CSCI 4470/6470 Algorithms, Spring 2019

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Department of Computer Science, UGA


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An Introduction to the Introduction

There are two ways of constructing a software design: One way is to make it so simple that there are obviously no deficiencies, and the other way is to make it so complicated that there are no obvious deficiencies.

- C.A.R. Hoare

Which way do we take in algorithm design?
An Introduction to the Introduction

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An Introduction to the Introduction

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Which way do we take in algorithm design?
An Introduction to the Introduction

Example 1
An Introduction to the Introduction

Example 1

Sequence Homology Reveals Functions

- Homology reveals evolution of structure/function

<table>
<thead>
<tr>
<th></th>
<th>TCTCAACGTAACACCT</th>
<th>ATACAGCGGCGC</th>
<th>CGT CATTTGGAATGTAGCGCTCCCGATATGAAGG</th>
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<tbody>
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- Homology reveals regulatory structure (E. Coli promoters)

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<th>GTACATAGGAAGGAGAGT</th>
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<td>T7 A2</td>
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An Introduction to the Introduction

• main task: sequence comparison (consider just 2 sequences)
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FOSB_MOUSE -MFQAEPGDYDS-GSRCSS-SPSAESQ--YLSSVDSFGSPPTAAASQE-CAGLGMEMGSF 54
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- analogy in text searching/matching

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- total number of number of alignments $\geq 2^n$. To find the most plausible one, cannot afford to try them all

- require to design "smarter" algorithms that run much faster

We will study various techniques for efficient algorithm design.
An Introduction to the Introduction

- main task: sequence comparison (consider just 2 sequences)

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  but allowing substitutions, insertions, deletions

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An Introduction to the Introduction

Example 2
An Introduction to the Introduction
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• 128 players in total;
• 127 matches were played to determine who was the champion.

Is it possible to just play fewer matches?
• 128 players in total;
An Introduction to the Introduction

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• 127 matches were played to determine who was the champion.
• Is it possible to just play fewer matches?
An Introduction to the Introduction

Sounds crazy ... but if I don't do it, my job won't be secure...

See if you can reduce the number of matches!
An Introduction to the Introduction

You were called to answer this question;
An Introduction to the Introduction

- You tried various match formats, but all need at least 127 matches;
An Introduction to the Introduction

- You tried various match formats, but all need at least 127 matches;
- Then you suspected that 127 matches were necessary,
An Introduction to the Introduction

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An Introduction to the Introduction

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Sir, this won’t happen. I have a mathematical proof that it needs 127 matches!

Hmmm, he is smart... I should give him more work to do.
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</tr>
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Tennis Tournament

Finding Maximum

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<th>Input:</th>
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You actually accomplished two tasks:
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You actually accomplished two tasks:

- tried various algorithms for **TENNIS TOURNAMENT** before giving up;
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You actually accomplished two tasks:

- tried various algorithms for TENNIS TOURNAMENT before giving up;
- proved that more efficient algorithms do not exist;
An Introduction to the Introduction

Same problems

Tennis Tournament

Finding Maximum

Input: 128 players
Output: Champion
Solution: a match scheme
Time Efficiency: number of matches

Input: $n$ numbers
Output: the maximum number
Solution: an algorithm
Time Efficiency: number of comparisons

You actually accomplished two tasks:

• tried various algorithms for Tennis Tournament before giving up;
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• actually you proved the “USTA algorithm” was already the optimal.
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**We will learn how to prove that an algorithm is already the best.**
An Introduction to the Introduction

Example 3

Toss a coin over the phone

• how does one person know the other is telling the truth?

• even if the person is, would the first person trust him?

How to accomplish this task?
Example 3

Toss a coin over the phone

I will let you choose, head or tail?

Uh...
Example 3

Toss a coin over the phone

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How to accomplish this task?
They decided the following protocol:

- A told B a huge Boolean circuit through the phone; along with an output $Y$ of binary bits (but NOT the input $X$);
- B guessed if the number of 0's in $X$ was odd or even;
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An Introduction to the Introduction

X = 110101

Y = 111010
An Introduction to the Introduction

from $Y$, it would be very difficult to figure out which $X$ had been used to generate $Y$ (an intractable problem);
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• So the best B can do is to randomly guess (odd/even), which has the same effect as guessing a coin toss.
An Introduction to the Introduction

- from $Y$, it would be very difficult to figure out which $X$ had been used to generate $Y$ (an intractable problem);
- So the best B can do is to randomly guess (odd/even), which has the same effect as guessing a coin toss.

We will investigate some intractable problems and the theory behind it.
Now we are back to the tennis tournament problem.
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Algorithms you tried, unfortunately

- Algorithm A
- Algorithm Z
- USTA algorithm

No algorithm uses fewer than 127 matches

USTA algorithm gives an upper bound ($\leq 127$)

The Tennis Tournament (128) Problem

Your proof gives a lower bound ($\geq 127$)
An Introduction to the Introduction

Definitions of complexity upper and lower bounds;
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- **Time complexity upper bound** is a time threshold *in* which all instances of a problem can be solved.
An Introduction to the Introduction

Definitions of complexity upper and lower bounds;

• **Time complexity upper bound** is a time threshold *in* which all instances of a problem *can* be solved.

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An Introduction to the Introduction

Definitions of complexity upper and lower bounds;

- **Time complexity upper bound** is a time threshold *in* which all instances of a problem *can* be solved.

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- These notions apply to **both problems and algorithms**.
An Introduction to the Introduction

In general, for a given problem $\Pi$,
An Introduction to the Introduction

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- an algorithm $A$ for $\Pi$ also gives a time upper bound $T_A$ for $\Pi$, where $T_A$ is the time complexity of algorithm $A$;
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  a proof that $\Pi$ cannot have algorithms faster than time $S$, with $S > T$, also the corresponding algorithms (that offer the upper bound) are called optimal for $\Pi$. 
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An Introduction to the Introduction
An Introduction to the Introduction

Time complexity situation for problem $\Pi$

Algorithm A

Algorithm B

$T_A$

$T_B$

S

$T$

gap, the smaller the better

2

1

0
An Introduction to the Introduction

So to design good algorithms for computational problems, our goals are
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1. to achieve tighter upper bounds
An Introduction to the Introduction

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1. to achieve tighter upper bounds
   • we need be familiar with techniques for algorithm complexity analysis;
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So to design good algorithms for computational problems, our goals are

1. to achieve tighter upper bounds
   • we need be familiar with techniques for algorithm complexity analysis;
   • we need to master efficient algorithm design skills;
An Introduction to the Introduction

So to design good algorithms for computational problems, our goals are
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So to design good algorithms for computational problems, our goals are

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   • we need to be familiar with techniques for algorithm complexity analysis;
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2. and to achieve tighter lower bounds,
   • we need to know the methods for lower bound proofs.
The Introduction

What is this course about (and why is it needed)?
• about basic yet indispensable skills for problem solving
• algorithm design leads to writing code, i.e., creative thinking without programming languages

How different is this course from other algorithm courses?
• design technique-oriented, not application-oriented
• emphasis on guaranteed performance (typically in efficiency)

Goals to achieve
• to learn to measure performance of algorithms
• to master some fundamental algorithmic techniques
• to study advanced algorithmic skills
• to understand computational intractability
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Part I. Foundations
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- Chapter 1. The role of algorithms in computing
- Chapter 2. Getting started
- Chapter 3. Growth of functions
- Chapter 4. Solving recurrences
- Chapter 5. Probabilistic analysis and randomized algorithms
Part I. Foundations

The theme of the course
Part I. Foundations

The theme of the course

- Goal: learning techniques to design efficient algorithms
Part I. Foundations

The theme of the course

• Goal: learning techniques to design efficient algorithms

• mean: through developing skills to analyze algorithms
The theme of the course

- Goal: learning techniques to design efficient algorithms
- mean: through developing skills to analyze algorithms

Design and analysis of algorithms are closely related.
Example: the Fibonacci sequence.

\[ f(n) = \begin{cases} 
  f(n - 1) + f(n - 2) & \text{if } n \geq 3 \\
  1, & \text{otherwise}
\end{cases} \]
Example: the Fibonacci sequence.

\[ f(n) = \begin{cases} 
  f(n - 1) + f(n - 2) & \text{if } n \geq 3 \\
  1, & \text{otherwise}
\end{cases} \]

That is:

| $n$  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | ...
|------|----|----|----|----|----|----|----|----|----|...
| $f(n)$ | 1  | 1  | 2  | 3  | 5  | 8  | 13 | 21 | 34 | ...
Part I. Foundations

Problem 1: Computing the $n$th Fibonacci number:
Problem 1: Computing the \( n \)th Fibonacci number:

\begin{itemize}
  \item \textbf{Input:} \( n \geq 1 \);
  \item \textbf{Output:} the \( n \)th number in the Fibonacci sequence.
\end{itemize}
Problem 1: Computing the \( n \)th Fibonacci number:

**Input:** \( n \geq 1; \)
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Two different types of algorithms: *recursive* and *iterative*
Problem 1: Computing the $n$th Fibonacci number:

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Two different types of algorithms: *recursive* and *iterative*

- **recursive**: task decomposition, top-down, recursive calls;
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Two different types of algorithms: *recursive* and *iterative*

- **recursive**: task decomposition, top-down, recursive calls;
- **iterative**: more tightly coupled tasks, bottom-up approaches;
Part I. Foundations

\section*{Rec-Fibonacci}

\begin{verbatim}
if n = 1 or n = 2, return (1);
else \text{T}_1 = \text{Rec-Fibonacci}(n - 1);
\text{T}_2 = \text{Rec-Fibonacci}(n - 2);
return (\text{T}_1 + \text{T}_2);
\end{verbatim}

But how efficient is it? Or how slow is it?
Its execution is via a run-time stack:

- Suitable for execution of subroutines
- But oblivious, cannot remember any completed subroutine.
Rec-Fibonacci\( (n) \)

\textbf{if} \( n = 1 \) or \( n = 2 \),
Rec-Fibonacci($n$)

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Part I. Foundations

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Part I. Foundations

**Rec-Fibonacci**

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Part I. Foundations

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its execution is via a \textit{run-time stack}

\begin{itemize}
\item suitable for execution of subroutines
\item but oblivious, cannot remember any completed subroutine.
\end{itemize}
Part I. Foundations

Repeated computations everywhere!
Part I. Foundations

The size of tree is the number of recursive calls;
Part I. Foundations

The size of tree is the number of recursive calls; How big is it?
Part I. Foundations

\[ \text{small triangle} \leq \text{size of tree} \leq \text{large triangle} \]

roughly:

\[ 2^n \leq \text{size of tree} \leq 2^{n+1} \]
Part I. Foundations

small triangle \leq \text{size of tree} \leq \text{large triangle}
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small triangle \leq \text{size of tree} \leq \text{large triangle}

\text{roughly: } 2^{\frac{n}{2}} \leq \text{size of tree} \leq 2^n
Iterative-Fibonacci($n$)

Iterative-Fibonacci($n$) is a simple dynamic programming algorithm.
Iterative-Fibonacci($n$)

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\text{if } n = 1 \text{ or } n = 2 \text{ return (1);}
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Iterative-Fibonacci($n$)

if $n = 1$ or $n = 2$ return (1);
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   $M[1] = 1, M[2] = 1$
Iterative-Fibonacci(n)

if \( n = 1 \) or \( n = 2 \) return (1);
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  \( M[1] = 1 \), \( M[2] = 1 \)
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    \( M[i] = M[i - 1] + M[i - 2] \)

How fast is it?
\[
T_{total} = \max\{T_{if}, T_{else}\}
\]
where
\[
T_{if} = c_1, \quad T_{else} = c_2 + T_{for} = c_2 + d \times (n - 2)
\]
\[
T_{total} \leq c_1 + c_2 + d \times (n - 2),
\]
a linear function in \( n \)
Iterative-Fibonacci$(n)$

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Part I. Foundations

**Iterative-Fibonacci** \( n \)

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\text{for } i = 3 \text{ to } n \text{ do} & \\
M[i] = M[i - 1] + M[i - 2] & \\
\text{return } (M[n])
\end{align*}
\]

How fast is it?

\[T_{total} = \max\{T_{if}, T_{else}\}\]

where $T_{if} = c_1$, $T_{else} = c_2 + T_{for}$
Part I. Foundations

**Iterative-Fibonacci**($n$)

```plaintext
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Iterative-Fibonacci\( (n) \) is a simple dynamic programming algorithm.
Part I. Foundations

Actually, all the algorithm \textsc{Iterative-Fibonacci}(n) does is:
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To fill out a table of size $n$, with
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- each entry being filled out exactly once, and
Part I. Foundations

Actually, all the algorithm \textsc{Iterative-Fibonacci}(n) does is:

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\begin{itemize}
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\end{itemize}
Part I. Foundations

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- each entry being filled out exactly once, and
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So the total time \textsc{Iterative-Fibonacci}(n) uses is

\[ T(n) = c \times n \]
Chapter 1. The role of algorithms in computing
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What is an Algorithm: a well-defined, finite procedure that takes an input and produces an output.
Chapter 1. The role of algorithms in computing

What is an Algorithm: a well-defined, finite procedure that takes an input and produces an output.

Example 2: An algorithm skeleton;

Algorithm Maximum;

\[
\text{Input: list } X = \{a_1, \cdots, a_n\};
\]

\[
\text{Body that is a series of instructions;}
\]

\[
\text{Output: } y, \text{ the maximum of } a_1, \cdots, a_n.
\]
Alternatively, an algorithm specifies a finite process to compute a function or a relation.
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e.g., algorithm MAXIMUM computes the following function:

\[ f_{\text{max}}(X) = y, \text{ where } \forall a \in X, y \geq a, \]
Alternatively, an algorithm specifies a finite process to compute a function or a relation.

e.g., algorithm \texttt{MAXIMUM} computes the following function:

\[
    f_{\text{max}}(X) = y, \text{ where } \forall a \in X, y \geq a,
\]

For some problems, the functions computed are predicates, i.e., output \( y \in \{\text{TRUE, FALSE}\} \)
Chapter 1. The Role of Algorithms in Computing

Algorithms as a technology to resolve efficiency issues
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Algorithms as a technology to resolve efficiency issues

Efficient use of computer resources such as time and space is necessary.
Chapter 1. The Role of Algorithms in Computing

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Two typical situations:
Chapter 1. The Role of Algorithms in Computing

Algorithms as a technology to resolve efficiency issues

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- very large input data for “easy” problems;
Chapter 1. The Role of Algorithms in Computing

Algorithms as a technology to resolve efficiency issues

Efficient use of computer resources such as time and space is necessary.

Two typical situations:

• very large input data for “easy” problems;
• moderately large input data for “hard” problems.
Chapter 2. Getting Started

Chapter 2. Getting started

The Sorting Problem

Input: \(n\) numbers \(\langle a_1, \cdots, a_n \rangle\);

Output: a reordering \(\langle a'_1, \cdots, a'_n \rangle\) of the input such that
\(a'_1 \leq a'_2 \leq \cdots \leq a'_n\).

Insertion Sort

idea: an iterative process to produce a new list such that
at each iteration, the new list consists of two sublists,
• a sorted sublist followed by an unsorted sublist,
• the leftmost number of the unsorted is being inserted into the sorted.

As the process goes, the sorted sublist gets longer, the unsorted sublist
gets shorter, until the unsorted becomes empty.
Chapter 2. Getting Started

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The Sorting Problem

**INPUT:** \( n \) numbers \( \langle a_1, \cdots, a_n \rangle \);
The Sorting Problem

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Chapter 2. Getting Started

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The Sorting Problem

**INPUT:** $n$ numbers $\langle a_1, \ldots , a_n \rangle$;

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Chapter 2. Getting Started

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The Sorting Problem

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Chapter 2. Getting Started

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The Sorting Problem

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- the leftmost number of the **unsorted** is being inserted into the **sorted**.

As the process goes, the **sorted sublist** gets longer, the **unsorted sublist** gets shorter, until the **unsorted** becomes empty.
Chapter 2. Getting Started

Algorithm `INSERTION-SORT(A)`
Chapter 2. Getting Started

Algorithm \textsc{Insertion-Sort}(A)

1. \textbf{for } $j = 2$ \textbf{to } $\text{length}[A]$ \textbf{do}
Chapter 2. Getting Started

Algorithm INSERTION-SORT($A$)

1. $\textbf{for } j = 2 \textbf{ to } length[A] \textbf{ do}$
2. $\hspace{1em} key = A[j]$
Chapter 2. Getting Started

Algorithm \textsc{Insertion-Sort}(A)

1 \hspace{1em} \textbf{for} \ j = 2 \ \textbf{to} \ \text{length}[A] \ \textbf{do}
2 \hspace{1em} \textit{key} = A[j]
3 \hspace{1em} \{ \text{Insert } A[j] \ \text{into sorted } A[1..j-1] \}
Chapter 2. Getting Started

Algorithm \textsc{Insertion-Sort}(A)

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2. \hspace{1em} \textit{key} = A[j]
3. \hspace{1em} \{ \text{Insert } A[j] \text{ into sorted } A[1..j - 1] \}
4. \hspace{1em} \textit{i} = j - 1

Analysis of the algorithm:
- (correctness proof): to show that the algorithm is as desired;
- (efficiency proof): to show a guaranteed efficiency of the algorithm.
Chapter 2. Getting Started

Algorithm \textsc{Insertion-Sort}(A)

1. \textbf{for} \( j = 2 \) \textbf{to} length[A] \textbf{do}
2. \hspace{1em} key = A[j]
3. \hspace{1em} \{Insert \( A[j] \) into sorted \( A[1..j-1]\)\}
4. \hspace{1em} \( i = j - 1 \)
5. \hspace{1em} \textbf{while} \( i > 0 \) and \( A[i] > key \)

Analysis of the algorithm:
- \textbf{Correctness proof}: to show that the algorithm is as desired;
- \textbf{Efficiency proof}: to show a guaranteed efficiency of the algorithm.
Chapter 2. Getting Started

Algorithm INSERTION-SORT(A)

1  for $j = 2$ to $\text{length}[A]$ do
2       key = $A[j]$
3       {Insert $A[j]$ into sorted $A[1..j - 1]$}
4       $i = j - 1$
5  while $i > 0$ and $A[i] > key$
6     do $A[i + 1] = A[i]$
Algorithm \textsc{Insertion-Sort}(A)

1. \textbf{for} \( j = 2 \) \textbf{to} \( \text{length}[A] \) \textbf{do}
2. \hspace{0.5cm} \text{key} = A[j]
3. \hspace{0.5cm} \{\text{Insert } A[j] \text{ into sorted } A[1..j-1]\}
4. \hspace{0.5cm} i = j - 1
5. \hspace{0.5cm} \textbf{while} \( i > 0 \) \textbf{and} \( A[i] > \text{key} \) \textbf{do}
6. \hspace{1.5cm} A[i + 1] = A[i]
7. \hspace{1.5cm} i = i - 1
Chapter 2. Getting Started

Algorithm **INSERTION-SORT**(A)

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Chapter 2. Getting Started

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Analysis of the algorithm:

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- (efficiency proof): to show a guaranteed efficiency of the algorithm
Chapter 2. Getting Started

Correctness proof: this is to prove
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Chapter 2. Getting Started

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Chapter 2. Getting Started

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If the algorithm consists of sequential blocks of instructions, the task is to prove the correct transformation by each block.
Correctness proof: this is to prove

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If the algorithm consists of sequential blocks of instructions,
the task is to prove the correct transformation by each block.

This means we need to prove that every sequential statement in the algorithm transforms the given pre-condition to the given post-condition.
Chapter 2. Getting Started

The most difficult task is to do this for a loop statement.
Chapter 2. Getting Started

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at each iteration, the sublist $A[1..j - 1]$ consists of the elements originally in the positions $[1..j-1]$ but in sorted order.
Chapter 2. Getting Started

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In Insertion-Sort, the loop invariant is

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However, finding loop invariants is difficult!
Chapter 2. Getting Started

Efficiency analysis: This is to show that...
Chapter 2. Getting Started

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- For all cases of input, the needed computation resources for the algorithm.
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- For all cases of input, the needed computation resources for the algorithm.
- resources can be CPU time and memory space used in the computation.
- however, the unit measured is not real time or memory unit.
Chapter 2. Getting Started

Time of an algorithm $A(x)$

Input Instances

$x$
Chapter 2. Getting Started

Time of an algorithm $A(x)$

Input Instances  worst case
Chapter 2. Getting Started

Time of an algorithm $A(x)$

**Upper bound** for algorithm A, bounding all cases of instances

- Input Instances
- worst case
- $x$
Resource measurement based on

• random-access machine (RAM)
• counting primitive operations: addition, subtraction, floor, ceiling, multiplication, jump, memory movement, these operations differs in time by a constant multiplicative factor.
• speed between different machines: a constant multiplicative factor.
Chapter 2. Getting Started

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Chapter 2. Getting Started

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Chapter 2. Getting Started

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5. \hspace{1em} \textbf{while} \textit{i} > 0 \textbf{and} \( A[i] > \textbf{key} \)
6. \hspace{2em} \textbf{do} \( A[i + 1] = A[i] \)
7. \hspace{2em} \text{\textit{i} = \textit{i} - 1}
8. \hspace{1em} \( A[i + 1] = \textbf{key} \)

Assume \( t_j \) to be the number of times \textbf{while} is executed for every \( j \).

\[
T(n) = c_1 n + c_2 (n-1) + c_4 (n-1) + c_5 \sum_{j=2}^{n} t_j + c_6 \sum_{j=2}^{n} (t_j - 1) + c_7 \sum_{j=2}^{n} (t_j - 1) + c_8 (n-1)
\]
Chapter 2. Getting Started

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for some constants \(a, b, c\), for example, \(a \geq c_5 + c_6 + c_7\), \(b \geq c_1 + c_2 + c_4 + c_8\).

Because \(t_j = j\) in the worst case (e.g., list is reversely sorted).

\[ T(n) \leq a n^2 (n+1) + b n + c - a \leq x n^2 + y n + z \]

for some constants \(x, y, z\).
$T(n) = c_1 n + c_2 (n-1) + c_4 (n-1) + c_5 \sum_{j=2}^{n} t_j + c_6 \sum_{j=2}^{n} (t_j - 1) + c_7 \sum_{j=2}^{n} (t_j - 1) + c_8 (n-1)$

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Chapter 2. Getting Started

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Chapter 2. Getting Started

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Chapter 2. Getting Started

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Chapter 2. Getting Started

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\[ T(n) \leq a \frac{n}{2} (n + 1) + bn + c - a \leq xn^2 + yn + z \]

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Chapter 2. Getting Started

So we have proved:

\[ T(n) \leq n^2 + yn + z \]

for some constants \( x, y, z \) \( \leq \) means in all cases for which \( \leq \) holds;

\( n^2 + yn + z \) is a complexity upper bound for \( T(n) \).
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Chapter 2. Getting Started

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Chapter 2. Getting Started

Important complexity issues:

1. size of input $n$: the number of bits encoding input $x$, i.e., $n = |x|$. It is inaccurate for $n$ to represent the number of items in the input.

Consider to sort 4 items $\langle x_1, x_2, x_3, x_4 \rangle$ of values in the scale of $2^N$, for some very large $N$.

- If $n$ is the number of items, $n = 4$, then any sorting algorithm would run in constant time.
- However, since $x_1, x_2, x_3, x_4$ are of very large values, a single comparison $x_1 \leq x_2$ would need a time proportional to $N$.

Hence, if $n = |\langle x_1, x_2, x_3, x_4 \rangle|$, then $n \approx N$.

To sort the 4 items, a constant number of comparisons is needed, each taking a time linear in $N$ (i.e., total time is linear in $n$).
Chapter 2. Getting Started

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Chapter 2. Getting Started

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Chapter 2. Getting Started

Running time

$T(n)$: the number of primitive operations executed,

worst-case running time: the running time upper bound for all inputs.

order of growth: $T(n) = an^2 + bn + c$ grows the same rate as $an^2$ (if $a > 0$).
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Chapter 3. Growth of Functions

Big-O: \( O(n^2) \) contains all functions of growth rate \( \leq cn^2 \).

So for function \( T(n) = an^2 + bn + c \), \( T(n) \in O(n^2) \), but written as \( T(n) = O(n^2) \).

In general, \( O(g(n)) = \{ f(n) : \exists c > 0, k > 0 \text{ such that } 0 \leq f(n) \leq cg(n), \text{ for all } n \geq k \} \).
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Chapter 3. Growth of Functions

For example: the following functions are all of the order of $O(n^2)$:

(1) $3n^2$

(2) $5 \cdot 3n^2 + 6n \log_2 n + 90$

(3) $0.001n^2 - 20n - 5000$

(4) $3n \log_2 n^2 + 6n$

(5) $\sqrt{n} - 20 \log_2 n$

(6) $\log_2 n + 56$

(7) $345$

But the following are not:

(8) $3n^2 \log_2 n - 400n$

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8. $3n^2 \log_2 n - 400n$
9. $n^{2.001}$
Chapter 4. Solving Recurrences

Example: binary search algorithm

Algorithm

Binary Search \((A, p, r, \text{key})\)

1. if \(p > r\) return (NULL)
2. else
3. \(q = \lfloor \frac{p + r}{2} \rfloor\)
4. if \(A[q] = \text{key}\) return (k)
5. else
6. if \(A[q] > \text{key}\) Binary Search \((A, \text{key}, p, q - 1)\)
7. else Binary Search \((A, \text{key}, q + 1, r)\)

Let \(n = r - p + 1\). It has the time with recurrence,

\[ T(n) \leq T(n/2) + c \]

and \(T(0) \leq c\) where \(c > 0\) is a constant.

Note: we can also set base case \(T(1) \leq c\).
Chapter 4. Solving Recurrences

Example: binary search algorithm

Algorithm Binary Search($A$, $p$, $r$, key)

1. if $p > r$ return (NULL)
2. else
3. \( q = \left\lfloor \frac{p+r}{2} \right\rfloor \)
4. if $A[q] = key$ return ($k$)
5. else
6. if $A[q] > key$ Binary Search ($A$, $key$, $p$, $q - 1$)
7. else
8. Binary Search ($A$, $key$, $q + 1$, $r$)

Let $n = r - p + 1$. It has the time with recurrence,
\[
T(n) \leq T(n/2) + c
\]
and $T(0) \leq c$ where $c > 0$ is a constant.

Note: we can also set base case $T(1) \leq c$. 
Chapter 4. Solving Recurrences

Example: binary search algorithm

Algorithm \textsc{Binary Search}(A, p, r, key)

1. \textbf{if} $p > r$ \textbf{return} (NULL)
2. \textbf{else}
3. \hspace{1em} $q = \lfloor \frac{p+r}{2} \rfloor$
4. \hspace{1em} \textbf{if} $A[q] = key$ \textbf{return} $(k)$
5. \hspace{1em} \textbf{else}
6. \hspace{2em} \textbf{if} $A[q] > key$ \textsc{Binary Search} $(A, key, p, q - 1)$
7. \hspace{2em} \textbf{else}
8. \hspace{3em} \textsc{Binary Search} $(A, key, q + 1, r)$

Let $n = r - p + 1$. It has the time with recurrence,

$$T(n) \leq T\left(\frac{n}{2}\right) + c \text{ and } T(0) \leq c$$

where $c > 0$ is a constant.
Chapter 4. Solving Recurrences

Example: binary search algorithm

Algorithm \textsc{Binary Search}(A, p, r, key)

1. \textbf{if} \( p > r \) \textbf{return} (NULL)
2. \textbf{else}
3. \( q = \lfloor \frac{p+r}{2} \rfloor \)
4. \textbf{if} \( A[q] = key \) \textbf{return} (k)
5. \textbf{else}
6. \textbf{if} \( A[q] > key \) \textsc{Binary Search} (A, key, p, q − 1)
7. \textbf{else}
8. \textsc{Binary Search} (A, key, q + 1, r)

Let \( n = r - p + 1 \). It has the time with recurrence,

\[ T(n) \leq T\left( \frac{n}{2} \right) + c \text{ and } T(0) \leq c \]

where \( c > 0 \) is a constant.

\textbf{Note:} we can also set base case \( T(1) \leq c \).
Chapter 4. Solving Recurrences

We have recurrence $T(n) \leq T(n/2) + c$ and $T(1) \leq c$ for the Binary Search algorithm. The recurrence can be resolved by unfolding the recursive terms. The process to unfold the recurrence $T(n)$ can be:

1. simple, straightforward
2. induction based (called substitution method)
3. graph based (called recursive tree method).
Chapter 4. Solving Recurrences

We have recurrence

\[ T(n) \leq T\left(\frac{n}{2}\right) + c \text{ and } T(1) \leq c \]

for the Binary Search algorithm.
Chapter 4. Solving Recurrences

We have recurrence

\[ T(n) \leq T\left(\frac{n}{2}\right) + c \] and \( T(1) \leq c \)

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We have recurrence

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for the \textsc{Binary Search} algorithm.

The recurrence can be resolved by unfolding the recursive terms. The process to unfold the recurrence \( T(n) \) can be:

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We have recurrence

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for the \texttt{Binary Search} algorithm.

The recurrence can be resolved by unfolding the recursive terms. The process to unfold the recurrence \( T(n) \) can be:

1. simple, straightforward
2. \texttt{induction based} (called \texttt{substitution method})
Chapter 4. Solving Recurrences

We have recurrence

\[ T(n) \leq T\left(\frac{n}{2}\right) + c \quad \text{and} \quad T(1) \leq c \]

for the Binary Search algorithm.

The recurrence can be resolved by unfolding the recursive terms. The process to unfold the recurrence \( T(n) \) can be:

1. simple, straightforward
2. induction based (called substitution method)
3. graph based (called recursive tree method).
Chapter 4. Solving Recurrences

1. Simple, straightforward unfolding:

\[ T(n) \leq T(n^2) + c \]

\[ T(n) \leq T(n^2) + c \]

\[ T(n) \leq T(n^2^2) + c \]

\[ T(n) \leq T(n^2^3) + c \]

\[ \ldots \]

\[ T(n^2^h) \leq T(n^2^{h+1}) + c \]

where \( n^2^{h+1} = 1 \), \( 2^{h+1} = n \), \( h+1 = \log_2 n \).
Chapter 4. Solving Recurrences

1. Simple, straightforward unfolding:
Chapter 4. Solving Recurrences

1. Simple, straightforward unfolding:
   
   use $T(n) \leq T\left(\frac{n}{2}\right) + c$ as a template;
Chapter 4. Solving Recurrences

1. Simple, straightforward unfolding:

   use $T(n) \leq T\left(\frac{n}{2}\right) + c$ as a template;

   $T(n) \leq T\left(\frac{n}{2}\right) + c$
Chapter 4. Solving Recurrences

1. Simple, straightforward unfolding:

   use $T(n) \leq T\left(\frac{n}{2}\right) + c$ as a template;

   $T(n) \leq T\left(\frac{n}{2}\right) + c$
   $T\left(\frac{n}{2}\right) \leq T\left(\frac{n}{2^2}\right) + c$
1. Simple, straightforward unfolding:

use $T(n) \leq T\left(\frac{n}{2}\right) + c$ as a template;

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$T\left(\frac{n}{2}\right) \leq T\left(\frac{n}{2^2}\right) + c$
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1. Simple, straightforward unfolding:

use \( T(n) \leq T\left(\frac{n}{2}\right) + c \) as a template;

\[
\begin{align*}
T(n) & \leq T\left(\frac{n}{2}\right) + c \\
T\left(\frac{n}{2}\right) & \leq T\left(\frac{n}{2^2}\right) + c \\
T\left(\frac{n}{2^2}\right) & \leq T\left(\frac{n}{2^3}\right) + c \\
\ldots & \ldots
\end{align*}
\]
1. Simple, straightforward unfolding:

use $T(n) \leq T\left(\frac{n}{2}\right) + c$ as a template;

$T(n) \leq T\left(\frac{n}{2}\right) + c$
$T\left(\frac{n}{2}\right) \leq T\left(\frac{n}{2^2}\right) + c$
$T\left(\frac{n}{2^2}\right) \leq T\left(\frac{n}{2^3}\right) + c$

\[ \ldots \]
$T\left(\frac{n}{2^h}\right) \leq T\left(\frac{n}{2^{h+1}}\right) + c$ \quad where $\frac{n}{2^{h+1}} = 1$, $2^{h+1} = n$, $h + 1 = \log_2 n$
Chapter 4. Solving Recurrences

1. Simple, straightforward unfolding:

use $T(n) \leq T\left(\frac{n}{2}\right) + c$ as a template;

\[
\begin{align*}
T(n) &\leq T\left(\frac{n}{2}\right) + c \\
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T\left(\frac{n}{2^2}\right) &\leq T\left(\frac{n}{2^3}\right) + c \\
\cdots \\
T\left(\frac{n}{2^h}\right) &\leq T\left(\frac{n}{2^{h+1}}\right) + c
\end{align*}
\]

where $\frac{n}{2^{h+1}} = 1$, $2^{h+1} = n$, $h + 1 = \log_2 n$
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T(n) & \leq T\left(\frac{n}{2}\right) + c \\
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T\left(\frac{n}{2^2}\right) & \leq T\left(\frac{n}{2^3}\right) + c \\
& \vdots \\
T\left(\frac{n}{2^h}\right) & \leq T\left(\frac{n}{2^{h+1}}\right) + c
\end{align*}
\]

where \( \frac{n}{2^{h+1}} = 1 \), \( 2^{h+1} = n \), \( h + 1 = \log_2 n \)

\[
T(n) \leq T\left(\frac{n}{2^{h+1}}\right) + (h + 1)c
\]
there are \( h + 1 \) inequalities
Chapter 4. Solving Recurrences

1. Simple, straightforward unfolding:

use $T(n) \leq T\left(\frac{n}{2}\right) + c$ as a template;

$T(n) \leq T\left(\frac{n}{2}\right) + c$

$T\left(\frac{n}{2}\right) \leq T\left(\frac{n}{2^2}\right) + c$

$T\left(\frac{n}{2^2}\right) \leq T\left(\frac{n}{2^3}\right) + c$

\[\ldots\ldots\]

$T\left(\frac{n}{2^h}\right) \leq T\left(\frac{n}{2^{h+1}}\right) + c$

where $\frac{n}{2^{h+1}} = 1$, $2^{h+1} = n$, $h + 1 = \log_2 n$

\[+\]

\[T(n) \leq T\left(\frac{n}{2^{h+1}}\right) + (h + 1)c\]

\[\leq c + c \log_2 n\]
1. Simple, straightforward unfolding:

use $T(n) \leq T\left(\frac{n}{2}\right) + c$ as a template;

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T(n) \leq T\left(\frac{n}{2}\right) + c \\
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T\left(\frac{n}{2^2}\right) \leq T\left(\frac{n}{2^3}\right) + c \\
\ldots \\
T\left(\frac{n}{2^h}\right) \leq T\left(\frac{n}{2^{h+1}}\right) + c \quad \text{where} \quad \frac{n}{2^{h+1}} = 1, \ 2^{h+1} = n, \ h + 1 = \log_2 n
\]

\[
T(n) \leq T\left(\frac{n}{2^{h+1}}\right) + (h + 1)c \\
\leq c + c \log_2 n \\
= O(\log_2 n)
\]

proved this last equation!
Chapter 4. Solving Recurrences

1. Simple, straightforward unfolding:

use $T(n) \leq T\left(\frac{n}{2}\right) + c$ as a template;

\[
T(n) \leq T\left(\frac{n}{2}\right) + c \\
T\left(\frac{n}{2}\right) \leq T\left(\frac{n}{2^2}\right) + c \\
T\left(\frac{n}{2^2}\right) \leq T\left(\frac{n}{2^3}\right) + c \\
\ldots \\
T\left(\frac{n}{2^h}\right) \leq T\left(\frac{n}{2^{h+1}}\right) + c \\
\text{where } \frac{n}{2^{h+1}} = 1, \ 2^{h+1} = n, \ h + 1 = \log_2 n
\]

\[
T(n) \leq T\left(\frac{n}{2^{h+1}}\right) + (h + 1)c \\
\leq c + c \log_2 n \\
= O(\log_2 n) \quad \text{proved this last equation!}
\]
Chapter 4. Solving Recurrences

Another example:

Algorithm Merge Sort \((A, p, r)\)

1. if \(p < r\)
2. then
3. \(q = \lfloor \frac{p + r}{2} \rfloor\)
4. Merge Sort \((A, p, q)\)
5. Merge Sort \((A, q + 1, r)\)

Merging 2 Lists \((A, p, q, r)\)

Analysis of the algorithm.

- Assume \(n = r - p + 1\), a power of 2; also assume \(T(n)\) is time for \(\text{Merge Sort}(A, p, r)\). Then

\[
\begin{align*}
\text{t}_1, 2 & = c \\
\text{t}_3 & = T(n/2) \\
\text{t}_4 & = T(n/2) \\
\text{t}_5 & \leq n \quad \text{(why?)}
\end{align*}
\]

Then

\[
T(n) = \text{t}_1, 2 + \text{t}_3 + \text{t}_4 + \text{t}_5 \leq 2T(n/2) + n + c
\]

Base case: \(T(1) = c\).
Another example:

Algorithm Merge Sort \((A, p, r)\)

1. if \(p < r\)
2. then \(q = \lfloor \frac{p + r}{2} \rfloor\)
3. Merge Sort \((A, p, q)\)
4. Merge Sort \((A, q + 1, r)\)
5. Merging2Lists \((A, p, q, r)\)

Analysis of the algorithm.

- Assume \(n = r - p + 1\), a power of 2; also assume \(T(n)\) is time for \(\text{Merge Sort}(A, p, r)\).

Then

- \(t_1, t_2 = c \cdot t_3 = t_4 = T(n/2)\)
- \(t_5 \leq n\) (why?)

\(T(n) = t_1 + t_2 + t_3 + t_4 + t_5 \leq 2T(n/2) + n + c\) base case: \(T(1) = c\).
Chapter 4. Solving Recurrences

Another example:

Algorithm `MERGE SORT(A, p, r)`

1. if $p < r$
2. then $q = \lfloor \frac{p+r}{2} \rfloor$
3. `MERGE SORT(A, p, q)`
4. `MERGE SORT(A, q + 1, r)`
5. `MERGING2LISTS(A.p, q, r)`

Analysis of the algorithm.

- Assume $n = r - p + 1$, a power of 2; also assume $T(n)$ is time for `MERGE SORT(A, p, r)`.

Then

- $t_1, t_2 = c$
- $t_3 = t_4 = T(n/2)$
- $t_5 \leq n$ (why?)

$T(n) = t_1, t_2 + t_3 + t_4 + t_5 \leq 2T(n/2) + n + c$

base case: $T(1) = c$. 
Another example:

Algorithm $\text{Merge Sort}(A, p, r)$

1. if $p < r$
2. then $q = \left\lfloor \frac{p+r}{2} \right\rfloor$
3. $\text{Merge Sort}(A, p, q)$
4. $\text{Merge Sort}(A, q + 1, r)$
5. $\text{Merging2Lists}(A, p, q, r)$

Analysis of the algorithm.

- Assume $n = r - p + 1$, a power of 2;
  - also assume $T(n)$ is time for $\text{Merge Sort}(A, p, r)$. Then
Another example:

Algorithm \textsc{Merge Sort}(A, p, r)

1. \textbf{if} $p < r$
2. \textbf{then} $q = \lfloor \frac{p+r}{2} \rfloor$
3. \textsc{Merge Sort}(A, $p$, $q$)
4. \textsc{Merge Sort}(A, $q+1$, $r$)
5. \textsc{Merging2Lists}(A, $p$, $q$, $r$)

Analysis of the algorithm.

- Assume $n = r - p + 1$, a power of 2;
- also assume $T(n)$ is time for \textsc{Merge Sort}(A, $p$, $r$). Then

- $t_{1,2} = c$
Chapter 4. Solving Recurrences

Another example:

Algorithm $\text{MERGE SORT}(A, p, r)$

1. if $p < r$
2. then $q = \lfloor \frac{p+r}{2} \rfloor$
3. $\text{MERGE SORT}(A, p, q)$
4. $\text{MERGE SORT}(A, q+1, r)$
5. $\text{MERGING2LISTS}(A.p, q, r)$

Analysis of the algorithm.

- Assume $n = r - p + 1$, a power of 2;
  also assume $T(n)$ is time for $\text{MERGE SORT}(A, p, r)$. Then

- $t_{1,2} = c$
- $t_3 = t_4 = T\left(\frac{n}{2}\right)$
Another example:

Algorithm \texttt{Merge Sort}(A, p, r)

1. \textbf{if} \( p < r \)
2. \hspace{1em} \textbf{then} \( q = \lfloor \frac{p+r}{2} \rfloor \)
3. \hspace{1em} \texttt{Merge Sort}(A, p, q)
4. \hspace{1em} \texttt{Merge Sort}(A, q + 1, r)
5. \hspace{1em} \texttt{Merging2Lists}(A.p, q, r)

Analysis of the algorithm.

- Assume \( n = r - p + 1 \), a power of 2;
  also assume \( T(n) \) is time for \texttt{Merge Sort}(A, p, r). Then

- \( t_{1,2} = c \)
- \( t_3 = t_4 = T\left( \frac{n}{2} \right) \)
- \( t_5 \leq n \) (why?)
Another example:

Algorithm **MERGE SORT**(*A, p, r*)

1. if \( p < r \)
2. then \( q = \lfloor \frac{p + r}{2} \rfloor \)
3. **MERGE SORT**(*A, p, q*)
4. **MERGE SORT**(*A, q + 1, r*)
5. **MERGING2LISTS**(*A, p, q, r*)

Analysis of the algorithm.

- Assume \( n = r - p + 1 \), a power of 2;
- also assume \( T(n) \) is time for **MERGE SORT**(*A, p, r*). Then

- \( t_{1,2} = c \)
- \( t_3 = t_4 = T\left(\frac{n}{2}\right) \)
- \( t_5 \leq n \) (why?)

\[
T(n) = t_{1,2} + t_3 + t_4 + t_5 \leq 2T\left(\frac{n}{2}\right) + n + c
\]
Chapter 4. Solving Recurrences

Another example:

Algorithm MERGE SORT(A, p, r)
1. if $p < r$
2. then $q = \left\lceil \frac{p+r}{2} \right\rceil$
3. MERGE SORT(A, p, q)
4. MERGE SORT(A, q + 1, r)
5. MERGING2LISTS(A, p, q, r)

Analysis of the algorithm.

- Assume $n = r - p + 1$, a power of 2;
  also assume $T(n)$ is time for MERGE SORT(A, p, r). Then

- $t_{1,2} = c$
- $t_3 = t_4 = T(\frac{n}{2})$
- $t_5 \leq n$ (why?)

\[
T(n) = t_{1,2} + t_3 + t_4 + t_5 \leq 2T(\frac{n}{2}) + n + c
\]

base case: $T(1) = c$. 
Chapter 4. Solving Recurrences

Solve recurrence

\[ T(n) \leq 2T(n^2) + n + c \]

with base case \[ T(1) = c \] with a simple method:

\[
T(n) \leq 2T(n^2) + n + c \\
\leq 2T(n^2) + n^2 + c \\
\leq 2T(n^2^2) + n^2 + c \\
\leq 2T(n^2^3) + n^2^2 + c \\
\vdots \\
\leq 2T(n^2^{h+1}) + n^2^h + c
\]

where \( n^{2h+1} = 1 \)

Multiplying \( 2, 2^2, \ldots \) to the second, third, \ldots inequalities, respectively,

\[
T(n) \leq 2T(n^2) + n + c \\
2T(n^2) \leq 2^2T(n^2^2) + 2n^2 + 2c \\
2^2T(n^2^2) \leq 2^3T(n^2^3) + 2^2n^2^2 + 2^2c \\
\vdots \\
2^{h}T(n^{2h}) \leq 2^{h+1}T(n^{2^{h+1}}) + 2^{h}n^{2h} + 2^{h}c
\]

where \( n^{2h+1} = 1 \)
Chapter 4. Solving Recurrences

Solve recurrence

\[ T(n) \leq 2T\left(\frac{n}{2}\right) + n + c \quad \text{with base case } T(1) = c \]

with a simple method:
Chapter 4. Solving Recurrences

Solve recurrence

\[ T(n) \leq 2T\left(\frac{n}{2}\right) + n + c \quad \text{with base case } T(1) = c \]

with a simple method:

\[ T(n) \leq 2T\left(\frac{n}{2}\right) + n + c \]
Chapter 4. Solving Recurrences

Solve recurrence

\[ T(n) \leq 2T\left(\frac{n}{2}\right) + n + c \] with base case \( T(1) = c \)

with a simple method:

\[
\begin{align*}
T(n) &\leq 2T\left(\frac{n}{2}\right) + n + c \\
T\left(\frac{n}{2}\right) &\leq 2T\left(\frac{n}{2^2}\right) + \frac{n}{2} + c \\
T\left(\frac{n}{2^h}\right) &\leq 2T\left(\frac{n}{2^{h+1}}\right) + \frac{n}{2^h} + c
\end{align*}
\]
Chapter 4. Solving Recurrences

Solve recurrence

\[ T(n) \leq 2T\left(\frac{n}{2}\right) + n + c \]

with base case \( T(1) = c \)

with a simple method:

\[
\begin{align*}
T(n) & \leq 2T\left(\frac{n}{2}\right) + n + c \\
T\left(\frac{n}{2}\right) & \leq 2T\left(\frac{n}{2^2}\right) + \frac{n}{2} + c \\
T\left(\frac{n}{2^2}\right) & \leq 2T\left(\frac{n}{2^3}\right) + \frac{n}{2^2} + c
\end{align*}
\]
Chapter 4. Solving Recurrences

Solve recurrence

\[ T(n) \leq 2T\left(\frac{n}{2}\right) + n + c \]  with base case \( T(1) = c \)

with a simple method:

\[ T(n) \leq 2T\left(\frac{n}{2}\right) + n + c \]
\[ T\left(\frac{n}{2}\right) \leq 2T\left(\frac{n}{2^2}\right) + \frac{n}{2} + c \]
\[ T\left(\frac{n}{2^2}\right) \leq 2T\left(\frac{n}{2^3}\right) + \frac{n}{2^2} + c \]
\[ \ldots \]
Solve recurrence

\[ T(n) \leq 2T\left(\frac{n}{2}\right) + n + c \]  with base case \( T(1) = c \)

with a simple method:

\[ T(n) \leq 2T\left(\frac{n}{2}\right) + n + c \]
\[ T\left(\frac{n}{2}\right) \leq 2T\left(\frac{n}{2^2}\right) + \frac{n}{2} + c \]
\[ T\left(\frac{n}{2^2}\right) \leq 2T\left(\frac{n}{2^3}\right) + \frac{n}{2^2} + c \]
\[ \ldots \]
\[ T\left(\frac{n}{2^h}\right) \leq 2T\left(\frac{n}{2^{h+1}}\right) + \frac{n}{2^h} + c \]  where \( \frac{n}{2^{h+1}} = 1 \)
Solve recurrence

\[ T(n) \leq 2T\left(\frac{n}{2}\right) + n + c \quad \text{with base case } T(1) = c \]

with a simple method:

\[
\begin{align*}
T(n) & \leq 2T\left(\frac{n}{2}\right) + n + c \\
T\left(\frac{n}{2}\right) & \leq 2T\left(\frac{n}{2^2}\right) + \frac{n}{2} + c \\
T\left(\frac{n}{2^2}\right) & \leq 2T\left(\frac{n}{2^3}\right) + \frac{n}{2^2} + c \\
& \vdots \\
T\left(\frac{n}{2^h}\right) & \leq 2T\left(\frac{n}{2^{h+1}}\right) + \frac{n}{2^h} + c \quad \text{where } \frac{n}{2^{h+1}} = 1
\end{align*}
\]

multiplying \(2, 2^2, \ldots\) to the second, third, \ldots inequalities, respectively,
**Chapter 4. Solving Recurrences**

Solve recurrence

\[ T(n) \leq 2T\left(\frac{n}{2}\right) + n + c \quad \text{with base case } T(1) = c \]

with a simple method:

\[
\begin{align*}
T(n) & \leq 2T\left(\frac{n}{2}\right) + n + c \\
T\left(\frac{n}{2}\right) & \leq 2T\left(\frac{n}{2^2}\right) + \frac{n}{2} + c \\
T\left(\frac{n}{2^2}\right) & \leq 2T\left(\frac{n}{2^3}\right) + \frac{n}{2^2} + c \\
\ldots
\end{align*}
\]

\[ T\left(\frac{n}{2^h}\right) \leq 2T\left(\frac{n}{2^{h+1}}\right) + \frac{n}{2^h} + c \quad \text{where } \frac{n}{2^{h+1}} = 1 \]

multiplying \(2, 2^2, \ldots\) to the second, third, \ldots inequalities, respectively,

\[ T(n) \leq 2T\left(\frac{n}{2}\right) + n + c \]
Chapter 4. Solving Recurrences

Solve recurrence

\[ T(n) \leq 2T\left(\frac{n}{2}\right) + n + c \] with base case \( T(1) = c \)

with a simple method:

\[ T(n) \leq 2T\left(\frac{n}{2}\right) + n + c \]
\[ T\left(\frac{n}{2}\right) \leq 2T\left(\frac{n}{2^2}\right) + \frac{n}{2} + c \]
\[ T\left(\frac{n}{2^2}\right) \leq 2T\left(\frac{n}{2^3}\right) + \frac{n}{2^2} + c \]
\[ \ldots \]
\[ T\left(\frac{n}{2^h}\right) \leq 2T\left(\frac{n}{2^{h+1}}\right) + \frac{n}{2^h} + c \] where \( \frac{n}{2^{h+1}} = 1 \)

multiplying \( 2, 2^2, \ldots \) to the second, third, \ldots inequalities, respectively,

\[ T(n) \leq 2T\left(\frac{n}{2}\right) + n + c \]
\[ 2T\left(\frac{n}{2}\right) \leq 2^2T\left(\frac{n}{2^2}\right) + 2 \times \frac{n}{2} + 2c \]
Chapter 4. Solving Recurrences

Solve recurrence

\[ T(n) \leq 2T\left(\frac{n}{2}\right) + n + c \]  with base case \( T(1) = c \)

with a simple method:

\[
\begin{align*}
T(n) & \leq 2T\left(\frac{n}{2}\right) + n + c \\
T\left(\frac{n}{2}\right) & \leq 2T\left(\frac{n}{2^2}\right) + \frac{n}{2} + c \\
T\left(\frac{n}{2^2}\right) & \leq 2T\left(\frac{n}{2^3}\right) + \frac{n}{2^2} + c \\
\vdots \\
T\left(\frac{n}{2^h}\right) & \leq 2T\left(\frac{n}{2^{h+1}}\right) + \frac{n}{2^h} + c \quad \text{where} \quad \frac{n}{2^{h+1}} = 1
\end{align*}
\]

by multiplying \( 2, 2^2, \ldots \) to the second, third, \ldots inequalities, respectively,

\[
\begin{align*}
T(n) & \leq 2T\left(\frac{n}{2}\right) + n + c \\
2T\left(\frac{n}{2}\right) & \leq 2^2T\left(\frac{n}{2^2}\right) + 2 \times \frac{n}{2} + 2c \\
2^2T\left(\frac{n}{2^2}\right) & \leq 2^3T\left(\frac{n}{2^3}\right) + 2^2 \times \frac{n}{2^2} + 2^2c
\end{align*}
\]
Chapter 4. Solving Recurrences

Solve recurrence

\[ T(n) \leq 2T\left( \frac{n}{2} \right) + n + c \quad \text{with base case } T(1) = c \]

with a simple method:

\[ T(n) \leq 2T\left( \frac{n}{2} \right) + n + c \]
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\[ T\left( \frac{n}{2^2} \right) \leq 2T\left( \frac{n}{2^3} \right) + \frac{n}{2^2} + c \]
\[ \ldots \]
\[ T\left( \frac{n}{2^h} \right) \leq 2T\left( \frac{n}{2^{h+1}} \right) + \frac{n}{2^h} + c \quad \text{where } \frac{n}{2^{h+1}} = 1 \]

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\[ 2^2T\left( \frac{n}{2^2} \right) \leq 2^3T\left( \frac{n}{2^3} \right) + 2^2 \times \frac{n}{2^2} + 2^2c \]
\[ \ldots \]
Chapter 4. Solving Recurrences

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T\left(\frac{n}{2}\right) & \leq 2T\left(\frac{n}{4}\right) + \frac{n}{2} + c \\
T\left(\frac{n}{4}\right) & \leq 2T\left(\frac{n}{8}\right) + \frac{n}{4} + c \\
& \vdots \\
T\left(\frac{n}{2^h}\right) & \leq 2T\left(\frac{n}{2^{h+1}}\right) + \frac{n}{2^h} + c \\
\end{align*}
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where \( \frac{n}{2^{h+1}} = 1 \)

multiplying 2, \(2^2, \ldots\) to the second, third, \ldots inequalities, respectively,

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& \vdots \\
2^hT\left(\frac{n}{2^h}\right) & \leq 2^{h+1}T\left(\frac{n}{2^{h+1}}\right) + 2^h \times \frac{n}{2^h} + 2^hc \\
\end{align*}
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Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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\end{align*}
\]

\[ T(n) \leq 2^{h+1}T\left(\frac{n}{2^{h+1}}\right) + (h + 1)n + c \sum_{i=0}^{h} 2^i \]
Chapter 4. Solving Recurrences

\[ T(n) \leq 2h + 1 \]
\[ T(n) + (h + 1) n + c h \sum_{i=0}^{2^h} = cn + n \log_2 n + c(n - 1) = O(n \log_2 n) \]

We need to prove the last equality, i.e., find constants \( a \) and \( k \) such that
\[ n \log_2 n + 2cn - c \leq an \log_2 n \]
when \( n > k \).

Choose \( a = 2 \). Then to make (1) holds, we need \( \log_2 n > 2c \). So \( k = 2^{2c} \) suffices.

That is, \( n \log_2 n + 2cn - c \leq 2n \log_2 n \) when \( n > k = 2^{2c} \).

So \( T(n) = O(n \log_2 n) \).
Chapter 4. Solving Recurrences

\[ T(n) \leq 2^{h+1}T\left(\frac{n}{2^{h+1}}\right) + (h + 1)n + c \sum_{i=0}^{h} 2^i \]
Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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when \( n > k \).
Chapter 4. Solving Recurrences

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T(n) \leq 2^{h+1} T(1) + n \log_2 n + c(2^{h+1} - 1) \\
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Chapter 4. Solving Recurrences

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T(n) \leq 2^{h+1}T(1) + n \log_2 n + c(2^{h+1} - 1) \\
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Chapter 4. Solving Recurrences

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So \( T(n) = O(n \log_2 n) \).
Chapter 4. Solving Recurrences

When $n$ is not a power of 2

• choose $m \cdot n$ such that $m \cdot n$ is a power of 2 and the smallest such that $n \leq m \cdot n$;

• $T(n) \leq T(m \cdot n)$, why?
  Assume $T$ to be monotonic;

• use the analysis we just did, $T(m \cdot n) = O(m \cdot n \log_2 m \cdot n)$;
  that is, $\exists c, k, T(m \cdot n) \leq cm \cdot n \log_2 m \cdot n$ when $m \cdot n \geq k$;

• but $m \cdot n < 2^n$, why?
  Because $m \cdot n < 2^n$;

• So $T(n) \leq T(m \cdot n) \leq cm \cdot n \log_2 m \cdot n \leq 2cn \log_2 (2^n) \leq 2cn \log_2 n = 4cn \log_2 n$, here $c' = 4c$,
  when $m \cdot n \geq k (\geq 4)$, i.e., when $n \geq \lceil k/2 \rceil (\geq 2)$;

• therefore, $T(n) = O(n \log_2 n)$. 

Chapter 4. Solving Recurrences

When \( n \) is not a power of 2
When $n$ is not a power of 2

- choose $m_n$ such that $m_n$ is a power of 2
Chapter 4. Solving Recurrences

When $n$ is not a power of 2

- choose $m_n$ such that $m_n$ is a power of 2 and the smallest such that $n \leq m_n$;
Chapter 4. Solving Recurrences

When $n$ is not a power of 2

- choose $m_n$ such that $m_n$ is a power of 2 and the smallest such that $n \leq m_n$;
- $T(n) \leq T(m_n)$, why?
When $n$ is not a power of 2

- Choose $m_n$ such that $m_n$ is a power of 2 and the smallest such that $n \leq m_n$;

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Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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- choose $m_n$ such that $m_n$ is a power of 2 and the smallest such that $n \leq m_n$;

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- but $m_n < 2n$, why?
Chapter 4. Solving Recurrences

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- choose \( m_n \) such that \( m_n \) is a power of 2 and the smallest such that \( n \leq m_n \);

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- but \( m_n < 2n \), why? because \( \frac{m_n}{2} < n \);
Chapter 4. Solving Recurrences

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- So \( T(n) \)
Chapter 4. Solving Recurrences

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• So $T(n) \leq T(m_n)$
Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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  $= 4cn \log_2 n = c'n \log_2 n$, here $c' = 4c$, 
When $n$ is not a power of 2

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- choose $m_n$ such that $m_n$ is a power of 2 and the smallest such that $n \leq m_n$;
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- use the analysis we just did, $T(m_n) = O(m_n \log_2 m_n)$;
  that is, $\exists c, k, T(m_n) \leq cm_n \log_2 m_n$ when $m_n \geq k$;
- but $m_n < 2n$, why? because $\frac{m_n}{2} < n$;
- So $T(n) \leq T(m_n) \leq cm_n \log_2 m_n \leq 2cn \log_2(2n) \leq 2cn \log_2 n^2$
  $= 4cn \log_2 n = c'n \log_2 n$, here $c' = 4c$,
  when $m_n \geq k(\geq 4)$, i.e., when $n \geq \lceil \frac{k}{2} \rceil(\geq 2)$;
When \( n \) is not a power of 2

- choose \( m_n \) such that \( m_n \) is a power of 2 and the smallest such that \( n \leq m_n \);
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  \( = 4cn \log_2 n = c'n \log_2 n \), here \( c' = 4c \),
  when \( m_n \geq k(\geq 4) \), i.e., when \( n \geq \lceil \frac{k}{2} \rceil (\geq 2) \);
- therefore, \( T(n) = O(n \log_2 n) \).
Chapter 4. Solving Recurrences

Methods for solving recurrences
Chapter 4. Solving Recurrences

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1. Substitution method (based on math induction)
Chapter 4. Solving Recurrences

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First we recall the principle of the math induction:

To prove a property $\mathcal{P}(n)$ for every natural number $n \geq 1$, it suffices to prove

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Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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"The principle for dominos to fall".
Chapter 4. Solving Recurrences

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- other variants: e.g., $\mathcal{P}(k) \rightarrow \mathcal{P}(2k)$
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$$P(k) \rightarrow P(2k) \land P(2k + 1)$$
Theorem: Arithmetic sequence of first $n$ terms

\[ 1 + 2 + 3 + \cdots + n = \frac{n}{2}(n + 1) \]
Chapter 4. Solving Recurrences

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step 1: base case: $n = 1$, left = 1, right = $\frac{1}{2}(1 + 1) = 1$;
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Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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So

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Algorithm MERGE SORT($A, p, r$)

1. \textbf{if} $p < r$
2. \textbf{then} $q = \lfloor \frac{p+r}{2} \rfloor$
3. MERGE SORT($A, p, q$)
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$T(n) \leq 2T(n/2) + n + c$ with base case $T(1) = c$
Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

**MergeSort** has the time complexity:

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Chapter 4. Solving Recurrences

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**Chapter 4. Solving Recurrences**

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Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

step 2. assumption: for $\frac{n}{2}$, the claim is true, i.e.,

$$T\left(\frac{n}{2}\right) \leq a \frac{n}{2} \log_2 \frac{n}{2}, \quad \text{when } n \geq 4$$
Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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$$T(n) \leq 2T\left(\frac{n}{2}\right) + n + c = 2T\left(\frac{n}{2}\right) + n + c \leq 2a \frac{n}{2} \log_2 \frac{n}{2} + n + c$$
Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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\[
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Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

A little review on logarithm functions:

- $\log_a n + \log_a m = \log_a nm$;
- $\log_a n^b = b \log_a n$, especially $\log_a 1^n = -\log_a n$;
- $a^{\log_a n} = n$;
- $\log_m a^n = (\log_a n)^m \neq \log_a n^m$.
Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

Solving recurrence with the **substitution method** (guess then verify)

\[ T(n) = \frac{3}{2} T(\lfloor \frac{2n}{3} \rfloor) + n, \quad \text{where} \ T(1) = 2 \]
Chapter 4. Solving Recurrences

Solving recurrence with the substitution method (guess then verify)

\[ T(n) = \frac{3}{2} T(\lceil \frac{2n}{3} \rceil) + n, \text{ where } T(1) = 2 \]

Guess \( T(n) \leq cn \log_2 n \), for some constant \( c \) to be determined later.
Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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Verify with induction:

Step 1, base case: \( T(1) = 2 \leq cn \log_2 n = 0 \) does not hold for the guessed inequality.
Chapter 4. Solving Recurrences

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Verify with induction:

Step 1, base case: \( T(1) = 2 \leq cn \log_2 n = 0 \) does not hold for the guessed inequality.

Instead, we choose \( T(2) \) to be the base case. From the recurrence, we have

\[ T(2) = \frac{3}{2} T(\lfloor \frac{2n}{3} \rfloor) + n \]
Chapter 4. Solving Recurrences

Solving recurrence with the substitution method (guess then verify)

\[ T(n) = \frac{3}{2} T(\lceil \frac{2n}{3} \rceil) + n, \text{ where } T(1) = 2 \]

Guess \( T(n) \leq cn \log_2 n \), for some constant \( c \) to be determined later.

Verify with induction:

Step 1, base case: \( T(1) = 2 \leq cn \log_2 n = 0 \) does not hold for the guessed inequality.

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\[ T(2) = \frac{3}{2} T(\lceil \frac{2n}{3} \rceil) + n = \frac{3}{2} T(\lceil \frac{4}{3} \rceil) + 2 \]
Chapter 4. Solving Recurrences

Solving recurrence with the **substitution method** (guess then verify)

\[
T(n) = \frac{3}{2} T(\lfloor \frac{2n}{3} \rfloor) + n, \text{ where } T(1) = 2
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\]

We make \(T(2) = 5 \leq cn \log_2 n\) holds for \(c = 3\), and when \(n \geq k = 2\),
Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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\[
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\]
Step 3, substitute it in the recurrence, we get

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Chapter 4. Solving Recurrences

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T(n) = \frac{3}{2} T(\lfloor \frac{2n}{3} \rfloor) + n \leq \frac{3}{2} (c \lfloor \frac{2n}{3} \rfloor \log_2 \lfloor \frac{2n}{3} \rfloor) + n
\]

\[
\leq \frac{3}{2} (c \frac{2n}{3} \log_2 \frac{2n}{3}) + n
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Step 3, substitute it in the recurrence, we get

\[ T(n) = \frac{3}{2} T\left(\left\lfloor \frac{2n}{3} \right\rfloor \right) + n \leq \frac{3}{2} \left( c \left\lfloor \frac{2n}{3} \right\rfloor \log_2 \left\lfloor \frac{2n}{3} \right\rfloor \right) + n \]

\[ \leq \frac{3}{2} \left( c \frac{2n}{3} \log_2 \frac{2n}{3} \right) + n \leq cn \left( \log_2 n + \log_2 \frac{2}{3} \right) + n \]
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\[ \leq cn \left( \log_2 n - \log_2 \frac{3}{2} \right) + n \]
Step 3, substitute it in the recurrence, we get

\[ T(n) = \frac{3}{2} T(\lfloor \frac{2n}{3} \rfloor) + n \leq \frac{3}{2} (c \lfloor \frac{2n}{3} \rfloor \log_2 \lfloor \frac{2n}{3} \rfloor) + n \]

\[ \leq \frac{3}{2} (c \frac{2n}{3} \log_2 \frac{2n}{3}) + n \leq cn (\log_2 n + \log_2 \frac{2}{3}) + n \]

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Chapter 4. Solving Recurrences

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Since \( \log_2 \frac{3}{2} > 0.5 \),
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But we already know $c = 3$ from the base case proof,
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Step 3, substitute it in the recurrence, we get

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\[ T(n) \leq cn \log_2 n - cn \log_2 \frac{3}{2} + n \leq cn \log_2 n \]
2. Changing variables

Example:

\[ T(n) = 2T(\sqrt{n}) + \log_2 n \]

Define \( m = \log_2 n \), i.e., \( n = 2^m \)

\[ T(2^m) = 2T(2^{m/2}) + m \]

solve it, we have

\[ S(m) = O(m \log m) \]

so

\[ T(n) = T(2^m) = O(m \log m) = O(\log n \log \log n) \]
Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

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Chapter 4. Solving Recurrences

3. Recursive tree method
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(1) $T(n)$ is a tree with non-recursive terms as the root and recursive terms as its children.
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(1) $T(n)$ is a tree with non-recursive terms as the root and recursive terms as its children.

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3. Recursive tree method

By unfolding the recurrence to make a recursive-tree.

(1) $T(n)$ is a tree with non-recursive terms as the root and recursive terms as its children.

(2) for each child, replace it with then non-recursive terms and produce children that are then recursive terms

(3) repeat (2), expand the tree until all children are the base case.
Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$
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Example \( T(n) = 3T(n/4) + n^2 \), with base case \( T(1) = 1 \)

\[
\begin{array}{ccc}
  l_0: & T(n/4) & T(n) \\
  l_1: & T(n/4) & T(n/4) & n^2 \\
\end{array}
\]
Chapter 4. Solving Recurrences

Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

\[
\begin{align*}
l_0: & & T(n) \\
l_1: & & T(n/4) & & T(n/4) & & T(n/4) \\
l_2: & & T(n/4) & & T(n/4) & & T(n/4) & & T(n/4) & & T(n/4) & & T(n/4) & & T(n/4)
\end{align*}
\]
Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

$l_0$: $T(n)$
$l_1$: $T(n/4)$ $T(n/4)$ $T(n/4)$ $T(n/4)$ $T(n/4)$ $3(\frac{n}{4})^2$
$l_2$: $T(\frac{n}{4^2})$ $T(\frac{n}{4^2})$ $T(\frac{n}{4^2})$ $T(\frac{n}{4^2})$ $T(\frac{n}{4^2})$ $3^2(\frac{n}{4^2})^2$
$l_3$: .......
Chapter 4. Solving Recurrences

Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

\[ l_0: \]
\[ l_1: \quad T(n/4) \quad T(n/4) \quad \text{T}(n) \]
\[ l_2: \quad T(n/4^2) \quad T(n/4^2) \quad T(n/4^2) \quad \text{T}(n/4^2) \quad \text{T}(n/4^2) \quad \text{T}(n/4^2) \quad \text{T}(n/4^2) \quad 3\left(\frac{n}{4}\right)^2 \]
\[ l_3: \ldots \]
\[ l_4: \ldots \]

\[ l_m: \quad \text{where } n/4^m = 1, \text{ i.e., } m = \log_4 n. \]

Then $T(n)$ is the sum
\[ T(n) = n^2 \left[ 1 + 3\left(\frac{1}{4}\right)^2 + 3\left(\frac{1}{4^2}\right)^2 + 3\left(\frac{1}{4^3}\right)^2 + \cdots + 3\left(\frac{1}{4^m}\right)^2 \right] + 3mT(1) \]
\[ T(n) = n^2 \left[ 1 + \frac{3}{16} + \left(\frac{3}{16}\right)^2 + \left(\frac{3}{16}\right)^3 + \cdots + \left(\frac{3}{16}\right)^m \right] + 3m \leq n^2 \text{ for all } n > 0. \]
Chapter 4. Solving Recurrences

Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

$l_0$: $T(n)$

$l_1$: $T(n/4)$ $T(n/4)$ $T(n)$

$l_2$: $T(n/4^2)$ $T(n/4^2)$ $T(n/4^2)$ $T(n/4^2)$ $T(n/4^2)$ $T(n/4^2)$ $T(n/4^2)$ $T(n/4^2)$ $T(n/4^2)$

$l_3$: 

$l_4$: 

$l_{m-1}$: 

$l_m$: 

$n^2$ $3(n/4)^2$ $3^2(n/4^2)^2$ $3^3(n/4^3)^2$
Chapter 4. Solving Recurrences

Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

- $l_0$: $T(n)$
- $l_1$: $T(n/4)\quad T(n/4)\quad T(n/4)$
- $l_2$: $T(n/4^2)\quad T(n/4^2)\quad T(n/4^2)\quad T(n/4^2)\quad T(n/4^2)\quad T(n/4^2)$
- $l_3$: $\ldots$
- $l_4$: $\ldots$
- $l_{m-1}$: $\ldots$

where $n_{4}^{m-1} = 1$, i.e., $m = \log_{4} n$.

Then $T(n)$ is the sum $T(n) = n^2[1 + 3(\frac{n}{4})^2 + 3^2(\frac{n}{4^2})^2 + 3^3(\frac{n}{4^3})^2 + \ldots + 3^{m-1}(\frac{n}{4^{m-1}})^2]$ 

$\leq n^2(1 - 3^{m-1}(\frac{n}{4^{m-1}})^2)$ for all $n > 0$. 
Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

\begin{align*}
l_0: & \quad T(n) \\
l_1: & \quad T(n/4) \\
l_2: & \quad T(n/4), T(n/4), T(n/4) \\
l_3: & \quad \ldots \ldots \\
l_4: & \quad \ldots \ldots \\
l_{m-1}: & \quad \ldots \ldots \\
l_m: & \quad T(1), T(1), T(1), T(1), T(1), \ldots ,
\end{align*}

\begin{align*}
l_0: & \quad T(n) \\
l_1: & \quad T(n/4) \\
l_2: & \quad T(n/4), T(n/4), T(n/4) \\
l_3: & \quad \ldots \ldots \\
l_4: & \quad \ldots \ldots \\
l_{m-1}: & \quad \ldots \ldots \\
l_m: & \quad T(1), T(1), T(1), T(1), T(1), \ldots ,
\end{align*}
Chapter 4. Solving Recurrences

Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

$l_0$: $T(n)

l_1$: $T(n/4) T(n/4)

l_2$: $T(n/4^2) T(n/4^2) T(n/4^2)

l_3$: .......

l_4$: .......

$l_{m-1}$: .......

$l_m$: $T(1), T(1), T(1), T(1), T(1), \ldots$

where $\frac{n}{4^m} = 1$, i.e., $m = \log_4 n$. 

**Chapter 4. Solving Recurrences**

Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

| $l_0$: | $T(n)$ | $l_1$: | $T(n/4)$ | $T(n/4)$ | $l_2$: | $T(n/4^2)$ | $T(n/4^2)$ | $T(n/4^2)$ | $T(n/4^2)$ | $l_3$: | $T(1)$, $T(1)$, $T(1)$, $T(1)$, $T(1)$, …,
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<td>$l_4$:</td>
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<td></td>
<td>$l_{m-1}$:</td>
<td></td>
<td></td>
<td></td>
<td>$3^{m-1}(\frac{n}{4^{m-1}})^2$</td>
<td>$l_m$:</td>
<td>$T(1)$</td>
</tr>
<tr>
<td>$l_m$:</td>
<td>$T(1)$, $T(1)$, $T(1)$, $T(1)$, $T(1)$, …,</td>
<td></td>
<td></td>
<td></td>
<td>$3^{m-1}(\frac{n}{4^{m-1}})^2$</td>
<td>$T(1)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

where $\frac{n}{4^m} = 1$, i.e., $m = \log_4 n$.

Then $T(n)$ is the sum

$$T(n) = n^2[1 + 3(\frac{1}{4})^2 + 3^2(\frac{1}{4^2})^2 + 3^3(\frac{1}{4^3})^2 + \cdots + 3^{m-1}(\frac{1}{4^{m-1}})^2] + 3^m T(1)$$
Chapter 4. Solving Recurrences

Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

$l_0$: $T(n) = n^2$

$l_1$: $T(n/4) = 3n^2$  

$l_2$: $T(n/4^2) = 3^2(n/4)^2$  

$l_3$: $T(n/4^3) = 3^3(n/4^3)^2$  

$l_4$: $T(n/4^4) = 3^4(n/4^4)^2$  

$l_m$: $T(n/4^m) = 3^{m-1}(n/4^m)^2$  

where $\frac{n}{4^m} = 1$, i.e., $m = \log_4 n$.

Then $T(n)$ is the sum

$T(n) = n^2[1 + 3(\frac{1}{4})^2 + 3^2(\frac{1}{4^2})^2 + 3^3(\frac{1}{4^3})^2 + \cdots + 3^{m-1}(\frac{1}{4^m-1})^2] + 3^mT(1)$

$T(n) = n^2[1 + 3(\frac{1}{4})^2 + 3^2(\frac{1}{4^2})^2 + 3^3(\frac{1}{4^3})^2 + \cdots + 3^{m-1}(\frac{1}{4^m-1})^2] + 3^m \times 1$
Chapter 4. Solving Recurrences

Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

$l_0$: $T(n)$

$l_1$: $T(n/4)$

$l_2$: $T(n/4^2)$, $T(n/4^2)$, $T(n/4^2)$

$l_3$: .......

$l_4$: .......

$l_{m-1}$: .......

$l_m$: $T(1), T(1), T(1), T(1), T(1), \ldots$,

where $\frac{n}{4^m} = 1$, i.e., $m = \log_4 n$.

Then $T(n)$ is the sum

$T(n) = n^2[1 + 3\left(\frac{1}{4}\right)^2 + 3^2\left(\frac{1}{4^2}\right)^2 + 3^3\left(\frac{1}{4^3}\right)^2 + \cdots + 3^{m-1}\left(\frac{1}{4^{m-1}}\right)^2] + 3^m T(1)$

$T(n) = n^2[1 + 3\left(\frac{1}{4}\right)^2 + 3^2\left(\frac{1}{4^2}\right)^2 + 3^3\left(\frac{1}{4^3}\right)^2 + \cdots + 3^{m-1}\left(\frac{1}{4^{m-1}}\right)^2] + 3^m \times 1$

$= n^2[1 + 3\left(\frac{1}{4}\right)^2 + 3^2\left(\frac{1}{4^2}\right)^2 + 3^3\left(\frac{1}{4^3}\right)^2 + \cdots + 3^{m-1}\left(\frac{1}{4^{m-1}}\right)^2] + 3^m \left(\frac{n}{4^m}\right)^2$
Chapter 4. Solving Recurrences

Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

\[
\begin{align*}
l_0: & \quad T(n) \\
l_1: & \quad T(n/4) \\
l_2: & \quad T(n/4^2) T(n/4^2) T(n/4^2) \\
l_3: & \quad \ldots \ldots \\
l_{m-1}: & \quad \ldots \ldots \\
l_m: & \quad T(1), T(1), T(1), T(1), T(1), \ldots ,
\end{align*}
\]

where \( \frac{n}{4^m} = 1 \), i.e., \( m = \log_4 n \).

Then $T(n)$ is the sum

\[
\begin{align*}
T(n) &= n^2 \left[ 1 + 3 \left( \frac{1}{4} \right)^2 + 3^2 \left( \frac{1}{4^2} \right)^2 + 3^3 \left( \frac{1}{4^3} \right)^2 + \cdots + 3^{m-1} \left( \frac{1}{4^{m-1}} \right)^2 \right] + 3^m T(1) \\
&= n^2 \left[ 1 + 3 \left( \frac{1}{4} \right)^2 + 3^2 \left( \frac{1}{4^2} \right)^2 + 3^3 \left( \frac{1}{4^3} \right)^2 + \cdots + 3^{m-1} \left( \frac{1}{4^{m-1}} \right)^2 \right] + 3^m \times 1 \\
&= n^2 \left[ 1 + 3 \left( \frac{1}{4} \right)^2 + 3^2 \left( \frac{1}{4^2} \right)^2 + 3^3 \left( \frac{1}{4^3} \right)^2 + \cdots + 3^{m-1} \left( \frac{1}{4^{m-1}} \right)^2 \right] + 3^m \left( \frac{n}{4^m} \right)^2 \\
&= n^2 \left[ 1 + \frac{3}{16} + \left( \frac{3}{16} \right)^2 + \left( \frac{3}{16} \right)^3 + \cdots + \left( \frac{3}{16} \right)^m \right]
\end{align*}
\]
Chapter 4. Solving Recurrences

Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

\[
\begin{align*}
  l_0: & & T(n) & & n^2 \\
  l_1: & & T(n/4) & & T(n/4) & & 3(n/4)^2 \\
  l_2: & & T(n/4^2) & & T(n/4^2) & & T(n/4^2) & & 3^2(n/4^2)^2 \\
  l_3: & & \cdots & & \cdots & & \cdots & & \cdots & & \cdots \\
  l_{m-1}: & & \cdots & & \cdots & & \cdots & & \cdots & & \cdots \\
  l_m: & & T(1), T(1), T(1), T(1), T(1), \ldots, & & 3^{m-1}(n/4^{m-1})^2 & & T(1) \\
\end{align*}
\]

where $\frac{n}{4^m} = 1$, i.e., $m = \log_4 n$.

Then $T(n)$ is the sum

\[
T(n) = n^2\left[1 + 3\left(\frac{1}{4}\right)^2 + 3^2\left(\frac{1}{4^2}\right)^2 + 3^3\left(\frac{1}{4^3}\right)^2 + \cdots + 3^{m-1}\left(\frac{1}{4^{m-1}}\right)^2\right] + 3^m T(1)
\]

\[
T(n) = n^2\left[1 + 3\left(\frac{1}{4}\right)^2 + 3^2\left(\frac{1}{4^2}\right)^2 + 3^3\left(\frac{1}{4^3}\right)^2 + \cdots + 3^{m-1}\left(\frac{1}{4^{m-1}}\right)^2\right] + 3^m \times 1 \\
= n^2\left[1 + 3\left(\frac{1}{4}\right)^2 + 3^2\left(\frac{1}{4^2}\right)^2 + 3^3\left(\frac{1}{4^3}\right)^2 + \cdots + 3^{m-1}\left(\frac{1}{4^{m-1}}\right)^2\right] + 3^m \left(\frac{n}{4^m}\right)^2 \\
= n^2\left[1 + \frac{3}{16} + \left(\frac{3}{16}\right)^2 + \left(\frac{3}{16}\right)^3 + \cdots + \left(\frac{3}{16}\right)^m\right] \\
= n^2\left(\frac{1 - \left(\frac{3}{16}\right)^{m+1}}{1 - \frac{3}{16}}\right)
\]
Chapter 4. Solving Recurrences

Example \( T(n) = 3T(n/4) + n^2 \), with base case \( T(1) = 1 \)

\[
\begin{align*}
l_0: & \quad T(n) \\
l_1: & \quad T(n/4) \\
l_2: & \quad T(n/4) T(n/4) T(n/4) \\
l_3: & \quad \ldots \ldots \\
l_{m-1}: & \quad \ldots \ldots \\
l_m: & \quad T(1), T(1), T(1), T(1), T(1), \ldots ,
\end{align*}
\]

where \( \frac{n}{4^m} = 1 \), i.e., \( m = \log_4 n \).

Then \( T(n) \) is the sum

\[
T(n) = n^2 \left[ 1 + 3 \left( \frac{1}{4} \right)^2 + 3^2 \left( \frac{1}{4^2} \right)^2 + 3^3 \left( \frac{1}{4^3} \right)^2 + \cdots + 3^{m-1} \left( \frac{1}{4^{m-1}} \right)^2 \right] + 3^m T(1)
\]

\[
= n^2 \left[ 1 + 3 \left( \frac{1}{4} \right)^2 + 3^2 \left( \frac{1}{4^2} \right)^2 + 3^3 \left( \frac{1}{4^3} \right)^2 + \cdots + 3^{m-1} \left( \frac{1}{4^{m-1}} \right)^2 \right] + 3^m \times 1
\]

\[
= n^2 \left[ 1 + \frac{3}{16} + \left( \frac{3}{16} \right)^2 + \left( \frac{3}{16} \right)^3 + \cdots + \left( \frac{3}{16} \right)^m \right]
\]

\[
= n^2 \left( \frac{\frac{1 - \left( \frac{3}{16} \right)^{m+1}}{1 - \frac{3}{16}}} \right)
\]

\[
\leq n^2 \left( \frac{1}{1 - \frac{3}{16}} \right)
\]
Chapter 4. Solving Recurrences

Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

\[ l_0: \quad T(n) \quad n^2 \]
\[ l_1: \quad T(n/4) \quad T(n/4) \quad T(n/4) \quad 3(\frac{n}{4})^2 \]
\[ l_2: \quad T(\frac{n}{4^2}) \quad T(\frac{n}{4^2}) \quad T(\frac{n}{4^2}) \quad 3^2(\frac{n}{4^2})^2 \]
\[ l_3: \quad \ldots \ldots \]
\[ l_4: \quad \ldots \ldots \]
\[ l_{m-1}: \quad \ldots \ldots \]
\[ l_m: \quad T(1), T(1), T(1), T(1), T(1), \ldots, \quad 3^{m-1}(\frac{n}{4^{m-1}})^2 \]

where $\frac{n}{4^m} = 1$, i.e., $m = \log_4 n$.

Then $T(n)$ is the sum

\[
T(n) = n^2[1 + 3(\frac{1}{4})^2 + 3^2(\frac{1}{4^2})^2 + 3^3(\frac{1}{4^3})^2 + \cdots + 3^{m-1}(\frac{1}{4^{m-1}})^2] + 3^m T(1)
\]

\[
T(n) = n^2[1 + 3(\frac{1}{4})^2 + 3^2(\frac{1}{4^2})^2 + 3^3(\frac{1}{4^3})^2 + \cdots + 3^{m-1}(\frac{1}{4^{m-1}})^2] + 3^m \times 1
\]

\[
= n^2[1 + 3(\frac{1}{4})^2 + 3^2(\frac{1}{4^2})^2 + 3^3(\frac{1}{4^3})^2 + \cdots + 3^{m-1}(\frac{1}{4^{m-1}})^2] + 3^m \left(\frac{n}{4^m}\right)^2
\]

\[
= n^2[1 + \frac{3}{16} + \left(\frac{3}{16}\right)^2 + \left(\frac{3}{16}\right)^3 + \cdots + \left(\frac{3}{16}\right)^m]
\]

\[
= n^2\left(\frac{1-\left(\frac{3}{16}\right)^{m+1}}{1-\frac{3}{16}}\right)
\]

\[
\leq n^2\left(\frac{1}{1-\frac{3}{16}}\right)
\]

\[
= \frac{16}{13} n^2
\]

for all $n > 0$. 
Chapter 4. Solving Recurrences

Recursive tree notation in the textbook:

\[ T(n) = 3T(n/4) + n^2, \text{ with base case } T(1) = 1 \]

\( l_0: \quad n^2 \)
Recursive tree notation in the textbook:

\[ T(n) = 3T(n/4) + n^2, \text{ with base case } T(1) = 1 \]

- \( l_0: \)  
  - \( (n/4)^2 \)
  - \( n^2 \)
  - \( (n/4)^2 \)

- \( l_1: \)  
  - \( (n/4)^2 \)
  - \( (n/4)^2 \)
  - \( (n/4)^2 \)
Chapter 4. Solving Recurrences

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\[
\begin{align*}
l_0: & \quad n^2 \\
l_1: & \quad (n/4)^2 \quad (n/4)^2 \\
l_2: & \quad (n/4)^2 \quad (n/4)^2 \quad (n/4)^2 \\
l_m: & \quad \ldots \quad \ldots \quad \ldots
\end{align*}
\]
Chapter 4. Solving Recurrences

Recursive tree notation in the textbook:

\[ T(n) = 3T(n/4) + n^2, \text{ with base case } T(1) = 1 \]

\[ l_0: \]
\[ l_1: \]
\[ l_2: \quad \left(\frac{n}{4^2}\right)^2 \left(\frac{n}{4^2}\right)^2 \left(\frac{n}{4^2}\right)^2 \left(\frac{n}{4^2}\right)^2 \left(\frac{n}{4^2}\right)^2 \left(\frac{n}{4^2}\right)^2 \left(\frac{n}{4^2}\right)^2 \left(\frac{n}{4^2}\right)^2 \]
\[ l_3: \ldots \ldots \]

\[ l_m: \]

3 \text{ nodes of } T(1)
Recursive tree notation in the textbook:

\[ T(n) = 3T(n/4) + n^2, \]  
with base case \( T(1) = 1 \)

\[ \begin{align*}
  l_0: & \quad \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } n^2 \\
  l_1: & \quad \left(\frac{n}{4^2}\right)^2 \left(\frac{n}{4^2}\right)^2 \left(\frac{n}{4^2}\right)^2 \left(\frac{n}{4^2}\right)^2 \left(\frac{n}{4^2}\right)^2 \left(\frac{n}{4^2}\right)^2 \left(\frac{n}{4^2}\right)^2 \left(\frac{n}{4^2}\right)^2 \\
  l_2: & \quad \left(\frac{n}{4^2}\right)^2 \left(\frac{n}{4^2}\right)^2 \left(\frac{n}{4^2}\right)^2 \left(\frac{n}{4^2}\right)^2 \left(\frac{n}{4^2}\right)^2 \left(\frac{n}{4^2}\right)^2 \left(\frac{n}{4^2}\right)^2 \left(\frac{n}{4^2}\right)^2 \\
  l_3: & \quad \ldots \ldots \\
  l_4: & \quad \ldots \ldots \\
\end{align*} \]
Chapter 4. Solving Recurrences

Recursive tree notation in the textbook:

\[ T(n) = 3T(n/4) + n^2, \text{ with base case } T(1) = 1 \]

\[ l_0: \frac{n^2}{\left(\frac{n}{4}\right)^2} \]
\[ l_1: \left(\frac{n}{4^2}\right)^2 \left(\frac{n}{4^2}\right)^2 \left(\frac{n}{4^2}\right)^2 \]  \( n^2 \)
\[ l_2: \left(\frac{n}{4^2}\right)^2 \left(\frac{n}{4^2}\right)^2 \left(\frac{n}{4^2}\right)^2 \]  \( \left(\frac{n}{4^2}\right)^2 \)  \( \left(\frac{n}{4^2}\right)^2 \)  \( \left(\frac{n}{4^2}\right)^2 \)  \( \left(\frac{n}{4^2}\right)^2 \)  \( \left(\frac{n}{4^2}\right)^2 \)
\[ l_3: \ldots \]
\[ l_4: \ldots \]
\[ l_{m-1}: \ldots \]

\( 3^3 \) nodes of \( \left(\frac{n}{4^3}\right)^2 \)
Chapter 4. Solving Recurrences

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\[ T(n) = 3T(n/4) + n^2, \text{ with base case } T(1) = 1 \]

\begin{align*}
  l_0: & \quad \text{n}^2 \\
  l_1: & \quad \left(\frac{n}{4}\right)^2 \left(\frac{n}{4}\right)^2 \left(\frac{n}{4}\right)^2 \left(\frac{n}{4}\right)^2 \\
  l_2: & \quad \left(\frac{n}{4^2}\right)^2 \left(\frac{n}{4^2}\right)^2 \left(\frac{n}{4^2}\right)^2 \left(\frac{n}{4^2}\right)^2 \\
  l_3: & \quad \ldots \ldots \\
  l_4: & \quad \ldots \ldots \\
  l_{m-1}: & \quad \ldots \ldots \\
\end{align*}

\[ 3^3 \text{ nodes of } \left(\frac{n}{4^3}\right)^2 \]

\[ 3^{m-1} \text{ of } \left(\frac{n}{4^{m-1}}\right)^2 \]
Chapter 4. Solving Recurrences

Recursive tree notation in the textbook:

\[ T(n) = 3T(n/4) + n^2, \text{ with base case } T(1) = 1 \]

- \( l_0: \) \( n^2 \)
- \( l_1: \) \( \left(\frac{n}{4}\right)^2 \) \( \left(\frac{n}{4}\right)^2 \)
- \( l_2: \) \( \left(\frac{n}{4^2}\right)^2 \) \( \left(\frac{n}{4^2}\right)^2 \) \( \left(\frac{n}{4^2}\right)^2 \) \( \left(\frac{n}{4^2}\right)^2 \) \( \left(\frac{n}{4^2}\right)^2 \)
- \( l_3: \ldots \)
- \( l_4: \ldots \)
- \( l_{m-1}: \ldots \)
- \( l_m: T(1), T(1), T(1), T(1), T(1), \ldots \)

- \( 3^3 \) nodes of \( \left(\frac{n}{4^3}\right)^2 \)
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Chapter 4. Solving Recurrences

Recursive tree notation in the textbook:

\[ T(n) = 3T(n/4) + n^2, \] with base case \( T(1) = 1 \)

\[ l_0: \quad n^2 \]
\[ l_1: \quad (n/4)^2 (n/4)^2 (n/4)^2 (n/4)^2 \]
\[ l_2: \quad \left( \frac{n}{4^2} \right)^2 \left( \frac{n}{4^2} \right)^2 \left( \frac{n}{4^2} \right)^2 \left( \frac{n}{4^2} \right)^2 \left( \frac{n}{4^2} \right)^2 \left( \frac{n}{4^2} \right)^2 \]
\[ l_3: \quad \ldots \ldots \]
\[ l_4: \quad \ldots \ldots \]
\[ l_{m-1}: \quad \ldots \ldots \]
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\[ 3^3 \text{ nodes of } \left( \frac{n}{4^3} \right)^2 \]
\[ 3^{m-1} \text{ of } \left( \frac{n}{4^{m-1}} \right)^2 \]
\[ 3^m \text{ nodes of } T(1) \]
Chapter 5. Probabilistic Analysis of Algorithms

Chapter 5. Probabilistic analysis and randomized algorithms
Chapter 5. Probabilistic Analysis of Algorithms

Chapter 5. Probabilistic analysis and randomized algorithms

• Estimate efficiency of algorithms on a majority of inputs, not all inputs;
Chapter 5. Probabilistic analysis and randomized algorithms

- Estimate efficiency of algorithms on a majority of inputs, not all inputs;
- Performance is “average” cases, not the worst case;
Chapter 5. Probabilistic Analysis of Algorithms

Chapter 5. Probabilistic analysis and randomized algorithms

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Chapter 5. Probabilistic Analysis of Algorithms

Chapter 5. Probabilistic analysis and randomized algorithms

- Estimate efficiency of algorithms on a majority of inputs, not all inputs;
- Performance is “average” cases, not the worst case;
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- Close relationship with randomized algorithms
A good example: Quick Sort algorithm [Hoare’1959]
A good example: **Quick Sort** algorithm [Hoare’1959]

- It has the worst case time $T_{wc}(n) \geq an^2$ for some constant $a > 0$. 

- Alternatively, we can enforce the desired distribution by using randomness (tossing coins) in the algorithms. That is, we use randomized algorithms.
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- It has the worst case time $T_{wc}(n) \geq an^2$ for some constant $a > 0$.
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Chapter 5. Probabilistic Analysis of Algorithms

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assumption: the input data are of the uniform distribution.
Chapter 5. Probabilistic Analysis of Algorithms

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Chapter 5. Probabilistic Analysis of Algorithms

In both probabilistic analysis of deterministic algorithms and analysis of randomized algorithms:

- actions in the algorithm are considered *random events*
Chapter 5. Probabilistic Analysis of Algorithms

In both probabilistic analysis of deterministic algorithms and analysis of randomized algorithms:

- actions in the algorithm are considered random events
- such random events are driven by random data

So the running time comes with a probability; you can compute the expected time (i.e., averaged time)
In both probabilistic analysis of deterministic algorithms and analysis of randomized algorithms:

- Actions in the algorithm are considered random events.
- Such random events are driven by random data, which are either input data or randomly tossed coins.

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Chapter 5. Probabilistic Analysis of Algorithms

In both probabilistic analysis of deterministic algorithms and analysis of randomized algorithms

- actions in the algorithm are considered random events
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Chapter 5. Probabilistic Analysis of Algorithms

In both probabilistic analysis of deterministic algorithms and analysis of randomized algorithms

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Chapter 5. Probabilistic Analysis of Algorithms

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Chapter 5. Probabilistic Analysis of Algorithms

There are two types of randomized algorithms:

- Las Vegas algorithms

- Monte Carlo algorithms
  - on 'NO' instances, 100% accuracy; \( \text{Prob}(\text{to answer 'NO' on 'NO' instance}) = 1 \)
  - on 'YES' instances, \( \geq 75\% \) accuracy; \( \text{Prob}(\text{to answer 'YES' on 'YES' instance}) \geq 0.75 \)

Accuracy \( 75\% \) can be improved to \( 99.99\% \) with multiple trials.

• Las Vegas algorithms is as powerful as Monte Carlo algorithms, if not more.
There are two types of randomized algorithms:

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Chapter 5. Probabilistic Analysis of Algorithms

There are two types of randomized algorithms:

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  - always gives answer correctly;
  - running time comes with a probability distribution

- Monte Carlo algorithms
  - on 'NO' instances, 100% accuracy;
  - on 'YES' instances, ≥ 75% accuracy;
  - accuracy 75% can be improved to 99.99% with multiple trials.

Las Vegas algorithms is as powerful as Monte Carlo algorithms, if not more.
Chapter 5. Probabilistic Analysis of Algorithms

There are two types of randomized algorithms:

- **Las Vegas algorithms**
  - always gives answer correctly;
  - running time comes with a probability distribution
    e.g., for QUICKSORT, $\text{Prob}(T(n) \leq cn \log n) \geq 0.75$

- **Monte Carlo algorithms**
  - on 'NO' instances, 100% accuracy; $\text{Prob}(\text{to answer } '\text{NO}' \text{ on } '\text{NO}' \text{ instance}) = 1$
  - on 'YES' instances, $\geq 75$% accuracy; $\text{Prob}(\text{to answer } '\text{YES}' \text{ on } '\text{YES}' \text{ instance}) \geq 0.75$

Accuracy $75$% can be improved to $99.99$% with multiple trials.

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