CSCI 4470/6470 Algorithms, Spring 2019

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Department of Computer Science, UGA


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An Introduction to the Introduction

There are two ways of constructing a software design: One way is to make it so simple that there are obviously no deficiencies, and the other way is to make it so complicated that there are no obvious deficiencies.

- C.A.R. Hoare

Which way do we take in algorithm design?
An Introduction to the Introduction

There are two ways of constructing a software design: One way is to make it so simple that there are obviously no deficiencies, and the other way is to make it so complicated that there are no obvious deficiencies.

- C.A.R. Hoare
An Introduction to the Introduction

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Which way do we take in algorithm design?
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Example 1
An Introduction to the Introduction

Example 1

Sequence Homology Reveals Functions

- Homology reveals evolution of structure/function
  - FOS_RAT
  - FOS_MOUSE
  - FOS_CHICK
  - FOSB_MOUSE
  - FOSB_HUMAN
  - Consensus

- Homology reveals regulatory structure (E. Coli promoters)
  - tyr tRNA
  - rm D1
  - rm X1
  - rm (DRE)2
  - rm E1
  - rm A1
  - rm A2
  - λ Pr
  - λ Pl
  - T7 A3
  - T7 A1
  - T7 A2
  - fd VIII
An Introduction to the Introduction

- main task: sequence comparison (consider just 2 sequences)
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```
FOS_CHICK:  MMYQGFAGYEAPSSRCSSASPAGDSLTYYPFADSFSSMGSPVNSQDFCTDLAVSSANF 60
FOSB_MOUSE: -MFQAFPGDYDS-GSRCSS-SPSAESQ--YLSSVDSFGSPPTAASQE-CAGLGEMPGSF 54
```
An Introduction to the Introduction

- main task: sequence comparison (consider just 2 sequences)

- analogy in text searching/matching

- total number of number of alignments $\geq 2^n$.

- to find the most plausible one, cannot afford to try them all

- require to design "smarter" algorithms that run much faster

We will study various techniques for efficient algorithm design.
An Introduction to the Introduction

• main task: sequence comparison (consider just 2 sequences)

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<th>FOS_CHEK</th>
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</table>

• analogy in text searching/matching
  but allowing substitutions, insertions, deletions
An Introduction to the Introduction

- main task: sequence comparison (consider just 2 sequences)

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An example of sequence comparison:

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\text{FOS\_CHICK: MMYQGFAGEYEAPSSRCSSASPAGDSLTYYPADSFSSMGSPVNSQDFCTDLAVSSANF 60}
\]

\[
\text{FOSB\_MOUSE: -MFQAFPGDYDS-GSRCSS-SPSAESQ--YLSSVDSFGSPPTAASQE-CAGLGEMPGSF 54}
\]

• analogy in text searching/matching
  but allowing substitutions, insertions, deletions

• total number of number of alignments \( \geq 2^n \).
  to find the most plausible one, cannot afford to try them all

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Example 2
### An Introduction to the Introduction

**Example 2**

#### US Open 2016 Men's Singles

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**Champion:**

Wawrinka, Stan SUI

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8-7(1) 1-6 4-1 7-5 6-3
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<td>126. MURRAY, Andy GBR [2]</td>
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An Introduction to the Introduction

- 128 players in total;
- 127 matches were played to determine who was the champion.
- Is it possible to just play fewer matches?
128 players in total;

Is it possible to just play fewer matches?
• 128 players in total;
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128 players in total;
127 matches were played to determine who was the champion.
Is it possible to just play fewer matches?
An Introduction to the Introduction

Sounds crazy ... but if I don't do it, my job won't be secure...

See if you can reduce the number of matches!

Office of Director
USTA
An Introduction to the Introduction

Sounds crazy ... but if I don’t do it, my job won’t be secure...

See if you can reduce the number of matches!

You were called to answer this question;
An Introduction to the Introduction

- You tried various match formats, but all need at least 127 matches;
An Introduction to the Introduction

- You tried various match formats, but all need at least 127 matches;
- Then you suspected that 127 matches were necessary,
An Introduction to the Introduction

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An Introduction to the Introduction

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Sir, this won’t happen. I have a mathematical proof that it needs 127 matches!

Hmmm, he is smart... I should give him more work to do.
An Introduction to the Introduction

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Same problems

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Same problems

Input: 128 players
Output: Champion
Solution: a match scheme
Time Efficiency: number of matches

Tennis Tournament

Finding Maximum

Input: $n$ numbers
Output: the maximum number
Solution: an algorithm
Time Efficiency: number of comparisons

You actually accomplished two tasks:
An Introduction to the Introduction

Same problems

Tennis Tournament
Finding Maximum

Input:
Output:
Solution:
Time Efficiency:

128 players
Champion
a match scheme
number of matches

n numbers
the maximum number
an algorithm
number of comparisons

You actually accomplished two tasks:

• tried various algorithms for Tennis Tournament before giving up;
An Introduction to the Introduction

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- **tried various algorithms** for **Tennis Tournament** before giving up;
- **proved that more efficient algorithms do not exist**;
## An Introduction to the Introduction

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### An Introduction to the Introduction

#### Same problems

- **Input:** 128 players
- **Output:** Champion
- **Solution:** a match scheme
- **Time Efficiency:** number of matches

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- actually you proved the “USTA algorithm” was already the optimal.

We will learn how to prove that an algorithm is already the best.
An Introduction to the Introduction

Example 3

Toss a coin over the phone

- how does one person know the other is telling the truth?
- even if the person is, would the first person trust him?

How to accomplish this task?
An Introduction to the Introduction

Example 3

Toss a coin over the phone

I will let you choose, head or tail?

Uh...
An Introduction to the Introduction

Example 3

Toss a coin over the phone

• how does one person know the other is telling the truth?
Example 3

Toss a coin over the phone

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An Introduction to the Introduction

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How to accomplish this task?
They decided the following protocol:
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An Introduction to the Introduction

They decided the following protocol:

- A told B a huge Boolean circuit through the phone; along with an output $Y$ of binary bits (but NOT the input $X$);
An Introduction to the Introduction

They decided the following protocol:

- A told B a huge Boolean circuit through the phone; along with an output $Y$ of binary bits (but NOT the input $X$);
- B guessed if the number of 0's in $X$ was odd or even;
An Introduction to the Introduction

$X = 110101$

$Y = 111010$
An Introduction to the Introduction

- from \( Y \), it would be very difficult to figure out which \( X \) had been used to generate \( Y \) (an intractable problem);
from $Y$, it would be very difficult to figure out which $X$ had been used to generate $Y$ (an intractable problem);

So the best $B$ can do is to randomly guess (odd/even), which has the same effect as guessing a coin toss.
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• So the best B can do is to randomly guess (odd/even), which has the same effect as guessing a coin toss.

We will investigate some intractable problems and the theory behind it.
An Introduction to the Introduction

Now we are back to the tennis tournament problem.
An Introduction to the Introduction

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An Introduction to the Introduction

Number of matches

- Algorithm A
- Algorithm Z
- USTA algorithm

Algorithms you tried, unfortunately

USTA algorithm gives an upper bound (≤ 127)

The Tennis Tournament (128) Problem

No algorithm uses fewer than 127 matches

Your proof gives a lower bound (≥ 127)
An Introduction to the Introduction

Definitions of complexity upper and lower bounds;
Definitions of complexity upper and lower bounds;

- **Time complexity upper bound** is a time threshold *in* which all instances of a problem *can* be solved.
An Introduction to the Introduction

Definitions of complexity upper and lower bounds;

• **Time complexity upper bound** is a time threshold *in* which all instances of a problem *can* be solved.

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Definitions of complexity upper and lower bounds:

- **Time complexity upper bound** is a time threshold *in* which all instances of a problem can be solved.

- **Time complexity lower bound** is a time threshold *below* which some instances of the problem cannot be solved.

- These notions apply to *both problems and algorithms*. 
An Introduction to the Introduction

In general, for a given problem $\Pi$, 

• an algorithm $A$ for $\Pi$ also gives a time upper bound $T_A$ for $\Pi$, where $T_A$ is the time complexity of algorithm $A$;

• a faster algorithm $B$ for $\Pi$, with $T_B < T_A$, gives a tighter upper bound $T_B$ for $\Pi$;

• a proof that $\Pi$ cannot have algorithms faster than time $S$, with $S > T$, gives a tighter lower bound $S$ for $\Pi$;

• for problem $\Pi$, time lower bounds $\leq$ time upper bounds; when a lower bound is the same as an upper bound, both the lower and upper bounds are called optimal.

Also the corresponding algorithms (that offer the upper bound) are called optimal for $\Pi$. 
An Introduction to the Introduction

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An Introduction to the Introduction

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An Introduction to the Introduction

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An Introduction to the Introduction

Time complexity situation for problem $\Pi$

- Algorithm A
- Algorithm B
- a proved lower bound
- a proved lower bound

$T_A$

$T_B$

$S$

$T$

gap, the smaller the better

2
1
0
An Introduction to the Introduction

So to design good algorithms for computational problems, our goals are
So to design good algorithms for computational problems, our goals are

1. to achieve tighter upper bounds
An Introduction to the Introduction

So to design good algorithms for computational problems, our goals are

1. to achieve tighter upper bounds
   - we need be familiar with techniques for algorithm complexity analysis;

2. and to achieve tighter lower bounds,
   - we need to know the methods for lower bound proofs.
An Introduction to the Introduction

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   - we need be familiar with techniques for algorithm complexity analysis;
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The Introduction

What is this course about (and why is it needed)?

• about basic yet indispensable skills for problem solving
• algorithm design leads to writing code, i.e., creative thinking without programming languages

How different is this course from other algorithm courses?

• design technique-oriented, not application-oriented
• emphasis on guaranteed performance (typically in efficiency)

Goals to achieve

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Part I. Foundations
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- Chapter 1. The role of algorithms in computing
- Chapter 2. Getting started
- Chapter 3. Growth of functions
- Chapter 4. Solving recurrences
- Chapter 5. Probabilistic analysis and randomized algorithms
Part I. Foundations

The theme of the course
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- Goal: learning techniques to design efficient algorithms
Part I. Foundations

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- Goal: learning techniques to design efficient algorithms
- mean: through developing skills to analyze algorithms
Part I. Foundations

The theme of the course

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Design and analysis of algorithms are closely related.
Example: the Fibonacci sequence.

\[ f(n) = \begin{cases} 
  f(n - 1) + f(n - 2) & \text{if } n \geq 3 \\
  1, & \text{otherwise}
\end{cases} \]
Example: the Fibonacci sequence.

\[ f(n) = \begin{cases} 
  f(n - 1) + f(n - 2) & \text{if } n \geq 3 \\
  1, & \text{otherwise}
\end{cases} \]

That is:

\[
\begin{array}{ccccccccccc}
  n & : & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \ldots \\
  f(n) & : & 1 & 1 & 2 & 3 & 5 & 8 & 13 & 21 & 34 & \ldots 
\end{array}
\]
Problem 1: Computing the $n$th Fibonacci number:
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**Input:** $n \geq 1$;
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Two different types of algorithms: *recursive* and *iterative*
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- *recursive*: task decomposition, top-down, recursive calls;
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Two different types of algorithms: *recursive* and *iterative*

- recursive: task decomposition, top-down, recursive calls;
- iterative: more tightly coupled tasks, bottom-up approaches;
Rec-Fibonacci(n)
Part I. Foundations

Rec-Fibonacci(n)

if \( n = 1 \) or \( n = 2 \),
Rec-Fibonacci($n$)

if $n = 1$ or $n = 2$, return (1);

But how efficient is it? Or how slow is it?
its execution is via a run-time stack
• suitable for execution of subroutines
• but oblivious, cannot remember any completed subroutine.
Part I. Foundations

**Rec-Fibonacci**

\[ \text{Rec-Fibonacci}(n) = \begin{cases} 1 & \text{if } n = 1 \text{ or } n = 2, \\ \text{Rec-Fibonacci}(n - 1) + \text{Rec-Fibonacci}(n - 2) & \text{otherwise}. \end{cases} \]

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if $n = 1$ or $n = 2$, return (1);
else
    $T_1 = \text{Rec-Fibonacci}(n - 1)$;
    $T_2 = \text{Rec-Fibonacci}(n - 2)$;
    return ($T_1 + T_2$);

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Part I. Foundations

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```
Part I. Foundations

Repeated computations everywhere!
The size of tree is the number of recursive calls;
Part I. Foundations

The size of tree is the number of recursive calls; How big is it?
Part I. Foundations

\[ \frac{2^{-n}}{2} \leq \text{size of tree} \leq 2^{-n} \]
Part I. Foundations

\[ \text{small triangle} \leq \text{size of tree} \leq \text{large triangle} \]
Part I. Foundations

small triangle \leq \text{size of tree} \leq \text{large triangle}

\text{roughly: } 2^{\frac{n}{2}} \leq \text{size of tree} \leq 2^n
Iterative-Fibonacci \( (n) \)
Part I. Foundations

Iterative-Fibonacci($n$)

if $n = 1$ or $n = 2$ return (1);
Iterative-Fibonacci($n$)

if $n = 1$ or $n = 2$ return (1);
else
  $M[1] = 1$, $M[2] = 1$

How fast is it?

$T_{total} = \max\{T_{if}, T_{else}\}$ where

$T_{if} = c_1$,
$T_{else} = c_2 + T_{for} = c_2 + d \times (n - 2)$

$T_{total} \leq c_1 + c_2 + d \times (n - 2)$, a linear function in $n$
Iterative-Fibonacci(n)

if $n = 1$ or $n = 2$ return (1);
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Iterative-Fibonacci(n) is a simple dynamic programming algorithm.
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Part I. Foundations

**Iterative-Fibonacci\((n)\)**

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T_{\text{total}} = \max\{T_{if}, T_{else}\}
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Iterative-Fibonacci\( (n) \)

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\[ T_{total} \leq c_1 \]
Part I. Foundations

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Iterative-Fibonacci($n$) is a simple dynamic programming algorithm.
Part I. Foundations

Actually, all the algorithm \textsc{Iterative-Fibonacci}(n) does is:
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To fill out a table of size \( n \), with
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So the total time $\text{ITERATIVE-FIBONACCI}(n)$ uses is
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- each entry being filled out exactly once, and
- filling out an entry takes a constant, say $c$ steps.

So the total time \textsc{Iterative-Fibonacci}(n) uses is

$$T(n) = c \times n$$
Chapter 1. The role of algorithms in computing
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**What is an Algorithm**: a well-defined, finite procedure that takes an input and produces an output.
Chapter 1. The role of algorithms in computing

What is an Algorithm: a well-defined, finite procedure that takes an input and produces an output.

Example 2: An algorithm skeleton;

Algorithm Maximum;

INPUT: list $X = \{a_1, \cdots, a_n\}$;

Body that is a series of instructions;

OUTPUT: $y$, the maximum of $a_1, \cdots, a_n$. 
Chapter 1. The Role of Algorithms in Computing

Alternatively, an algorithm specifies a finite process to compute a function or a relation.
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E.g., algorithm MAXIMUM computes the following function:

\[ f_{\text{max}}(X) = y, \text{ where } \forall a \in X, y \geq a, \]
Alternatively, an algorithm specifies a finite process to compute a function or a relation.

e.g., algorithm \texttt{MAXIMUM} computes the following function:

\[ f_{\text{max}}(X) = y, \text{ where } \forall a \in X, y \geq a, \]

For some problems, the functions computed are predicates, i.e., output \( y \in \{\text{TRUE}, \text{FALSE}\} \)
Chapter 1. The Role of Algorithms in Computing

Algorithms as a technology to resolve efficiency issues
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Algorithms as a technology to resolve efficiency issues

Efficient use of computer resources such as time and space is necessary.
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Two typical situations:
Chapter 1. The Role of Algorithms in Computing

Algorithms as a technology to resolve efficiency issues

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Two typical situations:

- very large input data for “easy” problems;
Chapter 1. The Role of Algorithms in Computing

Algorithms as a technology to resolve efficiency issues

Efficient use of computer resources such as time and space is necessary.

Two typical situations:

* very large input data for “easy” problems;
* moderately large input data for “hard” problems.
Chapter 2. Getting Started

Chapter 2. Getting started

The Sorting Problem

Input: \( n \) numbers \( \langle a_1, \cdots, a_n \rangle \); Output: a reordering \( \langle a'_1, \cdots, a'_n \rangle \) of the input such that \( a'_1 \leq a'_2 \leq \cdots \leq a'_n \).

Insertion Sort

idea: an iterative process to produce a new list such that at each iteration, the new list consists of two sublists,

- a sorted sublist followed by an unsorted sublist,
- the leftmost number of the unsorted is being inserted into the sorted.

As the process goes, the sorted sublist gets longer, the unsorted sublist gets shorter, until the unsorted becomes empty.
Chapter 2. Getting Started

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The Sorting Problem

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Chapter 2. Getting started

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\textbf{Output:} a reordering $\langle a'_1, \cdots, a'_n \rangle$ of the input such that

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Chapter 2. Getting Started

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at each iteration, the new list consists of two sublists,

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- the leftmost number of the unsorted is being inserted into the sorted.

As the process goes, the sorted sublist gets longer, the unsorted sublist gets shorter, until the unsorted becomes empty.
Algorithm INSERTION-SORT(A)

1. for $j = 2$ to length[A]
2.     key = A[j]
3.     {Insert A[j] into sorted A[1..j−1]}
4.     $i = j − 1$
5.     while $i > 0$ and A[i] > key
7.         $i = i − 1$
8.     A[i+1] = key

Analysis of the algorithm:
• (correctness proof): to show that the algorithm is as desired;
• (efficiency proof): to show a guaranteed efficiency of the algorithm.
Algorithm \textsc{Insertion-Sort}(A)

1. \textbf{for} $j = 2$ \textbf{to} \textit{length}[A] \textbf{do}
Algorithm \textsc{Insertion-Sort}(A)

1 \hspace{1em} \textbf{for} \hspace{0.5em} j = 2 \hspace{0.5em} \textbf{to} \hspace{0.5em} \text{length}[A] \hspace{0.5em} \textbf{do}
2 \hspace{1em} \text{key} = A[j]
Chapter 2. Getting Started

Algorithm **INSERTION-SORT**(\(A\))

1. \textbf{for} \(j = 2\) to \textit{length}[\(A\)] \textbf{do}  
2. \hspace{1em} \(key = A[j]\)  
3. \hspace{1em} \{\text{Insert } A[j] \text{ into sorted } A[1..j - 1]\}\
Algorithm **INSERTION-SORT**\((A)\)

1. **for** \(j = 2\) to length\([A]\) **do**
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2 \hspace{1em} \textit{key} = A[j] \hspace{2em}
3 \hspace{1em} \{ \text{Insert } A[j] \text{ into sorted } A[1..j - 1] \} \hspace{2em}
4 \hspace{1em} i = j - 1 \hspace{2em}
5 \hspace{1em} \textbf{while} \hspace{0.5em} i > 0 \hspace{0.5em} \text{and} \hspace{0.5em} A[i] > \textit{key} \hspace{2em}

Chapter 2. Getting Started

Algorithm `INSERTION-SORT(A)`

1. \textbf{for} $j = 2$ \textbf{to} length[$A$] \textbf{do}
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5. \hspace{1em} \textbf{while} $i > 0$ and $A[i] > key$
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Analysis of the algorithm:
• (correctness proof): to show that the algorithm is as desired;
• (efficiency proof): to show a guaranteed efficiency of the algorithm.
Chapter 2. Getting Started

Algorithm **Insertion-Sort**(\(A\))

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Chapter 2. Getting Started

Algorithm \textsc{Insertion-Sort}(A)

\begin{algorithmic}[1]
\State \textbf{for} \( j = 2 \) \textbf{to} \( \text{length}[A] \) \textbf{do}
\State \hspace{1em} \emph{key} = \( A[j] \)
\State \hspace{1em} \{\emph{Insert} \( A[j] \) \emph{into sorted} \( A[1..j - 1] \}\}
\State \hspace{1em} \emph{i} = \( j - 1 \)
\State \hspace{1em} \textbf{while} \( i > 0 \) \textbf{and} \( A[i] > \emph{key} \) \textbf{do}
\State \hspace{2em} \emph{do} \( A[i + 1] = A[i] \)
\State \hspace{2em} \hspace{1em} \emph{i} = \( i - 1 \)
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Chapter 2. Getting Started

Correctness proof: this is to prove
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Chapter 2. Getting Started

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the task is to prove the correct transformation by each block.
Correctness proof: this is to prove

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If the algorithm consists of sequential blocks of instructions,
the task is to prove the correct transformation by each block.

This means we need to prove that every sequential statement in the algorithm transforms the given pre-condition to the given post-condition.
Chapter 2. Getting Started

The most difficult task is to do this for a loop statement.
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Chapter 2. Getting Started

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In Insertion-Sort, the loop invariant is

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However, finding loop invariants is difficult!
Chapter 2. Getting Started

Efficiency analysis: This is to show that

• For all cases of input, the needed computation resources for the algorithm.
• Resources can be CPU time and memory space used in the computation.
• However, the unit measured is not real time or memory unit.
Chapter 2. Getting Started

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Chapter 2. Getting Started

Time of an algorithm $A(x)$

Input Instances

$x$
Chapter 2. Getting Started

Time of an algorithm $A(x)$

Input Instances  worst case

$x$
Chapter 2. Getting Started

**Upper bound** for algorithm A, bounding all cases of instances
Resource measurement based on:

- Random-access machine (RAM)
- Counting primitive operations: addition, subtraction, floor, ceiling, multiplication, jump, memory movement

These operations differ in time by a constant multiplicative factor.

Speed between different machines: a constant multiplicative factor.
Chapter 2. Getting Started

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Assume $t_j$ to be the number of times \textbf{while} is executed for every $j$.

$T(n) = c_1n + c_2(n-1) + c_4(n-1) + c_5\sum_{j=2}^{n} t_j + c_6\sum_{j=2}^{n} (t_j - 1) + c_7\sum_{j=2}^{n} (t_j - 1) + c_8(n-1)$
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6. \hspace{2em} \textbf{do} \textit{A}[\textit{i} + 1] = \textit{A}[\textit{i}]
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Chapter 2. Getting Started

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Chapter 2. Getting Started

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Chapter 2. Getting Started

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Chapter 2. Getting Started

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Chapter 2. Getting Started

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Chapter 2. Getting Started

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\[ T(n) \leq a \frac{n}{2} (n + 1) + bn + c - a \leq xn^2 + yn + z \]

for some constants \( x, y, z \).
Chapter 2. Getting Started

So we have proved:

\[ T(n) \leq x_n^2 + y_n + z \]

for some constants \( x, y, z \) \( \leq \) means in all cases for which \( \leq \) holds; 

\( x_n^2 + y_n + z \) is a complexity upper bound for \( T(n) \).
Chapter 2. Getting Started

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Chapter 2. Getting Started

Important complexity issues:

1. size of input $n$: the number of bits encoding input $x$, i.e., $n = |x|$. It is inaccurate for $n$ to represent the number of items in the input.

Consider to sort 4 items $\langle x_1, x_2, x_3, x_4 \rangle$ of values in the scale of $2^N$, for some very large $N$.

• If $n$ is the number of items, $n = 4$, then any sorting algorithm would run in constant time.

• However, since $x_1, x_2, x_3, x_4$ are of very large values, a single comparison $x_1 \leq x_2$ would need a time proportional to $N$.

Hence, if $n = |\langle x_1, x_2, x_3, x_4 \rangle|$, then $n \approx N$.

To sort the 4 items, a constant number of comparisons is needed, each taking a time linear in $N$ (i.e., total time is linear in $n$).
Chapter 2. Getting Started

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Chapter 2. Getting Started

2. Running time

$T(n)$: the number of primitive operations executed,

worst-case running time: the running time upper bound for all inputs.

order of growth: $T(n) = an^2 + bn + c$ grows the same rate as $an^2$ (if $a > 0$).
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Chapter 3. Growth of Functions

Big-O: \( O(n^2) \) contains all functions of growth rate \( \leq cn^2 \).

So for function \( T(n) = an^2 + bn + c \), \( T(n) \in O(n^2) \), but written as \( T(n) = O(n^2) \).

In general, \( O(g(n)) = \{ f(n) : \exists c > 0, k > 0 \text{ such that } 0 \leq f(n) \leq cg(n), \text{ for all } n \geq k \} \).
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Chapter 3. Growth of Functions

For example: the following functions are all of the order of $O(n^2)$:

(1) $3n^2$

(2) $5.3n^2 + 6n \log_2 n + 90$

(3) $0.001n^2 - 200n - 5000$

(4) $3n \log_2 n^2 + 6n$

(5) $\sqrt{n} - 20 \log_2 n$

(6) $\log_2 n + 56$

(7) $345$

But the following are not:

(8) $3n^2 \log_2 n - 400n$

(9) $n^2$.001
Chapter 3. Growth of Functions

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For example: the following functions are all of the order of $O(n^2)$:

(1) $3n^2$
(2) $5.3n^2 + 6n \log_2 n + 90$
(3) $0.001n^2 - 200n - 5000$
(4) $3n \log_2^n + 6n$
(5) $\sqrt{n} - 20 \log_2 n$
(6) $\log_2 n + 56$
(7) $345$

But the following are not:
Chapter 3. Growth of Functions

For example: the following functions are all of the order of $O(n^2)$:

(1) $3n^2$
(2) $5.3n^2 + 6n \log_2 n + 90$
(3) $0.001n^2 − 200n − 5000$
(4) $3n \log_2 n + 6n$
(5) $\sqrt{n} − 20 \log_2 n$
(6) $\log_2 n + 56$
(7) $345$

But the following are not:

(8) $3n^2 \log_2 n − 400n$
(9) $n^{2.001}$
Chapter 4. Solving Recurrences

Example: Binary Search Algorithm

Algorithm Binary Search \((A, p, r, \text{key})\)

1. \(\text{if } p > r \text{ then return } (\text{null})\)
2. \(\text{else}\)
3. \(q = \lfloor \frac{p + r}{2} \rfloor\)
4. \(\text{if } A[q] = \text{key} \text{ then return } (\text{key})\)
5. \(\text{else}\)
6. \(\text{if } A[q] > \text{key} \text{ then } \text{Binary Search } (A, p, q-1, \text{key})\)
7. \(\text{else} \text{ Binary Search } (A, q+1, r, \text{key})\)

Let \(T(n)\) be the worst case time complexity for Binary Search, where parameter \(n = r - p + 1\).

The time \(T(n)\) can be upper-bounded with the recurrence, 
\[ T(n) \leq T(\frac{n}{2}) + c \]
and \(T(0) \leq c\) where \(c > 0\) is a constant.

Note: we can also set base case \(T(1) \leq c\).
Example: binary search algorithm

Algorithm **Binary Search**\((A, p, r, key)\)

1. if \( p > r \) return (NULL)
2. else
3. \( q = \left\lfloor \frac{p + r}{2} \right\rfloor \)
4. if \( A[q] = key \) return \((k)\)
5. else
6. if \( A[q] > key \) Binary Search \((A, p, q - 1, key)\)
7. else
8. Binary Search \((A, q + 1, r, key)\)
Chapter 4. Solving Recurrences

Example: binary search algorithm

Algorithm \textsc{Binary Search}(A, p, r, \text{key})

1. \textbf{if} \ p > r \ \textbf{return} \ (\text{NULL})
2. \textbf{else}
3. \quad q = \lceil \frac{p + r}{2} \rceil
4. \quad \textbf{if} \ A[q] = \text{key} \ \textbf{return} \ (k)
5. \quad \textbf{else}
6. \quad \quad \textbf{if} \ A[q] > \text{key} \ \textbf{Binary Search} \ (A, p, q - 1, \text{key})
7. \quad \quad \textbf{else}
8. \quad \quad \quad \textbf{Binary Search} \ (A, q + 1, r, \text{key})

Let \( T(n) \) be the worst case time complexity for \textsc{Binary Search}, where parameter \( n = r - p + 1 \).
Example: binary search algorithm

Algorithm **Binary Search** \((A, p, r, key)\)

1. `if p > r return (NULL)`
2. `else`
3. \(q = \lfloor \frac{p+r}{2} \rfloor\)
4. `if A[q] = key return (k)`
5. `else`
6. `if A[q] > key Binary Search (A, p, q - 1, key)`
7. `else`
8. `Binary Search (A, q + 1, r, key)`

Let \(T(n)\) be the worst case time complexity for **Binary Search**, where parameter \(n = r - p + 1\).

The time \(T(n)\) can be upper-bounded with the recurrence,

\[
T(n) \leq T\left(\frac{n}{2}\right) + c \quad \text{and} \quad T(0) \leq c
\]

where \(c > 0\) is a constant.
Chapter 4. Solving Recurrences

Example: binary search algorithm

Algorithm $\text{Binary Search}(A, p, r, key)$

1. if $p > r$ return (NULL)
2. else
3. \[ q = \left\lfloor \frac{p + r}{2} \right\rfloor \]
4. if $A[q] = key$ return $(k)$
5. else
6. if $A[q] > key$ $\text{Binary Search}(A, p, q - 1, key)$
7. else
8. $\text{Binary Search}(A, q + 1, r, key)$

Let $T(n)$ be the worst case time complexity for $\text{Binary Search}$, where parameter \( n = r - p + 1 \).

The time $T(n)$ can be upper-bounded with the recurrence,

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T(n) \leq T\left(\frac{n}{2}\right) + c \quad \text{and} \quad T(0) \leq c
\]

where $c > 0$ is a constant.

Note: we can also set base case $T(1) \leq c$. 
Chapter 4. Solving Recurrences

We have recurrence
\[ T(n) \leq T(n^2) + c \]
and
\[ T(1) \leq c \]
for the Binary Search algorithm.

The recurrence can be resolved by unfolding the recursive terms. The process to unfold the recurrence \( T(n) \) can be:

1. simple, straightforward
2. induction based (called substitution method)
3. graph based (called recursive tree method).
We have recurrence

\[ T(n) \leq T\left(\frac{n}{2}\right) + c \] and \( T(1) \leq c \)

for the Binary Search algorithm.
Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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1. simple, straightforward
We have recurrence

\[ T(n) \leq T\left(\frac{n}{2}\right) + c \]  
and \( T(1) \leq c \)

for the Binary Search algorithm.

The recurrence can be resolved by unfolding the recursive terms. The process to unfold the recurrence \( T(n) \) can be:

1. simple, straightforward
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Chapter 4. Solving Recurrences

We have recurrence

\[ T(n) \leq T\left(\frac{n}{2}\right) + c \] and \[ T(1) \leq c \]

for the Binary Search algorithm.

The recurrence can be resolved by unfolding the recursive terms. The process to unfold the recurrence \( T(n) \) can be:

1. simple, straightforward
2. induction based (called substitution method)
3. graph based (called recursive tree method).
Chapter 4. Solving Recurrences

1. Simple, straightforward unfolding:
   
   \[ T(n) \leq T(n^2) + c \]

   \[ T(n) \leq T(n^2) + cT(n^2) \leq T(n^4) + c \]

   \[ \ldots \]

   \[ T(n^{2h}) \leq T(n^{2h+1}) + c \]

   where \( n^{2h+1} = 1 \), \( 2^{h+1} = n \), \( h + 1 = \log_2 n \).

   There are \( h + 1 \) inequalities.

   \[ T(n) \leq T(n^{2h+1}) + (h + 1)c \]

   \[ T(n) = O(\log_2 n) \]

   proved this last equation!
Chapter 4. Solving Recurrences

1. Simple, straightforward unfolding:
Chapter 4. Solving Recurrences

1. Simple, straightforward unfolding:
   
   use $T(n) \leq T\left(\frac{n}{2}\right) + c$ as a template;
Chapter 4. Solving Recurrences

1. Simple, straightforward unfolding:

   use \( T(n) \leq T\left(\frac{n}{2}\right) + c \) as a template;

   \[ T(n) \leq T\left(\frac{n}{2}\right) + c \]
1. Simple, straightforward unfolding:

use $T(n) \leq T\left(\frac{n}{2}\right) + c$ as a template;

\begin{align*}
T(n) &\leq T\left(\frac{n}{2}\right) + c \\
T\left(\frac{n}{2}\right) &\leq T\left(\frac{n}{2^2}\right) + c
\end{align*}
Chapter 4. Solving Recurrences

1. Simple, straightforward unfolding:

use $T(n) \leq T\left(\frac{n}{2}\right) + c$ as a template;

$$
T(n) \leq T\left(\frac{n}{2}\right) + c \\
T\left(\frac{n}{2}\right) \leq T\left(\frac{n}{2^2}\right) + c \\
T\left(\frac{n}{2^2}\right) \leq T\left(\frac{n}{2^3}\right) + c
$$
Chapter 4. Solving Recurrences

1. Simple, straightforward unfolding:

   use $T(n) \leq T\left(\frac{n}{2}\right) + c$ as a template;

   $T(n) \leq T\left(\frac{n}{2}\right) + c$
   $T\left(\frac{n}{2}\right) \leq T\left(\frac{n}{2^2}\right) + c$
   $T\left(\frac{n}{2^2}\right) \leq T\left(\frac{n}{2^3}\right) + c$

   .......

   $T\left(\frac{n}{2^h}\right) \leq T\left(\frac{n}{2^{h+1}}\right) + c$

   $h + 1 = \log_2 n$
1. Simple, straightforward unfolding:

use $T(n) \leq T\left(\frac{n}{2}\right) + c$ as a template;

$$
T(n) \leq T\left(\frac{n}{2}\right) + c \\
T\left(\frac{n}{2}\right) \leq T\left(\frac{n}{2^2}\right) + c \\
T\left(\frac{n}{2^2}\right) \leq T\left(\frac{n}{2^3}\right) + c \\
\vdots \\
T\left(\frac{n}{2^h}\right) \leq T\left(\frac{n}{2^{h+1}}\right) + c
$$

where $\frac{n}{2^{h+1}} = 1$, $2^{h+1} = n$, $h + 1 = \log_2 n$
1. Simple, straightforward unfolding:

use \( T(n) \leq T\left(\frac{n}{2}\right) + c \) as a template;

\[
\begin{align*}
T(n) & \leq T\left(\frac{n}{2}\right) + c \\
T\left(\frac{n}{2}\right) & \leq T\left(\frac{n}{2^2}\right) + c \\
T\left(\frac{n}{2^2}\right) & \leq T\left(\frac{n}{2^3}\right) + c \\
& \quad \quad \quad \vdots \\
T\left(\frac{n}{2^h}\right) & \leq T\left(\frac{n}{2^{h+1}}\right) + c
\end{align*}
\]

where \( \frac{n}{2^{h+1}} = 1, 2^{h+1} = n, h + 1 = \log_2 n \)
1. Simple, straightforward unfolding:

use $T(n) \leq T\left(\frac{n}{2}\right) + c$ as a template;

$T(n) \leq T\left(\frac{n}{2}\right) + c$

$T\left(\frac{n}{2}\right) \leq T\left(\frac{n}{2^2}\right) + c$

$T\left(\frac{n}{2^2}\right) \leq T\left(\frac{n}{2^3}\right) + c$

......

$T\left(\frac{n}{2^h}\right) \leq T\left(\frac{n}{2^{h+1}}\right) + c$ where $\frac{n}{2^{h+1}} = 1$, $2^{h+1} = n$, $h + 1 = \log_2 n$

$T(n) \leq T\left(\frac{n}{2^{h+1}}\right) + (h + 1)c$ there are $h + 1$ inequalities
1. Simple, straightforward unfolding:

use $T(n) \leq T\left(\frac{n}{2}\right) + c$ as a template;

\[
\begin{align*}
T(n) &\leq T\left(\frac{n}{2}\right) + c \\
T\left(\frac{n}{2}\right) &\leq T\left(\frac{n}{2^2}\right) + c \\
T\left(\frac{n}{2^2}\right) &\leq T\left(\frac{n}{2^3}\right) + c \\
&\vdots \\
T\left(\frac{n}{2^h}\right) &\leq T\left(\frac{n}{2^{h+1}}\right) + c \\
\end{align*}
\]

where $\frac{n}{2^{h+1}} = 1$, $2^{h+1} = n$, $h + 1 = \log_2 n$

\[
T(n) \leq T\left(\frac{n}{2^{h+1}}\right) + (h + 1)c
\]

\[
\leq c + c \log_2 n
\]

there are $h + 1$ inequalities

1. Simple, straightforward unfolding:

use \( T(n) \leq T\left(\frac{n}{2}\right) + c \) as a template;

\[
T(n) \leq T\left(\frac{n}{2}\right) + c \\
T\left(\frac{n}{2}\right) \leq T\left(\frac{n}{2^2}\right) + c \\
T\left(\frac{n}{2^2}\right) \leq T\left(\frac{n}{2^3}\right) + c \\
\cdots \\
T\left(\frac{n}{2^h}\right) \leq T\left(\frac{n}{2^{h+1}}\right) + c \quad \text{where } \frac{n}{2^{h+1}} = 1, \ 2^{h+1} = n, \ h + 1 = \log_2 n \\
+ \\
\frac{n}{2^{h+1}}
\]

\[
T(n) \leq T\left(\frac{n}{2^{h+1}}\right) + (h + 1)c \quad \text{there are } h + 1 \ \text{inequalities} \\
\leq c + c \log_2 n \\
= O(\log_2 n)
\]
1. Simple, straightforward unfolding:

use $T(n) \leq T\left(\frac{n}{2}\right) + c$ as a template;

\[
\begin{align*}
T(n) & \leq T\left(\frac{n}{2}\right) + c \\
T\left(\frac{n}{2}\right) & \leq T\left(\frac{n}{2^2}\right) + c \\
T\left(\frac{n}{2^2}\right) & \leq T\left(\frac{n}{2^3}\right) + c \\
& \ldots \ldots \\
T\left(\frac{n}{2^h}\right) & \leq T\left(\frac{n}{2^{h+1}}\right) + c \quad \text{where } \frac{n}{2^{h+1}} = 1, 2^{h+1} = n, h + 1 = \log_2 n
\end{align*}
\]

\[
T(n) \leq T\left(\frac{n}{2^{h+1}}\right) + (h + 1)c \quad \text{there are } h + 1 \text{ inequalities}
\]

\[
\begin{align*}
& \leq c + c \log_2 n \\
& = O(\log_2 n) \quad \text{proved this last equation!}
\end{align*}
\]
Chapter 4. Solving Recurrences

Another example: Algorithm Merge Sort \((A, p, r)\)

1. if \(p < r\)
2. then \(q = \left\lfloor \frac{p + r}{2} \right\rfloor\)
3. Merge Sort \((A, p, q)\)
4. Merge Sort \((A, q + 1, r)\)
5. Merging 2 Lists \((A, p, q, r)\)

Analysis of the algorithm.

- Assume \(n = r - p + 1\), a power of 2; also assume \(T(n)\) is time for Merge Sort \((A, p, r)\).
- \(t_1, t_2 = c \cdot t_3 = \cdot t_4 = T(n/2)\)
- \(t_5 \leq n\) (why?)

\[ T(n) = t_1, t_2 + t_3 + t_4 + t_5 \leq 2T(n/2) + n + c\]

Base case: \(T(1) = c\).
Chapter 4. Solving Recurrences

Another example:

Algorithm Merge Sort \((A, p, r)\)

1. if \(p < r\) then
2. \(q = \lfloor \frac{p + r}{2} \rfloor\)
3. Merge Sort \((A, p, q)\)
4. Merge Sort \((A, q + 1, r)\)
5. Merging2Lists \((A, p, q, r)\)

Analysis of the algorithm.

- Assume \(n = r - p + 1\), a power of 2; also assume \(T(n)\) is time for Merge Sort \((A, p, r)\).

\[ T(n) = t_1 + t_2 + t_3 + t_4 + t_5 \leq 2T(n/2) + n + c \]

base case: \(T(1) = c\).
Another example:

Algorithm `Merge Sort(A, p, r)`

1. if $p < r$
2. then $q = \left\lfloor \frac{p+r}{2} \right\rfloor$
3. `Merge Sort(A, p, q)`
4. `Merge Sort(A, q + 1, r)`
5. `Merging2Lists(A.p, q, r)`
Chapter 4. Solving Recurrences

Another example:

Algorithm \textsc{Merge Sort}(A, p, r)

1. \textbf{if} \ p < r
2. \textbf{then} \ \ q = \lfloor \frac{p+r}{2} \rfloor
3. \textsc{Merge Sort}(A, p, q)
4. \textsc{Merge Sort}(A, q + 1, r)
5. \textsc{Merging2Lists}(A.p, q, r)

Analysis of the algorithm.

• Assume \( n = r - p + 1 \), a power of 2;
  also assume \( T(n) \) is time for \textsc{Merge Sort}(A, p, r). Then
Another example:

Algorithm \texttt{Merge Sort}(A, p, r)

1. \textbf{if} \ p < r
2. \textbf{then} \ q = \left\lfloor \frac{p+r}{2} \right\rfloor
3. \texttt{Merge Sort}(A, p, q)
4. \texttt{Merge Sort}(A, q + 1, r)
5. \texttt{Merging2Lists}(A.p, q, r)

Analysis of the algorithm.

- Assume \( n = r - p + 1 \), a power of \( 2 \);
  also assume \( T(n) \) is time for \texttt{Merge Sort}(A, p, r). Then

- \( t_{1,2} = c \)
Chapter 4. Solving Recurrences

Another example:

Algorithm \textsc{Merge Sort}(A, p, r)

1. \textbf{if} \; p < r
2. \textbf{then} \; q = \left\lfloor \frac{p+r}{2} \right\rfloor
3. \textsc{Merge Sort}(A, p, q)
4. \textsc{Merge Sort}(A, q + 1, r)
5. \textsc{Merging2Lists}(A,p, q, r)

Analysis of the algorithm.

- Assume \( n = r - p + 1 \), a power of 2;
  also assume \( T(n) \) is time for \textsc{Merge Sort}(A, p, r). Then
  
- \( t_{1,2} = c \)
- \( t_3 = t_4 = T(\frac{n}{2}) \)
Another example:

Algorithm $\text{MERGE\,SORT}(A, p, r)$

1. if $p < r$
2. then $q = \left\lceil \frac{p+r}{2} \right\rceil$
3. $\text{MERGE\,SORT}(A, p, q)$
4. $\text{MERGE\,SORT}(A, q + 1, r)$
5. $\text{MERGING2LISTS}(A.p, q, r)$

Analysis of the algorithm.

- Assume $n = r - p + 1$, a power of 2;
  
  also assume $T(n)$ is time for $\text{MERGE\,SORT}(A, p, r)$. Then

- $t_{1,2} = c$
- $t_3 = t_4 = T\left(\frac{n}{2}\right)$
- $t_5 \leq n$ (why?)
Another example:

Algorithm `MERGE SORT(A, p, r)`

1. if \( p < r \)
2. then \( q = \lfloor \frac{p + r}{2} \rfloor \)
3. `MERGE SORT(A, p, q)`
4. `MERGE SORT(A, q + 1, r)`
5. `MERGING2LISTS(A.p, q, r)`

Analysis of the algorithm.

- Assume \( n = r - p + 1 \), a power of 2;
  
  also assume \( T(n) \) is time for `MERGE SORT(A, p, r)`. Then

- \( t_{1,2} = c \)
- \( t_3 = t_4 = T\left(\frac{n}{2}\right) \)
- \( t_5 \leq n \) (why?)

\[
T(n) = t_{1,2} + t_3 + t_4 + t_5 \leq 2T\left(\frac{n}{2}\right) + n + c
\]
Another example:

Algorithm \textsc{Merge Sort}(A, p, r)

1. \quad \textbf{if} \ p < r
2. \quad \textbf{then} \ q = \left\lfloor \frac{p+r}{2} \right\rfloor
3. \quad \textsc{Merge Sort}(A, p, q)
4. \quad \textsc{Merge Sort}(A, q + 1, r)
5. \quad \textsc{Merging2Lists}(A,p, q, r)

Analysis of the algorithm.

- Assume \( n = r - p + 1 \), a power of 2;
  also assume \( T(n) \) is time for \textsc{Merge Sort}(A, p, r). Then

- \( t_{1,2} = c \)
- \( t_3 = t_4 = T\left(\frac{n}{2}\right) \)
- \( t_5 \leq n \ (\text{why?}) \)

\[
T(n) = t_{1,2} + t_3 + t_4 + t_5 \leq 2T\left(\frac{n}{2}\right) + n + c
\]

base case: \( T(1) = c \).
Chapter 4. Solving Recurrences

Solve recurrence $T(n) \leq 2T(n^2) + n + c$ with base case $T(1) = c$ with a simple method:

$$T(n) \leq 2T(n^2) + n + c$$

$$T(n^2) \leq 2T(n^4) + n^2 + c$$

$$T(n^4) \leq 2T(n^8) + n^4 + c$$

$$\vdots$$

$$T(n^{2^h}) \leq 2T(n^{2^{h+1}}) + n^{2^h} + c$$

where $n^{2^h} + 1 = 1$
Chapter 4. Solving Recurrences

Solve recurrence

\[ T(n) \leq 2T\left(\frac{n}{2}\right) + n + c \]

with base case \( T(1) = c \)

with a simple method:
Chapter 4. Solving Recurrences

Solve recurrence

\[ T(n) \leq 2T\left(\frac{n}{2}\right) + n + c \] with base case \( T(1) = c \)

with a simple method:

\[ T(n) \leq 2T\left(\frac{n}{2}\right) + n + c \]
Chapter 4. Solving Recurrences

Solve recurrence

\[ T(n) \leq 2T\left(\frac{n}{2}\right) + n + c \]

with base case \( T(1) = c \)

with a simple method:

\[
\begin{align*}
T(n) &\leq 2T\left(\frac{n}{2}\right) + n + c \\
T\left(\frac{n}{2}\right) &\leq 2T\left(\frac{n}{2^2}\right) + \frac{n}{2} + c
\end{align*}
\]
Chapter 4. Solving Recurrences

Solve recurrence

\[ T(n) \leq 2T\left(\frac{n}{2}\right) + n + c \] with base case \( T(1) = c \)

with a simple method:

\[
\begin{align*}
T(n) &\leq 2T\left(\frac{n}{2}\right) + n + c \\
T\left(\frac{n}{2}\right) &\leq 2T\left(\frac{n}{2^2}\right) + \frac{n}{2} + c \\
T\left(\frac{n}{2^2}\right) &\leq 2T\left(\frac{n}{2^3}\right) + \frac{n}{2^2} + c
\end{align*}
\]
Chapter 4. Solving Recurrences

Solve recurrence

\[ T(n) \leq 2T\left(\frac{n}{2}\right) + n + c \quad \text{with base case} \ T(1) = c \]

with a simple method:

\[
\begin{align*}
T(n) & \leq 2T\left(\frac{n}{2}\right) + n + c \\
T\left(\frac{n}{2}\right) & \leq 2T\left(\frac{n}{2^2}\right) + \frac{n}{2} + c \\
T\left(\frac{n}{2^2}\right) & \leq 2T\left(\frac{n}{2^3}\right) + \frac{n}{2^2} + c \\
\ldots
\end{align*}
\]
Chapter 4. Solving Recurrences

Solve recurrence

\[ T(n) \leq 2T\left( \frac{n}{2} \right) + n + c \]  with base case \( T(1) = c \)

with a simple method:

\[
\begin{align*}
T(n) & \leq 2T\left( \frac{n}{2} \right) + n + c \\
T\left( \frac{n}{2} \right) & \leq 2T\left( \frac{n}{2^2} \right) + \frac{n}{2} + c \\
T\left( \frac{n}{2^2} \right) & \leq 2T\left( \frac{n}{2^3} \right) + \frac{n}{2^2} + c \\
\vdots \\
T\left( \frac{n}{2^h} \right) & \leq 2T\left( \frac{n}{2^{h+1}} \right) + \frac{n}{2^h} + c \quad \text{where} \quad \frac{n}{2^{h+1}} = 1 
\end{align*}
\]
Chapter 4. Solving Recurrences

Solve recurrence

\[ T(n) \leq 2T\left(\frac{n}{2}\right) + n + c \]  

with base case \( T(1) = c \)

with a simple method:

\[ T(n) \leq 2T\left(\frac{n}{2}\right) + n + c \]
\[ T\left(\frac{n}{2}\right) \leq 2T\left(\frac{n}{2^2}\right) + \frac{n}{2} + c \]
\[ T\left(\frac{n}{2^2}\right) \leq 2T\left(\frac{n}{2^3}\right) + \frac{n}{2^2} + c \]
\[ \ldots \]
\[ T\left(\frac{n}{2^h}\right) \leq 2T\left(\frac{n}{2^{h+1}}\right) + \frac{n}{2^h} + c \]

where \( \frac{n}{2^{h+1}} = 1 \)

multiplying \( 2, 2^2, \ldots \) to the second, third, \ldots inequalities, respectively,
Chapter 4. Solving Recurrences

Solve recurrence

\[ T(n) \leq 2T\left(\frac{n}{2}\right) + n + c \quad \text{with base case } T(1) = c \]

with a simple method:

\[
\begin{align*}
T(n) &\leq 2T\left(\frac{n}{2}\right) + n + c \\
T\left(\frac{n}{2}\right) &\leq 2T\left(\frac{n}{2^2}\right) + \frac{n}{2} + c \\
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\vdots \\
T\left(\frac{n}{2^h}\right) &\leq 2T\left(\frac{n}{2^{h+1}}\right) + \frac{n}{2^h} + c \quad \text{where } \frac{n}{2^{h+1}} = 1
\end{align*}
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multiplying \(2, 2^2, \ldots\) to the second, third, \ldots inequalities, respectively,

\[ T(n) \leq 2T\left(\frac{n}{2}\right) + n + c \]
Solve recurrence

\[ T(n) \leq 2T\left(\frac{n}{2}\right) + n + c \]  

with base case \( T(1) = c \)

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Chapter 4. Solving Recurrences

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\[ T\left(\frac{n}{2^h}\right) \leq 2T\left(\frac{n}{2^{h+1}}\right) + \frac{n}{2^h} + c \quad \text{where } \frac{n}{2^{h+1}} = 1 \]

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Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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2^hT\left( \frac{n}{2^h} \right) & \leq 2^{h+1}T\left( \frac{n}{2^{h+1}} \right) + 2^h \times \frac{n}{2^h} + 2^h c \quad \text{where } \frac{n}{2^{h+1}} = 1 \\
+ & \\
T(n) & \leq 2^{h+1}T\left( \frac{n}{2^{h+1}} \right) + (h + 1)n + c \sum_{i=0}^{h} 2^i
\end{align*}
\]
Chapter 4. Solving Recurrences

\[ T(n) \leq 2^{h+1} T(n^{2h+1}) + (h+1)n + c h \sum_{i=0}^{2h+1} i \]

With \( n^{2h+1} = 1 \), we have \( n = 2^{h+1} \) or \( h+1 = \log_2 n \).

\[ T(n) \leq 2^{h+1} + n \log_2 n + c (2^{h+1} - 1) = cn + n \log_2 n + c(n-1) = O(n \log_2 n) \]

We need to prove the last equality, i.e., find constants \( a \) and \( k \) such that

\[ n \log_2 n + 2cn - c \leq an \log_2 n \]

when \( n > k \).

Choose \( a = 2 \). Then to make (1) holds, we need \( \log_2 n > 2c \). So \( k = 2^{2c} \) suffices.

That is, \( n \log_2 n + 2cn - c \leq 2n \log_2 n \) when \( n > k = 2^{2c} \).

So \( T(n) = O(n \log_2 n) \).
Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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So

\[ T(n) = O(n \log_2 n) \].
Chapter 4. Solving Recurrences

When $n$ is not a power of 2

• choose $m_n$ such that $m_n n$ is a power of 2 and the smallest such that $n \leq m_n$;

• $T(n) \leq T(m_n)$, why?

• assume $T$ to be monotonic;

• use the analysis we just did, $T(m_n) = O(m_n \log_2 m_n)$; that is, $\exists c, k, T(m_n) \leq cm_n \log_2 m_n$ when $m_n \geq k$;

• but $m_n < 2^n$, why?

• So $T(n) \leq T(m_n) \leq cm_n \log_2 m_n \leq 2cn \log_2 (2^n) \leq 2cn \log_2 n = c'n \log_2 n$, here $c' = 4c$, when $m_n \geq k$ ($\geq 4$), i.e., when $n \geq \lceil k/2 \rceil$ ($\geq 2$);

• therefore, $T(n) = O(n \log_2 n)$. 
Chapter 4. Solving Recurrences

When $n$ is not a power of 2
Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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When $n$ is not a power of 2

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Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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When $n$ is not a power of 2

- choose $m_n$ such that $m_n$ is a power of 2 and the smallest such that $n \leq m_n$;
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Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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- therefore, $T(n) = O(n \log_2 n)$. 
Chapter 4. Solving Recurrences

Methods for solving recurrences
Chapter 4. Solving Recurrences

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1. Substitution method (based on math induction)
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First we recall the principle of the math induction:

To prove a property $\mathcal{P}(n)$ for every natural number $n \geq 1$, it suffices to prove

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"The principle for dominos to fall".
Chapter 4. Solving Recurrences

Math induction comes with different forms or variants
Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

**Theorem**: Arithmetic sequence of first $n$ terms

$$1 + 2 + 3 + \cdots + n = \frac{n}{2}(n + 1)$$
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Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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So

$$1 + 2 + 3 + \cdots + n = \frac{n}{2}(n + 1)$$
Algorithm \textsc{Merge Sort}(A, p, r)

1. \textbf{if} \( p < r \)
2. \textbf{then} \( q = \left\lfloor \frac{p+r}{2} \right\rfloor \)
3. \textsc{Merge Sort}(A, p, q)
4. \textsc{Merge Sort}(A, q + 1, r)
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Chapter 4. Solving Recurrences

**MergeSort** has the time complexity:

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Use **substitution method** to prove that \( T(n) = O(n \log_2 n) \).
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Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

A little review on logarithm functions:

\begin{align*}
\log a^n + \log a^m &= \log a^{nm}; \\
\log a^n b &= b \log a^n, \text{ especially } \log a^1 n = -\log a n; \\
a^{\log a n} &= n; \\
\log a n &= \log b n \log b a = \frac{1}{\log b a} \log b n; \\
\log m a^n &= (\log a n)^m \neq \log a n^m.
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- \( \log_a n = \frac{\log_b n}{\log_b a} = \frac{1}{\log_b a} \log_b n; \)
- \( \log_a n = (\log_a n)^m \neq \log_a n^m. \)
Solving recurrence with the substitution method (guess then verify)

\[ T(n) = \frac{3}{2} T(\lfloor \frac{2n}{3} \rfloor) + n, \quad \text{where } T(1) = 2 \]
Chapter 4. Solving Recurrences

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Guess \( T(n) \leq cn \log_2 n \), for some constant \( c \) to be determined later.
Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

Solving recurrence with the **substitution method** (guess then verify)

\[
T(n) = \frac{3}{2} T(\left\lfloor \frac{2n}{3} \right\rfloor) + n, \text{ where } T(1) = 2
\]

**Guess** \(T(n) \leq cn \log_2 n\), for some constant \(c\) to be determined later.

**Verify with induction:**

Step 1, base case: \(T(1) = 2 \leq cn \log_2 n = 0\) does not hold for the guessed inequality.

Instead, we choose \(T(2)\) to be the base case. From the recurrence, we have

\[
T(2) = \frac{3}{2} T(\left\lfloor \frac{2n}{3} \right\rfloor) + n
\]
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\[ T(2) = \frac{3}{2} T(\lfloor \frac{2n}{3} \rfloor) + n = \frac{3}{2} T(\lfloor \frac{4}{3} \rfloor) + 2 \]
Chapter 4. Solving Recurrences

Solving recurrence with the **substitution method** (guess then verify)

\[ T(n) = \frac{3}{2} T(\lfloor \frac{2n}{3} \rfloor) + n, \quad \text{where } T(1) = 2 \]

**Guess** \( T(n) \leq cn \log_2 n \), for some constant \( c \) to be determined later.

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Step 1, base case: \( T(1) = 2 \leq cn \log_2 n = 0 \) **does not** hold for the guessed inequality.

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\[
T(2) = \frac{3}{2} T(\lfloor \frac{2 \cdot 2}{3} \rfloor) + 2 = \frac{3}{2} T(\lfloor \frac{4}{3} \rfloor) + 2 = \frac{3}{2} T(1) + 2 = 5
\]

We make \( T(2) = 5 \leq cn \log_2 n \) holds for \( c = 3 \), and when \( n \geq k = 2 \),
Chapter 4. Solving Recurrences

Solving recurrence with the substitution method (guess then verify)

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\[ T(\lfloor \frac{2n}{3} \rfloor) \leq c\lfloor \frac{2n}{3} \rfloor \log_2 \lfloor \frac{2n}{3} \rfloor \]
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Step 3, substitute it in the recurrence, we get

\[ T(n) = \frac{3}{2} T(\lfloor \frac{2n}{3} \rfloor) + n \leq \frac{3}{2} \left( c \lfloor \frac{2n}{3} \rfloor \log_2 \lfloor \frac{2n}{3} \rfloor \right) + n \]
Chapter 4. Solving Recurrences

Step 3, substitute it in the recurrence, we get

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\[ \leq \frac{3}{2} \left( c \frac{2n}{3} \log_2 \frac{2n}{3} \right) + n \]

Since \( \log_2 \frac{3}{2} > 0.5 \), when \( c \geq 2 \)
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Step 3, substitute it in the recurrence, we get

\[ T(n) = \frac{3}{2} T(\lceil \frac{2n}{3} \rceil) + n \leq \frac{3}{2} \left( c \lceil \frac{2n}{3} \rceil \log_2 \lceil \frac{2n}{3} \rceil \right) + n \]

\[ \leq \frac{3}{2} \left( c \frac{2n}{3} \log_2 \frac{2n}{3} \right) + n \leq cn(\log_2 n + \log_2 \frac{2}{3}) + n \]
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\[ \leq cn \left( \log_2 n - \log_2 \frac{3}{2} \right) + n \]
Step 3, substitute it in the recurrence, we get

\[ T(n) = \frac{3}{2} T\left(\lfloor \frac{2n}{3} \rfloor \right) + n \leq \frac{3}{2} (c \lfloor \frac{2n}{3} \rfloor \log_2 \lfloor \frac{2n}{3} \rfloor) + n \]

\[ \leq \frac{3}{2} (c \frac{2n}{3} \log_2 \frac{2n}{3}) + n \leq cn(\log_2 n + \log_2 \frac{2}{3}) + n \]

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\[
T(n) = \frac{3}{2} T\left(\left\lfloor \frac{2n}{3} \right\rfloor \right) + n \leq \frac{3}{2}\left(c\left\lfloor \frac{2n}{3} \right\rfloor \log_2\left\lfloor \frac{2n}{3} \right\rfloor \right) + n
\]

\[
\leq \frac{3}{2}\left(c\frac{2n}{3} \log_2 \frac{2n}{3} \right) + n \leq cn(\log_2 n + \log_2 \frac{2}{3}) + n
\]

\[
\leq cn(\log_2 n - \log_2 \frac{3}{2}) + n = cn \log_2 n - cn \log_2 \frac{3}{2} + n
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But we already know \( c = 3 \) from the base case proof, we have

\[
T(n) \leq cn \log_2 n - cn \log_2 \frac{3}{2} + n \leq cn \log_2 n
\]
Chapter 4. Solving Recurrences

2. Changing variables
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Example: \( T(n) = 2T(\sqrt{n}) + \log_2 n \)
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Example: $T(n) = 2T(\sqrt{n}) + \log_2 n$

Define $m = \log_2 n$, 

so $T(n) = O(m \log m) = O(\log n \log \log n)$. 

2. Changing variables

Example: $T(n) = 2T(\sqrt{n}) + \log_2 n$

Define $m = \log_2 n$, i.e., $n = 2^m$
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$T(2^m) = 2T(2^{m/2}) + m$
Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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so \( T(n) = T(2^m) = O(m \log m) = O(\log n \log \log n) \).
Chapter 4. Solving Recurrences

3. Recursive tree method
Chapter 4. Solving Recurrences

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By unfolding the recurrence to make a recursive-tree.
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(1) $T(n)$ is a tree with non-recursive terms as the root and recursive terms as its children.

(2) for each child, replace it with then non-recursive terms and produce children that are then recursive terms.
3. Recursive tree method

By unfolding the recurrence to make a recursive-tree.

(1) $T(n)$ is a tree with non-recursive terms as the root and recursive terms as its children.

(2) for each child, replace it with then non-recursive terms and produce children that are then recursive terms.

(3) repeat (2), expand the tree until all children are the base case.
Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$
Chapter 4. Solving Recurrences

Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

$l_0$: 

\begin{align*}
& T(n) \\
\end{align*}
Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

$l_0$: 

$l_1$: $T(n/4)$ $T(n/4)$ $T(n/4)$
Chapter 4. Solving Recurrences

Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

\begin{align*}
  l_0: & & T(n) \\
  l_1: & & T(n/4) & T(n/4) & T(n/4) & n^2
\end{align*}
Chapter 4. Solving Recurrences

Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

$l_0$:

$l_1$:

$T(n/4)$

$n^2$
Chapter 4. Solving Recurrences

Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

\[\begin{align*}
l_0: & \quad T(n) \\
l_1: & \quad T(n/4) \\
l_2: & \quad T(n/4) T(n/4) T(n/4) \\
\end{align*}\]
Chapter 4. Solving Recurrences

Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

\[
\begin{array}{cccc}
l_0: & T(n) \\
l_1: & T(n/4) & T(n/4) \\
l_2: & T(n/4^2) & T(n/4^2) & T(n/4^2) & T(n/4^2) & T(n/4^2) & T(n/4^2) & n^2 \\
\end{array}
\]
Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

$l_0$: $T(n)

l_1$: $T(n/4)$

$l_2$: $T(n/4)$ $T(n/4)$ $T(n/4)$ $T(n/4)$ $T(n/4)$ $T(n/4)$ $T(n/4)$ $T(n/4)$ $n^2$

$l_3$: $3(n/4)^2$
Chapter 4. Solving Recurrences

Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

$l_0$: $T(n)$

$l_1$: \[ T(n/4) \quad T(n/4) \quad T(n/4) \]

$l_2$: \[ T(n/4^2) \quad T(n/4^2) \quad T(n/4^2) \quad T(n/4^2) \quad T(n/4^2) \quad T(n/4^2) \]

$l_3$: \[ \ldots \ldots \]

\[ 3(n^2) \quad 3(n^2)^2 \quad 3^2(n^2)^2 \]

\[ n^2 \quad \log_4 n \quad \leq \quad n^2 \quad \frac{1}{1 - 3/4} \quad 16 \frac{1}{13} n^2 \]
Chapter 4. Solving Recurrences

Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

$l_0$: 

$l_1$: 

$l_2$: $T(n/4) T(n/4) T(n/4)$

$l_3$: ......

$l_4$: ......
Chapter 4. Solving Recurrences

Example \( T(n) = 3T(n/4) + n^2 \), with base case \( T(1) = 1 \)

- \( l_0 \): \( T(n) \) \( l_1 \): \( T(n/4) \)
- \( l_2 \): \( T(n/4) \) \( T(n/4) \) \( T(n/4) \) \( T(n/4) \) \( T(n/4) \) \( T(n/4) \) \( n^2 \)
- \( l_3 \): \( \ldots \ldots \)
- \( l_4 \): \( \ldots \ldots \)

where \( n/4 \leq 1 \), i.e., \( m = \log_4 n \). Then \( T(n) \) is the sum

\[
T(n) = n^2 \left[ 1 + 3 \left( \frac{1}{4} \right)^2 + 3 \left( \frac{1}{4} \right)^2 \left( \frac{1}{4} \right)^2 + \cdots + 3 \left( \frac{1}{4} \right)^{m-1} \right] + 3 \times 1 = n^2 \left[ 1 + \frac{3}{16} + \left( \frac{3}{16} \right)^2 + \cdots + \left( \frac{3}{16} \right)^{m} \right]
\]

\[
= n^2 \left[ 1 - \left( \frac{3}{16} \right)^{m+1} \right] \leq n^2 \left( 1 - \frac{3}{16} \right) = \frac{13}{16} n^2
\]

for all \( n > 0 \).
Chapter 4. Solving Recurrences

Example \( T(n) = 3T(n/4) + n^2 \), with base case \( T(1) = 1 \)

\[
\begin{array}{lllllllll}
    & & & & & & & & \\
\text{l}_0: & & & & & & & & \\
\text{l}_1: & T(n/4) & T(n/4) & & & & & & \\
\text{l}_2: & T(n/4) & T(n/4) & T(n/4) & & & & & & \\
\text{l}_3: & \ldots \ldots & & & & & & & & \\
\text{l}_4: & \ldots \ldots & & & & & & & & \\
\text{l}_5: & \ldots \ldots & & & & & & & & \\
\end{array}
\]
Chapter 4. Solving Recurrences

Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

$l_0$: $T(n)$
$l_1$: $T(n/4)$ $T(n/4)$ $T(n/4)$
$l_2$: $T(n/4)$ $T(n/4)$ $T(n/4)$ $T(n/4)$ $T(n/4)$ $T(n/4)$ $T(n/4)$ $n^2$
$l_3$: .......
$l_4$: .......
$l_5$: .......
$l_{m−1}$: .......

where $n_{4^m−2} = 1$, i.e., $m = \log_4 n$.

Then $T(n)$ is the sum

$$T(n) = n^2\left[1 + 3\left(\frac{n}{4}\right)^2 + 3^2\left(\frac{n}{4^2}\right)^2 + 3^3\left(\frac{n}{4^3}\right)^2 + \cdots + 3^{m−1}\left(\frac{n}{4^{m−2}}\right)^2\right] + 3mT(1)$$

$$= n^2\left[1 + 3\left(\frac{1}{16}\right) + \left(\frac{3}{16}\right)^2 + \left(\frac{3^2}{16}\right)^2 + \cdots + \left(\frac{3^{m−1}}{16}\right)^2\right] + 3mT(1)$$

$$\leq n^2\left[1 - \frac{\left(\frac{3}{16}\right)^m}{1 - \frac{3}{16}}\right] + 3mT(1)$$

for all $n > 0$. 

Chapter 4. Solving Recurrences

Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

\begin{align*}
l_0: & & T(n) \\
l_1: & & T(n/4) & & T(n/4) \\
l_2: & & T(n/4^2) & & T(n/4^2) & & T(n/4^2) \\
l_3: & & \ldots \ldots \\
l_4: & & \ldots \ldots \\
l_5: & & \ldots \ldots \\
l_{m-1}: & & \ldots \ldots \\
l_m: & & 3^{m-2}(n/4^{m-2})^2
\end{align*}
Chapter 4. Solving Recurrences

Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

$l_0$: $T(n)$

$l_1$: $T(n/4)$ $T(n/4)$ $n^2$

$l_2$: $T(n/4^2)$ $T(n/4^2)$ $T(n/4^2)$ $T(n/4^2)$ $T(n/4^2)$ $3(n/4)^2$

$l_3$: $T(n/4^3)$ $T(n/4^3)$ $T(n/4^3)$ $T(n/4^3)$ $T(n/4^3)$ $3^2(n/4)^2$

$l_4$: $T(n/4^4)$ $T(n/4^4)$ $T(n/4^4)$ $T(n/4^4)$ $T(n/4^4)$ $3^3(n/4)^2$

$l_5$: $T(n/4^5)$ $T(n/4^5)$ $T(n/4^5)$ $T(n/4^5)$ $T(n/4^5)$ $3^4(n/4)^2$

$l_{m-1}$: $T(n/4^{m-1})$ $T(n/4^{m-1})$ $T(n/4^{m-1})$ $T(n/4^{m-1})$ $T(n/4^{m-1})$ $3^{m-2}(n/4^{m-2})^2$

$l_m$: $T(1)$ $T(1)$ $T(1)$ $T(1)$ $T(1)$ $\ldots$ $T(1)$ $3^{m-1}(n/4^{m-1})^2$
Chapter 4. Solving Recurrences

Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

$l_0$: 

$l_1$: 

$l_2$: $T(n/4)$ $T(n/4)$ $T(n/4)$ $T(n/4)$ $T(n/4)$ $T(n/4)$ $n^2$

$l_3$: . . . .

$l_4$: . . . .

$l_5$: . . . .

$l_{m-1}$: . . . .

$l_m$: $T(1), T(1), T(1), T(1), T(1), . . . , T(1)$

where $\frac{n}{4^m} = 1$, i.e., $m = \log_4 n$. 

\[ 3^{m-2}(\frac{n}{4^{m-2}})^2 \]
\[ 3^{m-1}(\frac{n}{4^{m-1}})^2 \]
Chapter 4. Solving Recurrences

Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

$l_0$: 

$l_1$: $T(n/4)$

$l_2$: $T(n/4)$ $T(n/4)$ $T(n/4)$ $T(n/4)$ $T(n/4)$ $T(n/4)$ $n^2$

$l_3$: $T(n/4)$ $T(n/4)$ $T(n/4)$ $T(n/4)$ $T(n/4)$ $T(n/4)$ $3(\frac{n}{4})^2$

$l_4$: $T(n/4)$ $T(n/4)$ $T(n/4)$ $T(n/4)$ $T(n/4)$ $T(n/4)$ $3^2(\frac{n}{4^2})^2$

$l_5$: $T(n/4)$ $T(n/4)$ $T(n/4)$ $T(n/4)$ $T(n/4)$ $T(n/4)$ $3^3(\frac{n}{4^3})^2$

$l_{m-1}$: $T(n/4)$ $T(n/4)$ $T(n/4)$ $T(n/4)$ $T(n/4)$ $T(n/4)$ $3^{m-2}(\frac{n}{4^{m-2}})^2$

$l_m$: $T(n/4)$ $T(n/4)$ $T(n/4)$ $T(n/4)$ $T(n/4)$ $T(n/4)$ $3^{m-1}(\frac{n}{4^{m-1}})^2$

where $\frac{n}{4^m} = 1$, i.e., $m = \log_4 n$.

Then $T(n)$ is the sum

$T(n) = n^2[1 + 3(\frac{1}{4})^2 + 3^2(\frac{1}{4^2})^2 + 3^3(\frac{1}{4^3})^2 + \ldots + 3^{m-1}(\frac{1}{4^{m-1}})^2] + 3^m T(1)$
Chapter 4. Solving Recurrences

Example \( T(n) = 3T(n/4) + n^2 \), with base case \( T(1) = 1 \)

\[ \begin{align*}
  l_0: & \quad T(n) \\
  l_1: & \quad T(n/4) \quad T(n/4) \\
  l_2: & \quad T(n/4^2) \quad T(n/4^2) \quad T(n/4^2) \\
  l_3: & \quad \ldots. \\
  l_4: & \quad \ldots. \\
  l_5: & \quad \ldots. \\
  l_{m-1}: & \quad \ldots. \\
  l_m: & \quad T(1), T(1), T(1), T(1), T(1), \ldots, T(1) \\
\end{align*} \]

where \( \frac{n}{4^m} = 1 \), i.e., \( m = \log_4 n \).

Then \( T(n) \) is the sum

\[ T(n) = n^2[1 + 3\left(\frac{1}{4}\right)^2 + 3^2\left(\frac{1}{4^2}\right)^2 + 3^3\left(\frac{1}{4^3}\right)^2 + \cdots + 3^{m-1}\left(\frac{1}{4^{m-1}}\right)^2] + 3^m T(1) \]

\[ T(n) = n^2[1 + 3\left(\frac{1}{4}\right)^2 + 3^2\left(\frac{1}{4^2}\right)^2 + 3^3\left(\frac{1}{4^3}\right)^2 + \cdots + 3^{m-1}\left(\frac{1}{4^{m-1}}\right)^2] + 3^m \times 1 \]
Chapter 4. Solving Recurrences

Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

$l_0$: $T(n)$

$l_1$: $T(n/4)$

$l_2$: $T(n/4) T(n/4) T(n/4)$

$l_3$: $\ldots$

$l_4$: $\ldots$

$l_5$: $\ldots$

$l_{m-1}$: $\ldots$

$l_m$: $T(1), T(1), T(1), T(1), T(1), \ldots, T(1)$

where $\frac{n}{4^m} = 1$, i.e., $m = \log_4 n$.

Then $T(n)$ is the sum

\[ T(n) = n^2[1 + 3\left(\frac{1}{4}\right)^2 + 3^2\left(\frac{1}{4^2}\right)^2 + 3^3\left(\frac{1}{4^3}\right)^2 + \cdots + 3^{m-1}\left(\frac{1}{4^{m-1}}\right)^2] + 3^m T(1) \]

\[ T(n) = n^2[1 + 3\left(\frac{1}{4}\right)^2 + 3^2\left(\frac{1}{4^2}\right)^2 + 3^3\left(\frac{1}{4^3}\right)^2 + \cdots + 3^{m-1}\left(\frac{1}{4^{m-1}}\right)^2] + 3^m \times 1 \]

\[ = n^2[1 + 3\left(\frac{1}{4}\right)^2 + 3^2\left(\frac{1}{4^2}\right)^2 + 3^3\left(\frac{1}{4^3}\right)^2 + \cdots + 3^{m-1}\left(\frac{1}{4^{m-1}}\right)^2] + 3^m \left(\frac{n}{4^m}\right)^2 \]
Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

\[
\begin{align*}
  l_0: & \quad T(n) \\
  l_1: & \quad n \quad T(n/4) \\
  l_2: & \quad T(n/4) T(n/4) T(n/4) \\
  l_3: & \quad \ldots \\
  l_4: & \quad \ldots \\
  l_5: & \quad \ldots \\
  l_{m-1}: & \quad \ldots \\
  l_m: & \quad T(1), T(1), T(1), T(1), T(1), \ldots, T(1)
\end{align*}
\]

where $\frac{n}{4^m} = 1$, i.e., $m = \log_4 n$.

Then $T(n)$ is the sum

\[
T(n) = n^2\left[1 + 3\left(\frac{1}{4}\right)^2 + 3^2\left(\frac{1}{4^2}\right)^2 + 3^3\left(\frac{1}{4^3}\right)^2 + \cdots + 3^{m-1}\left(\frac{1}{4^{m-1}}\right)^2\right] + 3^m T(1)
\]

\[
T(n) = n^2\left[1 + 3\left(\frac{1}{4}\right)^2 + 3^2\left(\frac{1}{4^2}\right)^2 + 3^3\left(\frac{1}{4^3}\right)^2 + \cdots + 3^{m-1}\left(\frac{1}{4^{m-1}}\right)^2\right] + 3^m \times 1
\]

\[
= n^2\left[1 + 3\left(\frac{1}{4}\right)^2 + 3^2\left(\frac{1}{4^2}\right)^2 + 3^3\left(\frac{1}{4^3}\right)^2 + \cdots + 3^{m-1}\left(\frac{1}{4^{m-1}}\right)^2\right] + 3^m \left(\frac{n}{4^m}\right)^2
\]

\[
= n^2\left[1 + \frac{3}{16} + \left(\frac{3}{16}\right)^2 + \left(\frac{3}{16}\right)^3 + \cdots + \left(\frac{3}{16}\right)^m\right]
\]
Chapter 4. Solving Recurrences

Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

\[
\begin{align*}
  l_0: & \quad T(n) \\
  l_1: & \quad \frac{T(n)}{4} \\
  l_2: & \quad T\left(\frac{n}{4^2}\right) T\left(\frac{n}{4^2}\right) T\left(\frac{n}{4^2}\right) T\left(\frac{n}{4^2}\right) T\left(\frac{n}{4^2}\right) T\left(\frac{n}{4^2}\right) T\left(\frac{n}{4^2}\right) T\left(\frac{n}{4^2}\right) \quad n^2 \\
  l_3: & \quad \ldots \ldots \\
  l_4: & \quad \ldots \ldots \\
  l_5: & \quad \ldots \ldots \\
  l_{m-1}: & \quad \ldots \ldots \\
  l_m: & \quad T(1), T(1), T(1), T(1), T(1), \ldots, T(1)
\end{align*}
\]

where $\frac{n}{4^m} = 1$, i.e., $m = \log_4 n$.

Then $T(n)$ is the sum

\[
\begin{align*}
  T(n) &= n^2[1 + 3\left(\frac{1}{4}\right)^2 + 3^2\left(\frac{1}{4^2}\right)^2 + 3^3\left(\frac{1}{4^3}\right)^2 + \cdots + 3^{m-1}\left(\frac{1}{4^{m-1}}\right)^2] + 3^m T(1) \\
  T(n) &= n^2[1 + 3\left(\frac{1}{4}\right)^2 + 3^2\left(\frac{1}{4^2}\right)^2 + 3^3\left(\frac{1}{4^3}\right)^2 + \cdots + 3^{m-1}\left(\frac{1}{4^{m-1}}\right)^2] + 3^m \times 1 \\
  &= n^2[1 + 3\left(\frac{1}{4}\right)^2 + 3^2\left(\frac{1}{4^2}\right)^2 + 3^3\left(\frac{1}{4^3}\right)^2 + \cdots + 3^{m-1}\left(\frac{1}{4^{m-1}}\right)^2] + 3^m \left(\frac{n}{4^m}\right)^2 \\
  &= n^2[1 + \frac{3}{16} + \left(\frac{3}{16}\right)^2 + \left(\frac{3}{16}\right)^3 + \cdots + \left(\frac{3}{16}\right)^m] \\
  &= n^2\left(\frac{1 - \left(\frac{3}{16}\right)^{m+1}}{1 - \frac{3}{16}}\right)
\end{align*}
\]
Chapter 4. Solving Recurrences

Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

$l_0$: 

$l_1$: 

$l_2$: $T(n/4)$ $T(n/4)$ $T(n/4)$ $T(n/4)$ $T(n/4)$ $T(n/4)$ $T(n/4)$ $T(n/4)$ $n^2$

$l_3$: 

$l_4$: 

$l_5$: 

$l_{m-1}$: 

$l_m$: $T(1), T(1), T(1), T(1), T(1), \ldots, T(1)$

where $\frac{n}{4^m} = 1$, i.e., $m = \log_4 n$.

Then $T(n)$ is the sum

$T(n) = n^2[1 + 3\left(\frac{1}{4}\right)^2 + 3^2\left(\frac{1}{4^2}\right)^2 + 3^3\left(\frac{1}{4^3}\right)^2 + \cdots + 3^{m-1}\left(\frac{1}{4^{m-1}}\right)^2] + 3^m T(1)$

$T(n) = n^2[1 + 3\left(\frac{1}{4}\right)^2 + 3^2\left(\frac{1}{4^2}\right)^2 + 3^3\left(\frac{1}{4^3}\right)^2 + \cdots + 3^{m-1}\left(\frac{1}{4^{m-1}}\right)^2] + 3^m \times 1$

$= n^2[1 + 3\left(\frac{1}{4}\right)^2 + 3^2\left(\frac{1}{4^2}\right)^2 + 3^3\left(\frac{1}{4^3}\right)^2 + \cdots + 3^{m-1}\left(\frac{1}{4^{m-1}}\right)^2] + 3^m \left(\frac{n}{4^m}\right)^2$

$= n^2[1 + \frac{3}{16} + \left(\frac{3}{16}\right)^2 + \left(\frac{3}{16}\right)^3 + \cdots + \left(\frac{3}{16}\right)^m]$

$= n^2\left(\frac{1 - \left(\frac{3}{16}\right)^{m+1}}{1 - \frac{3}{16}}\right)$

$\leq n^2\left(\frac{1}{1 - \frac{3}{16}}\right)$
Chapter 4. Solving Recurrences

Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

\begin{align*}
l_0: & \quad T(n) \\
l_1: & \quad T(n/4) \\
l_2: & \quad T(n/4) T(n/4) T(n/4) T(n/4) T(n/4) T(n/4) T(n/4) T(n/4) T(n/4) \quad n^2 \\
l_3: & \quad \ldots \ldots \\
l_4: & \quad \ldots \ldots \\
l_5: & \quad \ldots \ldots \\
l_{m-1}: & \quad \ldots \ldots \\
l_m: & \quad T(1), T(1), T(1), T(1), T(1), \ldots, T(1)
\end{align*}

where $\frac{n}{4^m} = 1$, i.e., $m = \log_4 n$.

Then $T(n)$ is the sum

\begin{align*}
T(n) &= n^2[1 + 3(\frac{1}{4})^2 + 3^2(\frac{1}{4^2})^2 + 3^3(\frac{1}{4^3})^2 + \cdots + 3^{m-1}(\frac{1}{4^m})^2] + 3^m T(1) \\
T(n) &= n^2[1 + 3(\frac{1}{4})^2 + 3^2(\frac{1}{4^2})^2 + 3^3(\frac{1}{4^3})^2 + \cdots + 3^{m-1}(\frac{1}{4^m})^2] + 3^m \times 1 \\
&= n^2[1 + 3(\frac{1}{4})^2 + 3^2(\frac{1}{4^2})^2 + 3^3(\frac{1}{4^3})^2 + \cdots + 3^{m-1}(\frac{1}{4^m})^2] + 3^m(\frac{n}{4^m})^2 \\
&= n^2[1 + \frac{3}{16} + (\frac{3}{16})^2 + (\frac{3}{16})^3 + \cdots + (\frac{3}{16})^m] \\
&= n^2\left(\frac{\frac{1}{16} - (\frac{3}{16})^{m+1}}{1 - \frac{3}{16}}\right) \\
&\leq n^2\left(\frac{1}{1 - \frac{3}{16}}\right) \\
&= \frac{16}{13} n^2
\end{align*}

for all $n \geq 0$. 
Chapter 4. Solving Recurrences

Recursive tree notation in the textbook:

\[ T(n) = 3T(n/4) + n^2, \text{ with base case } T(1) = 1 \]

\[ l_0: \quad n^2 \]
Chapter 4. Solving Recurrences

Recursive tree notation in the textbook:

\[ T(n) = 3T(n/4) + n^2, \text{ with base case } T(1) = 1 \]

- \( l_0: \) 
  - \( (n/4)^2 \)
  - \( n^2 \)
  - \( (n/4)^2 \)

- \( l_1: \) 
  - \( (n/4)^2 \)
  - \( (n/4)^2 \)
  - \( (n/4)^2 \)
Chapter 4. Solving Recurrences

Recursive tree notation in the textbook:

\[ T(n) = 3T(n/4) + n^2, \text{ with base case } T(1) = 1 \]

\[ l_0: \]
\[ l_1: \]
\[ l_2: \]
\[ \left( \frac{n}{4^2} \right)^2 \left( \frac{n}{4^2} \right)^2 \left( \frac{n}{4^2} \right)^2 \]
\[ \left( \frac{n}{4^2} \right)^2 \left( \frac{n}{4^2} \right)^2 \left( \frac{n}{4^2} \right)^2 \]
\[ \left( \frac{n}{4^2} \right)^2 \left( \frac{n}{4^2} \right)^2 \left( \frac{n}{4^2} \right)^2 \]
\[ \left( \frac{n}{4^2} \right)^2 \left( \frac{n}{4^2} \right)^2 \left( \frac{n}{4^2} \right)^2 \]
\[ \left( \frac{n}{4^2} \right)^2 \left( \frac{n}{4^2} \right)^2 \left( \frac{n}{4^2} \right)^2 \]
\[ \left( \frac{n}{4^2} \right)^2 \left( \frac{n}{4^2} \right)^2 \left( \frac{n}{4^2} \right)^2 \]
\[ \left( \frac{n}{4^2} \right)^2 \left( \frac{n}{4^2} \right)^2 \left( \frac{n}{4^2} \right)^2 \]
\[ \left( \frac{n}{4^2} \right)^2 \left( \frac{n}{4^2} \right)^2 \left( \frac{n}{4^2} \right)^2 \]
\[ \left( \frac{n}{4^2} \right)^2 \left( \frac{n}{4^2} \right)^2 \left( \frac{n}{4^2} \right)^2 \]
Chapter 4. Solving Recurrences

Recursive tree notation in the textbook:

\[ T(n) = 3T(n/4) + n^2, \text{ with base case } T(1) = 1 \]

\( l_0: \)

\( l_1: \) \( (n/4)^2 \) \( (n/4)^2 \) \( (n/4)^2 \) \( (n/4)^2 \) \( (n/4)^2 \) \( (n/4)^2 \) \( (n/4)^2 \) \( (n/4)^2 \) \( (n/4)^2 \) \( (n/4)^2 \)

\( l_2: \) \( (n/4)^2 \) \( (n/4)^2 \) \( (n/4)^2 \) \( (n/4)^2 \) \( (n/4)^2 \) \( (n/4)^2 \) \( (n/4)^2 \) \( (n/4)^2 \) \( (n/4)^2 \) \( (n/4)^2 \) \( (n/4)^2 \) \( (n/4)^2 \) \( (n/4)^2 \)

\( l_3: \) \ldots
Recursive tree notation in the textbook:

\[ T(n) = 3T(n/4) + n^2, \text{ with base case } T(1) = 1 \]

\[ l_0: \quad n^2 \]
\[ l_1: \quad (n/4)^2 \quad (n/4)^2 \quad (n/4)^2 \quad (n/4)^2 \]
\[ l_2: \quad \left( \frac{n}{4^2} \right)^2 \quad \left( \frac{n}{4^2} \right)^2 \quad \left( \frac{n}{4^2} \right)^2 \quad \left( \frac{n}{4^2} \right)^2 \quad \left( \frac{n}{4^2} \right)^2 \quad \left( \frac{n}{4^2} \right)^2 \]
\[ l_3: \ldots \ldots \]
\[ l_4: \ldots \ldots \]

\(3^3\) nodes of \(\left( \frac{n}{4^3} \right)^2\)
Recursive tree notation in the textbook:

\[ T(n) = 3T(n/4) + n^2, \text{ with base case } T(1) = 1 \]

\[ l_0: \]
\[ l_1: \quad (n/4)^2 \quad (n/4)^2 \quad (n/4)^2 \quad (n/4)^2 \quad (n/4)^2 \quad (n/4)^2 \quad (n/4)^2 \quad (n/4)^2 \quad (n/4)^2 \quad (n/4)^2 \quad (n/4)^2 \quad (n/4)^2 \]

\[ l_2: \quad \left(\frac{n}{4^2}\right)^2 \quad \left(\frac{n}{4^2}\right)^2 \quad \left(\frac{n}{4^2}\right)^2 \quad \left(\frac{n}{4^2}\right)^2 \quad \left(\frac{n}{4^2}\right)^2 \quad \left(\frac{n}{4^2}\right)^2 \quad \left(\frac{n}{4^2}\right)^2 \quad \left(\frac{n}{4^2}\right)^2 \quad \left(\frac{n}{4^2}\right)^2 \quad \left(\frac{n}{4^2}\right)^2 \quad \left(\frac{n}{4^2}\right)^2 \quad \left(\frac{n}{4^2}\right)^2 \]

\[ l_3: \quad \ldots \]

\[ l_4: \quad \ldots \]

\[ l_{m-1}: \quad \ldots \]

\[ 3^3 \text{ nodes of } \left(\frac{n}{4^3}\right)^2 \]
Chapter 4. Solving Recurrences

Recursive tree notation in the textbook:

\[ T(n) = 3T(n/4) + n^2, \text{ with base case } T(1) = 1 \]

\[ l_0: \]
\[ l_1: \quad (n/4)^2 \quad (n/4)^2 \quad \frac{n^2}{4^2} \]
\[ l_2: \quad (n/4)^2 \quad (n/4)^2 \quad (n/4)^2 \quad \frac{n^2}{4^2} \]
\[ l_3: \quad \ldots \ldots \]
\[ l_4: \quad \ldots \ldots \]
\[ l_{m-1}: \quad \ldots \ldots \]

\[ 3^3 \text{ nodes of } \left(\frac{n}{4^3}\right)^2 \]
\[ 3^{m-1} \text{ of } \left(\frac{n}{4^{m-1}}\right)^2 \]
Chapter 4. Solving Recurrences

Recursive tree notation in the textbook:

\[ T(n) = 3T(n/4) + n^2, \text{ with base case } T(1) = 1 \]

\[ l_0: \quad n^2 \]
\[ l_1: \quad (n/4)^2 \quad (n/4)^2 \quad (n/4)^2 \]
\[ l_2: \quad \left( \frac{n}{4^2} \right)^2 \left( \frac{n}{4^2} \right)^2 \left( \frac{n}{4^2} \right)^2 \left( \frac{n}{4^2} \right)^2 \left( \frac{n}{4^2} \right)^2 \left( \frac{n}{4^2} \right)^2 \left( \frac{n}{4^2} \right)^2 \left( \frac{n}{4^2} \right)^2 \left( \frac{n}{4^2} \right)^2 \left( \frac{n}{4^2} \right)^2 \left( \frac{n}{4^2} \right)^2 \]
\[ l_3: \quad \ldots \ldots \]
\[ l_4: \quad \ldots \ldots \]
\[ l_{m-1}: \quad \ldots \ldots \]
\[ l_m: \quad T(1), T(1), T(1), T(1), T(1), \ldots, \]

\[ 3^3 \text{ nodes of } \left( \frac{n}{4^3} \right)^2 \]
\[ 3^{m-1} \text{ of } \left( \frac{n}{4^{m-1}} \right)^2 \]
\[ 3^m \text{ nodes of } T(1) \]
Chapter 4. Solving Recurrences

Recursive tree notation in the textbook:

\[ T(n) = 3T(n/4) + n^2, \text{ with base case } T(1) = 1 \]

- \( l_0: \quad n^2 \)
- \( l_1: \quad T(n/4), T(n/4), T(n/4), T(n/4) \)
- \( l_2: \quad (n/4)^2, (n/4)^2, (n/4)^2, (n/4)^2 \)
- \( l_3: \quad \ldots \ldots \)
- \( l_4: \quad \ldots \ldots \)
- \( l_{m-1}: \quad \ldots \ldots \)
- \( l_m: \quad T(1), T(1), T(1), T(1), T(1), \ldots \)

3\(^m\) nodes of \( T(1) \)

3\(^{m-1}\) of \( (\frac{n}{4^{m-1}})^2 \)

3\(^3\) nodes of \( (\frac{n}{4^3})^2 \)
Another example (page 91 textbook):

Assume time function $T(n)$ of some algorithm has the recurrence

$$T(n) = T\left(\frac{n}{3}\right) + T\left(\frac{2n}{3}\right) + n$$

with base case $T(1) = T(2) = T(3) = c > 0$, a constant.

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(1) using recursive tree method to derive an upper bound

(2) using substitution method to verify the upper bound
Chapter 4. Solving Recurrences

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\[
\begin{align*}
l_0: & \quad T(n) \\
l_1: & \quad T\left(\frac{n}{3}\right) \quad T\left(\frac{2n}{3}\right) \\
\end{align*}
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l_2: & \quad T\left(\frac{n}{3^2}\right) \quad T\left(\frac{2n}{3^2}\right) \quad T\left(\frac{2^2n}{3^2}\right) \\
& \quad \frac{n}{3} + \frac{2n}{3} = n
\end{align*}
Chapter 4. Solving Recurrences

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\begin{align*}
&l_0: \quad T(n) \\
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&l_2: \quad T\left(\frac{n}{3^2}\right) \quad T\left(\frac{2n}{3^2}\right) \quad T\left(\frac{2n}{3^2}\right) \quad T\left(\frac{4n}{3^2}\right) \quad \frac{n}{3} + \frac{2n}{3} = n \\
&\vdots \\
&l_{m-1}: \quad T\left(\frac{n}{3^{m-1}}\right) \quad \ldots \quad T\left(\frac{2^{m-1}n}{3^{m-1}}\right) \quad n
\end{align*}
(1) using recursive tree method to derive an upper bound

\begin{align*}
\ell_0 &: T(n) \\
\ell_1 &: T\left(\frac{n}{3}\right), T\left(\frac{2n}{3}\right) \\
\ell_2 &: T\left(\frac{n}{3^2}\right), T\left(\frac{2n}{3^2}\right), T\left(\frac{2^2 n}{3^2}\right) \\
& \quad \vdots \\
\ell_{m-1} &: T\left(\frac{n}{3^{m-1}}\right), \ldots \\
\ell_m &: T\left(\frac{n}{3^m}\right), T\left(\frac{2n}{3^m}\right), \ldots \\
\ell_{\ell + 1} &: \ldots \,, T\left(\frac{2^{\ell - 1} n}{3^{m-1}}\right), T\left(\frac{2^\ell n}{3^m}\right) < n
\end{align*}
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\ldots

\( l_{m-1}: \quad T\left(\frac{n}{3^{m-1}}\right) \ldots \quad T\left(\frac{2^{m-1}n}{3^{m-1}}\right) \quad n \)

\( l_m: \quad T\left(\frac{n}{3^m}\right) \quad T\left(\frac{2n}{3^m}\right) \ldots \quad T\left(\frac{2^mn}{3^m}\right) \quad n \)

\( l_{m+1}: \quad \ldots \quad T\left(\frac{2^{m+1}n}{3^{m+1}}\right) < n \)
Chapter 4. Solving Recurrences

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\[ \frac{n}{3} + \frac{2n}{3} = n \]
\[ \ldots \]
\[ l_{m-1}: \quad T\left(\frac{n}{3^{m-1}}\right) \ldots \]
\[ l_m: \quad T\left(\frac{n}{3^m}\right) \quad T\left(\frac{2n}{3^m}\right) \ldots \]
\[ l_{m+1}: \quad \ldots \]
\[ \ldots \]
\[ l_r: \quad \ldots \]
\[ T\left(\frac{2^m n}{3^m}\right) < n \]
\[ T\left(\frac{2^{m+1} n}{3^{m+1}}\right) < n \]
\[ T\left(\frac{2^r n}{3^r}\right) < n \]
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(1) using recursive tree method to derive an upper bound
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- the leftmost branch is the shortest; the rightmost is the longest;

- $\frac{n}{3m} = 1$, $m = \log_3 n$; $\frac{2^r n}{3r} = 1$, $r = \log_\frac{3}{2} n$.

- beginning from level $m + 1$, some nodes will gradually disappear, so the number of leaves is NOT $2^{\log_\frac{3}{2} n} \neq O(n \log_2 n)$ (why?),

- we do not need to give an accurate account for the sum of all quantities in blue color and those at leaves, but only an estimated upper bound.

- we estimate an upper bound to be $O(n \log_2 n)$.
Chapter 4. Solving Recurrences

(2) using the substitution method to verify the upper bound

We prove that $T(n) = O(n \log_2 n)$.

That is to prove: $\exists a, k > 0$ such that $T(n) \leq an \log_2 n$ when $n \geq k$.

Base case: for $n = 3$, $T(3) = c \leq a \cdot 3 \log_2 3$ will be true if we choose $a \geq c$ because $\log_2 3 > 1$.

Assume that $T(n^3) \leq a \cdot n^3 \log_2 n^3$; and $T(2n^3) \leq a \cdot 2n^3 \log_2 2n^3$;

Then $T(n) = T(n^3) + T(2n^3) + n \leq an^3 \log_2 n^3 + an^3 \log_2 2n^3 + n = an \log_2 n - an^3 (\log_2 3 + 2 \log_2 3) + n$.
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Then

$$T(n) = T(\frac{n}{3}) + T(\frac{2n}{3}) + n$$

$$\leq a \frac{n}{3} \log_2 \frac{n}{3} + a \frac{2n}{3} \log_2 \frac{2n}{3} + n$$

$$= \frac{an}{3} \log_2 n + \frac{an}{3} \log_2 \frac{1}{3} + \frac{2an}{3} \log_2 n + \frac{2an}{3} \log_2 \frac{2}{3} + n$$

$$= an \log_2 n - \frac{an}{3} (\log_2 3 + 2 \log_2 \frac{3}{2}) + n \quad (1)$$
Therefore,

\[ T(n) \leq an \log_2 n - \frac{an}{3} (\log_2 3 + \log_2 \frac{9}{4}) + n \]

\[ = an \log_2 n - \frac{an}{3} \log_2 \frac{27}{4} + n \]

\[ \leq an \log_2 n \]

when \( a \) is chosen such that \( a \geq \max\{3, c\} \) since it makes

\[ -\frac{an}{3} \log_2 \frac{27}{4} + n \leq 0 \]
Chapter 5. Probabilistic Analysis of Algorithms

Chapter 5. Probabilistic analysis and randomized algorithms
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Chapter 5. Probabilistic Analysis of Algorithms

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Chapter 5. Probabilistic Analysis of Algorithms

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  That is, we use randomized algorithms.
Chapter 5. Probabilistic Analysis of Algorithms

In both probabilistic analysis of deterministic algorithms and analysis of randomized algorithms,

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Chapter 5. Probabilistic Analysis of Algorithms

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There are two types of randomized algorithms:

- Las Vegas algorithms

- Monte Carlo algorithms
  - On 'NO' instances, 100% accuracy; \( \text{Prob} \left( \text{to answer ´NO´ on ´NO´ instance} \right) = 1 \)
  - On 'YES' instances, ≥ 75% accuracy; \( \text{Prob} \left( \text{to answer ´YES´ on ´YES´ instance} \right) ≥ 0.75 \)

Accuracy 75% can be improved to 99.99% with multiple trials.

Las Vegas algorithms is as powerful as Monte Carlo algorithms, if not more.
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