{for} each vertex \( v \in G.V \) \hspace{1cm} \text{initialization}
2. \hspace{1cm} \( v.d = \infty \)
3. \hspace{1cm} \( v.\pi = NULL \)
4. \hspace{1cm} \( s.d = 0 \)
5. \textbf{for} \( i = 1 \) to \( |V| - 1 \) \hspace{1cm} \text{relaxation}
6. \hspace{2cm} \textbf{for} each edge \( (u, v) \in G.E \)
7. \hspace{3cm} \textbf{if} \( v.d > u.d + w(u, v) \)
8. \hspace{3cm} \hspace{1cm} \( v.d = u.d + w(u, v) \)
9. \hspace{3cm} \hspace{1cm} \( v.\pi = u \)
10. \textbf{for} each edge \( (u, v) \in G.E \) \hspace{1cm} \text{checking negative weight cycle}
11. \hspace{2cm} \textbf{if} \( v.d > u.d + w(u, v) \)
12. \hspace{3cm} \hspace{1cm} \textbf{return} \ (FALSE) \)
13. \hspace{3cm} \textbf{return} \ (TRUE) \\

Running time : \( O(|V||E|) \)
Chapter 24. Single Source Shortest Paths

1. 

2. 

3. 

4. 

5. 

Chapter 24. Single Source Shortest Paths

Correctness of Bellman-Ford Algorithm
Correctness of Bellman-Ford Algorithm

We want to prove that, if a shortest path $s \leadsto v$ consists of $k$ edges, Bellman-Fold obtains value $v.d = \delta(s, v)$ after the $k$th round of relaxation (assuming there is no negative cycle).
Correctness of Bellman-Ford Algorithm

We want to prove that, if a shortest path $s \leadsto v$ consists of $k$ edges, Bellman-Ford obtains value $v.d = \delta(s, v)$ after the $k$th round of relaxation (assuming there is no negative cycle).

Proof idea: Induction on $k$. 
Chapter 24. Single Source Shortest Paths

Correctness of Bellman-Ford Algorithm

We want to prove that, if a shortest path $s \leadsto v$ consists of $k$ edges, Bellman-Ford obtains value $v.d = \delta(s, v)$ after the $k$th round of relaxation (assuming there is no negative cycle).

Proof idea: Induction on $k$.

• $k = 0$, $v$ can only be $s$. Proved!
Chapter 24. Single Source Shortest Paths

Correctness of Bellman-Ford Algorithm

We want to prove that, if a shortest path $s \rightsquigarrow v$ consists of $k$ edges, Bellman-Fold obtains value $v.d = \delta(s, v)$ after the $k$th round of relaxation (assuming there is no negative cycle).

Proof idea: Induction on $k$.

• $k = 0$, $v$ can only be $s$. Proved!

• Assume after after the $k$th round of relaxation, $u.d = \delta(s, u)$ for all vertices $u$ that have a shortest path consisting of $k$ edges.
Chapter 24. Single Source Shortest Paths

Correctness of Bellman-Ford Algorithm

We want to prove that, if a shortest path \( s \rightarrow v \) consists of \( k \) edges, Bellman-Fold obtains value \( v.d = \delta(s, v) \) after the \( k \)th round of relaxation (assuming there is no negative cycle).

Proof idea: Induction on \( k \).

- \( k = 0 \), \( v \) can only be \( s \). Proved!

- Assume after the \( k \)th round of relaxation, \( u.d = \delta(s, u) \) for all vertices \( u \) that have a shortest path consisting of \( k \) edges.

- Let \( v \) be any vertex that has a shortest path \( s \rightarrow u \rightarrow v \), consisting of \( k + 1 \) edges;
Chapter 24. Single Source Shortest Paths

Correctness of Bellman-Ford Algorithm

We want to prove that, if a shortest path $s \leadsto v$ consists of $k$ edges, Bellman-Fold obtains value $v.d = \delta(s, v)$ after the $k$th round of relaxation \textbf{(assuming there is no negative cycle)}.

Proof idea: Induction on $k$.

- $k = 0$, $v$ can only be $s$. Proved!
- Assume after after the $k$th round of relaxation, $u.d = \delta(s, u)$ for all vertices $u$ that have a shortest path consisting of $k$ edges.
- Let $v$ be any vertex that has a shortest path $s \leadsto u \rightarrow v$, consisting of $k + 1$ edges;
  Then $s \leadsto u$ is a shortest path for $u$ consisting of $k$ edges.
Chapter 24. Single Source Shortest Paths

Correctness of Bellman-Ford Algorithm

We want to prove that, if a shortest path $s \leadsto v$ consists of $k$ edges, Bellman-Fold obtains value $v.d = \delta(s, v)$ after the $k$th round of relaxation (assuming there is no negative cycle).

Proof idea: Induction on $k$.

- $k = 0$, $v$ can only be $s$. Proved!

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- Let $v$ be any vertex that has a shortest path $s \leadsto u \rightarrow v$, consisting of $k + 1$ edges;
  
  Then $s \leadsto u$ is a shortest path for $u$ consisting of $k$ edges (By what?);
Chapter 24. Single Source Shortest Paths

Correctness of Bellman-Ford Algorithm

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Proof idea: Induction on $k$.

- $k = 0$, $v$ can only be $s$. Proved!
- Assume after the $k$th round of relaxation, $u.d = \delta(s, u)$ for all vertices $u$ that have a shortest path consisting of $k$ edges.
- Let $v$ be any vertex that has a shortest path $s \leadsto u \rightarrow v$, consisting of $k + 1$ edges;
  Then $s \leadsto u$ is a shortest path for $u$ consisting of $k$ edges (By what?);
  Now by assumption, $u.d = \delta(s, u)$ after $k$ round of relaxation.
Chapter 24. Single Source Shortest Paths

Correctness of Bellman-Ford Algorithm

We want to prove that, if a shortest path \( s \rightarrow v \) consists of \( k \) edges, Bellman-Ford obtains value \( v.d = \delta(s, v) \) after the \( k \)th round of relaxation (assuming there is no negative cycle).

Proof idea: Induction on \( k \).

- \( k = 0 \), \( v \) can only be \( s \). Proved!

- Assume after the \( k \)th round of relaxation, \( u.d = \delta(s, u) \) for all vertices \( u \) that have a shortest path consisting of \( k \) edges.

- Let \( v \) be any vertex that has a shortest path \( s \rightarrow u \rightarrow v \), consisting of \( k + 1 \) edges;

Then \( s \rightarrow u \) is a shortest path for \( u \) consisting of \( k \) edges (By what?);

Now by assumption, \( u.d = \delta(s, u) \) after \( k \) round of relaxation.

After an additional relaxation, \( v.d = \delta(s, u) + w(u, v) \), which is \( \delta(s, v) \).
Chapter 24. Single Source Shortest Paths

**Theorem 24.4** Bellman-Ford algorithm is correct on weighted, directed graphs.
Chapter 24. Single Source Shortest Paths

Theorem 24.4 Bellman-Ford algorithm is correct on weighted, directed graphs.

Proof: By Previous page, we only need to show, when $G$ contains a negative weight cycle reachable from $s$, the algorithm returns FALSE.
Chapter 24. Single Source Shortest Paths

**Theorem 24.4** Bellman-Ford algorithm is correct on weighted, directed graphs.

**Proof:** By Previous page, we only need to show, when $G$ contains a negative weight cycle reachable from $s$, the algorithm returns FALSE.

Let the cycle to be $c = (v_0, v_1, \ldots, v_k)$, where $v_0 = v_k$ and
Chapter 24. Single Source Shortest Paths

**Theorem 24.4** Bellman-Ford algorithm is correct on weighted, directed graphs.

**Proof:** By Previous page, we only need to show, when $G$ contains a negative weight cycle reachable from $s$, the algorithm returns FALSE.

Let the cycle to be $c = (v_0, v_1, \ldots, v_k)$, where $v_0 = v_k$ and

$$\sum_{i=1}^{k} w(v_{i-1}, v_i) < 0$$
Theorem 24.4 Bellman-Ford algorithm is correct on weighted, directed graphs.

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\[
\sum_{i=1}^{k} w(v_{i-1}, v_i) < 0
\]

Assume for all \( i \), \( v_i.d \leq v_{i-1}.d + w(v_{i-1}, v_i) \).
Theorem 24.4 Bellman-Ford algorithm is correct on weighted, directed graphs.

Proof: By Previous page, we only need to show, when $G$ contains a negative weight cycle reachable from $s$, the algorithm returns FALSE.

Let the cycle to be $c = (v_0, v_1, \cdots, v_k)$, where $v_0 = v_k$ and

$$
\sum_{i=1}^{k} w(v_{i-1}, v_i) < 0
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Assume for all $i$, $v_i.d \leq v_{i-1}.d + w(v_{i-1}, v_i)$.

$$
\sum_{i=1}^{k} v_i.d \leq \sum_{i=1}^{k} v_{i-1}.d + w(v_{i-1}, v_i),
$$
Theorem 24.4 Bellman-Ford algorithm is correct on weighted, directed graphs.

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Assume for all $i$, $v_i.d \leq v_{i-1}.d + w(v_{i-1}, v_i)$.

$$\sum_{i=1}^{k} v_i.d \leq \sum_{i=1}^{k} v_{i-1}.d + w(v_{i-1}, v_i),$$
Chapter 24. Single Source Shortest Paths

**Theorem 24.4** Bellman-Ford algorithm is correct on weighted, directed graphs.

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$$\sum_{i=1}^{k} w(v_{i-1}, v_i) < 0$$

Assume for all $i$, $v_i.d \leq v_{i-1}.d + w(v_{i-1}, v_i)$.

$$\sum_{i=1}^{k} v_i.d \leq \sum_{i=1}^{k} v_{i-1}.d + w(v_{i-1}, v_i),$$

But

$$\sum_{i=1}^{k} v_i.d = \sum_{i=1}^{k} v_{i-1}.d,$$
Chapter 24. Single Source Shortest Paths

**Theorem 24.4** Bellman-Ford algorithm is correct on weighted, directed graphs.

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Let the cycle to be $c = (v_0, v_1, \cdots, v_k)$, where $v_0 = v_k$ and

$$
\sum_{i=1}^{k} w(v_{i-1}, v_i) < 0
$$

Assume for all $i$, $v_i.d \leq v_{i-1}.d + w(v_{i-1}, v_i)$.

$$
\sum_{i=1}^{k} v_i.d \leq \sum_{i=1}^{k} v_{i-1}.d + w(v_{i-1}, v_i),
$$

But

$$
\sum_{i=1}^{k} v_i.d = \sum_{i=1}^{k} v_{i-1}.d, \quad \text{implying} \quad \sum_{i=1}^{k} w(v_{i-1}, v_i) \geq 0
$$
Chapter 24. Single Source Shortest Paths

**Theorem 24.4** Bellman-Ford algorithm is correct on weighted, directed graphs.

**Proof:** By Previous page, we only need to show, when $G$ contains a negative weight cycle reachable from $s$, the algorithm returns FALSE.

Let the cycle to be $c = (v_0, v_1, \cdots, v_k)$, where $v_0 = v_k$ and

$$\sum_{i=1}^{k} w(v_{i-1}, v_i) < 0$$

Assume for all $i$, $v_i.d \leq v_{i-1}.d + w(v_{i-1}, v_i)$.

$$\sum_{i=1}^{k} v_i.d \leq \sum_{i=1}^{k} v_{i-1}.d + w(v_{i-1}, v_i),$$

But

$$\sum_{i=1}^{k} v_i.d = \sum_{i=1}^{k} v_{i-1}.d, \quad \text{implying} \quad \sum_{i=1}^{k} w(v_{i-1}, v_i) \geq 0$$

contradicting to $c$ being a negative cycle where $\sum_{i=1}^{k} w(v_{i-1}, v_i) < 0$
Chapter 24. Single Source Shortest Paths

Finding shortest paths on DAGs (directed acyclic graphs)
Finding shortest paths on DAGs (directed acyclic graphs)

- Algorithms can take the advantage of the non-cyclicity.
Finding shortest paths on DAGs (directed acyclic graphs)

- Algorithms can take the advantage of the non-cyclicity.
- How would your algorithm be?
Chapter 24. Single Source Shortest Paths

Finding shortest paths on DAGs (directed acyclic graphs)

- Algorithms can take the advantage of the non-cyclicity.
- How would your algorithm be?

*topological order of vertices!*
Chapter 24. Single Source Shortest Paths

**Dag-Shortest Paths**($G,w,s$)

1. Topologically sort the vertices of $G$.V
2. For each vertex $v \in G$.V
3. $v.d = \infty$
4. $v.\pi = \text{null}$
5. $s.d = 0$
6. For each $u \in G$.V, in the topologically sorted order
7. For each vertex $v \in \text{Adj}[u]$
8. If $v.d > u.d + w(u,v)$
9. $v.d = u.d + w(u,v)$
10. $v.\pi = u$
11. return ($d,\pi$)

• Should we improve lines 6-7?
• Running time: ?
Chapter 24. Single Source Shortest Paths

\textbf{Dag-Shortest Paths}(G, w, s)
1. topologically sort the vertices of \( G.V \)
Chapter 24. Single Source Shortest Paths

DAG-SHORTEST PATHS($G, w, s$)
1. topologically sort the vertices of $G.V$
2. for each vertex $v \in G.V$

Should we improve lines 6-7?

Running time: ?
Chapter 24. Single Source Shortest Paths

**DAG-Shortest Paths**\((G, w, s)\)
1. topologically sort the vertices of \(G. V\)
2. for each vertex \(v \in G. V\)
3. \(v.d = \infty\)
Chapter 24. Single Source Shortest Paths

**DAG-Shortest Paths**($G, w, s$)
1. topologically sort the vertices of $G.V$
2. for each vertex $v \in G.V$
3. \hspace{1em} $v.d = \infty$
4. \hspace{1em} $v.\pi = NULL$

---

**Should we improve lines 6-7?**

**Running time:**
Chapter 24. Single Source Shortest Paths

**DAG-Shortest Paths** ($G, w, s$)
1. topologically sort the vertices of $G.V$
2. for each vertex $v \in G.V$
3. $v.d = \infty$
4. $v.\pi = NULL$
5. $s.d = 0$

Should we improve lines 6-7?

Running time: ?
Chapter 24. Single Source Shortest Paths

DAG-Shortest Paths \((G, w, s)\)
1. topologically sort the vertices of \(G.V\)
2. for each vertex \(v \in G.V\)
3. \(v.d = \infty\)
4. \(v.\pi = NULL\)
5. \(s.d = 0\)
6. for each \(u \in G.V\), in the topologically sorted order

Should we improve lines 6-7?

Running time: ?
Chapter 24. Single Source Shortest Paths

**DAG-Shortest Paths**\((G, w, s)\)
1. topologically sort the vertices of \(G.V\)
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7. for each vertex \(v \in Adj[u]\)
Chapter 24. Single Source Shortest Paths

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1. topologically sort the vertices of \(G.V\)
2. for each vertex \(v \in G.V\)
   3. \(v.d = \infty\)
   4. \(v.\pi = NULL\)
   5. \(s.d = 0\)
3. for each \(u \in G.V\), in the topologically sorted order
4. for each vertex \(v \in Adj[u]\)
5. if \(v.d > u.d + w(u, v)\)

Dag-Shortest Paths ($G, w, s$)
1. topologically sort the vertices of $G.V$
2. for each vertex $v \in G.V$
3. \hspace{1em} $v.d = \infty$
4. \hspace{1em} $v.\pi = NULL$
5. \hspace{1em} $s.d = 0$
6. for each $u \in G.V$, in the topologically sorted order
7. \hspace{1em} for each vertex $v \in Adj[u]$
8. \hspace{2em} if $v.d > u.d + w(u, v)$
9. \hspace{2em} \hspace{1em} $v.d = u.d + w(u, v)$
Chapter 24. Single Source Shortest Paths

DAG-SHORTEST PATHS(\(G, w, s\))
1. topologically sort the vertices of \(G.V\)
2. for each vertex \(v \in G.V\)
3. \(v.d = \infty\)
4. \(v.\pi = NULL\)
5. \(s.d = 0\)
6. for each \(u \in G.V\), in the topologically sorted order
7. for each vertex \(v \in Adj[u]\)
8. if \(v.d > u.d + w(u, v)\)
9. \(v.d = u.d + w(u, v)\)
10. \(v.\pi = u\)
Chapter 24. Single Source Shortest Paths

Dag-Shortest Paths($G, w, s$)
1. topologically sort the vertices of $G.V$
2. for each vertex $v \in G.V$
3. \hspace{1cm} $v.d = \infty$
4. \hspace{1cm} $v.\pi = NULL$
5. \hspace{1cm} $s.d = 0$
6. for each $u \in G.V$, in the topologically sorted order
7. \hspace{1cm} for each vertex $v \in Adj[u]$
8. \hspace{2cm} if $v.d > u.d + w(u, v)$
9. \hspace{2cm} \hspace{1cm} $v.d = u.d + w(u, v)$
10. \hspace{2cm} \hspace{1cm} $v.\pi = u$
11. return $(d, \pi)$
Chapter 24. Single Source Shortest Paths

**DAG-Shortest Paths**$(G, w, s)$

1. topologically sort the vertices of $G.V$
2. for each vertex $v \in G.V$
3. $v.d = \infty$
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9. $v.d = u.d + w(u, v)$
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11. return $(d, \pi)$

- Should we improve lines 6-7?

Running time?:
Chapter 24. Single Source Shortest Paths

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1. topologically sort the vertices of \(G.V\)
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7. for each vertex \(v \in Adj[u]\)
8. if \(v.d > u.d + w(u, v)\)
9. \(v.d = u.d + w(u, v)\)
10. \(v.\pi = u\)
11. return \((d, \pi)\)

- Should we improve lines 6-7?
- Running time: ?
Chapter 24. Single Source Shortest Paths

note: the root is $s$. 
Dijkstra’s algorithm

On weighted, directed graphs in which each edge has non-negative weight.
Dijkstra’s algorithm

On weighted, directed graphs in which each edge has non-negative weight.

\[ \text{Dijkstra}(G, w, s) \]
Dijkstra’s algorithm

On weighted, directed graphs in which each edge has non-negative weight.

\textbf{DIJKSTRA}(G, w, s)
1. for each vertex \( v \in G.V \)
Chapter 24. Single Source Shortest Paths

Dijkstra’s algorithm

On weighted, directed graphs in which each edge has non-negative weight.

\textsc{Dijkstra}(G, w, s)
1. for each vertex \( v \in G.V \)
2. \( v.d = \infty \)
Dijkstra’s algorithm

On weighted, directed graphs in which each edge has non-negative weight.

Dijkstra(G, w, s)
1. for each vertex v ∈ G.V
2. v.d = ∞
3. v.π = NULL
4. s.d = 0
5. S = ∅
6. Q = G.V
7. while Q is not empty
8. u = Extract Min(Q)
9. S = S ∪ {u}
10. for each vertex v ∈ Adj[u]
11. if v.d > u.d + w(u, v)
12. v.d = u.d + w(u, v)
13. v.π = u
14. return (d, π)

Running time:?
Chapter 24. Single Source Shortest Paths

Dijkstra’s algorithm

On weighted, directed graphs in which each edge has non-negative weight.

\text{Dijkstra}(G, w, s)
1. for each vertex \( v \in G.V \)
2. \( v.d = \infty \)
3. \( v.\pi = \text{NULL} \)
4. \( s.d = 0 \)
Dijkstra’s algorithm
On weighted, directed graphs in which each edge has non-negative weight.

\text{Dijkstra}(G, w, s)
1. for each vertex \( v \in G.V \)
2. \( v.d = \infty \)
3. \( v.\pi = NULL \)
4. \( s.d = 0 \)
5. \( S = \emptyset \)
Dijkstra’s algorithm

On weighted, directed graphs in which each edge has non-negative weight.

Dijkstra(G, w, s)
1. for each vertex $v \in G.V$
2. $v.d = \infty$
3. $v.\pi = NULL$
4. $s.d = 0$
5. $S = \emptyset$
6. $Q = G.V$

Running time:?
Dijkstra’s algorithm

On weighted, directed graphs in which each edge has non-negative weight.

**Dijkstra**(*G*, *w*, *s*)
1. for each vertex *v* ∈ *G*.V
2. *v*.d = ∞
3. *v*.π = NULL
4. *s*.d = 0
5. *S* = ∅
6. *Q* = *G*.V
7. **while** *Q* is not empty

Running time?
Dijkstra’s algorithm

On weighted, directed graphs in which each edge has non-negative weight.

\[
\text{DIJKSTRA}(G, w, s)
\]
1. for each vertex \( v \in G.V \)
2. \( v.d = \infty \)
3. \( v.\pi = NULL \)
4. \( s.d = 0 \)
5. \( S = \emptyset \)
6. \( Q = G.V \)
7. while \( Q \) is not empty
8. \( u = \text{EXTRACT MIN} (Q) \)
Chapter 24. Single Source Shortest Paths

Dijkstra’s algorithm

On weighted, directed graphs in which each edge has non-negative weight.

\textsc{Dijkstra}(G, w, s)
1. for each vertex $v \in G.V$
2. $v.d = \infty$
3. $v.\pi = NULL$
4. $s.d = 0$
5. $S = \emptyset$
6. $Q = G.V$
7. \textbf{while} $Q$ is not empty
8. $u = \textsc{Extract Min} (Q)$
9. $S = S \cup \{u\}$
Chapter 24. Single Source Shortest Paths

Dijkstra’s algorithm

On weighted, directed graphs in which each edge has non-negative weight.

\[
\text{DIJKSTRA}(G, w, s)
\]

1. for each vertex \( v \in G.V \)
2. \( v.d = \infty \)
3. \( v.\pi = NULL \)
4. \( s.d = 0 \)
5. \( S = \emptyset \)
6. \( Q = G.V \)
7. while \( Q \) is not empty
8. \( u = \text{EXTRACT MIN}(Q) \)
9. \( S = S \cup \{u\} \)
10. for each vertex \( v \in Adj[u] \)
Chapter 24. Single Source Shortest Paths

Dijkstra’s algorithm

On weighted, directed graphs in which each edge has non-negative weight.

\textsc{Dijkstra}(G, w, s)

1. for each vertex \( v \in G.V \)
2. \( v.d = \infty \)
3. \( v.\pi = \text{NULL} \)
4. \( s.d = 0 \)
5. \( S = \emptyset \)
6. \( Q = G.V \)
7. while \( Q \) is not empty
8. \( u = \text{Extract Min} (Q) \)
9. \( S = S \cup \{u\} \)
10. for each vertex \( v \in \text{Adj}[u] \)
11. if \( v.d > u.d + w(u,v) \)

Running time:?
Dijkstra’s algorithm

On weighted, directed graphs in which each edge has non-negative weight.

\[
\text{DIJKSTRA}(G, w, s)
\]

1. for each vertex \( v \in G.V \)
2. \( v.d = \infty \)
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4. \( s.d = 0 \)
5. \( S = \emptyset \)
6. \( Q = G.V \)
7. while \( Q \) is not empty
8. \( u = \text{EXTRACT MIN} (Q) \)
9. \( S = S \cup \{u\} \)
10. for each vertex \( v \in Adj[u] \)
11. if \( v.d > u.d + w(u, v) \)
12. \( v.d = u.d + w(u, v) \)
Dijkstra’s algorithm

On weighted, directed graphs in which each edge has non-negative weight.

Dijkstra\((G, w, s)\)
1. for each vertex \(v \in G.V\)
2. \(v.d = \infty\)
3. \(v.\pi = NULL\)
4. \(s.d = 0\)
5. \(S = \emptyset\)
6. \(Q = G.V\)
7. while \(Q\) is not empty
8. \(u = \text{Extract Min} (Q)\)
9. \(S = S \cup \{u\}\)
10. for each vertex \(v \in Adj[u]\)
11. if \(v.d > u.d + w(u, v)\)
12. \(v.d = u.d + w(u, v)\)
13. \(v.\pi = u\)
Dijkstra’s algorithm

On weighted, directed graphs in which each edge has non-negative weight.

Dijkstra\( (G, w, s) \)
1. for each vertex \( v \in G.V \)
2. \( v.d = \infty \)
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4. \( s.d = 0 \)
5. \( S = \emptyset \)
6. \( Q = G.V \)
7. while \( Q \) is not empty
8. \( u = \text{EXTRACT MIN} (Q) \)
9. \( S = S \cup \{u\} \)
10. for each vertex \( v \in \text{Adj}[u] \)
11. if \( v.d > u.d + w(u, v) \)
12. \( v.d = u.d + w(u, v) \)
13. \( v.\pi = u \)
14. return \((d, \pi)\)
Chapter 24. Single Source Shortest Paths

Dijkstra’s algorithm

On weighted, directed graphs in which each edge has non-negative weight.

\textbf{Dijkstra}(G, w, s)
1. for each vertex \( v \in G.V \)
2. \( v.d = \infty \)
3. \( v.\pi = NULL \)
4. \( s.d = 0 \)
5. \( S = \emptyset \)
6. \( Q = G.V \)
7. \textbf{while} \( Q \) is not empty
8. \( u = \text{Extract Min} (Q) \)
9. \( S = S \cup \{u\} \)
10. \textbf{for} each vertex \( v \in \text{Adj}[u] \)
11. \textbf{if} \( v.d > u.d + w(u, v) \)
12. \( v.d = u.d + w(u, v) \)
13. \( v.\pi = u \)
14. \textbf{return} \( (d, \pi) \)

Running time:?
Chapter 24. Single Source Shortest Paths

Note: the black-colored vertices are in set $S$. 
Chapter 24. Single Source Shortest Paths

Correctness of algorithm Dijkstra

Theorem 24.6
Dijkstra's algorithm, run on a weighted, directed graph $G = (V,E)$ with non-negative weight function $w$ and source $s$, terminates with $u.d = \delta(s,u)$ for all vertices $u \in V$.

Proof: We need to show the while loop has loop invariant:
$u.d = \delta(s,u)$ for each $u \in S$

Assume $u$ to be the first such vertex that $u.d > \delta(s,u)$ when it is being added to $S$ then there must be a shortest path $p$: $s \Rightarrow x \Rightarrow y \Rightarrow u$, for some $x \in S$ and some $y \neq S$.

$y.d = \delta(s,y)$ when $u$ is being added to $S$. This is because $x \in S$, $x.d = \delta(s,x)$ when $x$ was added to $S$.

Edge $(x,y)$ was related at that time, and $y.d = \delta(s,y)$ by Convergence-property.

So when $u$ was chosen, $u.d \leq y.d = \delta(s,y) \leq \delta(s,u)$. Contradicts the choice of $u$.

So $u.d = \delta(s,u)$ when it is being included to $S$. 
Chapter 24. Single Source Shortest Paths

Correctness of algorithm \textbf{Dijkstra}

\textbf{Theorem 24.6} Dijkstra’s algorithm, run on a weighted, directed graph \( G = (V, E) \) with non-negative weight function \( w \) and source \( s \), terminates with \( u.d = \delta(s, u) \) for all vertices \( u \in V \).
Correctness of algorithm \textbf{Dijkstra}

**Theorem 24.6** Dijkstra’s algorithm, run on a weighted, directed graph $G = (V, E)$ with non-negative weight function $w$ and source $s$, terminates with $u.d = \delta(s, u)$ for all vertices $u \in V$.

**Proof:** We need to show the while loop has loop invariant:
Correctness of algorithm Dijkstra

**Theorem 24.6** Dijkstra’s algorithm, run on a weighted, directed graph $G = (V, E)$ with non-negative weight function $w$ and source $s$, terminates with $u.d = \delta(s, u)$ for all vertices $u \in V$.

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Chapter 24. Single Source Shortest Paths

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\( y.d = \delta(s, y) \) **when** \( u \) is being added to \( S \). This is because \( x \in S \), \( x.d = \delta(s, x) \) when \( x \) was added to \( S \).
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$y.d = \delta(s, y)$ when $u$ is being added to $S$. This is because $x \in S$, $x.d = \delta(s, x)$ when $x$ was added to $S$. Edge $(x, y)$ was related at that time, and $y.d = \delta(s, y)$ by **Convergence-property**.
Chapter 24. Single Source Shortest Paths

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**Chapter 24. Single Source Shortest Paths**

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When \( u \) was chosen, \( u.d \leq y.d = \delta(s, y) \leq \delta(s, u) \). **Contradicts** the choice of \( u \).
Chapter 24. Single Source Shortest Paths

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$y.d = \delta(s, y)$ when $u$ is being added to $S$. This is because $x \in S$, $x.d = \delta(s, x)$ when $x$ was added to $S$. Edge $(x, y)$ was related at that time, and $y.d = \delta(s, y)$ by Convergence-property. So

When $u$ was chosen, $u.d \leq y.d = \delta(s, y) \leq \delta(s, u)$. **Contradicts** the choice of $u$. So $u.d = \delta(s, u)$ when it is being included to $S$. 
Chapter 24. Single Source Shortest Paths

- Running time of Dijkstra?
Chapter 24. Single Source Shortest Paths

- Running time of Dijkstra?
- Can Dijkstra deals with negative edges or cycles?
Chapter 24. Single Source Shortest Paths

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- Fundamental differences between Bellman-Ford and Dijkstra?
Chapter 24. Single Source Shortest Paths

- Running time of Dijkstra?
- Can Dijkstra deals with negative edges or cycles?
- Fundamental differences between Bellman-Ford and Dijkstra?

\[
\text{Dijkstra}(G, w, s)
\]
1. for each vertex \( v \in G.V \)
2. \( v.d = \infty \)
3. \( v.\pi = \text{NULL} \)
4. \( s.d = 0 \)
5. \( S = \emptyset \)
6. \( Q = G.V \)
7. while \( Q \) is not empty
8. \( u = \text{EXTRACT MIN} (Q) \)
9. \( S = S \cup \{u\} \)
10. for each vertex \( v \in \text{Adj}[u] \)
11. if \( v.d > u.d + w(u, v) \)
12. \( v.d = u.d + w(u, v) \)
13. \( v.\pi = u \)
14. return \( (d, \pi) \)

\[
\text{Bellman-Ford}(G, w, s)
\]
1. for each vertex \( v \in G.V \)
2. \( v.d = \infty \)
3. \( v.\pi = \text{NULL} \)
4. \( s.d = 0 \)
5. for \( i = 1 \) to \(|V| - 1\)
6. for each edge \((u, v) \in G.E\)
7. if \( v.d > u.d + w(u, v) \)
8. \( v.d = u.d + w(u, v) \)
9. \( v.\pi = u \)
10. for each edge \((u, v) \in G.E\)
11. if \( v.d > u.d + w(u, v) \)
12. return \( \text{FALSE} \)
13. return \( \text{TRUE} \)
Chapter 24. Single Source Shortest Paths

• Fundamental differences between Dijkstra and MST-Prim?
Chapter 24. Single Source Shortest Paths

- Fundamental differences between Dijkstra and MST-Prim?
Chapter 24. Single Source Shortest Paths

- Fundamental differences between **Dijkstra** and **MST-Prim**?

**Dijkstra**($G, w, s$)
1. for each vertex $v \in G.V$
2. \hspace{1cm} $v.d = \infty$
3. \hspace{1cm} $v.\pi = NULL$
4. \hspace{1cm} $s.d = 0$
5. \hspace{1cm} $S = \emptyset$
6. \hspace{1cm} $Q = G.V$
7. while $Q$ is not empty
8. \hspace{1cm} $u = \text{EXTRACT MIN}(Q)$
9. \hspace{1cm} $S = S \cup \{ u \}$
10. for each vertex $v \in \text{Adj}[u]$
11. \hspace{1cm} if $v.d > u.d + w(u, v)$
12. \hspace{1cm} $v.d = u.d + w(u, v)$
13. \hspace{1cm} $v.\pi = u$
14. return $(d, \pi)$

**MST-Prim**($G, w, r$)
1. for each $u \in G.V$
2. \hspace{1cm} $u.key = \infty$ \hspace{1cm} \{ $u.key$ is too large \}
3. \hspace{1cm} $u.\pi = NULL$
4. \hspace{1cm} $r.key = 0$
5. \hspace{1cm} $Q = G.V$
6. while $Q \neq \emptyset$
7. \hspace{1cm} $u = \text{EXTRACT MIN}(Q)$
8. for each $v \in \text{Adj}[u]$
9. \hspace{1cm} if $v \in Q$ and $w(u, v) < v.key$
10. \hspace{1cm} then $v.\pi = u$
11. \hspace{1cm} $v.key = w(u, v)$
12. return $\pi$
Chapter 25. All-pairs shortest paths

All-Pair Shortest Paths Problem

Input: A weighted graph $G = (V,E)$ with edge weight function $w$;

Output: Shortest paths between every pair of vertices in $G$.

- Dijkstra would run in time $O(|V|^2 \log |V| + |V||E|)$ on non-negative edges.
- Bellman-Ford would run in time $O(|V|^2 |E|)$ for general graphs, but $O(|V|^4)$ on "dense" graphs.

New algorithms:
- A dynamic programming algorithm $O(|V|^4)$, improved to $O(|V|^3 \log |V|)$.
- Floyd-Warshall algorithm: $O(|V|^3)$.

Graph representation: adjacency matrix $W = (w_{ij})$. 
Chapter 25. All-pairs shortest paths

All Pair Shortest Paths Problem
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All Pair Shortest Paths Problem

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**Chapter 25. All-pairs shortest paths**

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**Chapter 25. All-pairs shortest paths**
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Chapter 25. All-pairs shortest paths

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Chapter 25. All-pairs shortest paths

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New algorithms
Chapter 25. All-pairs shortest paths

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Chapter 25. All-pairs shortest paths

A dynamic programming approach
Chapter 25. All-pairs shortest paths

A dynamic programming approach

- Optimal substructure
- Objective function
Chapter 25. All-pairs shortest paths

A dynamic programming approach

- Optimal substructure
- Objective function

Define $l_{ij}$ be the minimum weight of any path from $v_i$ to $v_j$. 

or alternatively, Define $l_{kij}$ be the minimum weight of any path from $v_i$ to $v_j$ in which intermediate vertices have indexes $\leq k$. 

does not work! having a data dependency issue.
Chapter 25. All-pairs shortest paths

A dynamic programming approach

- Optimal substructure
- Objective function

Define $l_{ij}$ be the minimum weight of any path from $v_i$ to $v_j$ that does not work! having a data dependency issue.
Chapter 25. All-pairs shortest paths

A dynamic programming approach

- Optimal substructure
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Define $l_{ij}$ be the minimum weight of any path from $v_{i}$ to $v_{j}$ does not work! having a data dependency issue.

Define $l_{ij}^{m}$ be the minimum weight of any path from $v_{i}$ to $v_{j}$ that contains at most $m$ edges.
A dynamic programming approach

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A dynamic programming approach

- Optimal substructure
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or alternatively,

Define \( l^k_{ij} \) be the minimum weight of any path from \( v_i \) to \( v_j \) in which intermediate vertices have indexes \( \leq k \).
Define $l_{ij}^m$ be the minimum weight of any path from $v_i$ to $v_j$ that contains at most $m$ edges.
Chapter 25. All-pairs shortest paths

Define \( l^m_{ij} \) be the minimum weight of any path from \( v_i \) to \( v_j \) that contains at most \( m \) edges.

\[
l^m_{ij} = \min(l^m_{ij}^{-1}, \min_{1 \leq k \leq n} \{l^m_{ik}^{-1} + w_{kj}\})
\]
Define $l_{ij}^m$ be the minimum weight of any path from $v_i$ to $v_j$ that contains at most $m$ edges.

$$l_{ij}^m = \min(l_{ij}^{m-1}, \min_{1 \leq k \leq n} \{l_{ik}^{m-1} + w_{kj}\})$$

If $w_{jj} = 0$, we can rewrite

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and base cases:

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Adjacency matrix $W = (w_{ij})$ is the default.
Chapter 25. All-pairs shortest paths

DP table filling algorithm:

1. $n = \text{rows}[L]$;
2. let $L'$ be an $n \times n$ table;
3. for $i = 1$ to $n$
4. for $j = 1$ to $n$
5. $L'[i,j] = \infty$ (in case $w_{a,a} \neq 0$)
6. for $k = 1$ to $n$
7. $L'[i,j] = \min\{L'[i,j], L[i,k] + w[k,j]\}$
8. return $(L')$
Chapter 25. All-pairs shortest paths

DP table filling algorithm:
For $L^1 = W$ and $m = 2, \ldots, n - 1$, compute table $L^m$ from table $L^{m-1}$;
DP table filling algorithm:

For $L^1 = W$ and $m = 2, \ldots, n - 1$, compute table $L^m$ from table $L^{m-1}$;
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Chapter 25. All-pairs shortest paths

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**Extended Shortest Paths**($L, W$)
Chapter 25. All-pairs shortest paths

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Chapter 25. All-pairs shortest paths

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Chapter 25. All-pairs shortest paths

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Chapter 25. All-pairs shortest paths

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Chapter 25. All-pairs shortest paths

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6. for $k = 1$ to $n$
7. $L'[i, j] = \min\{L'[i, j], L[i, k] + w[k, j]\}$
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6.     for $k = 1$ to $n$
7.         $L'[i, j] = \min\{L'[i, j], L[i, k] + w[k, j]\}$
8. return $(L')$
Chapter 25. All-pairs shortest paths

DP table filling algorithm:

For \( L^1 = W \) and \( m = 2, \ldots, n - 1 \), compute table \( L^m \) from table \( L^{m-1} \);
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Extended Shortest Paths \((L, W)\)
1. \( n = \text{rows}[L] \);
2. let \( L' \) be an \( n \times n \) table;
3. for \( i = 1 \) to \( n \)
4.   for \( j = 1 \) to \( n \)
5.     \( L'[i, j] = \infty \) \( \quad (L'[i, j] = L[i, j] \text{ in case } w_{a,a} \neq 0) \)
6.     for \( k = 1 \) to \( n \)
7.       \( L'[i, j] = \min\{L'[i, j], L[i, k] + w[k, j]\} \)
8. return \( (L') \)

Call Extended Shortest Paths for \( m = 2, 3, \ldots, n - 1 \)
Chapter 25. All-pairs shortest paths

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For $L^1 = W$ and $m = 2, \ldots, n - 1$, compute table $L^m$ from table $L^{m-1}$; technically two tables are enough.

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8. return ( $L'$)

Call **Extended Shortest Paths** for $m = 2, 3, \ldots, n - 1$

\[ L^m \leftarrow \text{Extended Shortest Paths}(L^{m-1}, W) \]
Chapter 25. All-pairs shortest paths

Running on an example:
Chapter 25. All-pairs shortest paths

Running on an example:

\[ W = L^1 = \text{the first matrix.} \]

\[ l_{0,0}^2 = \min \begin{cases} l_{0,0}^1 \\ l_{0,0}^1 + l_{0,0}^1 \\ l_{0,1}^1 + l_{1,0}^1 \\ l_{0,2}^1 + l_{2,0}^1 \end{cases} \]

value = 8

\[ k = 0, \text{ value } = 8 + 8 = 16 \]

\[ k = 1, \text{ value } = 1 + 6 = 7 \]

\[ k = 2, \text{ value } = 1 + 3 = 4^* \]
Chapter 25. All-pairs shortest paths

- running time: $\Theta(n^4)$. 

Faster All Pair Shortest Paths

1. $n =$ rows $[W]$;
2. $L =$ $W$;
3. $m =$ 1;
4. while $m < n - 1$
5. $L =$ Extended Shortest Paths $(L, L)$
6. $m =$ $2 \times m$
7. return $(L)$
Chapter 25. All-pairs shortest paths

- running time: $\Theta(n^4)$.

- improving the running time by repeatedly squaring:
Chapter 25. All-pairs shortest paths

- running time: $\Theta(n^4)$.
- improving the running time by repeatedly squaring:
  
  compute: $L^1, L^2, L^4, \ldots, L^{2^k}$.

Faster All Pair Shortest Paths (W)

1. $n = \text{rows}[W]$;
2. $L = W$;
3. $m = 1$;
4. while $m < n - 1$
5. $L = \text{Extended Shortest Paths}(L, L)$
6. $m = 2 \times m$;
7. return $(L)$.
Chapter 25. All-pairs shortest paths

- running time: $\Theta(n^4)$.
- improving the running time by repeatedly squaring:
  compute: $L^1, L^2, L^4, \ldots, L^{2^k}$.
  what is $k$ here?
Chapter 25. All-pairs shortest paths

- running time: $\Theta(n^4)$.
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  what is $k$ here? Let $2^k = n - 1$. Then $k = \lceil \log_2(n - 1) \rceil$. 
Chapter 25. All-pairs shortest paths

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Faster All Pair Shortest Paths($W$)
Chapter 25. All-pairs shortest paths

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**Faster All Pair Shortest Paths**($W$)

1. \ $n = rows[W]$;
Chapter 25. All-pairs shortest paths

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**Faster All Pair Shortest Paths** ($W$)

1. $n = \text{rows}[W]$;
2. $L = W$;
Chapter 25. All-pairs shortest paths

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Chapter 25. All-pairs shortest paths

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**Faster All Pair Shortest Paths($W$)**
1. $n = \text{rows}[W]$;
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Chapter 25. All-pairs shortest paths

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Chapter 25. All-pairs shortest paths

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Chapter 25. All-pairs shortest paths

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**Faster All Pair Shortest Paths**($W$)

1. $n = \text{rows}[W]$;
2. $L = W$;
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5. \hspace{5mm} $L = \text{Extended Shortest Paths}(L, L)$
6. \hspace{5mm} $m = 2 \times m$
7. return $(L)$
Chapter 25. All-pairs shortest paths

Floyd-Warshall algorithm

Define:
\[d(k)_{ij}\] to be the shortest path distance from \(v_i\) to \(v_j\) with no intermediate vertices of indexes higher than \(k\).

Thus
\[d(k)_{ij} = \min(d(k-1)_{ij}, d(k-1)_{ik} + d(k-1)_{kj})\]

with base case:
\[d(0)_{ij} = w_{ij}.\]
Chapter 25. All-pairs shortest paths

Floyd-Warshall algorithm

intermediate vertices on a path \( v_i \leadsto v_j \): those other than \( v_i \) and \( v_j \).
Chapter 25. All-pairs shortest paths

Floyd-Warshall algorithm

Intermediate vertices on a path $v_i \leadsto v_j$: those other than $v_i$ and $v_j$.

Define: $d_{i,j}^{(k)}$ to be the shortest path distance from $v_i$ to $v_j$
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Chapter 25. All-pairs shortest paths

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$$d_{ij}^{(k)} = \min(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)})$$

with base case: $d_{ij}^{(0)} = w_{ij}$. 

Floyd-Warshall ($W$)

1. $n = \text{rows}[W]$
2. $D(0) = W$
3. for $k = 1$ to $n$
4. for $i = 1$ to $n$
5. for $j = 1$ to $n$
6. $D(k)[i,j] = \min\{D(k-1)[i,j], D(k-1)[i,k] + D(k-1)[k,j]\}$
7. return ($D(n)$)
Chapter 25. All-pairs shortest paths

Floyd-Warshall algorithm

Intermediate vertices on a path $v_i \leadsto v_j$: those other than $v_i$ and $v_j$.

Define: $d^{(k)}_{ij}$ to be the shortest path distance from $v_i$ to $v_j$ with no intermediate vertices of indexes higher than $k$. Thus

$$d^{(k)}_{ij} = \min(d^{(k-1)}_{ij}, d^{(k-1)}_{ik} + d^{(k-1)}_{kj})$$

with base case: $d^{(0)}_{ij} = w_{ij}$.

$\text{Floyd-Warshall}(W)$
Chapter 25. All-pairs shortest paths

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Floyd-Warshall(\( W \))
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Chapter 25. All-pairs shortest paths

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2. $D^{(0)} = W$
Chapter 25. All-pairs shortest paths

Floyd-Warshall algorithm

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FLOYD-WARSHALL($W$)
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2. $D^{(0)} = W$
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Chapter 25. All-pairs shortest paths

Floyd-Warshall algorithm

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Floyd-Warshall($W$)
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4. for $i = 1$ to $n$
Chapter 25. All-pairs shortest paths

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\( \text{Floyd-Warshall}(W) \)
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4. \( \quad \text{for } i = 1 \text{ to } n \)
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6. $D^{(k)}[i, j] = \min\{D^{(k-1)}[i, j], D^{(k-1)}[i, k] + D^{(k-1)}[k, j]\}$
Chapter 25. All-pairs shortest paths

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Define: $d_{ij}^{(k)}$ to be the shortest path distance from $v_i$ to $v_j$ with no intermediate vertices of indexes higher than $k$. Thus

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with base case: $d_{ij}^{(0)} = w_{ij}$.

\begin{algorithm}
\textsc{Floyd-Warshall}(W)
\begin{algorithmic}
\State $n = \text{rows}[W]$
\State $D^{(0)} = W$
\For {$k = 1$ to $n$}
\For {$i = 1$ to $n$}
\For {$j = 1$ to $n$}
\State $D^{(k)}[i, j] = \min\{D^{(k-1)}[i, j], D^{(k-1)}[i, k] + D^{(k-1)}[k, j]\}$
\EndFor
\EndFor
\EndFor
\State \textbf{return} ($D^{(n)}$)
\end{algorithmic}
\end{algorithm}
Chapter 25. All-pairs shortest paths

\[ D^{(0)} = \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}, \quad \Pi^{(0)} = \begin{pmatrix} \text{NIL} & 1 & 1 & \text{NIL} & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 2 & 2 \\ \text{NIL} & 3 & \text{NIL} & \text{NIL} & \text{NIL} \\ 4 & \text{NIL} & 4 & \text{NIL} & \text{NIL} \\ \text{NIL} & \text{NIL} & \text{NIL} & 5 & \text{NIL} \end{pmatrix} \]

\[ D^{(1)} = \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}, \quad \Pi^{(1)} = \begin{pmatrix} \text{NIL} & 1 & 1 & \text{NIL} & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 2 & 2 \\ \text{NIL} & 3 & \text{NIL} & \text{NIL} & \text{NIL} \\ 4 & 1 & 4 & \text{NIL} & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 5 & \text{NIL} \end{pmatrix} \]

\[ D^{(2)} = \begin{pmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}, \quad \Pi^{(2)} = \begin{pmatrix} \text{NIL} & 1 & 1 & 2 & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 2 & 2 \\ \text{NIL} & 3 & \text{NIL} & 2 & 2 \\ 4 & 1 & 4 & \text{NIL} & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 5 & \text{NIL} \end{pmatrix} \]

\[ D^{(3)} = \begin{pmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}, \quad \Pi^{(3)} = \begin{pmatrix} \text{NIL} & 1 & 1 & 2 & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 2 & 2 \\ \text{NIL} & 3 & \text{NIL} & 2 & 2 \\ 4 & 3 & 4 & \text{NIL} & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 5 & \text{NIL} \end{pmatrix} \]

\[ D^{(4)} = \begin{pmatrix} 0 & 3 & 1 & 4 & -4 \\ 2 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix}, \quad \Pi^{(4)} = \begin{pmatrix} \text{NIL} & 1 & 4 & 2 & 1 \\ 4 & \text{NIL} & 4 & 2 & 1 \\ 4 & 3 & \text{NIL} & 2 & 1 \\ 4 & 3 & 4 & \text{NIL} & 1 \\ 4 & 3 & 4 & 5 & \text{NIL} \end{pmatrix} \]

\[ D^{(5)} = \begin{pmatrix} 0 & 1 & -3 & 2 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix}, \quad \Pi^{(5)} = \begin{pmatrix} \text{NIL} & 3 & 4 & 5 & 1 \\ 4 & \text{NIL} & 4 & 2 & 1 \\ 4 & 3 & \text{NIL} & 2 & 1 \\ 4 & 3 & 4 & \text{NIL} & 1 \\ 4 & 3 & 4 & 5 & \text{NIL} \end{pmatrix} \]
Chapter 25. All-pairs shortest paths

- Constructing a shortest path

\[
\begin{align*}
\pi(0)_{ij} &= \text{NULL if } i = j \text{ or } w_{ij} = \infty, \\
\pi(0)_{ij} &= i \text{ if } i \neq j \text{ and } w_{ij} < \infty.
\end{align*}
\]

\[
\begin{align*}
\pi(k)_{ij} &= \pi(k-1)_{ij} \text{ if } d(k-1)_{ij} \leq d(k-1)_{ik} + d(k-1)_{kj}, \\
\pi(k)_{ij} &= \pi(k-1)_{kj} \text{ if } d(k-1)_{ij} > d(k-1)_{ik} + d(k-1)_{kj}.
\end{align*}
\]
Chapter 25. All-pairs shortest paths

- Constructing a shortest path
- for each $v_i$ and each $v_j$, to remember the last step to reach $j$. 

$$\pi(0)_{ij} = \begin{cases} \text{NULL} & \text{if } i = j \text{ or } w_{ij} = \infty, \\ i & \text{if } i \neq j \text{ and } w_{ij} < \infty. \end{cases}$$

$$\pi(k)_{ij} = \begin{cases} \pi(k-1)_{ij} & \text{if } d(k-1)_{ij} \leq d(k-1)_{ik} + d(k-1)_{kj}, \\ \pi(k-1)_{kj} & \text{if } d(k-1)_{ij} > d(k-1)_{ik} + d(k-1)_{kj}. \end{cases}$$
Chapter 25. All-pairs shortest paths

• Constructing a shortest path

• for each $v_i$ and each $v_j$, to remember the last step to reach $j$. predecessor matrix $\pi$, recursively defined as
Chapter 25. All-pairs shortest paths

- Constructing a shortest path
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$$
Chapter 25. All-pairs shortest paths

- Constructing a shortest path
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predecessor matrix \( \pi \), recursively defined as

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Chapter 25. All-pairs shortest paths

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$$
\pi_{ij}^{(k)} = \pi_{ij}^{(k-1)} \text{ if } d_{ij}^{(k-1)} \leq d_{ik}^{(k-1)} + d_{kj}^{(k-1)}, \text{ or } \\
\pi_{ij}^{(k)} = \pi_{kj}^{(k-1)} \text{ if } d_{ij}^{(k-1)} > d_{ik}^{(k-1)} + d_{kj}^{(k-1)}
$$
Chapter 25. All-pairs shortest paths

Summary of shortest path algorithms

1. Bellman-Ford's algorithm (able to detect negative weight cycles)
2. DAG Shortest Paths (use topological sorting) [Lawler]
3. Dijkstra's algorithm (assuming non-negative weights)
4. Matrix multiplication (DP) [Lawler, folklore]
5. Floyd-Warshall algorithm (DP)
Chapter 25. All-pairs shortest paths

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Chapter 25. All-pairs shortest paths

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