Part VII. Selected Topics
Chapter 33.9 Exhaustive Search (Covered!)
Chapter 33.9 Exhaustive Search (Covered!)
Chapter 34 NP-Completeness
Chapter 33.9 Exhaustive Search (Covered!)

Chapter 34 NP-Completeness
Chapter 33.9. Exhaustive Search

To enumerate all possible solutions to the problem instance

- systematic examining all solutions
- without repeating solutions that have been examined
- stop when a satisfactory solution is found
Chapter 33.9. Exhaustive Search

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Chapter 33.9. Exhaustive Search

First, we need to be able to count total number of “things” to be enumerated.
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- without missing one (correctness)
- without over-counting (efficiency)
- A sophisticated counting often has recursive solution.
Chapter 33.9. Exhaustive Search

Examples of counting:

(1) The total number of permutations of \(1, 2, \ldots, n\) is \(P(n) = n \times P(n-1)\) with base case \(P(1) = 1\).

(2) The total number of ways to choose \(k\) from \(n\) items is \(\binom{n}{k} = \frac{n!}{(n-k)!k!}\) or, alternatively, \(\binom{n}{k} = \frac{n-k}{n} \binom{n-1}{k} + \binom{n-1}{k-1}\) with base cases: (?)
Chapter 33.9. Exhaustive Search

Examples of counting:

(1) total number of permutations of $(1, 2, \ldots, n)$ is

\[ P(n) = \]

\[ n \times P(n-1) = n \times (n-1) \times P(n-2) = \cdots = n! \]
Chapter 33.9. Exhaustive Search

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P(n) = n! \quad \text{(factorial)}
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Chapter 33.9. Exhaustive Search

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Chapter 33.9. Exhaustive Search

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with base cases: (?)
Example: Boolean Formula Satisfiability problem (SAT)

Input:

Output: "yes" if and only if \( f(x_1, x_2, \ldots, x_n) \) is satisfiable.

\( f(x_1, x_2, \ldots, x_n) \) is satisfiable if there is an assignment to boolean variables \( x_i \in \{T, F\}, i = 1, 2, \ldots, n \), such that \( f \) is evaluated to \( T \).

e.g, \( f(x_1, x_2, x_3) = (x_1 \lor x_2 \lor \neg x_3) \land (\neg x_1 \lor x_3) \) is satisfiable

\( g(x_1, x_2) = (x_1 \lor \neg x_2) \land (\neg x_1 \lor x_2) \land (x_1 \lor x_2) \land (\neg x_1 \lor \neg x_2) \) is not!
Chapter 33.9. Exhaustive Search

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\[ f(x_1, x_2, \ldots, x_n) \text{ is satisfiable if there is an assignment to boolean variables} \]

\[ x_i \in \{T, F\}, \text{ } i = 1, 2, \ldots, n, \]
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Chapter 33.9. Exhaustive Search

Use exhaustive search to solve the SAT problem.
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**INPUT:** boolean formula $f(x_1, x_2, \ldots, x_n)$,
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Chapter 33.9. Exhaustive Search

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How? What will you exhaustively search on?
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- **Enumerate all combinations of T and F for $x_1, \ldots, x_n$.**
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Chapter 33.9. Exhaustive Search

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- Can you solve it with a recursive algorithm?
- Can you solve it with an iterative algorithm?
Chapter 33.9. Exhaustive Search

Solve SAT problem with a recursive algorithm:

- what data will the recursion be applied to?
  - boolean formula \( f(x_1, \ldots, x_n) \)

- what is the terminating (base) case?
  - \( n=0 \), formula without variables

- what is the recursive case?
  - For \( f(x_1, \ldots, x_n) \), define:
    - \( f(x_1, \ldots, x_{n-1}, T) \) for true assignment
    - \( f(x_1, \ldots, x_{n-1}, F) \) for false assignment

Then, for all assignments:

- \( f(x_1, \ldots, x_{n-1}, T) \) is true if at least one variable is true
- \( f(x_1, \ldots, x_{n-1}, F) \) is false if all variables are false

Thus:

- \( f(x_1, \ldots, x_{n-1}, T) \Rightarrow g(x_1, \ldots, x_{n-1}) \)
- \( f(x_1, \ldots, x_{n-1}, F) \Rightarrow h(x_1, \ldots, x_{n-1}) \)
Chapter 33.9. Exhaustive Search

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  $f(x_1, \ldots, x_{n-1}, x_n) = f(x_1, \ldots, x_{n-1}, T) \lor f(x_1, \ldots, x_{n-1}, F)$

  $f(x_1, \ldots, x_{n-1}, T) = \Rightarrow g(x_1, \ldots, x_{n-1})$

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Chapter 33.9. Exhaustive Search

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  \[
  f(x_1, \ldots, x_{n-1}, x_n) =
  \begin{cases}
  f(x_1, \ldots, x_{n-1}, T) \\
  f(x_1, \ldots, x_{n-1}, F)
  \end{cases}
  \]
Chapter 33.9. Exhaustive Search

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  f(x_1, \ldots, x_{n-1}, T) \implies g(x_1, \ldots, x_{n-1})
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Chapter 33.9. Exhaustive Search

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\begin{align*}
  f(x_1, \ldots, x_{n-1}, x_n) &= f(x_1, \ldots, x_{n-1}, T) \lor f(x_1, \ldots, x_{n-1}, F) \\
  f(x_1, \ldots, x_{n-1}, T) &\implies g(x_1, \ldots, x_{n-1}) \\
  f(x_1, \ldots, x_{n-1}, F) &\implies h(x_1, \ldots, x_{n-1})
\end{align*}
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Chapter 33.9. Exhaustive Search

Solve SAT problem with a recursive algorithm:

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Chapter 33.9. Exhaustive Search

Algorithm SAT_SOLVER($f(x_1, \ldots, x_{n-1}, x_n)$)

1. if $n = 0$, return ($f$);
2. else $g(x_1, \ldots, x_{n-1}) = f(x_1, \ldots, x_{n-1}, T)$
3. $h(x_1, \ldots, x_{n-1}) = f(x_1, \ldots, x_{n-1}, F)$
4. return (SAT Solver($g(x_1, \ldots, x_{n-1})$) $\lor$ SAT Solver($h(x_1, \ldots, x_{n-1})$))

Does this algorithm exhaustively search all assignments to the variables?

• draw a search tree based on the algorithm.
• what does the tree look like?
• what does each path mean?
• how many paths?
• time?

$T(n) = 2T(n-1) + cn$, $T(0) = c$, $\Rightarrow T(n) = \Theta(2^n)$
Chapter 33.9. Exhaustive Search

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Algorithm SAT Solver\( (f(x_1, \ldots, x_{n-1}, x_n)) \)

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3. \( h(x_1, \ldots, x_{n-1}) = f(x_1, \ldots, x_{n-1}, F) \)
4. \textbf{return} \( \text{SAT Solver}(g(x_1, \ldots, x_{n-1})) \lor \text{SAT Solver}(h(x_1, \ldots, x_{n-1})) \)

Does this algorithm exhaustively search all assignments to the variables?

- draw a search tree based on the algorithm.
- what does the tree look like?
- what does each path mean?
- how many paths?
- time? \( T(n) = 2T(n-1) + cn \) \( T(0) = c \) \( \Rightarrow T(n) = \Theta(2^n) \)
Chapter 33.9. Exhaustive Search

Algorithm $\text{SAT Solver}(f(x_1, \ldots, x_{n-1}, x_n))$

1. if $n = 0$, return ($f$);
2. else $g(x_1, \ldots, x_{n-1}) = f(x_1, \ldots, x_{n-1}, \text{T})$
3. $h(x_1, \ldots, x_{n-1}) = f(x_1, \ldots, x_{n-1}, \text{F})$
4. return ($\text{SAT Solver}(g(x_1, \ldots, x_{n-1})) \lor$
   $\text{SAT Solver}(h(x_1, \ldots, x_{n-1})))$

Does this algorithm exhaustively search all assignments to the variables?
Algorithm SAT_SOLVER(f(x₁, ..., xₙ₋₁, xₙ))

1. if $n = 0$, return (f);
2. else $g(x₁, ..., xₙ₋₁) = f(x₁, ..., xₙ₋₁, T)$
3. $h(x₁, ..., xₙ₋₁) = f(x₁, ..., xₙ₋₁, F)$
4. return (SAT_SOLVER(g(x₁, ..., xₙ₋₁)) $\lor$ SAT_SOLVER(h(x₁, ..., xₙ₋₁)))

Does this algorithm exhaustively search all assignments to the variables?

- draw a search tree based on the algorithm.
Chapter 33.9. Exhaustive Search

Algorithm SAT Solver\( (f(x_1, \ldots, x_{n-1}, x_n)) \)

1. If\( n = 0 \), return \( f \);
2. Else \( g(x_1, \ldots, x_{n-1}) = f(x_1, \ldots, x_{n-1}, T) \)
3. \( h(x_1, \ldots, x_{n-1}) = f(x_1, \ldots, x_{n-1}, F) \)
4. Return\( (\text{SAT Solver}(g(x_1, \ldots, x_{n-1})) \lor \text{SAT Solver}(h(x_1, \ldots, x_{n-1}))) \)

Does this algorithm exhaustively search all assignments to the variables?

- Draw a search tree based on the algorithm.
- What does the tree look like?

\[ T(n) = 2T(n-1) + cn, \quad T(0) = c \implies T(n) = \Theta(2^n) \]
Chapter 33.9. Exhaustive Search

Algorithm SAT Solver\( (f(x_1, \ldots, x_{n-1}, x_n))\)

1. if \( n = 0 \), return \( (f) \);
2. else \( g(x_1, \ldots, x_{n-1}) = f(x_1, \ldots, x_{n-1}, \text{T}) \)
3. \( h(x_1, \ldots, x_{n-1}) = f(x_1, \ldots, x_{n-1}, \text{F}) \)
4. return (SAT Solver\( (g(x_1, \ldots, x_{n-1})) \lor \) SAT Solver\( (h(x_1, \ldots, x_{n-1}))))\)

Does this algorithm exhaustively search all assignments to the variables?

- draw a *search tree* based on the algorithm.
- what does the tree look like?
- what does each path mean?
Chapter 33.9. Exhaustive Search

Algorithm SAT Solver$(f(x_1, \ldots, x_{n-1}, x_n))$

1. if $n = 0$, return $(f)$;
2. else $g(x_1, \ldots, x_{n-1}) = f(x_1, \ldots, x_{n-1}, T)$
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Does this algorithm exhaustively search all assignments to the variables?

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Chapter 33.9. Exhaustive Search

Algorithm SAT Solver($f(x_1, \ldots, x_{n-1}, x_n)$)

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- draw a search tree based on the algorithm.
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Chapter 33.9. Exhaustive Search

Algorithm SAT Solver\((f(x_1, \ldots, x_{n-1}, x_n))\)

1. \textbf{if} \( n = 0 \), \textbf{return} \( (f) \);
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3. \( h(x_1, \ldots, x_{n-1}) = f(x_1, \ldots, x_{n-1}, F) \)
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Algorithm SAT Solver\( (f(x_1, \ldots, x_{n-1}, x_n)) \)

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4. \quad \textbf{return } (\text{SAT Solver}(g(x_1, \ldots, x_{n-1})) \lor \\
\quad \text{SAT Solver}(h(x_1, \ldots, x_{n-1})))

Does this algorithm exhaustively search all assignments to the variables?

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Chapter 33.9. Exhaustive Search

Solve SAT problem with iterative algorithms
Chapter 33.9. Exhaustive Search

Solve SAT problem with iterative algorithms

- How? what to iterate on?

\[ x_1 = F, x_2 = F, \ldots, x_n = F, \]

or simply \((F, F, \ldots, F)\)

- what to increment \((\ldots, F, T, \ldots, T)\) → \((\ldots, T, F, \ldots, F)\)

always flip the last bit.
Chapter 33.9. Exhaustive Search

Solve SAT problem with iterative algorithms

• How? what to iterate on?

  assignments
Solve SAT problem with iterative algorithms

• How? what to iterate on?
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• what is the initial value?
Solve SAT problem with iterative algorithms

- How? what to iterate on?
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Chapter 33.9. Exhaustive Search

Solve SAT problem with iterative algorithms

- How? what to iterate on?
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- what is the initial value?
  \[ x_1 = F, x_2 = F, \ldots, x_n = F, \text{ or simply } (F, F, \ldots, F) \]
Chapter 33.9. Exhaustive Search

Solve SAT problem with iterative algorithms

• How? what to iterate on?
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  \[ x_1 = F, x_2 = F, \ldots, x_n = F, \text{ or simply } (F, F, \ldots, F) \]

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Chapter 33.9. Exhaustive Search

Solve SAT problem with iterative algorithms

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- what is the initial value?
  
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  \((\ldots, F, T, \ldots, T)\)
Chapter 33.9. Exhaustive Search

Solve SAT problem with iterative algorithms

- How? what to iterate on?
  - assignments
- what is the initial value?
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- what to increment
  \[ (\ldots, F, T, \ldots, T) \rightarrow (\ldots, T, F, \ldots, F) \]
Chapter 33.9. Exhaustive Search

Solve SAT problem with iterative algorithms

- How? what to iterate on?
  assignments

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- what to increment
  \[(\ldots, F, T, \ldots, T) \rightarrow (\ldots, T, F, \ldots, F)\]
  always flip the last bit.
Chapter 33.9. Exhaustive Search

Algorithm

 SAT Solver-Enum

\[(f(x_1, \ldots, x_{n-1}, x_n))\]

1. for \(\langle x_1, \ldots, x_n \rangle = \langle F, \ldots, F \rangle\) to \(\langle T, \ldots, T \rangle\)
2. \(V = \text{Evaluate}(f, x_1, \ldots, x_n)\)
3. if \(V = T\), return \((T)\)
4. return \((F)\)

• for loop can be implemented by encoding vectors \(\langle F, \ldots, F \rangle\), \ldots, \(\langle T, \ldots, T \rangle\) with binary numbers then further with integers
• a decoding process is needed to converting integers back to vectors
Chapter 33.9. Exhaustive Search

Algorithm SAT SOLVER-ENUM\( (f(x_1, \ldots, x_{n-1}, x_n)) \)
Algorithm SAT SOLVER-ENUM($f(x_1, \ldots, x_{n-1}, x_n)$)

1. for $\langle x_1, \ldots, x_n \rangle = \langle F, \ldots, F \rangle$ to $\langle T, \ldots, T \rangle$
Algorithm \texttt{SAT Solver-Enum}(f(x_1, \ldots, x_{n-1}, x_n))

1. \textbf{for} $\langle x_1, \ldots, x_n \rangle = \langle F, \ldots, F \rangle \textbf{ to } \langle T, \ldots, T \rangle$
2. \hspace{1em} $V = \texttt{Evaluate}(f, x_1, \ldots, x_n)$
Chapter 33.9. Exhaustive Search

Algorithm $\text{SAT Solver-Enum}(f(x_1, \ldots, x_{n-1}, x_n))$

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Chapter 33.9. Exhaustive Search

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- \textbf{for} loop can be implemented by encoding vectors \( \langle F, \ldots, F \rangle, \ldots, \langle T, \ldots, T \rangle \) with...
Chapter 33.9. Exhaustive Search

Algorithm SAT Solver-Enum\((f(x_1, \ldots, x_{n-1}, x_n))\)

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Chapter 33.9. Exhaustive Search

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Chapter 33.9. Exhaustive Search

Iterative exhaustive search seems to be more convenient.

Another example: Travel Salesman Problem (TSP)

Related problem: Hamiltonian Cycle

Input: a graph $G = (V, E)$

Output: yes if and only if $G$ contains a Hamiltonian cycle (Hamiltonian path is a cycle going through every vertex exactly once.)

How to enumerate all cycles and validate?

- enumerate all permutations of $(12...n)$

- how to encode these permutations as integers?
Chapter 33.9. Exhaustive Search

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Chapter 33.9. Exhaustive Search

Exhaustive search could be non-trivial

Maximum Independent Set

Input: a graph $G = (V, E)$

Output: a subset $I \subseteq V$ such that

1. $\forall u, v \in I, (u, v) \notin E$,
2. $|I|$ is the maximum.

• trivial exhaustive search: check every subset of $V$ and verify
• non-trivial: use a search tree, achieving a better time upper bound.

taking advantage of the independent set
Chapter 33.9. Exhaustive Search

Exhaustive search could be non-trivial

Maximum Independent Set

\textbf{Input}: a graph } G = (V, E) \\
\textbf{Output}: a subset } I \subseteq V \text{ such that \\
\begin{enumerate}
\item \forall u, v \in I, (u, v) \not\in E,
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\end{enumerate}
Chapter 33.9. Exhaustive Search

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Chapter 33.9. Exhaustive Search

The algorithm follows a logical search tree
Chapter 33.9. Exhaustive Search

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- given a graph $G$, it picks an arbitrary vertex $v$ from $G$;
Chapter 33.9. Exhaustive Search

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• given a graph $G$, it picks an arbitrary vertex $v$ from $G$;
• exhaustively, there are two cases to consider:
Chapter 33.9. Exhaustive Search

The algorithm follows a logical search tree

- given a graph $G$, it picks an arbitrary vertex $v$ from $G$;
- exhaustively, there are two cases to consider:
  1. to **include** $v$ in the independent set;
  2. to exclude $v$ from the independent set;

resulting in two subgraphs $G_1$ and $G_2$ to be recursively considered,

  1. $G_1$ is the result of $G$ after $v$ and all its neighbors are removed;
  2. $G_2$ is the result of $G$ after $v$ is removed.

the algorithm terminates when the considered graph is empty.
Chapter 33.9. Exhaustive Search

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- given a graph $G$, it picks an arbitrary vertex $v$ from $G$;
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  1. $G_1$ is the result of $G$ after \( v \).
Chapter 33.9. Exhaustive Search

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Chapter 33.9. Exhaustive Search

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Chapter 33.9. Exhaustive Search

Algorithm MaxIndSet \((G)\)
Chapter 33.9. Exhaustive Search

Algorithm \texttt{MaxIndSet} \((G)\)

1. \textbf{if} \(G = \emptyset\) \textbf{return} \((\emptyset)\)
Chapter 33.9. Exhaustive Search

Algorithm \texttt{MaxIndSet} \((G)\)

1. \textbf{if} \(G = \emptyset\) \textbf{return} \((\emptyset)\)
2. \textbf{else} pick an arbitrary vertex \(v\) in \(G\)
Chapter 33.9. Exhaustive Search

Algorithm $\text{MaxIndSet} \ (G)$

1. if $G = \emptyset$ return $(\emptyset)$
2. else pick an arbitrary vertex $v$ in $G$
3. let $G_1$ be $G$ with $v$ and all its neighbors removed
Algorithm \textsc{MaxIndSet} \((G)\)

1. \textbf{if} \(G = \emptyset\) \textbf{return} \((\emptyset)\)
2. \textbf{else} pick an arbitrary vertex \(v\) in \(G\)
3. let \(G_1\) be \(G\) with \(v\) and all its neighbors removed
4. \(I_1 = \{v\} \cup \text{MaxIndSet} \,(G_1)\)
Algorithm $\text{MaxIndSet} \ (G)$

1. if $G = \emptyset$ return $(\emptyset)$
2. else pick an arbitrary vertex $v$ in $G$
3. let $G_1$ be $G$ with $v$ and all its neighbors removed
4. $I_1 = \{v\} \cup \text{MaxIndSet} \ (G_1)$
5. let $G_2$ be $G$ with $v$ removed
Algorithm \texttt{MaxIndSet}\,\,(G)

1. \quad \textbf{if} \; \ G = \emptyset \; \textbf{return} \; (\emptyset)  
2. \quad \textbf{else} \; \text{pick an arbitrary vertex} \; v \; \text{in} \; G  
3. \quad \text{let} \; G_1 \; \text{be} \; G \; \text{with} \; v \; \text{and all its neighbors removed}  
4. \quad I_1 = \{v\} \cup \texttt{MaxIndSet}\,\,(G_1)  
5. \quad \text{let} \; G_2 \; \text{be} \; G \; \text{with} \; v \; \text{removed}  
6. \quad I_2 = \texttt{MaxIndSet}\,\,(G_2)
Chapter 33.9. Exhaustive Search

Algorithm $\text{MaxIndSet} \ (G)$

1. if $G = \emptyset$ return ($\emptyset$)
2. else pick an arbitrary vertex $v$ in $G$
3. let $G_1$ be $G$ with $v$ and all its neighbors removed
4. $I_1 = \{v\} \cup \text{MaxIndSet} \ (G_1)$
5. let $G_2$ be $G$ with $v$ removed
6. $I_2 = \text{MaxIndSet} \ (G_2)$
7. if $|I_1| \geq |I_2|$ return ($I_1$)

• the algorithm is a search tree
• the time complexity: $T(n) = T(n) - m + T(n - 1) + cn^2$ where $m$ is the number of neighbors of $v$'s
Algorithm $\text{MaxIndSet} (G)$

1. if $G = \emptyset$ return $(\emptyset)$
2. else pick an arbitrary vertex $v$ in $G$
3. let $G_1$ be $G$ with $v$ and all its neighbors removed
4. $I_1 = \{v\} \cup \text{MaxIndSet} (G_1)$
5. let $G_2$ be $G$ with $v$ removed
6. $I_2 = \text{MaxIndSet} (G_2)$
7. if $|I_1| \geq |I_2|$ return $(I_1)$
8. else return $(I_2)$
Algorithm **\text{MaxIndSet} (G)**

1. \textbf{if} \ G = \emptyset \ \textbf{return} \ (\emptyset)
2. \textbf{else} pick an arbitrary vertex \ v \ in \ G
3. \ \ \ \text{let} \ G_1 \ \text{be} \ G \ \text{with} \ v \ \text{and all its neighbors removed}
4. \ I_1 = \{v\} \cup \text{MaxIndSet} (G_1)
5. \ \ \ \text{let} \ G_2 \ \text{be} \ G \ \text{with} \ v \ \text{removed}
6. \ I_2 = \text{MaxIndSet} (G_2)
7. \ \ \ \textbf{if} \ |I_1| \geq |I_2| \ \textbf{return} \ (I_1)
8. \ \ \ \textbf{else} \ \textbf{return} \ (I_2)

- the algorithm is a search tree
Algorithm \textsc{MaxIndSet} \((G)\)

1. \textbf{if} \(G = \emptyset\) \textbf{return} \((\emptyset)\)
2. \textbf{else} pick an arbitrary vertex \(v\) in \(G\)
3. \hspace{1em} let \(G_1\) be \(G\) with \(v\) and all its neighbors removed
4. \hspace{1em} \(I_1 = \{v\} \cup \textsc{MaxIndSet} (G_1)\)
5. \hspace{1em} let \(G_2\) be \(G\) with \(v\) removed
6. \hspace{1em} \(I_2 = \textsc{MaxIndSet} (G_2)\)
7. \hspace{1em} \textbf{if} \(|I_1| \geq |I_2|\) \textbf{return} \((I_1)\)
8. \hspace{1em} \textbf{else} \textbf{return} \((I_2)\)

- the algorithm is a search tree
- the time complexity:
Chapter 33.9. Exhaustive Search

Algorithm $\text{MaxIndSet (} G \text{)}$

1. if $G = \emptyset$ return $(\emptyset)$
2. else pick an arbitrary vertex $v$ in $G$
3. let $G_1$ be $G$ with $v$ and all its neighbors removed
4. $I_1 = \{v\} \cup \text{MaxIndSet (} G_1 \text{)}$
5. let $G_2$ be $G$ with $v$ removed
6. $I_2 = \text{MaxIndSet (} G_2 \text{)}$
7. if $|I_1| \geq |I_2|$ return $(I_1)$
8. else return $(I_2)$

- the algorithm is a search tree
- the time complexity: $T(|G|) = cn^2 + T(|G_1|) + T(|G_2|)$
Algorithm \textsc{MaxIndSet} \((G)\)

1. \textbf{if} \(G = \emptyset\) \textbf{return} \((\emptyset)\)
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- the algorithm is a search tree
- the time complexity: \(T(|G|) = cn^2 + T(|G_1|) + T(|G_2|)\)

\[
T(n) = T(n - 1 - m) + T(n - 1) + cn^2
\]

where \(m\) is the number of neighbors of \(v\)’s
Chapter 33.9. Exhaustive Search

\[ T(n) = T(n - 1 - m) + T(n - 1) + cn^2 \]
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\[ T(n) = T(n - 1 - m) + T(n - 1) + cn^2 \]

• \( m \geq 0 \),
$T(n) = T(n - 1 - m) + T(n - 1) + cn^2$

- $m \geq 0$, $T(n) \leq T(n - 1) + T(n - 1) + cn^2$, 

Chapter 33.9. Exhaustive Search
Chapter 33.9. Exhaustive Search

\[ T(n) = T(n - 1 - m) + T(n - 1) + cn^2 \]

- \( m \geq 0 \), \( T(n) \leq T(n - 1) + T(n - 1) + cn^2 \), \( \implies T(n) = O(2^n) \)
Chapter 33.9. Exhaustive Search

\[ T(n) = T(n - 1 - m) + T(n - 1) + cn^2 \]

- \( m \geq 0, \ T(n) \leq T(n - 1) + T(n - 1) + cn^2, \implies T(n) = O(2^n) \)
- Can we guarantee \( m \geq 1 \)
Chapter 33.9. Exhaustive Search

\[ T(n) = T(n - 1 - m) + T(n - 1) + cn^2 \]

- \( m \geq 0, \ T(n) \leq T(n - 1) + T(n - 1) + cn^2, \ \Rightarrow \ T(n) = O(2^n) \)

- Can we guarantee \( m \geq 1 \) so we have
  \[ T(n) \leq T(n - 2) + T(n - 1) + cn^2, \]

- Or even better, to guarantee \( m \geq 2 \)? if we can,
  \[ T(n) \leq T(n - 3) + T(n - 1) + cn^2, \]
  \[ \Rightarrow \ T(n) = O(1.6181^n) \]

- Can we guarantee \( m \geq 3 \)? possible but a little more complicated.
Chapter 33.9. Exhaustive Search

\[ T(n) = T(n - 1 - m) + T(n - 1) + cn^2 \]

- \( m \geq 0, \ T(n) \leq T(n - 1) + T(n - 1) + cn^2, \implies T(n) = O(2^n) \)

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Chapter 33.9. Exhaustive Search

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Chapter 33.9. Exhaustive Search

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Chapter 33.9. Exhaustive Search

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  \[ T(n) \leq T(n - 2) + T(n - 1) + cn^2, \implies T(n) = O(1.6181^n) \]
- Or even better, to guarantee \( m \geq 2 \)? if we can,
  \[ T(n) \leq T(n - 3) + T(n - 1) + cn^2, \implies T(n) = O(1.5^n) \]
Chapter 33.9. Exhaustive Search

\[ T(n) = T(n - 1 - m) + T(n - 1) + cn^2 \]

- \( m \geq 0, \ T(n) \leq T(n - 1) + T(n - 1) + cn^2, \implies T(n) = O(2^n) \)

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- Or even better, to guarantee \( m \geq 2 \)? if we can,
  \( T(n) \leq T(n - 3) + T(n - 1) + cn^2, \implies T(n) = O(1.5^n) \)
  
  use the substitution method to prove \( T(n) = O(1.5^n) \).
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\[ T(n) = T(n - 1 - m) + T(n - 1) + cn^2 \]

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use the substitution method to prove \( T(n) = O(1.5^n) \).

- Can we guarantee \( m \geq 3 \)?
Chapter 33.9. Exhaustive Search

\[ T(n) = T(n - 1 - m) + T(n - 1) + cn^2 \]

- \( m \geq 0, T(n) \leq T(n - 1) + T(n - 1) + cn^2 \), \( \implies T(n) = O(2^n) \)
- Can we guarantee \( m \geq 1 \) so we have
  \( T(n) \leq T(n - 2) + T(n - 1) + cn^2 \), \( \implies T(n) = O(1.6181^n) \)
- Or even better, to guarantee \( m \geq 2 \)? if we can,
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Chapter 33.9. Exhaustive Search

Let $T(n) \leq T(n - 3) + T(n - 1) + cn^2$, with $T(1) = O(1)$
Chapter 33.9. Exhaustive Search

Let $T(n) \leq T(n - 3) + T(n - 1) + cn^2$, with $T(1) = O(1)$

**Claim:** $T(n) = O(1.5^n)$
Chapter 33.9. Exhaustive Search

Let \( T(n) \leq T(n - 3) + T(n - 1) + cn^2 \), with \( T(1) = O(1) \)

Claim: \( T(n) = O(1.5^n) \)

Proof (use the substitution method)
Chapter 33.9. Exhaustive Search

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Assume that \( T(k) \leq 1.5^k \) for all \( k < n \).
Chapter 33.9. Exhaustive Search

Let $T(n) \leq T(n - 3) + T(n - 1) + cn^2$, with $T(1) = O(1)$

Claim: $T(n) = O(1.5^n)$

Proof (use the substitution method)

Assume that $T(k) \leq 1.5^k$ for all $k < n$. Then

$$T(n) \leq T(n - 3) + T(n - 1) + n^2$$
Chapter 33.9. Exhaustive Search

Let \( T(n) \leq T(n-3) + T(n-1) + cn^2 \), with \( T(1) = O(1) \)

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Assume that \( T(k) \leq 1.5^k \) for all \( k < n \). Then

\[
T(n) \leq T(n-3) + T(n-1) + n^2 \leq 1.5^{n-3} + 1.5^{n-1} + n^2
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Chapter 33.9. Exhaustive Search

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$$T(n) \leq T(n - 3) + T(n - 1) + n^2 \leq 1.5^{n-3} + 1.5^{n-1} + n^2$$

when $n > n_0$ ($n_0$ to be determined)
Chapter 33.9. Exhaustive Search

Let $T(n) \leq T(n - 3) + T(n - 1) + cn^2$, with $T(1) = O(1)$

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when $n > n_0$ ($n_0$ to be determined)

$$\leq 1.5^{n-3}(1 + 1.5^2 + \frac{n^2}{1.5^{n-3}})$$
Chapter 33.9. Exhaustive Search

Let $T(n) \leq T(n - 3) + T(n - 1) + cn^2$, with $T(1) = O(1)$

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$$\leq 1.5^{n-3}(1 + 1.5^2 + \frac{n^2}{1.5^{n-3}}) \leq 1.5^{n-3}(1 + 1.5^2 + 0.1)$$
Let \( T(n) \leq T(n - 3) + T(n - 1) + cn^2 \), with \( T(1) = O(1) \)

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when \( n > n_0 \) (\( n_0 \) to be determined)

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\leq 1.5^{n-3}(1 + 1.5^2 + \frac{n^2}{1.5^{n-3}}) \leq 1.5^{n-3}(1 + 1.5^2 + 0.1) = 1.5^{n-3}(1 + 2.25 + 0.1)
\]
Chapter 33.9. Exhaustive Search

Let \( T(n) \leq T(n - 3) + T(n - 1) + cn^2 \), with \( T(1) = O(1) \)

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\[
= 1.5^{n-3} \times 3.35
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Chapter 33.9. Exhaustive Search

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$$= 1.5^{n-3} \times 3.35 \leq 1.5^{n-3} \times 3.375$$
Let $T(n) \leq T(n - 3) + T(n - 1) + cn^2$, with $T(1) = O(1)$

Claim: $T(n) = O(1.5^n)$

Proof (use the substitution method)

Assume that $T(k) \leq 1.5^k$ for all $k < n$. Then

$$T(n) \leq T(n - 3) + T(n - 1) + n^2 \leq 1.5^{n-3} + 1.5^{n-1} + n^2$$

when $n > n_0$ ($n_0$ to be determined)

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$$= 1.5^{n-3} \times 3.35 \leq 1.5^{n-3} \times 3.375 = 1.5^{n-3} \times 1.5^3$$
Chapter 33.9. Exhaustive Search

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Chapter 33.9. Exhaustive Search

Let $T(n) \leq T(n - 3) + T(n - 1) + cn^2$, with $T(1) = O(1)$

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Now we decide $n_0$:

$$\frac{n^2}{1.5^{n-3}} \leq 0.1$$
Chapter 33.9. Exhaustive Search

Let \( T(n) \leq T(n - 3) + T(n - 1) + cn^2 \), with \( T(1) = O(1) \)

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\[
\frac{n^2}{1.5^{n-3}} \leq 0.1 \implies n^2 \leq 0.1 \times 1.5^{n-3}
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Chapter 33.9. Exhaustive Search

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Now we decide $n_0$:

$$\frac{n^2}{1.5^{n-3}} \leq 0.1 \implies n^2 \leq 0.1 \times 1.5^{n-3} \text{ holds when roughly } n \geq n_0 = 29$$
Chapter 33.9. Exhaustive Search

- Algorithms for SAT and \texttt{MAXINDS}\texttt{SET} run in exponential time $O(2^n)$ or $O(\gamma^n)$ for $1 < \gamma < 2$. 
Chapter 33.9. Exhaustive Search

- Algorithms for SAT and MAXINDSET run in exponential time $O(2^n)$ or $O(\gamma^n)$ for $1 < \gamma < 2$
- Search tree (solution search space) is large, inherently large
Chapter 33.9. Exhaustive Search

- Algorithms for SAT and MAXINDSET run in exponential time $O(2^n)$ or $O(\gamma^n)$ for $1 < \gamma < 2$
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- Search tree does not have obvious overlapping subproblems,
Chapter 33.9. Exhaustive Search

- Algorithms for SAT and \textsc{MaxIndSet} run in exponential time $O(2^n)$ or $O(\gamma^n)$ for $1 < \gamma < 2$
- search tree (solution search space) is large, \textit{inherently} large
- search tree does not have obvious overlapping subproblems, which otherwise would incur dynamic programming approaches.
Chapter 34. NP-Completeness
Chapter 34. NP-Completeness
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Chapter 34 NP-Completeness

1. Intractable problems
Chapter 34. NP-Completeness

Chapter 34 NP-Completeness

1. Intractable problems
   • investigating decision problems suffice
Chapter 34. NP-Completeness

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2. NP model
Chapter 34. NP-Completeness

Chapter 34 NP-Completeness

1. Intractable problems
   • investigating decision problems suffice

2. NP model
   • how to show a problem is in class NP
Chapter 34 NP-Completeness

1. Intractable problems
   • investigating decision problems suffice

2. NP model
   • how to show a problem is in class NP

3. NP-completeness framework
Chapter 34. NP-Completeness

Chapter 34 NP-Completeness

1. Intractable problems
   - investigating decision problems suffice

2. NP model
   - how to show a problem is in class NP

3. NP-completeness framework
   - polynomial-time reduction and NP-completeness proof.
Chapter 34. NP-Completeness

1. Intractable problems
Chapter 34. NP-Completeness

1. Intractable problems
   - we have seen many problems solvable in polynomial time
Chapter 34. NP-Completeness

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- we have seen many problems solvable in polynomial time
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Chapter 34. NP-Completeness

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Chapter 34. NP-Completeness

1. Intractable problems

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• there are problems that do not seem to have polynomial time algorithms
  i.e., not solvable in time $O(n \log n)$, $O(n^3)$, or
1. Intractable problems

- we have seen many problems solvable in polynomial time e.g., sorting, SCC, MST
- there are problems that do not seem to have polynomial time algorithms i.e., not solvable in time $O(n \log n)$, $O(n^3)$, or $O(n^{100})$. 

 why would a time $O(n^{100})$-time algorithm be attractive?

only theoretical? practical significance as well
Chapter 34. NP-Completeness

1. Intractable problems

- we have seen many problems solvable in polynomial time
e.g., sorting, SCC, MST

- there are problems that do not seem to have polynomial time algorithms
  i.e., not solvable in time $O(n \log n)$, $O(n^3)$, or $O(n^{100})$.

- why would a time $O(n^{100})$-time algorithm be attractive?
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Chapter 34. NP-Completeness

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Chapter 34. NP-Completeness

Define: a Hamiltonian cycle in a graph is a circular path going through every vertex exactly once. Different from Eulerian cycle that goes through every edge exactly once.
Chapter 34. NP-Completeness

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Chapter 34. NP-Completeness

Travel Salesman Problem (TSP)

Input: an edge-weighted graph $G = (V, E)$;
Output: a Hamiltonian cycle of the minimum weight sum.

- intuitively, a circular path is a permutation of $(v_1, v_2, ..., v_n)$ or simply a permutation of $(1, 2, ..., n)$, where $|V| = n$.

so the problem has time upper bound $O(n! |E|)$, exponential time.

$n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1 \geq n \times (n-1) \times \cdots \times n^2 \geq (n^2)^n$.

- all known algorithms (solving TSP) are of exponential-time.
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Chapter 34. NP-Completeness

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Instead of considering the Travel Salesman Problem (TSP)
Input: an edge-weighted graph $G = (V,E)$
Output: a Hamiltonian cycle of the minimum weight sum.

We may consider a related problem: H-Cycle Weight Decision (HCW)
Input: an edge-weighted graph $G = (V,E)$, a weight value $K$
Output: "YES" if and only there is a Hamiltonian cycle of weight $\leq K$ in $G$.

• HCW appears to "easier" than TSP as an H-cycle is not produced in the answer.
• However, HCW may not be "easier"

Theorem 1: HCW is solvable in P-time if and only if TSP is solvable in P-time.
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Trivially, P-time algorithms for TSP
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How to prove: P-time algorithms for TSP $\iff$ P-time algorithms for HCW?
Theorem 1: HCW is solvable in P-time if and only if TSP is solvable in P-time.

Proof: (P-time algorithms for TSP $\iff$ P-time algorithms for HCW)
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**Proof**: (P-time algorithms for TSP \(\iff\) P-time algorithms for HCW)

- Assume P-time algorithm \(A\) for HCW such that \(A(G, K) = \text{"YES/"NO"}\)
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- assume P-time algorithm $A$ for HCW such that $A(G, K) = \text{"YES/"NO}$
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   How to make Step 1 P-time?
Chapter 34. NP-Completeness

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2. mark all edges in $G$ as “unvisited”;
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How to make Step 1 P-time?

Theorem 1 says problems HCW and TSP are “polynomially equivalent”.
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Chapter 34. NP-Completeness

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Chapter 34. NP-Completeness

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Chapter 34. NP-Completeness

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      else mark $(u, v)$ “critical”;

How to make Step 1 P-time?

Theorem 1 says problems HCW and TSP are “polynomially equivalent.”
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Proof: (P-time algorithms for TSP ⇐ P-time algorithms for HCW)

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   return (all “critical” edges)
Chapter 34. NP-Completeness

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- show algorithm $B$ runs in P-time.
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Theorem 1 says problems HCW and TSP are “polynomially equivalent”.
Consider another related problem:

$H$-Cycle Decision ($HC$)

Input: an edge-weighted graph $G = (V,E)$

Output: "YES" if and only if there is a Hamiltonian cycle in $G$.

$H$-Cycle Weight Decision ($HCW$)

Input: an edge-weighted graph $G = (V,E)$, a weight value $K$

Output: "YES" if and only if there is a Hamiltonian cycle of weight $\leq K$ in $G$.

• Which problem is seemingly "easier"?

Theorem 2: $HCW$ is P-time solvable if and only if $HC$ is P-time solvable.
Can you prove it?

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**H-Cycle Decision (HC)**
- **Input**: an edge-weighted graph $G = (V, E)$;
- **Output**: “YES” if and only there is a Hamiltonian cycle in $G$.

Compared with

**H-Cycle Weight Decision (HCW)**
- **Input**: an edge-weighted graph $G = (V, E)$, a weight value $K$;
- **Output**: “YES” if and only there is a Hamiltonian cycle of weight $\leq K$ in $G$.

- Which problem is seemingly “easier”?

**Theorem 2**: HCW is P-time solvable if and only if HC is P-time solvable.

  Can you prove it?

Theorem 2 says problems HCW and HC are “polynomially equivalent”.
Corollary 3: Problems TSP, HCW, and HC are all “polynomially equivalent”.

Max Independent Set (MaxIS)
Input: graph $G = (V,E)$;
Output: an independent set of vertices of the maximum size;

Independent Set (IS)
Input: graph $G = (V,E)$, integer $k$;
Output: “YES” if and only if $G$ has an independent set of size $\geq k$.

Theorem 4: $\text{MaxIS}$ is P-time solvable if and only if $\text{IS}$ is P-time solvable.
Can you prove the theorem?
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**Corollary 3:** Problems TSP, HCW, and HC are all “polynomially equivalent”.

There are other problems that have the similar situation.
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**Chapter 34. NP-Completeness**

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Can you prove the theorem?
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Similarly,
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**Min Vertex Cover (MinVC)**

**Input**: graph $G = (V, E)$;

**Theorem 5**: \( \text{MinVC} \) is P-time solvable if and only if \( \text{VC} \) is P-time solvable.

Can you prove the theorem?
Similarly,

**Min Vertex Cover (MinVC)**

**Input:** graph $G = (V, E)$;

**Output:** a vertex cover set of vertices of the minimum size;

**Theorem 5:**

MinVC is P-time solvable if and only if VC is P-time solvable.

Can you prove the theorem?
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Can you prove the theorem?
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Conclusions:

"Polynomial equivalency" can be established between optimization problems and decision problems. To study tractability of optimization problems, often it suffices to investigate decision problems. (Decision problems are also called languages.) One language can be defined for one decision problem, let $\Sigma = \{0, 1\}$:

$L_{HCW} = \{x \in \Sigma^*: x$ encodes $\langle G, k \rangle, G$ has a H-cycle of weight $\leq k\}$

Answering "yes" or "no" to input $\langle G, k \rangle$ for problem $HCW \iff$ answering "yes" or "no" to language membership question $x \in L_{HCW}$?
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Chapter 34. NP-Completeness

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Chapter 34. NP-Completeness

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3. However, “Polynomial equivalency” does not tell us the tractability of the problems.
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   **Corollary 6**: VC is P-time solvable \textbf{if and only if} IS is P-time solvable.

3. However, “Polynomial equivalency” does not tell us the tractability of the problems.

4. We need a rigorous framework to study tractability via the notion “Polynomial equivalency”.


Chapter 34. NP-Completeness

2. Nondeterministic algorithms
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Deterministic algorithms
Chapter 34. NP-Completeness

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Deterministic algorithms

- Given input data, a deterministic algorithm has its every step completely determined by the algorithm and data.
- All algorithms we have seen so far are deterministic.
- Every deterministic algorithm can be unfolded into a linear sequence of steps (when the input is given).

\[ M = -\infty \]
\[ n = 3 \]
\[ i = 1 \]
check 1 \leq 3
check \(-\infty < 10\)
\[ M = 10 \]
\[ i = 2 \]
check 2 \leq 3
check 10 < 30
\[ M = 30 \]
\[ i = 3 \]
check 3 \leq 3
check 30 < 20
\[ i = 4 \]
check 4 \leq 3
return (30)

MaxOfList(L)
1. \[ M = -\infty \]
2. \[ n = \text{length}(L) \]
3. for \( i = 1 \) to \( n \)
4. \[ \text{if} \ M < L[i] \]
5. \[ M = L[i] \]
6. return (M)

Unfolded when input \( L = (10, 30, 20) \)
A deterministic algorithm can be thought of a linear path of steps;

In a nondeterministic algorithm, when unfolded, there may be more than one possible successor.

A nondeterministic algorithm can be thought of a tree of steps.

Each step has more than one nondeterministic choice for its successor.

A path from root to a leaf is a sequence of nondeterministic choices; thus a nondeterministic execution of the algorithm.

The algorithm answers "YES" if one execution path leads to "YES".

The running time is the number of steps on a longest path.

If running time is $m$ steps, there may be $2^m$ paths.

Let us call this tree model of nondeterministic algorithms.
Chapter 34. NP-Completeness

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Chapter 34. NP-Completeness

Deterministic

\[ f(n) \rightarrow \cdots \rightarrow \text{yes or no} \]

Non Deterministic

\[ f(n) \rightarrow \cdots \rightarrow \text{yes or no} \]
Chapter 34. NP-Completeness

Use nondeterministic algorithms to solve problem SAT
Use nondeterministic algorithms to solve problem SAT in polynomial time.
Use **nondeterministic algorithms** to solve problem SAT in polynomial time.

Algorithm **NonDetSAT-Solver**

*Input: $\phi(x_1, \ldots, x_n)$*

1. Let $\phi_0 = \phi(x_1, \ldots, x_n)$
2. for $i = 1$ to $n$
3.  nondeterministically let $a_i = 0$ or $a_i = 1$;
4.  $\phi_i = \phi_{i-1}(x_i = a_i)$
5.  if ($\phi_n == 1$)  
6.        return YES
7.  else
8.        return NO
Use nondeterministic algorithms to solve problem SAT in polynomial time.

Algorithm NonDetSAT-Solver
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Algorithm NonDetSAT-Solver-1
Input: \( \phi(x_1, \ldots, x_n) \)

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3. if (\( \phi(a_1, \ldots, a_n) == 1 \))
4.   return Yes
5. else
6.   return No
Chapter 34. NP-Completeness

Algorithm \textsc{NonDetSAT-Solver-1}
\textbf{Input:} $\phi(x_1, \ldots, x_n)$
1. \textbf{for} $i = 1$ \textbf{to} $n$
2. \quad nondeterministically let $a_i = 0$ or $a_i = 1$;
3. \quad if ($\phi(a_1, \ldots, a_n) == 1$)
4. \quad \quad \textbf{return} \texttt{YES}
5. \quad \textbf{else}
6. \quad \quad \textbf{return} \texttt{NO}
Chapter 34. NP-Completeness

Algorithm NonDetSAT-Solver-1
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4. **return** YES
5. **else**
6. **return** NO

Answer is YES iff $\exists(a_1, \ldots, a_n)$
Chapter 34. NP-Completeness

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2. nondeterministically let \( a_i = 0 \) or \( a_i = 1 \);
3. **if** \( (\phi(a_1, \ldots, a_n) == 1) \)
4. **return** \( \text{YES} \)
5. **else**
6. **return** \( \text{NO} \)

Answer is \( \text{YES} \) **iff** \( \exists (a_1, \ldots, a_n) \ \phi(a_1, \ldots, a_n) = 1 \).
Chapter 34. NP-Completeness

1. Answer is Yes iff \( \exists (a_1, \ldots, a_n), \varphi(a_1, \ldots, a_n) = 1 \).

2. \( \varphi(x_1, \ldots, x_n) \) is satisfiable iff \( \exists \) witness \( (a_1, \ldots, a_n) \), \( \varphi(a_1, \ldots, a_n) = 1 \) can be verified.

3. \( \varphi(x_1, \ldots, x_n) \) is satisfiable iff \( \exists \) witness \( (a_1, \ldots, a_n) \), \( \mathcal{V}(\varphi, a_1, \ldots, a_n) \) can be verified to true in P-time.
Chapter 34. NP-Completeness

(1) Answer is \textbf{Yes}
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Chapter 34. NP-Completeness

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Chapter 34. NP-Completeness

(1) Answer is *YES* iff $\exists (a_1, \ldots, a_n)$, $\phi(a_1, \ldots, a_n) = 1$.

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(1) Answer is \textbf{Yes} iff \( \exists (a_1, \ldots, a_n), \phi(a_1, \ldots, a_n) = 1 \).

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   V(\phi, a_1, \ldots, a_n) \) can be verified to \textbf{true} in \textit{P-time}
Use nondeterministic algorithms to solve problem Hamiltonian Cycle.
Chapter 34. NP-Completeness

Use nondeterministic algorithms to solve problem Hamiltonian Cycle in polynomial time.

• The algorithm will answer "YES" iff there is a H-cycle in $G$.

What does the nondeterministic computation tree look like?
Use nondeterministic algorithms to solve problem **Hamiltonian Cycle** in polynomial time.

(1) starting from any vertex \( v \) in the graph;
Chapter 34. NP-Completeness

Use nondeterministic algorithms to solve problem Hamiltonian Cycle in polynomial time.

(1) starting from any vertex \( v \) in the graph;
(2) nondeterministically choose one of its (at most \( n - 1 \)) neighbors which has not been chosen;
Chapter 34. NP-Completeness

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(1) starting from any vertex $v$ in the graph;
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Chapter 34. NP-Completeness

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3. if all vertices have been chosen,
   return “YES” if these vertices are connected to form an H-cycle;
   return “NO”, otherwise;
Chapter 34. NP-Completeness

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Chapter 34. NP-Completeness

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- The algorithm will answer “YES” iff there is a H-cycle in \( G \).
- **What does the nondeterministic computation tree look like?**
Problems like Independent Set, Vertex Cover, HCW can all be solved with nondeterministic algorithms in polynomial time.
Problems like Independent Set, Vertex Cover, HCW can all be solved with nondeterministic algorithms in polynomial time.

Can you prove the claim?
Chapter 34. NP-Completeness

**Definition:** \( \mathcal{P} \) is the class of languages (i.e., decision problems) that can be solved by deterministic polynomial-time algorithms.
Chapter 34. NP-Completeness

**Definition**: $\mathcal{P}$ is the class of languages (i.e., decision problems) that can be solved by deterministic polynomial-time algorithms.

- class $\mathcal{P}$ contains problems like **Reachability**
Chapter 34. NP-Completeness

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Chapter 34. NP-Completeness

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Chapter 34. NP-Completeness
We consider the tree model of nondeterministic algorithms.

We may assume each step has exactly 2 nondeterministic choices (5 choices can be simulated with 4 nondeterministic steps). Each nondeterministic path can be represented with a binary string: 0 for branching left, 1 for right. We can assume the algorithm does all nondeterministic choices before other operations. So we can model the computation as (1) first choose a binary string nondeterministically, and (2) follow the specified path deterministically. The binary string is called certificate or witness; the deterministic computation part is called verification. Deterministic algorithms are when the certificate is empty.
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Every nondeterministic polynomial time computation is

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Let $\Pi \in \text{NP}$. Then there is a deterministic algorithm $A_{\Pi}$, and a constant $c > 0$, such that

1. if $x$ is a positive instance of $\Pi$, there is a binary string $y$ of length $n^c$, $A_{\Pi}(x, y) = \text{"YES"}$;
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and $A_{\Pi}$ runs in time $O(n^c)$.

We call $y$ a certificate/witness and $A_{\Pi}$ the verification algorithm.

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Chapter 34. NP-Completeness

Definition of $NP$ in terms of languages:

Let $L \subseteq \{0, 1\}^*$ be a language in the class $NP$. Then there is a deterministic algorithm $A_L$, and a constant $c > 0$, such that, for every $x \in \{0, 1\}^*$, $x \in L \iff \exists y, |y| \leq |x|^c, A_L(x, y) = 1$ and $A_L$ runs in polynomial time.

$A_L$ runs in polynomial time of what? in $m = |x, y| = |x| + |y| \leq n + nc$.

So if $A_L$ runs in polynomial time $m^d \leq (n + nc)^d \leq (2nc)^d = O(n^dc)$, also polynomial time of $n = |x|$. 

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Chapter 34. NP-Completeness

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Chapter 34. NP-Completeness
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non-det. moves

det. moves

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choose a certificate

$y = 10\ldots11\ldots$

det. algo.

A for verification
Proof that $HC \in \mathcal{NP}$.

• Certificate $y$ represents a sequence of ordered vertices;
• Algorithm $A$ is to verify that $y$ does form a H-cycle.

Details:
• $y = B_1B_2...B_n$, where $B_i$ is a binary representation of some vertex in $G$;
• How many bits does $B_i$ need?
  \[ \lceil \log_2 n \rceil \]
• Whether $y$ forms a H-cycle can be verified in time $O(|E|)$. 

\[ \text{Chapter 34. NP-Completeness} \]
Chapter 34. NP-Completeness

Proof that $HC \in \mathcal{NP}$.

We need to show there is a deterministic algorithm $A$ and a constant $c > 0$, such that for any $G$,

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$$G \in HC \iff \exists \ y, |y| \leq |G|^c, A(G, y) = \text{"YES"}$$

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Chapter 34. NP-Completeness

exercises:

Proof that Independent Set $\in$ NP.

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Notes
1. To prove a language is in the class NP by no means to prove that the language can be solved in polynomial time. Instead, it only shows the language is in the class NP.
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Chapter 34. NP-Completeness

3. NP-Completeness Framework

The notion of reduction (i.e., transformation) between languages:

- We use languages for decision problems.
- A language contains positive instances of the corresponding decision problem.

Define $L = \{ x : x \not\in L \}$ called the complement of $L$.

$L \cup \overline{L} = \{ 0, 1 \}^*$ called the universe.
3. NP-Completeness Framework

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Chapter 34. NP-Completeness

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Because the two problems are very relevant to each other, we have:

Theorem: \( L_{IS} \leq L_{VC} \)

Proof: we use the fact that complement set of an independent set is a vertex cover in the same graph.

We construct a mapping \( f \) that maps instance \( \langle G, k \rangle \) to instance \( \langle G, |G| - k \rangle \), i.e., \( f(\langle G, k \rangle) = \langle G, |G| - k \rangle \).

This is a reduction from \( L_{IS} \) to \( L_{VC} \).

Claim: \( G \) has an i.s. of size \( \geq k \) \( \iff \) \( G \) has an v.c. of size \( \leq |G| - k \).

(proof of the claim is on the next slide)

So \( L_{IS} \leq L_{VC} \).
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Chapter 34. NP-Completeness

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Let $G$ be such that has vertices $V = \{v_1, \ldots, v_n\}$.

Assume that $G$ has a i.s. of size $k_0$ for some $k_0 \geq k$. We further assume, without loss of generality, the i.s include vertices $\{v_1, \ldots, v_{k_0}\}$.
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Then vertices $\{v_{k_0+1}, \ldots, v_n\}$ form a v.c. for $G$. 

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Assume that $G$ has a i.s. of size $k_0$ for some $k_0 ≥ k$.
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Suppose otherwise, $\exists$ edge $(u, v)$ that is not covered, i.e.,
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Can you prove “⇐” ??
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An important motivation for reduction:
Chapter 34. NP-Completeness

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- a reduction transforms instances of the first problem to the instances of the second problem;
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So the combined algorithm (gray-color box) solves for $L_1$. 
Formally,
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A polynomial-time reduction from $L_1$ to $L_2$, denoted as $L_1 \leq_p L_2$, is some mapping function $f : \{0, 1\}^* \rightarrow \{0, 1\}^*$,
Chapter 34. NP-Completeness

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A polynomial-time reduction from \( L_1 \) to \( L_2 \), denoted as \( L_1 \leq_p L_2 \), is some mapping function \( f : \{0,1\}^* \rightarrow \{0,1\}^* \), such that

\[ x \in L_1 \iff f(x) \in L_2 \]

where \( f \) can be computed in time \( O(|x|^c) \) for some fixed \( c > 0 \). For example, \( L_{IS} \leq_p L_{V} \).
Chapter 34. NP-Completeness

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Chapter 34. NP-Completeness

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**Chapter 34. NP-Completeness**
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Chapter 34. NP-Completeness

**Theorem:** Let $L_1 \leq_P L_2$. If $L_2 \in \mathcal{P}$, then $L_1 \in \mathcal{P}$. 
Chapter 34. NP-Completeness

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**Proof:** Assume algorithm $F$ computes $f$, and algorithm $A_2$ solves for $L_2$. We need to show that the gray box runs in polynomial time if both $F$ and $A_2$ runs in polynomial time: Total time is the sum of time for $F$ and time for $A_2$.

$$O(|x|^c) + O(|f(x)|^d)$$

Now, what is the length of $f(x)$?

Because $F$ runs in time $O(|x|^c)$, the number of bits outputted by $F$ is $O(|x|^c)$.

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![Diagram of the gray box model](image)
Chapter 34. NP-Completeness

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Chapter 34. NP-Completeness

**Theorem:** Polynomial-time reductions compose (are transitive). That is, if $L_1 \leq_p L_2$ and $L_2 \leq_p L_3$, then $L_1 \leq_p L_3$.

Proof. Assume functions $f$ for $L_1 \leq_p L_2$; function $h$ for $L_2 \leq_p L_3$. For every $x \in \{0, 1\}^*$, $x \in L_1 \iff f(x) \in L_2 \iff h(f(x)) \in L_3$. That is $x \in L_1 \iff h(f(x)) \in L_3$. So the composite function $(h \circ f)$ realizes reduction $L_1 \leq_p L_3$.

But we need to show the reduction is $\leq_p$, i.e., a polynomial time reduction. Assume that algorithm $F$ computes $f$: $F(x) = f(x)$ in time $O(|x|^c)$ and algorithm $H$ computes $h$: $H(y) = h(y)$ in time $O(|y|^d)$. Let $y = f(x)$, the total time for computing $(h \circ f) = \text{time of } F \text{ and time of } H = O(|x|^c) + O(|f(x)|^d) = O(|x|^c) + O(|x|^cd) = O(|x|^{c+d})$. So $L_1 \leq_p L_3$. 
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Chapter 34. NP-Completeness

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Some conclusions:

• Using \( \leq_p \), languages in \( NP \) can be ordered partially;
• If those languages at the end of a \( \leq_p \) chain have polynomial-time algorithms, so does every language on the chain.
• Informally, those at the end of a \( \leq_p \) chain are called NP-hard.
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Chapter 34. NP-Completeness

Definition 1: \( L \) is NP-hard

Definition 2: \( L \) is NP-complete if (1) \( L \) is NP-hard and (2) \( L \) ∈ NP.
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**Properties of NP-hard problems**

- If $L$ is NP-hard and $L \in \mathcal{P}$, then $\mathcal{P} = \mathcal{NP}$.
  
  **Proof?**

- If $L$ is NP-hard and $L \leq_p L'$, then $L'$ is NP-hard.
  
  **Proof?**

**How to prove a language is NP-hard?**


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Definition 1: $L$ is **NP-hard** if for every language $L' \in NP$, $L' \leq_p L$.

Definition 2: $L$ is **NP-complete** if (1) $L$ is NP-hard and (2) $L \in NP$.

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Chapter 34. NP-Completeness

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Chapter 34. NP-Completeness

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Chapter 34. NP-Completeness

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Chapter 34. NP-Completeness

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**How to prove a language is NP-hard?**
Chapter 34. NP-Completeness

4. NP-Completeness Proofs
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Chapter 34. NP-Completeness

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- Instead, formulate a generic language that represents all languages in NP and prove that every language in $\mathcal{NP}$ can be reduced to the generic language in polynomial time.

- To obtain such a generic language, we need to consider the definition of languages in NP.
Chapter 34. NP-Completeness

Recall the definition of languages in \( \mathcal{NP} \):
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Let $L \subseteq \{0, 1\}^*$ be any language in the class $NP$. Then there is a deterministic algorithm $A_L$,
Chapter 34. NP-Completeness

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Chapter 34. NP-Completeness

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where $L_{tbd}$ is a language to be defined.
Chapter 34. NP-Completeness

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where $L_{tbd}$ is a language to be defined.

Can we identify $L_{tbd}$ and $f$?
Again we examine

\[ x \in L \iff \exists y, |y| \leq |x|^c, A_L(x, y) = 1 \]  

(1)
Chapter 34. NP-Completeness

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\[ x \in L \iff \exists y, |y| \leq |x|^c, A_L(x, y) = 1 \quad (1) \]

- \( A_L \) is a deterministic algorithm can be implemented with a boolean circuit \( B_L \) with two sets of input gates \( x = x_1 x_2 \ldots x_n \) and \( y = y_1 y_2 \ldots y_m \) such that

\[ A_L(x, y) = 1 \text{ if and only if } B_L(x, y) = 1 \quad (2) \]
Chapter 34. NP-Completeness

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\[ x \in L \iff \exists y, |y| \leq |x|^c, A_L(x, y) = 1 \]  \hspace{1cm} (1)

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Chapter 34. NP-Completeness

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\[ C^x_L \]
Chapter 34. NP-Completeness

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- Because \( x \) is given, circuit \( B_L \) can be made into circuit \( C^x_L \) such that

\[ B_L(x, y) = 1 \text{ if and only if } C^x_L(y) = 1 \]  \hspace{1cm} (3)
Chapter 34. NP-Completeness

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\[ B_L(x, y) = 1 \text{ if and only if } C_L^x(y) = 1 \quad (3) \]

- From (1), (2), and (3), we have

\[ x \in L \iff \exists y C_L^x(y) = 1 \quad (4) \]
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Now we have

\[ x \in L \iff \exists y \ C^x_L(y) = 1 \]  \hspace{1cm} (5)

• Define: a boolean circuit \( C \) is satisfiable if there exists at least one set of values \( y \) to its input gates such that \( C(y) = 1 \).

e.g., \( C(x_1, x_2) = (\neg x_1 \lor \neg x_2) \land (x_1 \lor x_2) \) is satisfiable; but \( D(x_1, x_2) = (x_1 \lor x_2) \land (x_1 \lor \neg x_2) \land (\neg x_1 \lor x_2) \land (\neg x_1 \lor \neg x_2) \) is not!

• Define the following language:

\[ CSAT = \{ C : \text{circuit } C \text{ is satisfiable} \} \]

• From (4), we have

\[ x \in L \iff C^x_L \in CSAT \]  \hspace{1cm} (6)

It remains to be shown

• that reducing algorithm \( A_L \) to circuit \( B_L \) is valid; and
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Chapter 34. NP-Completeness

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Chapter 34. NP-Completeness

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Chapter 34. NP-Completeness

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• that the reduction can be done in **polynomial time**.
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Unfold deterministic polynomial-time algorithm $A(x, y)$ with input $\langle x, y \rangle$
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Unfold deterministic polynomial-time algorithm \( A(x, y) \) with input \( \langle x, y \rangle \)
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The algorithm is physically implemented with a boolean circuit
Chapter 34. NP-Completeness

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Chapter 34. NP-Completeness

The algorithm is physically implemented with a boolean circuit.

And the circuit can be built from the algorithm in polynomial time.
The above discussion shows that $L_{CSAT}$ is NP-hard.
Chapter 34. NP-Completeness

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**Theorem:** Language $CSAT$ is NP-complete.
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**Proof:** It suffices to show the $CSAT$ is in NP. (Can you prove this?)
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**Theorem**: Language $CSAT$ is NP-complete.

**Proof**: It suffices to show the $CSAT$ is in NP. (Can you prove this?)

Actually, the following language SAT was first proved to be NP-complete [Cook’71] https://www.cs.toronto.edu/~sacook/homepage/1971.pdf

$$SAT = \{ \phi : \text{CNF boolean formula } \phi \text{ is satisfiable} \}$$
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**Cook’s Theorem**: SAT is NP-complete.

Cook’s reduction: characterizing a polynomial-time computation on nondeterministic Turing machine with a boolean formula, such that a nondeterministic path leading to the accept state corresponds to an assignment to the variables making the the formula TRUE.
It is very easy to convert a boolean formula to a boolean circuit. So
Chapter 34. NP-Completeness

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**Theorem:** $SAT \leq_p CSAT$. 
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how to convert a circuit to a boolean formula (from network to tree)?
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**how to convert a circuit to a boolean formula** (from network to tree)? simply replicating gates may blow-up the size of formula to exponential!
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**Theorem:** \( CSAT \leq_p SAT \).

is satisfiable if and only if formula \( \phi \) is satisfiable:
Chapter 34. NP-Completeness

**Theorem:** $CSAT \leq_p SAT$.

is satisfiable if and only if formula $\phi$ is satisfiable:

\[
\phi = x_{10} \land (x_4 \leftrightarrow \neg x_3) \\
\land (x_5 \leftrightarrow (x_1 \lor x_2)) \\
\land (x_6 \leftrightarrow \neg x_4) \\
\land (x_7 \leftrightarrow (x_1 \land x_2 \land x_4)) \\
\land (x_8 \leftrightarrow (x_5 \lor x_6)) \\
\land (x_9 \leftrightarrow (x_6 \lor x_7)) \\
\land (x_{10} \leftrightarrow (x_7 \land x_8 \land x_9)).
\]
Chapter 34. NP-Completeness

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$\phi$ can be transformed to an equivalent CNF formula.
Chapter 34. NP-Completeness

Landscape of NP problems and beyond
Chapter 34. NP-Completeness

Landscape of NP problems and beyond
Many problems/languages have been proved NP-complete (Karp70s)
Chapter 34. NP-Completeness

Examples of reduction techniques
Examples of reduction techniques

Example 1: SAT $\leq_p$ 3SAT
Examples of reduction techniques

Example 1: SAT $\leq_P 3$SAT

($z$)
Chapter 34. NP-Completeness

Examples of reduction techniques

Example 1: $\text{SAT} \leq_p \text{3SAT}$

$($z$) \rightarrow (z, x_1, x_2) \land (z, x_1, \neg x_2) \land (z, \neg x_1, x_2) \land (z, \neg x_1, \neg x_2)$
Examples of reduction techniques

Example 1: SAT $\leq_p$ 3SAT

$$(z) \iff (z, x_1, x_2) \land (z, x_1, \neg x_2) \land (z, \neg x_1, x_2) \land (z, \neg x_1, \neg x_2)$$

$$(y, z)$$
Examples of reduction techniques

Example 1: $\text{SAT} \leq_p \text{3SAT}$

\[
\begin{align*}
(z) &\rightarrow (z, x_1, x_2) \land (z, x_1, \neg x_2) \land (z, \neg x_1, x_2) \land (z, \neg x_1, \neg x_2) \\
(y, z) &\rightarrow (y, z, x_1) \land (y, z, \neg x_1)
\end{align*}
\]
Examples of reduction techniques

Example 1: SAT $\leq_p$ 3SAT

$$(z) \implies (z, x_1, x_2) \land (z, x_1, \neg x_2) \land (z, \neg x_1, x_2) \land (z, \neg x_1, \neg x_2)$$

$$(y, z) \implies (y, z, x_1) \land (y, z, \neg x_1)$$

$$(x, y, z)$$
Examples of reduction techniques

Example 1: SAT \leq_p 3SAT

\((z) \rightarrow (z, x_1, x_2) \land (z, x_1, \neg x_2) \land (z, \neg x_1, x_2) \land (z, \neg x_1, \neg x_2)\)

\((y, z) \rightarrow (y, z, x_1) \land (y, z, \neg x_1)\)

\((x, y, z) \rightarrow (x, y, z)\)
Chapter 34. NP-Completeness

Examples of reduction techniques

Example 1: SAT $\leq_p$ 3SAT

$\begin{align*}
(z) &\iff (z, x_1, x_2) \land (z, x_1, \neg x_2) \land (z, \neg x_1, x_2) \land (z, \neg x_1, \neg x_2) \\
(y, z) &\iff (y, z, x_1) \land (y, z, \neg x_1) \\
(x, y, z) &\iff (x, y, z) \\
(y, z, u, v) &\iff (y, z, x_1) \land (\neg x_1, u, v)
\end{align*}$
Chapter 34. NP-Completeness

Examples of reduction techniques

Example 1: SAT \( \leq_p \) 3SAT

\((z) \rightarrow (z, x_1, x_2) \land (z, x_1, \neg x_2) \land (z, \neg x_1, x_2) \land (z, \neg x_1, \neg x_2)\)

\((y, z) \rightarrow (y, z, x_1) \land (y, z, \neg x_1)\)

\((x, y, z) \rightarrow (x, y, z)\)

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Examples of reduction techniques

Example 1: SAT \( \leq_p \) 3SAT

\[
\begin{align*}
(z) & \implies (z, x_1, x_2) \land (z, x_1, \neg x_2) \land (z, \neg x_1, x_2) \land (z, \neg x_1, \neg x_2) \\
(y, z) & \implies (y, z, x_1) \land (y, z, \neg x_1) \\
(x, y, z) & \implies (x, y, z) \\
(y, z, u, v) & \implies (y, z, x_1) \land (\neg x_1, u, v) \\
(y, z, u, v, w) &
\end{align*}
\]
Examples of reduction techniques

Example 1: $SAT \leq_p 3SAT$

\[
\begin{align*}
(z) & \iff (z, x_1, x_2) \land (z, x_1, \neg x_2) \land (z, \neg x_1, x_2) \land (z, \neg x_1, \neg x_2) \\
(y, z) & \iff (y, z, x_1) \land (y, z, \neg x_1) \\
(x, y, z) & \iff (x, y, z) \\
(y, z, u, v) & \iff (y, z, x_1) \land (\neg x_1, u, v) \\
(y, z, u, v, w) & \iff (y, z, x_1) \land (\neg x_1, u, x_2) \land (\neg x_2, v, w)
\end{align*}
\]
Chapter 34. NP-Completeness

Example 2:

3SAT \leq_p IS

An assignment TRUE to one literal in each clause corresponds to an independent set in the transformed graph.
Example 2: \(3\text{SAT} \leq_p \text{IS}\)
Chapter 34. NP-Completeness

Example 2: \( 3\text{SAT} \leq_p \text{IS} \)

\[
(x_1 \lor x_2 \lor \overline{x_3}) \land (x_2 \lor x_3 \lor \overline{x_4}) \land (x_1 \lor \overline{x_2} \lor x_4)
\]
Example 2: $3\text{SAT} \leq_p \text{IS}$

\[(x_1 \lor x_2 \lor \overline{x_3}) \land (x_2 \lor x_3 \lor \overline{x_4}) \land (x_1 \lor \overline{x_2} \lor x_4)\

An assignment TRUE to one literal in each clause
Example 2: $3\text{SAT} \leq_p \text{IS}$

\[(x_1 \lor x_2 \lor \overline{x}_3) \land (x_2 \lor x_3 \lor \overline{x}_4) \land (x_1 \lor \overline{x}_2 \lor x_4)\]

An assignment TRUE to one literal in each clause corresponds to an independent set in the transformed graph.
Summary

Scope of the Final Exam

- Dynamic programming (4 steps)
- Greedy algorithms (greedy choice property and proof)
- Depth-First-Search algorithm, DFS search tree, time stamps
- Applications: topological sort
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Scope of the Final Exam (cont’)

- Minimum spanning tree concept/properties of MST: generic, Kruskal’s, and Prim’s
- Shortest path (single source and all pairs): concept/properties of shortest path, greedy algorithms, relaxation technique
  - Single source: Bellman-Ford’s, Dijkstra’s
  - All pairs: Floyd-Warshall
Summary

Scope of the Final Exam (cont’)

▶ Minimum spanning tree
Summary

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▶ NP-completeness theory
 definitions of NP class (certificate + verification)
 proof that a language is in NP
 reduction, polynomial-time reduction, properties
 definitions of NP-hard, NP-complete languages, properties
 NP-completeness proofs (simple, limited to previously known reductions)
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Scope of the Final Exam (cont’)

▶ NP-completeness theory

definitions of NP class (certificate + verification)

proof that a language is in NP
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