Part VII. Selected Topics
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Chapter 33.9 Exhaustive Search (Covered!)
Part VII. Selected Topics

Chapter 33.9 Exhaustive Search (Covered!)
Chapter 34 NP-Completeness
Part VII. Selected Topics

Chapter 33.9 Exhaustive Search (Covered!)

Chapter 34 NP-Completeness
Chapter 33.9. Exhaustive Search

To enumerate all possible solutions to the problem instance:

- systematically examine all solutions
- without repeating solutions that have been examined
- stop when a satisfactory solution is found
Chapter 33.9. Exhaustive Search

To enumerate all possible solutions to the problem instance
Chapter 33.9. Exhaustive Search

To enumerate all possible solutions to the problem instance

How?
Chapter 33.9. Exhaustive Search

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Chapter 33.9. Exhaustive Search

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Chapter 33.9. Exhaustive Search

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First, we need to be able to count total number of “things” to be enumerated.
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Chapter 33.9. Exhaustive Search

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First, we need to be able to count total number of “things” to be enumerated.

- without missing one (correctness)
- without over-counting (efficiency)
- A sophisticated counting often has recursive solution.
Chapter 33.9. Exhaustive Search

Examples of counting:

(1) total number of permutations of \(1, 2, \ldots, n\) is \(P(n) = n \times P(n-1)\) with base case \(P(1) = 1\).

(2) total number of ways to choose \(k\) from \(n\) items is \(\binom{n}{k} = \frac{n(n-1)\ldots(n-k+1)}{k!} = \frac{n!}{(n-k)!k!}\) or, alternatively, \(\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}\) with base cases: \(\)
Chapter 33.9. Exhaustive Search

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or, alternatively,

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

with base cases: $\binom{n}{0} = 1$, $\binom{0}{k} = 0$ for $k > 0$. 
Chapter 33.9. Exhaustive Search

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Chapter 33.9. Exhaustive Search

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Chapter 33.9. Exhaustive Search

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with base cases: (?)
Example: Boolean Formula Satisfiability problem (SAT)

Input: boolean formula $f(x_1, x_2, ..., x_n)$

Output: "yes" if and only if $f(x_1, x_2, ..., x_n)$ is satisfiable.

$f(x_1, x_2, ..., x_n)$ is satisfiable if there is an assignment to boolean variables $x_i \in \{T, F\}, i = 1, 2, ..., n$ such that $f$ is evaluated to $T$.

e.g, $f(x_1, x_2, x_3) = (x_1 \lor x_2 \lor \neg x_3) \land (\neg x_1 \lor x_3)$ is satisfiable

g(x_1, x_2) = (x_1 \lor \neg x_2) \land (\neg x_1 \lor x_2) \land (x_1 \lor x_2) \land (\neg x_1 \lor \neg x_2)$ is not!
Chapter 33.9. Exhaustive Search

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\begin{align*}
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\end{align*}
\]
Chapter 33.9. Exhaustive Search

Use exhaustive search to solve the SAT problem.

Input: boolean formula \( f(x_1, x_2, \ldots, x_n) \)

Output: “yes” if and only if \( f(x_1, x_2, \ldots, x_n) \) is satisfiable.

How?

What will you exhaustively search on?

• Enumerate all combinations of \( T \) and \( F \) for \( x_1, x_2, \ldots, x_n \).

• Can you solve it with a recursive algorithm?

• Can you solve it with an iterative algorithm?
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Chapter 33.9. Exhaustive Search

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Chapter 33.9. Exhaustive Search

Solve SAT problem with a recursive algorithm:

- what data will the recursion be applied to? boolean formula \( f(x_1, \ldots, x_n) \)
- what is the terminating (base) case? \( n=0 \), formula without variables
- what is the recursive case? 
  \[
  f(x_1, \ldots, x_{n-1}, x_n) = f(x_1, \ldots, x_{n-1}, T) \lor f(x_1, \ldots, x_{n-1}, F)
  \]

\[
  f(x_1, \ldots, x_{n-1}, T) = \Rightarrow g(x_1, \ldots, x_{n-1})
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Chapter 33.9. Exhaustive Search

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Chapter 33.9. Exhaustive Search

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Chapter 33.9. Exhaustive Search

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  $$f(x_1, \ldots, x_{n-1}, F) \implies h(x_1, \ldots, x_{n-1})$$
Chapter 33.9. Exhaustive Search

Algorithm SAT Solver\((f(x_1, \ldots, x_{n-1}, x_n))\)
Chapter 33.9. Exhaustive Search

Algorithm SAT Solver($f(x_1, \ldots, x_{n-1}, x_n)$)

1. if $n = 0$, return ($f$);
Chapter 33.9. Exhaustive Search

Algorithm SAT SOLVER\( (f(x_1, \ldots, x_{n-1}, x_n)) \)

1. \textbf{if} \( n = 0 \), \textbf{return} \( f \);
2. \textbf{else} \( g(x_1, \ldots, x_{n-1}) = f(x_1, \ldots, x_{n-1}, T) \)
Chapter 33.9. Exhaustive Search

Algorithm SAT Solver($f(x_1, \ldots, x_{n-1}, x_n)$)

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4. return (SAT Solver($g(x_1, \ldots, x_{n-1})$) \lor SAT Solver($h(x_1, \ldots, x_{n-1})$))
Chapter 33.9. Exhaustive Search

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3. \( h(x_1, \ldots, x_{n-1}) = f(x_1, \ldots, x_{n-1}, F) \)
4. \textbf{return } (\text{SAT Solver}(g(x_1, \ldots, x_{n-1})) \lor \text{SAT Solver}(h(x_1, \ldots, x_{n-1})))

Does this algorithm exhaustively search all assignments to the variables?
Algorithm SAT Solver \( f(x_1, \ldots, x_{n-1}, x_n) \)

1. if \( n = 0 \), return \( f \);
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3. \( h(x_1, \ldots, x_{n-1}) = f(x_1, \ldots, x_{n-1}, F) \)
4. return (SAT Solver\( (g(x_1, \ldots, x_{n-1})) \lor \)SAT Solver\( (h(x_1, \ldots, x_{n-1})) \))

Does this algorithm exhaustively search all assignments to the variables?

- draw a search tree based on the algorithm.
Chapter 33.9. Exhaustive Search

Algorithm SAT Solver($f(x_1, \ldots, x_{n-1}, x_n)$)

1. if $n = 0$, return ($f$);
2. else $g(x_1, \ldots, x_{n-1}) = f(x_1, \ldots, x_{n-1}, T)$
3. $h(x_1, \ldots, x_{n-1}) = f(x_1, \ldots, x_{n-1}, F)$
4. return (SAT Solver($g(x_1, \ldots, x_{n-1})$) $\lor$ SAT Solver($h(x_1, \ldots, x_{n-1})$))

Does this algorithm exhaustively search all assignments to the variables?

- draw a search tree based on the algorithm.
- what does the tree look like?
Chapter 33.9. Exhaustive Search

Algorithm SAT Solver($f(x_1, \ldots, x_{n-1}, x_n)$)

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Does this algorithm exhaustively search all assignments to the variables?

- draw a search tree based on the algorithm.
- what does the tree look like?
- what does each path mean?
Chapter 33.9. Exhaustive Search

Algorithm SAT Solver($f(x_1, \ldots, x_{n-1}, x_n)$)

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Does this algorithm exhaustively search all assignments to the variables?

- draw a search tree based on the algorithm.
- what does the tree look like?
- what does each path mean? how many paths?
Chapter 33.9. Exhaustive Search

Algorithm \texttt{SAT Solver}(f(x_1, \ldots, x_{n-1}, x_n))

1. \textbf{if} \ n = 0, \textbf{return} (f);
2. \textbf{else} \ g(x_1, \ldots, x_{n-1}) = f(x_1, \ldots, x_{n-1}, \text{T})
3. \hspace{1em} h(x_1, \ldots, x_{n-1}) = f(x_1, \ldots, x_{n-1}, \text{F})
4. \textbf{return} (\texttt{SAT Solver}(g(x_1, \ldots, x_{n-1})) \vee \texttt{SAT Solver}(h(x_1, \ldots, x_{n-1})))

Does this algorithm exhaustively search all assignments to the variables?

- draw a \textit{search tree} based on the algorithm.
- what does the tree look like?
- what does each path mean? how many paths?
- time?
Chapter 33.9. Exhaustive Search

Algorithm \texttt{SAT Solver}(f(x_1, \ldots, x_{n-1}, x_n))

1. \textbf{if} \ n = 0, \textbf{return} \ (f);
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3. \h(x_1, \ldots, x_{n-1}) = f(x_1, \ldots, x_{n-1}, \text{F})
4. \textbf{return} \ (\text{SAT Solver}(g(x_1, \ldots, x_{n-1})) \lor \text{SAT Solver}(h(x_1, \ldots, x_{n-1})))

Does this algorithm exhaustively search all assignments to the variables?

- draw a search tree based on the algorithm.
- what does the tree look like?
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- time? \( T(n) = 2T(n - 1) + cn, \ T(0) = c, \)
Algorithm \text{SAT Solver}(f(x_1, \ldots, x_{n-1}, x_n))

1. if \( n = 0 \), return \( f \);
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Does this algorithm exhaustively search all assignments to the variables?

- draw a search tree based on the algorithm.
- what does the tree look like?
- what does each path mean? how many paths?
- time? \( T(n) = 2T(n-1) + cn, \ T(0) = c, \implies T(n) = \Theta(2^n) \)
Chapter 33.9. Exhaustive Search

Solve SAT problem with iterative algorithms
Chapter 33.9. Exhaustive Search

Solve SAT problem with iterative algorithms

- How? what to iterate on?
Chapter 33.9. Exhaustive Search

Solve SAT problem with iterative algorithms

- How? what to iterate on?
  
  assignments
Chapter 33.9. Exhaustive Search

Solve SAT problem with iterative algorithms

- How? what to iterate on?
  assignments

- what is the initial value?
Chapter 33.9. Exhaustive Search

Solve SAT problem with iterative algorithms

- How? what to iterate on?
  - assignments
- what is the initial value?

\[
x_1 = F, \ x_2 = F, \ldots, \ x_n = F,
\]
Chapter 33.9. Exhaustive Search

Solve SAT problem with iterative algorithms

- How? what to iterate on?
  assignments
- what is the initial value?
  \[ x_1 = F, x_2 = F, \ldots, x_n = F, \text{ or simply } (F, F, \ldots, F) \]
Chapter 33.9. Exhaustive Search

Solve SAT problem with iterative algorithms

• How? what to iterate on?

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• what is the initial value?

  \[ x_1 = F, x_2 = F, \ldots, x_n = F, \text{ or simply } (F, F, \ldots, F) \]

• what to increment
Chapter 33.9. Exhaustive Search

Solve SAT problem with iterative algorithms

• How? what to iterate on?
  assignments

• what is the initial value?
  \( x_1 = F, x_2 = F, \ldots, x_n = F, \) or simply \((F, F, \ldots, F)\)

• what to increment
  \((\ldots, F, T, \ldots, T)\)
Chapter 33.9. Exhaustive Search

Solve SAT problem with iterative algorithms

- How? what to iterate on?
  - assignments
- what is the initial value?
  \[ x_1 = F, x_2 = F, \ldots, x_n = F, \text{ or simply } (F, F, \ldots, F) \]
- what to increment
  \[ (\ldots, F, T, \ldots, T) \rightarrow (\ldots, T, F, \ldots, F) \]
Chapter 33.9. Exhaustive Search

Solve SAT problem with iterative algorithms

- **How?** what to iterate on?

  assignments

- **what is the initial value?**

  \[ x_1 = F, x_2 = F, \ldots, x_n = F, \text{ or simply } (F, F, \ldots, F) \]

- **what to increment**

  \((\ldots, F, T, \ldots, T) \rightarrow (\ldots, T, F, \ldots, F)\)
  
  always flip the last bit.
Chapter 33.9. Exhaustive Search

1. $\text{for } \langle x_1, ..., x_n \rangle = \langle F, ..., F \rangle \text{ to } \langle T, ..., T \rangle$

2. $V = \text{Evaluate}(f, x_1, ..., x_n)$

3. if $V = T$, return $(T)$

4. return $(F)$

• For loop can be implemented by encoding vectors $\langle F, ..., F \rangle$, ..., $\langle T, ..., T \rangle$ with binary numbers then further with integers

• A decoding process is needed to converting integers back to vectors
Algorithm $\text{SAT Solver-Enum}(f(x_1, \ldots, x_{n-1}, x_n))$
Chapter 33.9. Exhaustive Search

Algorithm SAT Solver-Enum\(f(x_1, \ldots, x_{n-1}, x_n)\)

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Chapter 33.9. Exhaustive Search

Algorithm $\text{SAT Solver-Enum}(f(x_1, \ldots, x_{n-1}, x_n))$

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Chapter 33.9. Exhaustive Search

Algorithm SAT SOLVER-ENUM\( (f(x_1, \ldots, x_{n-1}, x_n)) \)

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Chapter 33.9. Exhaustive Search

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Chapter 33.9. Exhaustive Search

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Chapter 33.9. Exhaustive Search

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Chapter 33.9. Exhaustive Search

Iterative exhaustive search seems to be more convenient. Another example: Travel Salesman Problem (TSP). Related problem: Hamiltonian Cycle.

Input: a graph \( G = (V, E) \)

Output: yes if and only if \( G \) contains a Hamiltonian cycle (Hamiltonian path is a cycle going through every vertex exactly once.)

How to enumerate all cycles and validate?

- enumerate all permutations of \((1, 2, ..., n)\)
- how to encode these permutations as integers?
Chapter 33.9. Exhaustive Search

Iterative exhaustive search seems to be more convenient
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Another example: Travel Salesman Problem (TSP)
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Chapter 33.9. Exhaustive Search

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Chapter 33.9. Exhaustive Search

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- enumerate all permutations of $(12 \ldots n)$
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Chapter 33.9. Exhaustive Search

Exhaustive search could be non-trivial
Exhaustive search could be non-trivial

Maximum Independent Set

**Input:** a graph $G = (V, E)$

**Output:** a subset $I \subseteq V$ such that

1. $\forall u,v \in I, (u,v) \notin E$,
2. $|I|$ is the maximum.

- trivial exhaustive search: check every subset of $V$ and verify
- non-trivial: use a search tree, achieving a better time upper bound.
Chapter 33.9. Exhaustive Search

Exhaustive search could be non-trivial

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Chapter 33.9. Exhaustive Search

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  - use \( n \)-binary bits to encode a subset; totally \( 2^n \) subsets
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Chapter 33.9. Exhaustive Search

Exhaustive search could be non-trivial

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**Output:** a subset $I \subseteq V$ such that

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- trivial exhaustive search: check every subset of $V$ and verify
  use $n$-binary bits to encode a subset; totally $2^n$ subsets

- non-trivial: use a search tree, achieving a better time upper bound.
  taking advantage of the independent set
Chapter 33.9. Exhaustive Search

The algorithm follows a logical search tree

- Given a graph $G$, it picks an arbitrary vertex $v$ from $G$.
- Exhaustively, there are two cases to consider:
  1. To include $v$ in the independent set;
  2. To exclude $v$ from the independent set;
- Resulting in two subgraphs $G_1$ and $G_2$ to be recursively considered,
  1. $G_1$ is the result of $G$ after $v$ and all its neighbors are removed;
  2. $G_2$ is the result of $G$ after $v$ is removed.
- The algorithm terminates when the considered graph is empty.
Chapter 33.9. Exhaustive Search

The algorithm follows a logical search tree
• given a graph $G$, it picks an arbitrary vertex $v$ from $G$;
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Chapter 33.9. Exhaustive Search

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Chapter 33.9. Exhaustive Search

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Chapter 33.9. Exhaustive Search

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Chapter 33.9. Exhaustive Search

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Chapter 33.9. Exhaustive Search

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  (1) $G_1$ is the result of $G$ after $v$
Chapter 33.9. Exhaustive Search

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Chapter 33.9. Exhaustive Search

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Chapter 33.9. Exhaustive Search

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• the algorithm terminates when the considered graph is empty.
Algorithm MaxIndSet ($G$)

1. If $G = \emptyset$ return ($\emptyset$)
2. Else pick an arbitrary vertex $v$ in $G$
3. Let $G_1$ be $G$ with $v$ and all its neighbors removed
4. $I_1 = \{v\} \cup \text{MaxIndSet} (G_1)$
5. Let $G_2$ be $G$ with $v$ removed
6. $I_2 = \text{MaxIndSet} (G_2)$
7. If $|I_1| \geq |I_2|$ return ($I_1$)
8. Else return ($I_2$)

- The algorithm is a search tree
- The time complexity: $T(n) = cn^2 + T(|G_1|) + T(|G_2|)$ where $m$ is the number of neighbors of $v$'s
Chapter 33.9. Exhaustive Search

Algorithm MaxIndSet \((G)\)

1. if \(G = \emptyset\) return \((\emptyset)\)
Algorithm \textbf{MaxIndSet} \((G)\)

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Chapter 33.9. Exhaustive Search

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\(T(n) = T(n-1-m) + T(n-1) + cn^2\) where \(m\) is the number of neighbors of \(v\)’s
Algorithm \textsc{MaxIndSet} \((G)\)

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5. \hspace{1em} let \(G_2\) be \(G\) with \(v\) removed

- the algorithm is a search tree
- the time complexity:
  \(T(n) = T(n-1-m) + T(n-1) + cn^2\)
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5. let $G_2$ be $G$ with $v$ removed
6. $I_2 = \text{MaxIndSet} \ (G_2)$
Chapter 33.9. Exhaustive Search

Algorithm $\text{MaxIndSet} (G)$

1. if $G = \emptyset$ return ($\emptyset$)
2. else pick an arbitrary vertex $v$ in $G$
3. let $G_1$ be $G$ with $v$ and all its neighbors removed
4. $I_1 = \{v\} \cup \text{MaxIndSet} (G_1)$
5. let $G_2$ be $G$ with $v$ removed
6. $I_2 = \text{MaxIndSet} (G_2)$
7. if $|I_1| \geq |I_2|$ return ($I_1$)

$T(n) = T(n - 1 - m) + T(n - 1) + cn^2$ where $m$ is the number of neighbors of $v$'s
Chapter 33.9. Exhaustive Search

Algorithm \textsc{MaxIndSet} \((G)\)

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- the algorithm is a search tree
Chapter 33.9. Exhaustive Search

Algorithm \textsc{MaxIndSet} \((G)\)

1. \textbf{if} \quad \(G = \emptyset\) \textbf{return} \((\emptyset)\)
2. \textbf{else} \quad \text{pick an arbitrary vertex} \(v\) \text{ in} \(G\)
3. \quad \text{let} \quad \(G_1\) \quad \text{be} \quad G \quad \text{with} \quad v \quad \text{and all its neighbors removed}
4. \quad \(I_1 = \{v\} \cup \text{MaxIndSet} \((G_1)\)\)
5. \quad \text{let} \quad \(G_2\) \quad \text{be} \quad G \quad \text{with} \quad v \quad \text{removed}
6. \quad \(I_2 = \text{MaxIndSet} \((G_2)\)\)
7. \quad \textbf{if} \quad |I_1| \geq |I_2| \quad \textbf{return} \quad (I_1)\)
8. \quad \textbf{else} \quad \textbf{return} \quad (I_2)\)

- the algorithm is a search tree
- the time complexity:
Chapter 33.9. Exhaustive Search

Algorithm MaxIndSet \((G)\)

1. \(\textbf{if } G = \emptyset \textbf{ return } (\emptyset)\)
2. \(\textbf{else pick an arbitrary vertex } v \text{ in } G\)
3. \(\text{let } G_1 \text{ be } G \text{ with } v \text{ and all its neighbors removed}\)
4. \(I_1 = \{v\} \cup \text{MaxIndSet} \ (G_1)\)
5. \(\text{let } G_2 \text{ be } G \text{ with } v \text{ removed}\)
6. \(I_2 = \text{MaxIndSet} \ (G_2)\)
7. \(\text{if } |I_1| \geq |I_2| \textbf{ return } (I_1)\)
8. \(\textbf{else return } (I_2)\)

- the algorithm is a search tree
- the time complexity: \(T(|G|) = cn^2 + T(|G_1|) + T(|G_2|)\)
Chapter 33.9. Exhaustive Search

Algorithm **MaxIndSet** \((G)\)

1. **if** \(G = \emptyset\) **return** \((\emptyset)\)
2. **else** pick an arbitrary vertex \(v\) in \(G\)
3. let \(G_1\) be \(G\) with \(v\) and all its neighbors removed
4. \(I_1 = \{v\} \cup \text{MaxIndSet} (G_1)\)
5. let \(G_2\) be \(G\) with \(v\) removed
6. \(I_2 = \text{MaxIndSet} (G_2)\)
7. **if** \(|I_1| \geq |I_2|\) **return** \((I_1)\)
8. **else** **return** \((I_2)\)

- the algorithm is a search tree
- the time complexity: \(T(|G|) = cn^2 + T(|G_1|) + T(|G_2|)\)

\[ T(n) = T(n - 1 - m) + T(n - 1) + cn^2 \]

where \(m\) is the number of neighbors of \(v\)’s
Chapter 33.9. Exhaustive Search

\[ T(n) = T(n - 1 - m) + T(n - 1) + cn^2 \]
Chapter 33.9. Exhaustive Search

\[ T(n) = T(n - 1 - m) + T(n - 1) + cn^2 \]

- \( m \geq 0 \),
Chapter 33.9. Exhaustive Search

\[ T(n) = T(n - 1 - m) + T(n - 1) + cn^2 \]

- \( m \geq 0, \) \( T(n) \leq T(n - 1) + T(n - 1) + cn^2, \)
Chapter 33.9. Exhaustive Search

\[ T(n) = T(n - 1 - m) + T(n - 1) + cn^2 \]

- \( m \geq 0, \ T(n) \leq T(n - 1) + T(n - 1) + cn^2 \), \( \implies \ T(n) = O(2^n) \)
Chapter 33.9. Exhaustive Search

\[ T(n) = T(n - 1 - m) + T(n - 1) + cn^2 \]

- \( m \geq 0, \ T(n) \leq T(n - 1) + T(n - 1) + cn^2, \implies T(n) = O(2^n) \)
- Can we guarantee \( m \geq 1 \)
Chapter 33.9. Exhaustive Search

\[ T(n) = T(n - 1 - m) + T(n - 1) + cn^2 \]

- \( m \geq 0, T(n) \leq T(n - 1) + T(n - 1) + cn^2, \implies T(n) = O(2^n) \)

- Can we guarantee \( m \geq 1 \) so we have \( T(n) \leq T(n - 2) + T(n - 1) + cn^2 \),
Chapter 33.9. Exhaustive Search

\[ T(n) = T(n - 1 - m) + T(n - 1) + cn^2 \]

- \( m \geq 0, \ T(n) \leq T(n - 1) + T(n - 1) + cn^2, \implies T(n) = O(2^n) \)

- Can we guarantee \( m \geq 1 \) so we have
  \[ T(n) \leq T(n - 2) + T(n - 1) + cn^2, \implies T(n) = O(1.6181^n) \]
Chapter 33.9. Exhaustive Search

\[ T(n) = T(n - 1 - m) + T(n - 1) + cn^2 \]

- \( m \geq 0, T(n) \leq T(n - 1) + T(n - 1) + cn^2, \implies T(n) = O(2^n) \)
- Can we guarantee \( m \geq 1 \) so we have \( T(n) \leq T(n - 2) + T(n - 1) + cn^2, \implies T(n) = O(1.6181^n) \)
- Or even better, to guarantee \( m \geq 2 \)?
Chapter 33.9. Exhaustive Search

\[ T(n) = T(n - 1 - m) + T(n - 1) + cn^2 \]

- \( m \geq 0, \ T(n) \leq T(n - 1) + T(n - 1) + cn^2 \), \( \implies T(n) = O(2^n) \)

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- Or even better, to guarantee \( m \geq 2 \)? if we can,
  \( T(n) \leq T(n - 3) + T(n - 1) + cn^2 \),
Chapter 33.9. Exhaustive Search

\[ T(n) = T(n - 1 - m) + T(n - 1) + cn^2 \]

- \( m \geq 0, \ T(n) \leq T(n - 1) + T(n - 1) + cn^2, \implies T(n) = O(2^n) \)
- Can we guarantee \( m \geq 1 \) so we have
  \( T(n) \leq T(n - 2) + T(n - 1) + cn^2, \implies T(n) = O(1.6181^n) \)
- Or even better, to guarantee \( m \geq 2 \)? if we can,
  \( T(n) \leq T(n - 3) + T(n - 1) + cn^2, \implies T(n) = O(1.5^n) \)
Chapter 33.9. Exhaustive Search

\[ T(n) = T(n - 1 - m) + T(n - 1) + cn^2 \]

- \( m \geq 0, \ T(n) \leq T(n - 1) + T(n - 1) + cn^2, \implies T(n) = O(2^n) \)

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- Or even better, to guarantee \( m \geq 2 \)? if we can,
  \( T(n) \leq T(n - 3) + T(n - 1) + cn^2, \implies T(n) = O(1.5^n) \)

use the substitution method to prove \( T(n) = O(1.5^n) \).
Chapter 33.9. Exhaustive Search

\[ T(n) = T(n - 1 - m) + T(n - 1) + cn^2 \]

- \( m \geq 0, T(n) \leq T(n - 1) + T(n - 1) + cn^2, \Rightarrow T(n) = O(2^n) \)

- Can we guarantee \( m \geq 1 \) so we have
  \[ T(n) \leq T(n - 2) + T(n - 1) + cn^2, \Rightarrow T(n) = O(1.6181^n) \]

- Or even better, to guarantee \( m \geq 2 \)? if we can,
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use the substitution method to prove \( T(n) = O(1.5^n) \).

- Can we guarantee \( m \geq 3 \)?
Chapter 33.9. Exhaustive Search

\[ T(n) = T(n - 1 - m) + T(n - 1) + cn^2 \]

- \( m \geq 0, \ T(n) \leq T(n - 1) + T(n - 1) + cn^2, \implies T(n) = O(2^n) \)

- Can we guarantee \( m \geq 1 \) so we have
  \( T(n) \leq T(n - 2) + T(n - 1) + cn^2, \implies T(n) = O(1.6181^n) \)

- Or even better, to guarantee \( m \geq 2 \)? If we can,
  \( T(n) \leq T(n - 3) + T(n - 1) + cn^2, \implies T(n) = O(1.5^n) \)

  use the substitution method to prove \( T(n) = O(1.5^n) \).

- Can we guarantee \( m \geq 3 \)? Possible but a little more complicated.
Chapter 33.9. Exhaustive Search

Let $T(n) \leq T(n - 3) + T(n - 1) + cn^2$, with $T(1) = O(1)$
Let $T(n) \leq T(n - 3) + T(n - 1) + cn^2$, with $T(1) = O(1)$

Claim: $T(n) = O(1.5^n)$
Let \( T(n) \leq T(n - 3) + T(n - 1) + cn^2 \), with \( T(1) = O(1) \)

**Claim:** \( T(n) = O(1.5^n) \)

**Proof** (use the substitution method)
Let $T(n) \leq T(n - 3) + T(n - 1) + cn^2$, with $T(1) = O(1)$

**Claim**: $T(n) = O(1.5^n)$

**Proof** (use the substitution method)

Assume that $T(k) \leq 1.5^k$ for all $k < n$. 

Now we decide $n_0$:

$n^2 \leq 0.1 \Rightarrow n^2 \leq 0.1 \times 1.5^{n-3} \leq n^2$ when roughly $n \geq n_0 = 29$
Let $T(n) \leq T(n - 3) + T(n - 1) + cn^2$, with $T(1) = O(1)$

**Claim:** $T(n) = O(1.5^n)$

**Proof (use the substitution method)**

Assume that $T(k) \leq 1.5^k$ for all $k < n$. Then

$$T(n) \leq T(n - 3) + T(n - 1) + n^2$$
Chapter 33.9. Exhaustive Search

Let $T(n) \leq T(n - 3) + T(n - 1) + cn^2$, with $T(1) = O(1)$

Claim: $T(n) = O(1.5^n)$

Proof (use the substitution method)

Assume that $T(k) \leq 1.5^k$ for all $k < n$. Then

$$T(n) \leq T(n - 3) + T(n - 1) + n^2 \leq 1.5^{n-3} + 1.5^{n-1} + n^2$$
Chapter 33.9. Exhaustive Search

Let $T(n) \leq T(n-3) + T(n-1) + cn^2$, with $T(1) = O(1)$

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Assume that $T(k) \leq 1.5^k$ for all $k < n$. Then

$$T(n) \leq T(n-3) + T(n-1) + n^2 \leq 1.5^{n-3} + 1.5^{n-1} + n^2$$

when $n > n_0$ ($n_0$ to be determined)
Chapter 33.9. Exhaustive Search

Let $T(n) \leq T(n - 3) + T(n - 1) + cn^2$, with $T(1) = O(1)$

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when $n > n_0$ ($n_0$ to be determined)

$$\leq 1.5^{n-3}(1 + 1.5^2 + \frac{n^2}{1.5^{n-3}})$$
Let $T(n) \leq T(n - 3) + T(n - 1) + cn^2$, with $T(1) = O(1)$

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Assume that $T(k) \leq 1.5^k$ for all $k < n$. Then

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$$\leq 1.5^{n-3}(1 + 1.5^2 + \frac{n^2}{1.5^{n-3}}) \leq 1.5^{n-3}(1 + 1.5^2 + 0.1)$$
Chapter 33.9. Exhaustive Search

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when $n > n_0$ ($n_0$ to be determined)

$$\leq 1.5^{n-3}(1 + 1.5^2 + \frac{n^2}{1.5^{n-3}}) \leq 1.5^{n-3}(1 + 1.5^2 + 0.1) = 1.5^{n-3}(1 + 2.25 + 0.1)$$
Let $T(n) \leq T(n - 3) + T(n - 1) + cn^2$, with $T(1) = O(1)$

**Claim:** $T(n) = O(1.5^n)$

**Proof** (use the substitution method)

Assume that $T(k) \leq 1.5^k$ for all $k < n$. Then

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Chapter 33.9. Exhaustive Search

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$$= 1.5^{n-3} \times 3.35 \leq 1.5^{n-3} \times 3.375$$
Let $T(n) \leq T(n - 3) + T(n - 1) + cn^2$, with $T(1) = O(1)$

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Chapter 33.9. Exhaustive Search

Let $T(n) \leq T(n-3) + T(n-1) + cn^2$, with $T(1) = O(1)$

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Chapter 33.9. Exhaustive Search

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Now we decide $n_0$:

$$\frac{n^2}{1.5^{n-3}} \leq 0.1$$
Let $T(n) \leq T(n - 3) + T(n - 1) + cn^2$, with $T(1) = O(1)$

**Claim:** $T(n) = O(1.5^n)$

**Proof (use the substitution method)**

Assume that $T(k) \leq 1.5^k$ for all $k \leq n$. Then

$$T(n) \leq T(n - 3) + T(n - 1) + n^2 \leq 1.5^{n-3} + 1.5^{n-1} + n^2$$

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$$\frac{n^2}{1.5^{n-3}} \leq 0.1 \implies n^2 \leq 0.1 \times 1.5^{n-3}$$
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Chapter 33.9. Exhaustive Search

- Algorithms for SAT and MAXINDSET run in exponential time $O(2^n)$ or $O(\gamma^n)$ for $1 < \gamma < 2$
Chapter 33.9. Exhaustive Search

- Algorithms for SAT and \textbf{MAXINDSET} run in exponential time $O(2^n)$ or $O(\gamma^n)$ for $1 < \gamma < 2$

- Search tree (solution search space) is large, \textit{inherently} large
Chapter 33.9. Exhaustive Search

- Algorithms for SAT and \textsc{MaxIndSet} run in exponential time $O(2^n)$ or $O(\gamma^n)$ for $1 < \gamma < 2$
- search tree (solution search space) is large, \textit{inherently} large
- search tree does not have obvious overlapping subproblems,
Chapter 33.9. Exhaustive Search

- Algorithms for SAT and MaxIndSet run in exponential time $O(2^n)$ or $O(\gamma^n)$ for $1 < \gamma < 2$
- Search tree (solution search space) is large, inherently large
- Search tree does not have obvious overlapping subproblems, which otherwise would incur dynamic programming approaches.
Chapter 34. NP-Completeness

1. Intractable problems
   • investigating decision problems suffice

2. NP model
   • how to show a problem is in class NP

3. NP-completeness framework
   • polynomial-time reduction and NP-completeness proof.
Chapter 34. NP-Completeness

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Chapter 34. NP-Completeness

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- we have seen many problems solvable in polynomial time
Chapter 34. NP-Completeness

1. Intractable problems

- we have seen many problems solvable in polynomial time
  e.g., sorting, SCC, MST

- why would a time $O(n^{100})$-time algorithm be attractive?
  only theoretical or practical significance as well
Chapter 34. NP-Completeness

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- there are problems that do not seem to have polynomial time algorithms

why would a time $O(n^{100})$-time algorithm be attractive?
only theoretical?
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1. Intractable problems

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  i.e., not solvable in time $O(n \log n)$,
Chapter 34. NP-Completeness

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1. **Intractable problems**

- we have seen many problems **solvable in polynomial time**
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- there are problems that do **not** seem to have polynomial time algorithms
  i.e., **not solvable** in time $O(n \log n)$, $O(n^3)$, or $O(n^{100})$. 
Chapter 34. NP-Completeness

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- we have seen many problems solvable in polynomial time
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Chapter 34. NP-Completeness

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- we have seen many problems solvable in polynomial time e.g., sorting, SCC, MST

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Chapter 34. NP-Completeness

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- why would a time $O(n^{100})$-time algorithm be attractive?
  - only theoretical? practical significance as well
Chapter 34. NP-Completeness

Define: a Hamiltonian cycle in a graph is a circular path going through every vertex exactly once. Different from an Eulerian cycle that goes through every edge exactly once.
Chapter 34. NP-Completeness

**Define:** a Hamiltonian cycle in a graph is a circular path going through every vertex exactly once.
Chapter 34. NP-Completeness

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Chapter 34. NP-Completeness

Define: a Hamiltonian cycle in a graph is a circular path going through every vertex exactly once.

Different from Eulerian cycle that goes through every edge exactly once.
Define: a Hamiltonian cycle in a graph is a circular path going through every vertex exactly once.

Different from Eulerian cycle that goes through every edge exactly once.
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Travel Salesman Problem (TSP)

Input: an edge-weighted graph \( G = (V, E) \);

Output: a Hamiltonian cycle of the minimum weight sum.

• intuitively, a circular path is a permutation of \((v_1, v_2, ..., v_n)\) or simply a permutation of \((1, 2, ..., n)\), where \(|V| = n\).

so the problem has time upper bound \(O(n!|E|)\), exponential time.

\[ n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1 \geq n \times (n-1) \times \cdots \times n^2 \geq (n^2)^n \]

• all known algorithms (solving TSP) are of exponential-time.
Travel Salesman Problem (TSP)

Input: an edge-weighted graph $G = (V, E)$;
Travel Salesman Problem (TSP)

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**Travel Salesman Problem (TSP)**

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$$n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1 \geq n \times (n-1) \cdots \times \frac{n}{2} \geq \left(\frac{n}{2}\right)^{\frac{n}{2}}$$

- all known algorithms (solving TSP) are of exponential-time.
Chapter 34. NP-Completeness

Instead of considering the Travel Salesman Problem (TSP),

Input: an edge-weighted graph \( G = (V, E) \);

Output: a Hamiltonian cycle of the minimum weight sum.

We may consider a related problem: H-Cycle Weight Decision (HCW)

Input: an edge-weighted graph \( G = (V, E) \), a weight value \( K \);

Output: “YES” if and only if there is a Hamiltonian cycle of weight \( \leq K \) in \( G \).

- HCW appears to “easier” than TSP as an H-cycle is not produced in the answer.
- However, HCW may not be “easier”.

Theorem 1: HCW is solvable in P-time if and only if TSP is solvable in P-time.
Instead of considering

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Trivially,
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Trivially, P-time algorithms for TSP
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**Theorem 1**: HCW is solvable in P-time if and only if TSP is solvable in P-time.

Trivially, P-time algorithms for TSP $\implies$ P-time algorithms for HCW,
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**Theorem 1**: HCW is solvable in P-time if and only if TSP is solvable in P-time.

Trivially, P-time algorithms for TSP $\implies$ P-time algorithms for HCW, why?
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How to prove:
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How to prove: P-time algorithms for TSP $\iff$ P-time algorithms for HCW?
Theorem 1: HCW is solvable in P-time if and only if TSP is solvable in P-time.

Proof: (P-time algorithms for TSP $\iff$ P-time algorithms for HCW)
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**Theorem 1**: HCW is solvable in P-time if and only if TSP is solvable in P-time.

**Proof**: (P-time algorithms for TSP ⇐ P-time algorithms for HCW)

- assume P-time algorithm $A$ for HCW such that $A(G, K) = \text{"YES/"NO}$. 

  How to make Step 1 P-time?

Theorem 1 says problems HCW and TSP are "polynomially equivalent."
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Theorem 1: HCW is solvable in P-time if and only if TSP is solvable in P-time.

Proof: (P-time algorithms for TSP \iff P-time algorithms for HCW)

- assume P-time algorithm \( A \) for HCW such that \( A(G, K) = \text{"YES/"NO} \)
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**Proof**: (P-time algorithms for TSP $\iff$ P-time algorithms for HCW)

- assume P-time algorithm $A$ for HCW such that $A(G, K) =$ “YES”/“NO”
- construct a P-time algorithm $B(G)$ for TSP to behave as follows:

  1. on input $G$, for every possible values of $K$, call $A(G, K)$;
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Chapter 34. NP-Completeness

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How to make Step 1 P-time?
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  1. on input $G$, for every possible values of $K$, call $A(G, K)$; remember the smallest $k_{min}$ such that $A(G, k_{min}) = \text{"YES"}$.
  2. mark all edges in $G$ as “unvisited”;

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Chapter 34. NP-Completeness

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Chapter 34. NP-Completeness

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      if $A(G', k_{\text{min}}) = \text{"YES"}$
        then $G = G'$;
      else mark $(u, v)$ “critical”;

How to make Step 1 P-time? Theorem 1 says problems HCW and TSP are “polynomially equivalent”.
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     pick an “unvisited” edge $(u, v)$, mark it “visited”;
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     if $A(G', k_{min}) =$ “YES”
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     else mark $(u, v)$ “critical”;
     return (all “critical” edges)
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- show algorithm $B$ runs in P-time.
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Theorem 1 says problems HCW and TSP are “polynomially equivalent”.
Consider another related problem:

**H-Cycle Decision (HC)**

**Input:** an edge-weighted graph $G = (V,E)$

**Output:** "YES" if and only there is a Hamiltonian cycle in $G$.

**H-Cycle Weight Decision (HCW)**

**Input:** an edge-weighted graph $G = (V,E)$, a weight value $K$

**Output:** "YES" if and only there is a Hamiltonian cycle of weight $\leq K$ in $G$.

- Which problem is seemingly "easier"?

**Theorem 2:** $HCW$ is P-time solvable if and only if $HC$ is P-time solvable.

Can you prove it?

Theorem 2 says problems $HCW$ and $HC$ are "polynomially equivalent."
Consider another related problem:

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Compared with H-Cycle Weight Decision (HCW)

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Can you prove it?

Theorem 2 says problems HCW and HC are “polynomially equivalent”.
Corollary 3: Problems TSP, HCW, and HC are all “polynomially equivalent”.

Theorem 4: MaxIS is P-time solvable if and only if IS is P-time solvable. Can you prove the theorem?
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There are other problems that have the similar situation.
Corollary 3: Problems TSP, HCW, and HC are all “polynomially equivalent”.

There are other problems that have the similar situation.

Max Independent Set (MaxIS)
Input: graph $G = (V, E)$;
Corollary 3: Problems TSP, HCW, and HC are all “polynomially equivalent”.

There are other problems that have the similar situation.

Max Independent Set (MaxIS)

Input: graph $G = (V, E)$;
Output: an independent set of vertices of the maximum size;
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**Corollary 3**: Problems TSP, HCW, and HC are all "polynomially equivalent".

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- **Input**: graph $G = (V, E)$;
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- **Input**: graph $G = (V, E)$, integer $k$;
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**Theorem 4:** MaxIS is P-time solvable if and only if IS is P-time solvable.
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- **Input**: graph $G = (V, E)$;
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**Theorem 4**: MaxIS is P-time solvable if and only if IS is P-time solvable.

Can you prove the theorem?
Similarly,

Min Vertex Cover (MinVC)

Input: graph \( G = (V,E) \);
Output: a vertex cover set of vertices of the minimum size;

Vertex Cover (VC)

Input: graph \( G = (V,E) \), integer \( k \);
Output: "YES" if and only if \( G \) has a vertex cover of size \( \leq k \).

Theorem 5:
MinVC is P-time solvable if and only if VC is P-time solvable.

Can you prove the theorem?
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Min Vertex Cover (MinVC)
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Theorem 5: MinVC is P-time solvable if and only if VC is P-time solvable.
Can you prove the theorem?
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Theorem 5: MinVC is P-time solvable if and only if VC is P-time solvable. Can you prove the theorem?
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Conclusions:

1. "Polynomial equivalency" can be established between optimization problems and decision problems.

To study tractability of optimization problems, often it suffices to investigate decision problems. (Decision problems are also called languages.) One language can be defined for one decision problem, let

$$\Sigma = \{0, 1\}$$

$$L_{HCW} = \{x \in \Sigma^*: x \text{ encodes } \langle G, k \rangle, G \text{ has a H-cycle of weight } \leq k\}$$

Answering "yes" or "no" to input $$\langle G, k \rangle$$ for problem HCW $$\iff$$ answering "yes" or "no" to language membership question $$x \in L_{HCW}$$?
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Chapter 34. NP-Completeness

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**Corollary 6**: VC is P-time solvable **if and only if** IS is P-time solvable.

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**Corollary 6**: VC is P-time solvable if and only if IS is P-time solvable.

3. However, “Polynomial equivalency” does not tell us the tractability of the problems.

4. We need a rigorous framework to study tractability via the notion “Polynomial equivalency”.
Chapter 34. NP-Completeness

2. Nondeterministic algorithms

Deterministic algorithms

• Given input data, a deterministic algorithm has its every step completely determined by the algorithm and data.
• All algorithms we have seen so far are deterministic.
• Every deterministic algorithm can be unfolded into a linear sequence of steps (when the input is given).
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```
M = -\infty 
n = 3 
i = 1 
check 1 \leq 3 
check -\infty < 10 
M = 10 
i = 2 
check 2 \leq 3 
check 10 < 30 
M = 30 
i = 3 
check 3 \leq 3 
check 30 < 20 
i = 4 
check 4 \leq 3 
return (30)
```

MaxOfList(L)
1. \( M = -\infty \)
2. \( n = \text{length}(L) \)
3. for \( i = 1 \) to \( n \)
4. \( \text{if } M < L[i] \)
5. \( M = L[i] \)
6. return \( (M) \)

Unfolded when input \( L = (10, 30, 20) \)
Chapter 34. NP-Completeness

A deterministic algorithm can be thought of a linear path of steps;
A **deterministic algorithm** can be thought of a **linear path** of steps; each vertex uniquely determines its successor step.
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- the running time is the number of steps on the path.

In a nondeterministic algorithm, when unfolded, there may be more than one possible successor.

- a nondeterministic algorithm can be thought of a tree of steps.
- each step has more than one nondeterministic choice for its successor;
- a path from root to a leaf is a sequence of nondeterministic choices; thus a nondeterministic execution of the algorithm.
- The algorithm answers “YES” if one execution path leads to “YES”.
- the running time is the number of steps on a longest path.

If running time is $m$ steps, there may be $2^m$ paths.

Let us call this tree model of nondeterministic algorithms.
Chapter 34. NP-Completeness

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Chapter 34. NP-Completeness

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Chapter 34. NP-Completeness

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Chapter 34. NP-Completeness

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Let us call this tree model of nondeterministic algorithms.
Chapter 34. NP-Completeness

Use nondeterministic algorithms to solve problem SAT
Chapter 34. NP-Completeness

Use nondeterministic algorithms to solve problem SAT in polynomial time.
Use **nondeterministic algorithms** to solve problem SAT in polynomial time.

**Algorithm** NONDET SAT-SOLVER  
**Input:** $\phi(x_1, \ldots, x_n)$  
1. Let $\phi_0 = \phi(x_1, \ldots, x_n)$  
2. **for** $i = 1$ **to** $n$  
3. nondeterministically let $a_i = 0$ or $a_i = 1$;  
4. $\phi_i = \phi_{i-1}(x_i = a_i)$  
5. **if** ($\phi_n == 1$)  
6. **return** YES  
7. **else**  
8. **return** NO
Chapter 34. NP-Completeness

Use nondeterministic algorithms to solve problem SAT in polynomial time.

Algorithm **NonDetSAT-Solver**

**Input:** \( \phi(x_1, \ldots, x_n) \)

1. Let \( \phi_0 = \phi(x_1, \ldots, x_n) \)
2. for \( i = 1 \) to \( n \)
3.   nondeterministically let \( a_i = 0 \) or \( a_i = 1 \);
4.   \( \phi_i = \phi_{i-1}(x_i = a_i) \)
5. if \( \phi_n == 1 \)
6.   return YES
7. else
8.   return NO

Algorithm **NonDetSAT-Solver-1**

**Input:** \( \phi(x_1, \ldots, x_n) \)

1. for \( i = 1 \) to \( n \)
2.   nondeterministically let \( a_i = 0 \) or \( a_i = 1 \);
3. if \( \phi(a_1, \ldots, a_n) == 1 \)
4.   return YES
5. else
6.   return NO
Chapter 34. NP-Completeness

Algorithm NonDetSAT-Solver-1

Input: $\phi(x_1, \ldots, x_n)$

1. for $i = 1$ to $n$
2.   nondeterministically let $a_i = 0$ or $a_i = 1$;
3. if $(\phi(a_1, \ldots, a_n) == 1)$
4.   return Yes
5. else
6.   return No
Algorithm NonDetSAT-Solver-1

Input: $\phi(x_1, \ldots, x_n)$

1. for $i = 1$ to $n$
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Chapter 34. NP-Completeness

Algorithm \textsc{NonDetSAT-Solver-1}
\begin{itemize}
\item[] \textbf{Input:} $\phi(x_1, \ldots, x_n)$
\item[] 1. for $i = 1$ to $n$
\item[] 2. nondeterministically let $a_i = 0$ or $a_i = 1$;
\item[] 3. if ($\phi(a_1, \ldots, a_n) == 1$)
\item[] 4. \textbf{return} \textsc{Yes}
\item[] 5. \textbf{else}
\item[] 6. \textbf{return} \textsc{No}
\end{itemize}

Answer is \textsc{Yes} iff $\exists(a_1, \ldots, a_n)$
Chapter 34. NP-Completeness

Algorithm NonDetSAT-Solver-1

Input: $\phi(x_1, \ldots, x_n)$

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3. if ($\phi(a_1, \ldots, a_n) == 1$)
4.  return Yes
5. else
6.  return No

Answer is Yes iff $\exists (a_1, \ldots, a_n) \phi(a_1, \ldots, a_n) = 1.$
Chapter 34. NP-Completeness
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(1) Answer is \textbf{YES}
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(2) \( \phi(x_1, \ldots, x_n) \) is satisfiable iff
Chapter 34. NP-Completeness

(1) Answer is \textbf{YES} iff \( \exists (a_1, \ldots, a_n), \phi(a_1, \ldots, a_n) = 1 \).

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iff \( \exists \textit{witness}(a_1, \ldots, a_n), \phi(a_1, \ldots, a_n) = 1 \) can be verified
Chapter 34. NP-Completeness

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(1) Answer is \textbf{YES} iff $\exists (a_1, \ldots, a_n), \phi(a_1, \ldots, a_n) = 1$.

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(3) $\phi(x_1, \ldots, x_n)$ is satisfiable
\hspace{1cm} iff $\exists \text{witness}(a_1, \ldots, a_n),
\hspace{1cm} V(\phi, a_1, \ldots, a_n)$ can be verified to true in P-time
Use nondeterministic algorithms to solve problem Hamiltonian Cycle.
Chapter 34. NP-Completeness

Use nondeterministic algorithms to solve problem Hamiltonian Cycle in polynomial time.

• The algorithm will answer "YES" iff there is a H-cycle in G.

What does the nondeterministic computation tree look like?
Use nondeterministic algorithms to solve problem Hamiltonian Cycle in polynomial time.

1. Starting from any vertex \( v \) in the graph;

![Diagram of a graph with vertices a, b, c, d and edges labeled with weights 1, 30, 30, 99.](image)

(1) starting from any vertex \( v \) in the graph;
Chapter 34. NP-Completeness

Use nondeterministic algorithms to solve problem Hamiltonian Cycle in polynomial time.

(1) starting from any vertex $v$ in the graph;
(2) nondeterministically choose one of its (at most $n - 1$) neighbors which has not been chosen;
Chapter 34. NP-Completeness

Use nondeterministic algorithms to solve problem Hamiltonian Cycle in polynomial time.

(1) starting from any vertex \( v \) in the graph;
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    let the newly picked vertex be \( v \), go to step (2)

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{hamiltonian_cycle_graph.png}
\caption{Example graph for Hamiltonian Cycle problem.}
\end{figure}
Use nondeterministic algorithms to solve problem Hamiltonian Cycle in polynomial time.

1. starting from any vertex \( v \) in the graph;
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3. if all vertices have been chosen,
   return “YES” if these vertices are connected to form an H-cycle;
   return “NO”, otherwise;
Chapter 34. NP-Completeness

Use **nondeterministic algorithms** to solve problem **Hamiltonian Cycle** in polynomial time.

(1) starting from any vertex \( v \) in the graph;
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Chapter 34. NP-Completeness

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(3) if all vertices have been chosen,
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• The algorithm will answer “YES” iff there is a H-cycle in \( G \).

• What does the nondeterministic computation tree look like?
Problems like Independent Set, Vertex Cover, HCW can all be solved with nondeterministic algorithms in polynomial time.
Problems like Independent Set, Vertex Cover, HCW can all be solved with nondeterministic algorithms in polynomial time.

Can you prove the claim?
Chapter 34. NP-Completeness

**Definition:** \( \mathcal{P} \) is the class of languages (i.e., decision problems) that can be solved by deterministic polynomial-time algorithms.
Chapter 34. NP-Completeness

Definition: \( \mathcal{P} \) is the class of languages (i.e., decision problems) that can be solved by deterministic polynomial-time algorithms.

- class \( \mathcal{P} \) contains problems like \textsc{Reachability}
Chapter 34. NP-Completeness

**Definition:** $\mathcal{P}$ is the class of languages (i.e., decision problems) that can be solved by deterministic polynomial-time algorithms.

- class $\mathcal{P}$ contains problems like **Reachability** and many others.
Chapter 34. NP-Completeness

**Definition:** \( \mathcal{P} \) is the class of languages (i.e., decision problems) that can be solved by **deterministic polynomial-time algorithms**.

- class \( \mathcal{P} \) contains problems like \textsc{Reachability} and many others.

**Definition:** \( \mathcal{NP} \) is the class of languages (i.e., decision problems) that can be solved by **nondeterministic polynomial-time algorithms**.
Chapter 34. NP-Completeness

**Definition:** $\mathcal{P}$ is the class of languages (i.e., decision problems) that can be solved by deterministic polynomial-time algorithms.

- class $\mathcal{P}$ contains problems like \textsc{Reachability} and many others.

**Definition:** $\mathcal{NP}$ is the class of languages (i.e., decision problems) that can be solved by nondeterministic polynomial-time algorithms.

- class $\mathcal{NP}$ contains problems like \textsc{VC}, \textsc{HC}, \textsc{IS} and many others.

Because every deterministic algorithm is a special case of a nondeterministic algorithm, $\mathcal{P} \subseteq \mathcal{NP}$.
Chapter 34. NP-Completeness

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Because every deterministic algorithm is a special case of a nondeterministic algorithm,

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Chapter 34. NP-Completeness

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Chapter 34. NP-Completeness
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- we can assume the algorithm does all nondeterministic choices before other operations.

The binary string is called certificate or witness; the deterministic computation part is called verification.
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Deterministic algorithms are when the certificate is empty.
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Alternative view of nondeterministic polynomial-time computation
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Alternative view of nondeterministic polynomial-time computation

Every nondeterministic polynomial time computation is

\begin{itemize}
  \item to nondeterministically choose a binary string of a polynomial length,
  \item then to compute deterministically in polynomial time.
\end{itemize}

Let $\Pi \in \text{NP}$. Then there is a deterministic algorithm $A_\Pi$, and a constant $c > 0$, such that

\begin{enumerate}
  \item if $x$ is a positive instance of $\Pi$, there is a binary string $y$ of length $n^c$, $A_\Pi(x, y) = \text{"YES"}$;
  \item if $x$ is a negative instance of $\Pi$, for all binary string $y$ of length $n^c$, $A_\Pi(x, y) = \text{"NO"}$;
  \item and $A_\Pi$ runs in time $O(n^c)$.
\end{enumerate}

We call $y$ a certificate/witness and $A_\Pi$ the verification algorithm.

$P$ is defined with certificate $y = \epsilon$, i.e., empty string.
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We call \( y \) a **certificate**/**witness** and \( A_{\Pi} \) the **verification algorithm**.

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$\mathcal{P}$ is defined with certificate $y = \epsilon$, i.e., empty string.
Definition of $\mathcal{NP}$ in terms of languages:

Let $L \subseteq \{0, 1\}^*$ be a language in the class $\mathcal{NP}$. Then there is a deterministic algorithm $A_L$, and a constant $c > 0$, such that, for every $x \in \{0, 1\}^*$, $x \in L \iff \exists y, |y| \leq |x| c$, $A_L(x, y) = 1$ and $A_L$ runs in polynomial time.

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Chapter 34. NP-Completeness

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where $|x|$ denotes the length of the string $x$. The algorithm $A_L$ runs in polynomial time of $m = |x,y| = |x| + |y| \leq n + n c \leq (2n c)^d = O(n^{dc})$, also polynomial time of $n = |x|$. Class $\mathcal{P}$ is defined with certificate $y = \epsilon$, i.e., empty string.
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Chapter 34. NP-Completeness

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Chapter 34. NP-Completeness

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Chapter 34. NP-Completeness

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Chapter 34. NP-Completeness

The diagram illustrates concepts related to NP-Completeness. The arrows represent moves: non-deterministic (dashed) and deterministic (solid). The nodes are labeled with 'S', 'Y', and 'N' representing start, yes, and no respectively. The transitions between the nodes demonstrate the complexity and decision-making process in NP-Complete problems.
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- Non-deterministic moves
- Deterministic moves

Non-deterministic choice: choose a certificate $y = 10...11...$

Deterministic algorithm: A for verification
Chapter 34. NP-Completeness

Proof that $HC \in NP$. 

We need to show there is a deterministic algorithm $A$ and a constant $c > 0$, such that for any $G$, $G \in HC \iff \exists y, |y| \leq |G|^c, A(G, y) = "YES"$.

We can design that

- certificate $y$ represents a sequence of ordered vertices;
- algorithm $A$ is to verify that $y$ does form a H-cycle.

Details:

- $y = B_1 B_2 \ldots B_n$, where $B_i$ is a binary representation of some vertex in $G$;
- How many bits does $B_i$ need? $\lceil \log_2 n \rceil$;
- whether $y$ forms a H-cycle can be verified in time $O(|E|)$. 
Proof that $HC \in \mathcal{NP}$.

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Chapter 34. NP-Completeness

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- $y = B_1B_2\ldots B_n$, where $B_i$ is a binary representation of some vertex in $G$; How many bits does $B_i$ need? \(\lceil \log_2 n \rceil\)
Chapter 34. NP-Completeness

Proof that $HC \in \mathcal{NP}$.

We need to show there is a deterministic algorithm $A$ and a constant $c > 0$, such that for any $G$,

$$G \in HC \iff \exists y, |y| \leq |G|^c, A(G, y) = "YES"$$

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- certificate $y$ represents a sequence of ordered vertices;
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exercises:

Proof that Independent Set $\in$ NP.

Proof that Vertex Cover $\in$ NP.

Notes
1. To prove a language is in the class NP by no means proves that the language can be solved in polynomial time. Instead, it only shows the language is in the class NP.
2. There is a difference between deciding $x \in L$ and checking $A_L(x, y) = 1$.
3. As between convicting a suspect vs checking an evidence against the suspect.
Chapter 34. NP-Completeness

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3. NP-Completeness Framework
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The notion of reduction (i.e., transformation) between languages
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Chapter 34. NP-Completeness

Let \( L_1 \) and \( L_2 \) are two languages over the alphabet \( \{0, 1\} \).
Chapter 34. NP-Completeness

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Two example problems:

3SAT

Input: CNF formula \( \phi(x_1, \ldots, x_n) \) with 3 literals in each clause;

Output: "YES" if and only if there is an assignment to variables \( x_1, \ldots, x_n \) to make \( \phi(x_1, \ldots, x_n) \) TRUE.

Independent Set (IS)

Input: graph \( G = (V,E) \), integer \( k \);

Output: "YES" if and only if \( G \) has an independent set of size \( \geq k \).

We claim: 3SAT \( \leq \) IS

How to transform an input for 3SAT to an input for IS?
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Chapter 34. NP-Completeness

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Chapter 34. NP-Completeness

How to transform a boolean formula $\phi$ into a pair $\langle G,k \rangle$, such that $\phi \in \text{SAT} \iff \langle G,k \rangle \in \text{IS}$.

An assignment TRUE to one literal in each clause corresponds to an independent set in the transformed graph.
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Chapter 34. NP-Completeness

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$$(x_1 \lor x_2 \lor \overline{x_3}) \land (x_2 \lor x_3 \lor \overline{x_4}) \land (x_1 \lor \overline{x_2} \lor x_4)$$
Chapter 34. NP-Completeness

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**Input:** graph $G = (V, E)$, integer $k$;

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Consider their corresponding languages:
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Because the two problems are very relevant to each other, we have:

**Theorem:** \( L_{IS} \leq L_{VC} \)

**Proof:** we use the fact that complement set of an independent set is a vertex cover in the same graph.

We construct a mapping \( f \) that maps instance \( \langle G, k \rangle \) to instance \( \langle G, |G| - k \rangle \), i.e., \( f(\langle G, k \rangle) = \langle G, |G| - k \rangle \). This is a reduction from \( L_{IS} \) to \( L_{VC} \).

**Claim:** \( G \) has an i.s. of size \( \geq k \) \( \iff \) \( G \) has an v.c. of size \( \leq |G| - k \) (proof of the claim is on the next slide)

So \( L_{IS} \leq L_{VC} \).
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Chapter 34. NP-Completeness

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Let $G$ be such that has vertices $V = \{v_1, \ldots, v_n\}$.

Assume that $G$ has a i.s. of size $k_0$ for some $k_0 \geq k$.
We further assume, without loss of generality, the i.s include vertices $\{v_1, \ldots, v_{k_0}\}$. 
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Then vertices $\{v_{k_0+1}, \ldots, v_n\}$ form a v.c. for $G$. 

Can you prove “⇐”??
Chapter 34. NP-Completeness

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Suppose otherwise, $\exists$ edge $(u, v)$ that is not covered, i.e.,
neither $u \in \{v_{k_0+1}, \ldots, v_n\}$
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Thus, $u, v \in \{v_1, \ldots, v_{k_0}\}$, the independent set.
Proof: “⇒”
(to prove that \( G \) has i.s. of size \( \geq k \) implies \( G \) has v.c of size \( \geq k \))

Let \( G \) be such that has vertices \( V = \{v_1, \ldots, v_n\} \).

Assume that \( G \) has a i.s. of size \( k_0 \) for some \( k_0 \geq k \).
We further assume, without loss of generality, the i.s include vertices 
\( \{v_1, \ldots, v_{k_0}\} \).

Then vertices \( \{v_{k_0+1}, \ldots, v_n\} \) form a v.c. for \( G \).

Suppose otherwise, \( \exists \) edge \((u, v)\) that is not covered, i.e.,
- neither \( u \in \{v_{k_0+1}, \ldots, v_n\} \)
- nor \( v \in \{v_{k_0+1}, \ldots, v_n\} \).

Thus, \( u, v \in \{v_1, \ldots, v_{k_0}\} \), the independent set.
But \((u, v)\) is an edge, contradicts that \( \{v_1, \ldots, v_{k_0}\} \) is an i.s.
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Proof: “⇒”
(to prove that $G$ has i.s. of size $\geq k$ implies $G$ has v.c of size $\geq k$)

Let $G$ be such that has vertices $V = \{v_1, \ldots, v_n\}$.

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Can you prove “⇐” ??
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An important motivation for reduction:
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So the combined algorithm (gray-color box) solves for $L_1$. 
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A polynomial-time reduction from $L_1$ to $L_2$, denoted as $L_1 \leq_p L_2$, is some mapping function $f : \{0, 1\}^* \rightarrow \{0, 1\}^*$,
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![Diagram showing a polynomial-time reduction from $L_1$ to $L_2$]
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For example, $\text{IS} \leq_p \text{CLIQUE}$. 
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**Theorem:** Let $L_1 \leq_p L_2$. If $L_2 \in \mathcal{P}$, then $L_1 \in \mathcal{P}$. 
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**Proof:** Assume algorithm $F$ computes $f$, and algorithm $A_2$ solves for $L_2$. We need to show that the gray box runs in polynomial time if both $F$ and $A_2$ run in polynomial time:

Total time is the sum of time for $F$ and time for $A_2$.

$$O(|x|_c) + O(|f(x)|_d)$$

now, what is the length of $f(x)$?

Because $F$ runs in time $O(|x|_c)$, the number of bits outputted by $F$ is $O(|x|_c)$.

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$$O(|x|^c) + O(|f(x)|^d) = O(|x|^c + O((|x|^c)^d)) = O(|x|^{cd})$$
Theorem: Polynomial-time reductions compose (are transitive). That is if \( L_1 \leq_p L_2 \) and \( L_2 \leq_p L_3 \), then \( L_1 \leq_p L_3 \).

Proof. Assume functions \( f \) for \( L_1 \leq_p L_2 \); function \( h \) for \( L_2 \leq_p L_3 \).

For every \( x \in \{0, 1\}^* \), \( x \in L_1 \iff f(x) \in L_2 \iff h(f(x)) \in L_3 \).

That is \( x \in L_1 \iff h(f(x)) \in L_3 \).

So composite function \((h \circ f)\) realizes reduction \( L_1 \leq_p L_3 \).

But we need to show the reduction is \( \leq_p \), i.e., a polynomial time reduction.

Assume that algorithm \( F \) computes \( f: F(x) = f(x) \) in time \( O(|x|^c) \) and algorithm \( H \) computes \( h: H(y) = h(y) \) in time \( O(|y|^d) \).

Let \( y = f(x) \), the total time for computing \((h \circ f)\) = time of \( F \) and time of \( H \) = \( O(|x|^c) + O(|f(x)|^d) \) = \( O(|x|^c) + O(|x|^cd) \) = \( O(|x|^c + |x|^cd) \).

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For every $x \in \{0, 1\}^*$, $x \in L_1 \iff f(x) \in L_2 \iff h(f(x)) \in L_3$.

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Assume that algorithm $F$ computes $f$: $F(x) = f(x)$ in time $O(\|x\|^c)$ and algorithm $H$ computes $h$: $H(y) = h(y)$ in time $O(\|y\|^d)$.

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Proof. Assume functions $f$ for $L_1 \leq_p L_2$; function $h$ for $L_2 \leq_p L_3$. For every $x \in \{0, 1\}^*$, $x \in L_1 \iff f(x) \in L_2 \iff h(f(x)) \in L_3$. That is $x \in L_1 \iff h(f(x)) \in L_3$. So composite function $(h \circ f)$ realizes reduction $L_1 \leq_p L_3$. But we need to show the reduction is $\leq_p$, i.e., a polynomial time reduction. Assume that algorithm $F$ computes $f$: $F(x) = f(x)$ in time $O(|x|^c)$ and algorithm $H$ computes $h$: $H(y) = h(y)$ in time $O(|y|^d)$. Let $y = f(x)$, the total time for computing $(h \circ f) = \text{time of } F \text{ and time of } H = O(|x|^c) + O(|f(x)|^d) = O(|x|^c) + O(|x|^cd)$. So $L_1 \leq_p L_3$. 

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$$x \in L_1 \iff f(x) \in L_2 \iff h(f(x)) \in L_3$$

That is $x \in L_1 \iff h(f(x)) \in L_3$

So composite function $(h \circ f)$ realizes reduction $L_1 \leq L_3$.

But we need to show the reduction is $\leq_p$, i.e., a polynomial time reduction.

Assume that algorithm $F$ computes $f$: $F(x) = f(x)$ in time $O(|x|^c)$
and algorithm $H$ computes $h$: $H(y) = h(y)$ in time $O(|y|^d)$

Let $y = f(x)$, the total time for computing $(h \circ f) =$ time of $F$ and time of $H$

$$= O(|x|^c) + O(|f(x)|^d) = O(|x|^c) + O(|x|^c)^d) = O(|x|^c) + O(|x|^{cd})$$
**Theorem**: Polynomial-time reductions compose (are transitive). That is, if \( L_1 \leq_p L_2 \) and \( L_2 \leq_p L_3 \), then \( L_1 \leq_p L_3 \).

**Proof**. Assume functions \( f \) for \( L_1 \leq_p L_2 \); function \( h \) for \( L_2 \leq_p L_3 \).

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So \( L_1 \leq_p L_3 \).
Some conclusions:

• Using $\leq_p$, languages in $NP$ can be ordered partially;
• If those languages at the end of a $\leq_p$ chain have polynomial-time algorithms, so does every language on the chain.
• Informally, those at the end of a $\leq_p$ chain are called NP-hard.
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Chapter 34. NP-Completeness

Definition 1: $L$ is **NP-hard**
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**Definition 1:** $L$ is **NP-hard** if for every language $L' \in \mathcal{NP}$, $L' \leq^p L$.

**Definition 2:** $L$ is **NP-complete** if (1) $L$ is NP-hard and (2) $L \in \mathcal{NP}$.

**Properties of NP-hard problems**

- If $L$ is NP-hard and $L \in \mathcal{P}$, then $\mathcal{P} = \mathcal{NP}$.

**Proof?**

- If $L$ is NP-hard and $L' \leq^p L'$, then $L'$ is NP-hard.

**Proof?**

**How to prove a language is NP-hard?**
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**How to prove a language is NP-hard?**
To prove a language $L$ is NP-complete, we need to show it is NP-hard. That is, we need to show for every language $L' \in \text{NP}$, $L' \leq_p L$.

Apparently, it is not possible to enumerate all languages in NP and prove that everyone is polynomial-time reducible to $L$.

Instead, formulate a generic language that represents all languages in NP and prove that every language in NP can be reduced to the generic language in polynomial time.

To obtain such a generic language, we need to consider the definition of languages in NP.
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To prove a language $L$ is NP-complete, we need to show it is NP-hard. That is, we need to show

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- Apparently, it is not possible to enumerate all languages in NP and prove that everyone is polynomial-time reducible to $L$.
- Instead, formulate a generic language that represents all languages in NP and prove that every language in $\mathcal{NP}$ can be reduced to the generic language in polynomial time.
- To obtain such a generic language, we need to consider the definition of languages in NP.
Chapter 34. NP-Completeness

Recall the definition of languages in $\mathcal{NP}$:

Let $L \subseteq \{0, 1\}^*$ be any language in the class $\mathcal{NP}$. Then there is a deterministic algorithm $A_L$, and a constant $c > 0$, such that, for every $x \in \{0, 1\}^*$, $x \in L \iff \exists y, |y| \leq |x|^c, A_L(x, y) = 1$ and $A_L$ runs in polynomial time.

The "iff" relationship looks a little like the relationship in a reduction $x \in L \iff \exists y, |y| \leq |x|^c, A_L(x, y) = 1 \leftrightarrow x \in L \iff f(x) \in L_{tbd}$ where $L_{tbd}$ is a language to be defined.

Can we identify $L_{tbd}$ and $f$?
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Again we examine
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(1)
Chapter 34. NP-Completeness

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\[ x \in L \iff \exists y, |y| \leq |x|^c, A_L(x, y) = 1 \] (1)

- \( A_L \) is a deterministic algorithm can be implemented with a boolean circuit \( B_L \) with two sets of input gates \( x = x_1x_2 \ldots x_n \) and \( y = y_1y_2 \ldots y_m \) such that

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Because \( x \) is given, circuit \( B_L \) can be made into circuit \( C^x_L \) such that

\[ B_L(x, y) = 1 \text{ if and only if } C^x_L(y) = 1 \]  \hspace{1cm} (3)

From (1), (2), and (3), we have

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\[ x \in L \iff \exists y \ C^x_L(y) = 1 \]  

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• Define: a boolean circuit \( C \) is satisfiable if there exists at least one set of values \( y \) to its input gates such that \( C(y) = 1 \).

\[ e.g., \ C(x_1, x_2) = (\neg x_1 \lor \neg x_2) \land (x_1 \lor x_2) \text{ is satisfiable;} \]

\[ \text{but } \ D(x_1, x_2) = (x_1 \lor x_2) \land (x_1 \lor \neg x_2) \land (\neg x_1 \lor x_2) \land (\neg x_1 \lor \neg x_2) \text{ is not!} \]

• Define the following language:

\[ \text{CSAT} = \{ C: \text{circuit } C \text{ is satisfiable} \} \]

• From (4), we have \( x \in L \iff C^x_L \in \text{CSAT} \)  

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It remains to be shown
• that reducing algorithm \( A_L \) to circuit \( B_L \) is valid;
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• Define: a boolean circuit \( C \) is **satisfiable** if there exists at least one set of values \( y \) to its input gates such that \( C(y) = 1 \).

  e.g., \( C(x_1, x_2) = (\neg x_1 \lor \neg x_2) \land (x_1 \lor x_2) \) is satisfiable;
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Chapter 34. NP-Completeness

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Chapter 34. NP-Completeness

Unfold deterministic polynomial-time algorithm $A(x, y)$ with input $\langle x, y \rangle$
Chapter 34. NP-Completeness

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Chapter 34. NP-Completeness

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**Theorem**: Language $CSAT$ is NP-complete.
Chapter 34. NP-Completeness

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Chapter 34. NP-Completeness

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**Cook’s Theorem:** SAT is NP-complete.

Cook’s reduction: characterizing a polynomial-time computation on nondeterministic Turing machine with a boolean formula, such that a **nondeterministic path** leading to the accept state **corresponds** to an **assignment to the variables** making the the formula TRUE.
It is very easy to convert a boolean formula to a boolean circuit. So
Chapter 34. NP-Completeness

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Chapter 34. NP-Completeness

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Chapter 34. NP-Completeness

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Chapter 34. NP-Completeness

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$$
\phi = x_{10} \land (x_4 \leftrightarrow \neg x_3) \\
\land (x_5 \leftrightarrow (x_1 \lor x_2)) \\
\land (x_6 \leftrightarrow \neg x_4) \\
\land (x_7 \leftrightarrow (x_1 \land x_2 \land x_4)) \\
\land (x_8 \leftrightarrow (x_5 \lor x_6)) \\
\land (x_9 \leftrightarrow (x_6 \lor x_7)) \\
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Chapter 34. NP-Completeness

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\]

$\phi$ can be transformed to an equivalent CNF formula.
Chapter 34. NP-Completeness

Landscape of NP problems and beyond
Chapter 34. NP-Completeness

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Chapter 34. NP-Completeness

Many problems/languages have been proved NP-complete (Karp70s)
Examples of reduction techniques
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Example 1: $SAT \leq_p 3SAT$
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Chapter 34. NP-Completeness

Examples of reduction techniques

Example 1: $\text{SAT} \leq_p \text{3SAT}$

$$(z) \iff (z, x_1, x_2) \land (z, x_1, \neg x_2) \land (z, \neg x_1, x_2) \land (z, \neg x_1, \neg x_2)$$
Examples of reduction techniques

Example 1: SAT $\leq_p$ 3SAT

$(z) \rightarrow (z, x_1, x_2) \land (z, x_1, \neg x_2) \land (z, \neg x_1, x_2) \land (z, \neg x_1, \neg x_2)$

$(y, z)$
Chapter 34. NP-Completeness

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\[(y, z) \implies (y, z, x_1) \land (y, z, \neg x_1)\]
Chapter 34. NP-Completeness

Examples of reduction techniques

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$$(z) \quad \mapsto \quad (z, x_1, x_2) \land (z, x_1, \neg x_2) \land (z, \neg x_1, x_2) \land (z, \neg x_1, \neg x_2)$$

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(y, z) \implies (y, z, x_1) \land (y, z, \neg x_1) \\
(x, y, z) \implies (x, y, z)
\]
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$$(y, z) \iff (y, z, x_1) \land (y, z, \neg x_1)$$

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Example 2: 3SAT

An assignment TRUE to one literal in each clause corresponds to an independent set in the transformed graph.
Example 2: $3\text{SAT} \leq_p \text{IS}$
Example 2: $3SAT \leq_p IS$

$$(x_1 \lor x_2 \lor \overline{x_3}) \land (x_2 \lor x_3 \lor \overline{x_4}) \land (x_1 \lor \overline{x_2} \lor x_4)$$
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Summary

Scope of the Final Exam

- Recursive formulation of problem solutions
- Dynamic programming (4 steps)
- Greedy algorithms (greedy choice property and proof)
- Depth-First-Search algorithm, DFS search tree, time stamps
- Applications topological sort
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- Minimum spanning tree
  concept/properties of MST

- Shortest path (single source and all pairs)
  concept/properties of shortest path, greedy algorithms, relaxation technique
  single source: Bellman-Ford’s, Dijkstra’s
  all pairs: Floyd-Warshall
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- NP-completeness theory
  - definitions of NP class (certificate + verification)
  - proof that a language is in NP
  - reduction, polynomial-time reduction, properties
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