CSCI 4470/6470 Algorithms, Spring 2019

Liming Cai
Department of Computer Science, UGA


January 22, 2019
An Introduction to the Introduction

There are two ways of constructing a software design: one way is to make it so simple that there are obviously no deficiencies, and the other way is to make it so complicated that there are no obvious deficiencies.

-C.A.R. Hoare

Which way do we take in algorithm design?
There are two ways of constructing a software design: One way is to make it so simple that there are obviously no deficiencies, and the other way is to make it so complicated that there are no obvious deficiencies.

- C.A.R. Hoare
An Introduction to the Introduction

There are two ways of constructing a software design: One way is to make it so simple that there are obviously no deficiencies, and the other way is to make it so complicated that there are no obvious deficiencies.

- C.A.R. Hoare

Which way do we take in algorithm design?
An Introduction to the Introduction

Example 1
An Introduction to the Introduction

Example 1

Sequence Homology Reveals Functions

- Homology reveals evolution of structure/function

<table>
<thead>
<tr>
<th>Homology reveals regulatory structure (E. Coli promoters)</th>
</tr>
</thead>
<tbody>
<tr>
<td>tyr: tRNA</td>
</tr>
<tr>
<td>rpm D1</td>
</tr>
<tr>
<td>rpm X1</td>
</tr>
<tr>
<td>rpm (DCE)2</td>
</tr>
<tr>
<td>rpm E1</td>
</tr>
<tr>
<td>rpm A1</td>
</tr>
<tr>
<td>rpm A2</td>
</tr>
<tr>
<td>APr</td>
</tr>
<tr>
<td>PL</td>
</tr>
<tr>
<td>T7A3</td>
</tr>
<tr>
<td>T7A1</td>
</tr>
<tr>
<td>T7A2</td>
</tr>
<tr>
<td>fd VIII</td>
</tr>
</tbody>
</table>

Consensus
An Introduction to the Introduction

- main task: sequence comparison (consider just 2 sequences)
An Introduction to the Introduction

• main task: sequence comparison (consider just 2 sequences)

```
FOS_CHICK:  MMYQGFAGEYEAPSSRCSSASPAGDSLTYYPSFADSFSSMGSPVNSQDFCTDLAVSSANF 60
FOSB_MOUSE: MFQAFPGDYDS-GSRCSS-SPSAESQ--YLSSVDSFGSPPTAAASQE-CAGLGEQPGSF 54
```
An Introduction to the Introduction

• main task: sequence comparison (consider just 2 sequences)

• analogy in text searching/matching

An`  `Introduction to the Introduction

• main task: sequence comparison (consider just 2 sequences)

• analogy in text searching/matching

An`  `Introduction to the Introduction

• main task: sequence comparison (consider just 2 sequences)

• analogy in text searching/matching
An Introduction to the Introduction

- main task: sequence comparison (consider just 2 sequences)

- analogy in text searching/matching but allowing substitutions, insertions, deletions

```
FOS_CHICK  MMYQGFAGYEAPSSRCSSASPAGDSLTYYPSPADSFSSMVNSQDFCTDLAVSANSNF 60
FOSB_MOUSE -MFQAFPGDYDS-GSRCSS-SPSASEQ--YLSSVDSFGSPPTAASQE-CAGLGMPSGF 54
```
An Introduction to the Introduction

- main task: sequence comparison (consider just 2 sequences)

- analogy in text searching/matching but allowing substitutions, insertions, deletions

- total number of number of alignments $\geq 2^n$. 
An Introduction to the Introduction

• main task: sequence comparison (consider just 2 sequences)

• analogy in text searching/matching
  but allowing substitutions, insertions, deletions

• total number of number of alignments $\geq 2^n$.
  to find the most plausible one, cannot afford to try them all
An Introduction to the Introduction

• main task: sequence comparison (consider just 2 sequences)

<table>
<thead>
<tr>
<th>FOS_CHICK</th>
<th>MTVQGFTAGEYEAPSSRCASSRPAGDSLTVYPSFADSFSMGPVNSQDFCTDLAVSSANF</th>
</tr>
</thead>
<tbody>
<tr>
<td>FOSB_MOUSE</td>
<td>MFQAFPGDYDS-GSRCSS-SPSAESQ--YLSSVDSFGSPTAAASQE-CAGILGEMPGSF</td>
</tr>
</tbody>
</table>

• analogy in text searching/matching
  but allowing substitutions, insertions, deletions

• total number of number of alignments $\geq 2^n$.
  to find the most plausible one, cannot afford to try them all

• require to design “smarter” algorithms that run much faster
An Introduction to the Introduction

• main task: sequence comparison (consider just 2 sequences)

• analogy in text searching/matching
  but allowing substitutions, insertions, deletions

• total number of number of alignments $\geq 2^n$.
  to find the most plausible one, cannot afford to try them all

• require to design “smarter” algorithms that run much faster

We will study various techniques for efficient algorithm design.
An Introduction to the Introduction

Example 2
An Introduction to the Introduction

Example 2

---

**US Open 2016 Men's Singles**

<table>
<thead>
<tr>
<th>Round 1</th>
<th>Round 2</th>
<th>Round 3</th>
<th>Round 4</th>
<th>Quarterfinals</th>
<th>Semifinals</th>
</tr>
</thead>
<tbody>
<tr>
<td>3. Vesely, Jan CZE</td>
<td>(O)</td>
<td>G. Pella</td>
<td>M. Youzhny</td>
<td>6-4/6-1/7-6(3)</td>
<td></td>
</tr>
<tr>
<td>4. Myhnen, Sascha IND</td>
<td>(O)</td>
<td>G. Pella</td>
<td>M. Youzhny</td>
<td>6-4/6-1/7-6(3)</td>
<td></td>
</tr>
<tr>
<td>5. Fratangelo, Blem USA</td>
<td>(W)</td>
<td>M. Youzhny</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6. Pella, Guido ARG</td>
<td>(W)</td>
<td>M. Youzhny</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7. Youzhny, Mikhail RUS</td>
<td>(W)</td>
<td>M. Youzhny</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8. Klizan, Martin SVK</td>
<td>(W)</td>
<td>M. Youzhny</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9. Isner, John USA</td>
<td>(O)</td>
<td>M. Youzhny</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10. Tiafoe, Francesca USA</td>
<td>(O)</td>
<td>M. Youzhny</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>11. Thompson, Jordan AUS</td>
<td>(O)</td>
<td>M. Youzhny</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>12. Darcis, Steve BEL</td>
<td>(O)</td>
<td>M. Youzhny</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>13. Lacko, Lukas SVK</td>
<td>(O)</td>
<td>M. Youzhny</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>14. Escobedo, Emnesto USA</td>
<td>(W)</td>
<td>M. Youzhny</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>15. Edmund, Kyle GBR</td>
<td>(W)</td>
<td>M. Youzhny</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>16. Gasquet, Richard FRA</td>
<td>(O)</td>
<td>M. Youzhny</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>17. Tsonga, Jo-Wilfried FRA</td>
<td>(O)</td>
<td>M. Youzhny</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>18. Andreescu, Guido ARG</td>
<td>(O)</td>
<td>M. Youzhny</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>19. Duckworth, James AUS</td>
<td>(O)</td>
<td>M. Youzhny</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>20. Haase, Robin NED</td>
<td>(O)</td>
<td>M. Youzhny</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>21. Pospisil, Vasek CAN</td>
<td>(O)</td>
<td>M. Youzhny</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>22. Kovacevic, Jozef SVK</td>
<td>(O)</td>
<td>M. Youzhny</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>23. Nishikori, Yoshito JPN</td>
<td>(L)</td>
<td>M. Youzhny</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>24. Anderson, Kevin RSA</td>
<td>(L)</td>
<td>M. Youzhny</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>25. Sock, Jack USA</td>
<td>(L)</td>
<td>M. Youzhny</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>26. Fritz, Taylor USA</td>
<td>(L)</td>
<td>M. Youzhny</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>27. Zverev, Misha GER</td>
<td>(O)</td>
<td>M. Youzhny</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>28. Herbert, Pierre-Hugues FRA</td>
<td>(O)</td>
<td>M. Youzhny</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>29. Stakhovsky, Sergi UKR</td>
<td>(O)</td>
<td>M. Youzhny</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>30. Elia, Gastap PUK</td>
<td>(O)</td>
<td>M. Youzhny</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>31. Dutra Silva, Rogerio BRA</td>
<td>(O)</td>
<td>M. Youzhny</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>32. Cilic, Marin CRO</td>
<td>(O)</td>
<td>M. Youzhny</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>33. Nadal, Rafael ESP</td>
<td>(O)</td>
<td>M. Youzhny</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>34. Istomin, Denis UZB</td>
<td>(O)</td>
<td>M. Youzhny</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>35. Robert, Stephane FRA</td>
<td>(O)</td>
<td>M. Youzhny</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>36. Seppi, Andreas ITA</td>
<td>(O)</td>
<td>M. Youzhny</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>37. Bellucci, Thomaz BRA</td>
<td>(O)</td>
<td>M. Youzhny</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>38. Kuznetsov, Andrey RUS</td>
<td>(O)</td>
<td>M. Youzhny</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>39. Benneteau, Julien FRA</td>
<td>(O)</td>
<td>M. Youzhny</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>40. Ramos-Vinolas, Albert ESP</td>
<td>(O)</td>
<td>M. Youzhny</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>41. Pouille, Lucas FRA</td>
<td>(O)</td>
<td>M. Youzhny</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>42. Kukushkin, Mikhail KAZ</td>
<td>(O)</td>
<td>M. Youzhny</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>43. Cielezar, Guillaume FRA</td>
<td>(O)</td>
<td>M. Youzhny</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>44. Chudinelli, Marco SUI</td>
<td>(O)</td>
<td>M. Youzhny</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>45. Beker, Brian USA</td>
<td>(O)</td>
<td>M. Youzhny</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>46. Delbonis, Federico ARG</td>
<td>(O)</td>
<td>M. Youzhny</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>47. Garcia Lopez, Guillermo ESP</td>
<td>(O)</td>
<td>M. Youzhny</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>48. Bautista Agut, Roberto ESP</td>
<td>(O)</td>
<td>M. Youzhny</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>49. Monfils, Gael FRA</td>
<td>(O)</td>
<td>M. Youzhny</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>50. Muller, Gilles LUX</td>
<td>(O)</td>
<td>M. Youzhny</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>51. Satrav, Jan GIE</td>
<td>(O)</td>
<td>M. Youzhny</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>52. McDonald, Mackenzie USA</td>
<td>(W)</td>
<td>M. Youzhny</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>53. Fucsovics, Marton HUN</td>
<td>(W)</td>
<td>M. Youzhny</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>54. Almagro, Nicolas ESP</td>
<td>(W)</td>
<td>M. Youzhny</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>55. Sela, Dudi ISR</td>
<td>(W)</td>
<td>M. Youzhny</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>56. Cuevas, Pablo RUB</td>
<td>(W)</td>
<td>M. Youzhny</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>57. Paire, Benoit FRA</td>
<td>(W)</td>
<td>M. Youzhny</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>58. Lajovic, Dusan SRB</td>
<td>(W)</td>
<td>M. Youzhny</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>59. Bagdatis, Marcos CYP</td>
<td>(W)</td>
<td>M. Youzhny</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>60. Bagnis, Fabrizio ARG</td>
<td>(W)</td>
<td>M. Youzhny</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>61. Mannarino, Adrian FRA</td>
<td>(W)</td>
<td>M. Youzhny</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>62. Harrison, Ryan USA</td>
<td>(W)</td>
<td>M. Youzhny</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>63. Brown, Dustin AUS</td>
<td>(W)</td>
<td>M. Youzhny</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>64. Raonic, Milos CAN</td>
<td>(W)</td>
<td>M. Youzhny</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Quarterfinals**

- N. Djokovic [1] 6-2/6-1/6-4
- K. Edmund
- J. Tsonga [9] 6-4/1-6/6-3
- N. Djokovic [1] 6-3/6-2

**Semifinals**

- N. Djokovic [1] 6-2/6-4/6-4
- J. Tsonga [9] 6-4/1-6/6-3
- J. Tsonga [9] 6-4/1-6/6-3
- J. Tsonga [9] 6-4/1-6/6-3

**Champion:**

Wawrinka, Stan SUI
8-7(1)/1-6/4/1-6/6-3
An Introduction to the Introduction
An Introduction to the Introduction

- 128 players in total;
- 127 matches were played to determine who was the champion.

Is it possible to just play fewer matches?
• 128 players in total;
An Introduction to the Introduction

- 128 players in total;
- 127 matches were played to determine who was the champion.
• 128 players in total;
• 127 matches were played to determine who was the champion.
• Is it possible to just play fewer matches?
An Introduction to the Introduction

You were called to answer this question;

Sounds crazy ... but if I don't do it, my job won't be secure...

See if you can reduce the number of matches!
An Introduction to the Introduction

You were called to answer this question;

Sounds crazy ... but if I don’t do it, my job won’t be secure...

See if you can reduce the number of matches!
An Introduction to the Introduction

• You tried various match formats, but all need at least 127 matches;
An Introduction to the Introduction

• You tried various match formats, but all need at least 127 matches;
• Then you suspected that 127 matches were necessary,
An Introduction to the Introduction

- You tried various match formats, but all need at least 127 matches;
- Then you suspected that 127 matches were necessary, and finally came up with a mathematical proof for that.
An Introduction to the Introduction

- You tried various match formats, but all need at least 127 matches;
- Then you suspected that 127 matches were necessary, and finally came up with a mathematical proof for that.

Sir, this won’t happen. I have a mathematical proof that it needs 127 matches!

Hmmm, he is smart... I should give him more work to do.
An Introduction to the Introduction

- You tried various match formats, but all need at least 127 matches;
- Then you suspected that 127 matches were necessary, and finally came up with a mathematical proof for that.
An Introduction to the Introduction

<table>
<thead>
<tr>
<th>Same problems</th>
<th>TENNIS TOURNAMENT</th>
<th>FINDING MAXIMUM</th>
</tr>
</thead>
<tbody>
<tr>
<td>Input:</td>
<td>128 players</td>
<td>( n ) numbers</td>
</tr>
<tr>
<td>Output:</td>
<td>Champion</td>
<td>the maximum number</td>
</tr>
<tr>
<td>Solution:</td>
<td>a match scheme</td>
<td>an algorithm</td>
</tr>
<tr>
<td>Time Efficiency:</td>
<td>number of matches</td>
<td>number of comparisons</td>
</tr>
</tbody>
</table>
An Introduction to the Introduction

<table>
<thead>
<tr>
<th>Same problems</th>
<th><strong>Tennis Tournament</strong></th>
<th>Finding Maximum</th>
</tr>
</thead>
<tbody>
<tr>
<td>Input:</td>
<td>128 players</td>
<td>$n$ numbers</td>
</tr>
<tr>
<td>Output:</td>
<td>Champion</td>
<td>the maximum number</td>
</tr>
<tr>
<td>Solution:</td>
<td>a match scheme</td>
<td>an algorithm</td>
</tr>
<tr>
<td>Time Efficiency:</td>
<td>number of matches</td>
<td>number of comparisons</td>
</tr>
</tbody>
</table>

You actually accomplished two tasks:
An Introduction to the Introduction

<table>
<thead>
<tr>
<th>Same problems</th>
<th>Tennis Tournament</th>
<th>Finding Maximum</th>
</tr>
</thead>
<tbody>
<tr>
<td>Input:</td>
<td>128 players</td>
<td>$n$ numbers</td>
</tr>
<tr>
<td>Output:</td>
<td>Champion</td>
<td>the maximum number</td>
</tr>
<tr>
<td>Solution:</td>
<td>a match scheme</td>
<td>an algorithm</td>
</tr>
<tr>
<td>Time Efficiency:</td>
<td>number of matches</td>
<td>number of comparisons</td>
</tr>
</tbody>
</table>

You actually accomplished two tasks:

- tried various algorithms for **Tennis Tournament** before giving up;
An Introduction to the Introduction

<table>
<thead>
<tr>
<th>Same problems</th>
<th><strong>TENNIS TOURNAMENT</strong></th>
<th><strong>FINDING MAXIMUM</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>Input:</td>
<td>128 players</td>
<td>$n$ numbers</td>
</tr>
<tr>
<td>Output:</td>
<td>Champion</td>
<td>the maximum number</td>
</tr>
<tr>
<td>Solution:</td>
<td>a match scheme</td>
<td>an algorithm</td>
</tr>
<tr>
<td>Time Efficiency:</td>
<td>number of matches</td>
<td>number of comparisons</td>
</tr>
</tbody>
</table>

You actually accomplished two tasks:

- **tried various algorithms** for **TENNIS TOURNAMENT** before giving up;
- **proved that more efficient algorithms do not exist**;
An Introduction to the Introduction

<table>
<thead>
<tr>
<th>Same problems</th>
<th><strong>Tennis Tournament</strong></th>
<th><strong>Finding Maximum</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>Input:</td>
<td>128 players</td>
<td>$n$ numbers</td>
</tr>
<tr>
<td>Output:</td>
<td>Champion</td>
<td>the maximum number</td>
</tr>
<tr>
<td>Solution:</td>
<td>a match scheme</td>
<td>an algorithm</td>
</tr>
<tr>
<td>Time Efficiency:</td>
<td>number of matches</td>
<td>number of comparisons</td>
</tr>
</tbody>
</table>

You actually accomplished two tasks:

- **tried various algorithms** for **Tennis Tournament** before giving up;
- proved that more efficient algorithms do not exist;
- actually you proved the “USTA algorithm” was already the optimal.
An Introduction to the Introduction

<table>
<thead>
<tr>
<th>Same problems</th>
<th>Tennis Tournament</th>
<th>Finding Maximum</th>
</tr>
</thead>
<tbody>
<tr>
<td>Input:</td>
<td>128 players</td>
<td>$n$ numbers</td>
</tr>
<tr>
<td>Output:</td>
<td>Champion</td>
<td>the maximum number</td>
</tr>
<tr>
<td>Solution:</td>
<td>a match scheme</td>
<td>an algorithm</td>
</tr>
<tr>
<td>Time Efficiency:</td>
<td>number of matches</td>
<td>number of comparisons</td>
</tr>
</tbody>
</table>

You actually accomplished two tasks:

- tried various algorithms for Tennis Tournament before giving up;
- proved that more efficient algorithms do not exist;
- actually you proved the “USTA algorithm” was already the optimal.

We will learn how to prove that an algorithm is already the best.
Example 3
An Introduction to the Introduction

Example 3

Toss a coin over the phone

I will let you choose, head or tail?

Uh...
An Introduction to the Introduction

Example 3

Toss a coin over the phone

• how does one person know the other is telling the truth?
An Introduction to the Introduction

Example 3

Toss a coin over the phone

- how does one person know the other is telling the truth?
- even if the person is, would the first person trust him?
An Introduction to the Introduction

Example 3

Toss a coin over the phone

- how does one person know the other is telling the truth?
- even if the person is, would the first person trust him?

How to accomplish this task?
They decided the following protocol:

- A told B a huge Boolean circuit through the phone; along with an output $Y$ of binary bits (but NOT the input $X$);
- B guessed if the number of 0's in $X$ was odd or even.
An Introduction to the Introduction

They decided the following protocol:

• A told B a huge Boolean circuit through the phone;

X = 110101

Y = 111010
An Introduction to the Introduction

They decided the following protocol:

- A told B a huge Boolean circuit through the phone; along with an output $Y$ of binary bits (but NOT the input $X$);
They decided the following protocol:

- A told B a huge Boolean circuit through the phone; along with an output $Y$ of binary bits (but NOT the input $X$);
- B guessed if the number of 0's in $X$ was odd or even;
An Introduction to the Introduction

X = 110101

Y = 111010
An Introduction to the Introduction

- from $Y$, it would be very difficult to figure out which $X$ had been used to generate $Y$ (an intractable problem);
An Introduction to the Introduction

- from $Y$, it would be very difficult to figure out which $X$ had been used to generate $Y$ (an intractable problem);

- So the best $B$ can do is to randomly guess (odd/even), which has the same effect as guessing a coin toss.
An Introduction to the Introduction

- from $Y$, it would be very difficult to figure out which $X$ had been used to generate $Y$ (an intractable problem);
- So the best B can do is to randomly guess (odd/even), which has the same effect as guessing a coin toss.

We will investigate some intractable problems and the theory behind it.
An Introduction to the Introduction

Now we are back to the tennis tournament problem.
An Introduction to the Introduction

Now we are back to the tennis tournament problem.

Same problems

**TENNIS TOURNAMENT**

Input: 128 players
Output: Champion
Solution: a match scheme
Time Efficiency: number of matches

**FINDING MAXIMUM**

Input: \(n\) numbers
Output: the maximum number
Solution: an algorithm
Time Efficiency: number of comparisons
An Introduction to the Introduction

Now we are back to the tennis tournament problem.

Same problems

<table>
<thead>
<tr>
<th><strong>TENNIS TOURNAMENT</strong></th>
<th><strong>FINDING MAXIMUM</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>Input: 128 players</td>
<td>Output: n numbers</td>
</tr>
<tr>
<td>Output: Champion</td>
<td></td>
</tr>
<tr>
<td>Solution: a match scheme</td>
<td></td>
</tr>
<tr>
<td>Time Efficiency:</td>
<td>number of matches</td>
</tr>
<tr>
<td></td>
<td>number of comparisons</td>
</tr>
</tbody>
</table>

You actually accomplished two tasks:
Now we are back to the tennis tournament problem.

<table>
<thead>
<tr>
<th>Same problems</th>
<th>TENNIS TOURNAMENT</th>
<th>FINDING MAXIMUM</th>
</tr>
</thead>
<tbody>
<tr>
<td>Input:</td>
<td>128 players</td>
<td>$n$ numbers</td>
</tr>
<tr>
<td>Output:</td>
<td>Champion</td>
<td>the maximum number</td>
</tr>
<tr>
<td>Solution:</td>
<td>a match scheme</td>
<td>an algorithm</td>
</tr>
<tr>
<td>Time Efficiency:</td>
<td>number of matches</td>
<td>number of comparisons</td>
</tr>
</tbody>
</table>

You actually accomplished two tasks:

- **tried various algorithms** for **TENNIS TOURNAMENT** before giving up;
Now we are back to the tennis tournament problem.

<table>
<thead>
<tr>
<th>Same problems</th>
<th><strong>TENNIS TOURNAMENT</strong></th>
<th><strong>FINDING MAXIMUM</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>Input:</td>
<td>128 players</td>
<td>$n$ numbers</td>
</tr>
<tr>
<td>Output:</td>
<td>Champion</td>
<td>the maximum number</td>
</tr>
<tr>
<td>Solution:</td>
<td>a match scheme</td>
<td>an algorithm</td>
</tr>
<tr>
<td>Time Efficiency:</td>
<td>number of matches</td>
<td>number of comparisons</td>
</tr>
</tbody>
</table>

You actually accomplished two tasks:

- tried various algorithms for **TENNIS TOURNAMENT** before giving up;
- proved that more efficient algorithms do not exist;
An Introduction to the Introduction

Now we are back to the tennis tournament problem.

<table>
<thead>
<tr>
<th>Same problems</th>
<th>TENNIS TOURNAMENT</th>
<th>FINDING MAXIMUM</th>
</tr>
</thead>
<tbody>
<tr>
<td>Input: 128 players</td>
<td>n numbers</td>
<td>Output: Champion</td>
</tr>
<tr>
<td>Solution: a match scheme</td>
<td>an algorithm</td>
<td>Time Efficiency: number of matches</td>
</tr>
</tbody>
</table>

You actually accomplished two tasks:

- tried various algorithms for TENNIS TOURNAMENT before giving up;
- proved that more efficient algorithms do not exist;
- actually you proved the “USTA algorithm” was already the optimal.
An Introduction to the Introduction

Number of matches

- Algorithm A
- Algorithm Z
- USTA algorithm

Algorithms you tried, unfortunately

No algorithm uses fewer than 127 matches

USTA algorithm gives an upper bound (≤ 127)

The Tennis Tournament (128) Problem

Your proof gives a lower bound (≥ 127)
An Introduction to the Introduction

Definitions of complexity upper and lower bounds;
An Introduction to the Introduction

Definitions of complexity upper and lower bounds;

• **Time complexity upper bound** is a time threshold *in* which all instances of a problem *can* be solved.
Definitions of complexity upper and lower bounds;

- **Time complexity upper bound** is a time threshold *in* which all instances of a problem can be solved.

- **Time complexity lower bound** is a time threshold *below* which some instances of the problem cannot be solved.
Definitions of complexity upper and lower bounds;

- **Time complexity upper bound** is a time threshold *in* which all instances of a problem *can* be solved.

- **Time complexity lower bound** is a time threshold *below* which some instances of the problem *cannot* be solved.

- These notions apply to both problems and algorithms.
An Introduction to the Introduction

In general, for a given problem $\Pi$, 

• an algorithm $A$ for $\Pi$ also gives a time upper bound $T_A$ for $\Pi$, where $T_A$ is the time complexity of algorithm $A$;

• a faster algorithm $B$ for $\Pi$, with $T_B < T_A$, gives a tighter upper bound $T_B$ for $\Pi$;

• a proof that $\Pi$ cannot have algorithms faster than time $T$, gives a lower bound $T$ for $\Pi$;

• a proof that $\Pi$ cannot have algorithms faster than time $S$, with $S > T$, gives a tighter lower bound $S$ for $\Pi$.

• for problem $\Pi$, time lower bounds $\leq$ time upper bounds; when a lower bound is the same as an upper bound, both the lower and upper bounds are called optimal. Also the corresponding algorithms (that offer the upper bound) are called optimal for $\Pi$. 
An Introduction to the Introduction

In general, for a given problem $\Pi$,

- an algorithm $A$ for $\Pi$ also gives a time upper bound $T_A$ for $\Pi$, where $T_A$ is the time complexity of algorithm $A$;
An Introduction to the Introduction

In general, for a given problem $\Pi$,

- an algorithm $A$ for $\Pi$ also gives a time upper bound $T_A$ for $\Pi$, where $T_A$ is the time complexity of algorithm $A$;

  a faster algorithm $B$ for $\Pi$, with $T_B < T_A$, gives a tighter upper bound $T_B$ for $\Pi$;
In general, for a given problem $\Pi$,

- an algorithm $A$ for $\Pi$ also gives a time *upper bound* $T_A$ for $\Pi$, where $T_A$ is the time complexity of algorithm $A$;
  - a faster algorithm $B$ for $\Pi$, with $T_B < T_A$, gives a *tighter upper bound* $T_B$ for $\Pi$;
- a proof that $\Pi$ cannot have algorithms faster than time $T$,
In general, for a given problem $\Pi$,

- an algorithm $A$ for $\Pi$ also gives a time upper bound $T_A$ for $\Pi$, where $T_A$ is the time complexity of algorithm $A$;
  
  a faster algorithm $B$ for $\Pi$, with $T_B < T_A$, gives a tighter upper bound $T_B$ for $\Pi$;

- a proof that $\Pi$ cannot have algorithms faster than time $T$, gives a lower bound $T$ for $\Pi$;
In general, for a given problem $\Pi$,

- an algorithm $A$ for $\Pi$ also gives a time upper bound $T_A$ for $\Pi$, where $T_A$ is the time complexity of algorithm $A$;
  
  a faster algorithm $B$ for $\Pi$, with $T_B < T_A$, gives a tighter upper bound $T_B$ for $\Pi$;

- a proof that $\Pi$ cannot have algorithms faster than time $T$, gives a lower bound $T$ for $\Pi$;
  
  a proof that $\Pi$ cannot have algorithms faster than time $S$, with $S > T$,
In general, for a given problem $\Pi$,

- an algorithm $A$ for $\Pi$ also gives a time upper bound $T_A$ for $\Pi$, where $T_A$ is the time complexity of algorithm $A$;

  a faster algorithm $B$ for $\Pi$, with $T_B < T_A$, gives a tighter upper bound $T_B$ for $\Pi$;

- a proof that $\Pi$ cannot have algorithms faster than time $T$, gives a lower bound $T$ for $\Pi$;

  a proof that $\Pi$ cannot have algorithms faster than time $S$, with $S > T$, gives a tighter lower bound $S$ for $\Pi$. 

An Introduction to the Introduction

In general, for a given problem \( \Pi \),

- an algorithm \( A \) for \( \Pi \) also gives a time upper bound \( T_A \) for \( \Pi \), where \( T_A \) is the time complexity of algorithm \( A \);
  
a faster algorithm \( B \) for \( \Pi \), with \( T_B < T_A \), gives a tighter upper bound \( T_B \) for \( \Pi \);

- a proof that \( \Pi \) cannot have algorithms faster than time \( T \), gives a lower bound \( T \) for \( \Pi \);
  
a proof that \( \Pi \) cannot have algorithms faster than time \( S \), with \( S > T \), gives a tighter lower bound \( S \) for \( \Pi \).

- for problem \( \Pi \), time lower bounds \( \leq \) time upper bounds;
In general, for a given problem $\Pi$,

- an algorithm $A$ for $\Pi$ also gives a time upper bound $T_A$ for $\Pi$, where $T_A$ is the time complexity of algorithm $A$;

  a faster algorithm $B$ for $\Pi$, with $T_B < T_A$, gives a tighter upper bound $T_B$ for $\Pi$;

- a proof that $\Pi$ cannot have algorithms faster than time $T$, gives a lower bound $T$ for $\Pi$;

  a proof that $\Pi$ cannot have algorithms faster than time $S$, with $S > T$, gives a tighter lower bound $S$ for $\Pi$.

- for problem $\Pi$, time lower bounds $\leq$ time upper bounds;

  when a lower bound is the same as an upper bound, both the lower and upper bounds are called optimal.
An Introduction to the Introduction

In general, for a given problem \( \Pi \),

- an algorithm \( A \) for \( \Pi \) also gives a time upper bound \( T_A \) for \( \Pi \), where \( T_A \) is the time complexity of algorithm \( A \);
  
a faster algorithm \( B \) for \( \Pi \), with \( T_B < T_A \), gives a tighter upper bound \( T_B \) for \( \Pi \);

- a proof that \( \Pi \) cannot have algorithms faster than time \( T \), gives a lower bound \( T \) for \( \Pi \);
  
a proof that \( \Pi \) cannot have algorithms faster than time \( S \), with \( S > T \), gives a tighter lower bound \( S \) for \( \Pi \).

- for problem \( \Pi \), time lower bounds \( \leq \) time upper bounds;

  when a lower bound is the same as an upper bound, both the lower and upper bounds are called optimal.
Also the corresponding algorithms (that offer the upper bound) are called optimal for \( \Pi \).
An Introduction to the Introduction

Time complexity situation for problem \( \Pi \)

- Algorithm A
- Algorithm B
- A proved lower bound
- A proved lower bound

\[ T_A \]
\[ T_B \]
\[ S \]
\[ T \]

2
1
0

gap, the smaller the better
An Introduction to the Introduction

So to design good algorithms for computational problems, our goals are
An Introduction to the Introduction

So to design good algorithms for computational problems, our goals are
1. to achieve tighter upper bounds
An Introduction to the Introduction

So to design good algorithms for computational problems, our goals are

1. to achieve tighter upper bounds
   - we need be familiar with techniques for algorithm complexity analysis;
An Introduction to the Introduction

So to design good algorithms for computational problems, our goals are

1. to achieve tighter upper bounds
   - we need be familiar with techniques for algorithm complexity analysis;
   - we need to master efficient algorithm design skills;
So to design good algorithms for computational problems, our goals are

1. to achieve tighter upper bounds
   • we need be familiar with techniques for algorithm complexity analysis;
   • we need to master efficient algorithm design skills;
2. and to achieve tighter lower bounds,
An Introduction to the Introduction

So to design good algorithms for computational problems, our goals are

1. to achieve tighter upper bounds
   - we need to be familiar with techniques for algorithm complexity analysis;
   - we need to master efficient algorithm design skills;

2. and to achieve tighter lower bounds,
   - we need to know the methods for lower bound proofs.
The Introduction

What is this course about (and why is it needed)?

- about basic yet indispensable skills for problem solving
- algorithm design leads to writing code, i.e., creative thinking without programming languages

How different is this course from other algorithm courses?

- design technique-oriented, not application-oriented
- emphasis on guaranteed performance (typically in efficiency)

Goals to achieve

- to learn to measure performance of algorithms
- to master some fundamental algorithmic techniques
- to study advanced algorithmic skills
- to understand computational intractability
The Introduction

- What is this course about (and why is it needed)?
The Introduction

- What is this course about (and why is it needed)?
  - about basic yet indispensable skills for problem solving

- How different is this course from other algorithm courses?
  - design technique-oriented, not application-oriented
  - emphasis on guaranteed performance (typically in efficiency)

- Goals to achieve
  - to learn to measure performance of algorithms
  - to master some fundamental algorithmic techniques
  - to study advanced algorithmic skills
  - to understand computational intractability
The Introduction

- What is this course about (and why is it needed)?
  - about basic yet indispensable skills for problem solving
  - algorithm design leads to writing code,
- How different is this course from other algorithm courses?
  - design technique-oriented, not application-oriented
  - emphasis on guaranteed performance (typically in efficiency)
- Goals to achieve
  - to learn to measure performance of algorithms
  - to master some fundamental algorithmic techniques
  - to study advanced algorithmic skills
  - to understand computational intractability
The Introduction

▶ What is this course about (and why is it needed)?

• about basic yet indispensable skills for problem solving

• algorithm design leads to writing code,
  i.e., creative thinking without programming languages
The Introduction

▶ What is this course about (and why is it needed)?
• about basic yet indispensable skills for problem solving
• algorithm design leads to writing code,
  i.e., creative thinking without programming languages

▶ How different is this course from other algorithm courses?
The Introduction

▶ What is this course about (and why is it needed)?
  • about basic yet indispensable skills for problem solving
  • algorithm design leads to writing code,
    i.e., creative thinking without programming languages

▶ How different is this course from other algorithm courses?
  • design technique-oriented, not application-oriented
The Introduction

- What is this course about (and why is it needed)?
  - about basic yet indispensable skills for problem solving
  - algorithm design leads to writing code,
    i.e., creative thinking without programming languages

- How different is this course from other algorithm courses?
  - design technique-oriented, not application-oriented
  - emphasis on guaranteed performance (typically in efficiency)
The Introduction

► What is this course about (and why is it needed)?
  • about basic yet indispensable skills for problem solving
  • algorithm design leads to writing code, i.e., creative thinking without programming languages

► How different is this course from other algorithm courses?
  • design technique-oriented, not application-oriented
  • emphasis on guaranteed performance (typically in efficiency)

► Goals to achieve
The Introduction

► What is this course about (and why is it needed)?
  • about basic yet indispensable skills for problem solving
  • algorithm design leads to writing code,
    i.e., creative thinking without programming languages

► How different is this course from other algorithm courses?
  • design technique-oriented, not application-oriented
  • emphasis on guaranteed performance (typically in efficiency)

► Goals to achieve
  • to learn to measure performance of algorithms
The Introduction

► What is this course about (and why is it needed)?
  • about basic yet indispensable skills for problem solving
  • algorithm design leads to writing code,
    i.e., creative thinking without programming languages

► How different is this course from other algorithm courses?
  • design technique-oriented, not application-oriented
  • emphasis on guaranteed performance (typically in efficiency)

► Goals to achieve
  • to learn to measure performance of algorithms
  • to master some fundamental algorithmic techniques
The Introduction

 ► What is this course about (and why is it needed)?
   • about basic yet indispensable skills for problem solving
   • algorithm design leads to writing code,
     i.e., creative thinking without programming languages

 ► How different is this course from other algorithm courses?
   • design technique-oriented, not application-oriented
   • emphasis on guaranteed performance (typically in efficiency)

 ► Goals to achieve
   • to learn to measure performance of algorithms
   • to master some fundamental algorithmic techniques
   • to study advanced algorithmic skills
The Introduction

▶ What is this course about (and why is it needed)?
  • about basic yet indispensable skills for problem solving
  • algorithm design leads to writing code,
    i.e., creative thinking without programming languages

▶ How different is this course from other algorithm courses?
  • design technique-oriented, not application-oriented
  • emphasis on guaranteed performance (typically in efficiency)

▶ Goals to achieve
  • to learn to measure performance of algorithms
  • to master some fundamental algorithmic techniques
  • to study advanced algorithmic skills
  • to understand computational intractability
Part I. Foundations
Part I. Foundations

- Chapter 1. The role of algorithms in computing
- Chapter 2. Getting started
- Chapter 3. Growth of functions
- Chapter 4. Solving recurrences
- Chapter 5. Probabilistic analysis and randomized algorithms
Part I. Foundations

The theme of the course
Part I. Foundations

The theme of the course

• Goal: learning techniques to design efficient algorithms
Part I. Foundations

The theme of the course

- Goal: learning techniques to design efficient algorithms
- Mean: through developing skills to analyze algorithms
Part I. Foundations

The theme of the course

• Goal: learning techniques to design efficient algorithms
• mean: through developing skills to analyze algorithms

Design and analysis of algorithms are closely related.
Example: the Fibonacci sequence.

\[ f(n) = \begin{cases} 
  f(n - 1) + f(n - 2) & \text{if } n \geq 3 \\
  1, & \text{otherwise}
\end{cases} \]
Example: the Fibonacci sequence.

\[ f(n) = \begin{cases} 
  f(n - 1) + f(n - 2) & \text{if } n \geq 3 \\
  1, & \text{otherwise}
\end{cases} \]

That is:

\[ \begin{array}{ccccccccccc}
  n & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \ldots \\
  f(n) & 1 & 1 & 2 & 3 & 5 & 8 & 13 & 21 & 34 & \ldots 
\end{array} \]
Part I. Foundations

Problem 1: Computing the $n$th Fibonacci number:
Part I. Foundations

Problem 1: Computing the $n$th Fibonacci number:

**Input:** $n \geq 1$;

**Output:** the $n$th number in the Fibonacci sequence.
Part I. Foundations

Problem 1: Computing the $n$th Fibonacci number:

**INPUT:** $n \geq 1$

**OUTPUT:** the $n$th number in the Fibonacci sequence.

Two different types of algorithms: *recursive* and *iterative*
Part I. Foundations

Problem 1: Computing the $n$th Fibonacci number:

\textbf{Input}: $n \geq 1$;
\textbf{Output}: the $n$th number in the Fibonacci sequence.

Two different types of algorithms: \textit{recursive} and \textit{iterative}

- recursive: task decomposition, top-down, recursive calls;
Part I. Foundations

Problem 1: Computing the \( n \)th Fibonacci number:

**INPUT:** \( n \geq 1 \);  
**OUTPUT:** the \( n \)th number in the Fibonacci sequence.

Two different types of algorithms: *recursive* and *iterative*

- **recursive:** task decomposition, top-down, recursive calls;
- **iterative:** more tightly coupled tasks, bottom-up approaches;
Rec-Fibonacci($n$)

But how efficient is it? Or how slow is it?

• suitable for execution of subroutines
• but oblivious, cannot remember any completed subroutine.
Rec-Fibonacci($n$)

if $n = 1$ or $n = 2$, 

But how efficient is it? Or how slow is it? Its execution is via a run-time stack

• suitable for execution of subroutines
• but oblivious, cannot remember any completed subroutine.
Part I. Foundations

**Rec-Fibonacci**$(n)$

\[
\text{if } n = 1 \text{ or } n = 2, \text{ return } (1); 
\]
Rec-Fibonacci($n$)

\textbf{if} $n = 1$ or $n = 2$, \textbf{return} (1);
\textbf{else}

\hspace{10pt} T_1 = \text{Rec-Fibonacci}(n - 1);
**Part I. Foundations**

Rec-Fibonacci($n$)

```plaintext
if $n = 1$ or $n = 2$, return (1);
else
    $T_1 = \text{Rec-Fibonacci}(n - 1);
    T_2 = \text{Rec-Fibonacci}(n - 2);
    return (T_1 + T_2);
```

But how efficient is it? Or how slow is it?

Its execution is via a run-time stack:

• Suitable for execution of subroutines
• But oblivious, cannot remember any completed subroutine.
Part I. Foundations

**Rec-Fibonacci**

\[
\text{Rec-Fibonacci}(n) \\
\text{if } n = 1 \text{ or } n = 2, \text{ return } (1); \\
\text{else} \\
T_1 = \text{Rec-Fibonacci}(n - 1); \\
T_2 = \text{Rec-Fibonacci}(n - 2); \\
\text{return } (T_1 + T_2);
\]

But how efficient is it? Or how slow is it? Its execution is via a run-time stack:

- Suitable for execution of subroutines
- But oblivious, cannot remember any completed subroutine.
Part I. Foundations

\textbf{Rec-Fibonacci}(n)

\begin{itemize}
  \item \textbf{if} \textit{n} = 1 \textbf{or} \textit{n} = 2, \textbf{return} (1);
  \item \textbf{else}
  \begin{itemize}
    \item \textit{T}_1 = \textbf{Rec-Fibonacci}(n - 1);
    \item \textit{T}_2 = \textbf{Rec-Fibonacci}(n - 2);
    \item \textbf{return} (\textit{T}_1 + \textit{T}_2);
  \end{itemize}
\end{itemize}

But how efficient is it? Or how slow is it?
Rec-Fibonacci \( (n) \)

\[
\text{if } n = 1 \text{ or } n = 2, \text{ return } (1); \\
\text{else} \\
T_1 = \text{Rec-Fibonacci}(n - 1); \\
T_2 = \text{Rec-Fibonacci}(n - 2); \\
\text{return } (T_1 + T_2); 
\]

But how efficient is it? Or how slow is it?

its execution is via a *run-time stack*
Rec-Fibonacci($n$)

if $n = 1$ or $n = 2$, return (1);
else

    $T_1 = \text{Rec-Fibonacci}(n - 1);
    T_2 = \text{Rec-Fibonacci}(n - 2);

return ($T_1 + T_2$);

But how efficient is it? Or how slow is it?
its execution is via a run-time stack

- suitable for execution of subroutines
Rec-Fibonacci(n)

if \( n = 1 \) or \( n = 2 \), return (1);
else
    \( T_1 = \text{Rec-Fibonacci}(n - 1); \)
    \( T_2 = \text{Rec-Fibonacci}(n - 2); \)
    return \( (T_1 + T_2); \)

But how efficient is it? Or how slow is it?

its execution is via a run-time stack

- suitable for execution of subroutines
- but oblivious, cannot remember any completed subroutine.
Part I. Foundations

**RECURSIVE FIBONACCI**

```
if n = 1 or n = 2, return 1;
else
    T1 = RECURSIVE FIBONACCI(n - 1);
    T2 = RECURSIVE FIBONACCI(n - 2);
    return (T1 + T2);
```
Part I. Foundations

Repeated computations everywhere!
Part I. Foundations

The size of tree is the number of recursive calls;
Part I. Foundations

The size of a tree is the number of recursive calls; How big is it?
Part I. Foundations

\[ \text{small triangle} \leq \text{size of tree} \leq \text{large triangle} \]

\[ 2^n \leq \text{size of tree} \leq 2^{n+1} \]
Part I. Foundations

small triangle \leq \text{size of tree} \leq \text{large triangle}
small triangle $\leq$ size of tree $\leq$ large triangle

roughly: $2^{\frac{n}{2}} \leq$ size of tree $\leq 2^n$
Part I. Foundations

**Iterative-Fibonacci**

Iterative-Fibonacci(n)

---

How fast is it?

\[ T_{\text{total}} = \max\{ T_{\text{if}}, T_{\text{else}} \} \]

where

\[ T_{\text{if}} = c_1, \quad T_{\text{else}} = c_2 + T_{\text{for}} = c_2 + d(n-2) \]

\[ T_{\text{total}} \leq c_1 + c_2 + d(n-2) \]

Iterative-Fibonacci(n) is a simple dynamic programming algorithm.
Part I. Foundations

Iterative-Fibonacci\((n)\)

\[
\text{if } n = 1 \text{ or } n = 2 \text{ return } (1);
\]
Iterative-Fibonacci($n$)

if $n = 1$ or $n = 2$ return (1);
else
  $M[1] = 1$, $M[2] = 1$
**Iterative-Fibonacci**

```plaintext
Iterative-Fibonacci(n)

if n = 1 or n = 2 return (1);
else
    M[1] = 1, M[2] = 1
    for i = 3 to n do
        M[i] = M[i - 1] + M[i - 2]
```

How fast is it?

\[
T_{total} = \max\{T_{if}, T_{else}\}
\]

where

\[
T_{if} = c_1,
\]

\[
T_{else} = c_2 + T_{for} = c_2 + d \times (n - 2)
\]

\[
T_{total} \leq c_1 + c_2 + d (n - 2),
\]

a linear function in \(n\).
Part I. Foundations

Iterative-Fibonacci\((n)\)

\[
\text{if } n = 1 \text{ or } n = 2 \text{ return } (1); \\
\text{else} \\
\text{for } i = 3 \text{ to } n \text{ do} \\
\quad M[i] = M[i - 1] + M[i - 2] \\
\text{return } (M[n])
\]
Iterative-Fibonacci(n)

if \( n = 1 \) or \( n = 2 \) return (1);
else
  \( M[1] = 1, M[2] = 1 \)
  for \( i = 3 \) to \( n \) do
    \( M[i] = M[i - 1] + M[i - 2] \)
  return \( (M[n]) \)

How fast is it?
Part I. Foundations

Iterative-Fibonacci($n$)

\[
\text{if } n = 1 \text{ or } n = 2 \text{ return } (1); \\
\text{else} \\
M[1] = 1, \ M[2] = 1 \\
\text{for } i = 3 \text{ to } n \text{ do} \\
\quad M[i] = M[i - 1] + M[i - 2] \\
\text{return } (M[n])
\]

How fast is it?

\[
T_{total} = \max\{T_{if}, T_{else}\}
\]
Iterative-Fibonacci($n$)

if $n = 1$ or $n = 2$ return (1);
else
  $M[1] = 1$, $M[2] = 1$
  for $i = 3$ to $n$ do
    $M[i] = M[i - 1] + M[i - 2]$
  return ($M[n]$)

How fast is it?

$$T_{total} = \max\{T_{if}, T_{else}\}$$

where $T_{if} = c_1$, 
Part I. Foundations

**Iterative-Fibonacci** \((n)\)

\[
\text{if } n = 1 \text{ or } n = 2 \text{ return } (1);
\]

\[
\text{else}
\]

\[
\]

\[
\text{for } i = 3 \text{ to } n \text{ do}
\]

\[
M[i] = M[i - 1] + M[i - 2]
\]

\[
\text{return } (M[n])
\]

How fast is it?

\[
T_{total} = \max\{T_{if}, T_{else}\}
\]

where \(T_{if} = c_1, \, T_{else} = c_2 + T_{for}\)
**Iterative-Fibonacci**

```
if \( n = 1 \) or \( n = 2 \) return \( (1) \);
else
    \( M[1] = 1, M[2] = 1 \)
    for \( i = 3 \) to \( n \) do
        \( M[i] = M[i - 1] + M[i - 2] \)
    return \( (M[n]) \)
```

How fast is it?

\[
T_{total} = \max\{T_{if}, T_{else}\}
\]

where \( T_{if} = c_1 \), \( T_{else} = c_2 + T_{for} = c_2 + d \times (n - 2) \)
Iterative-Fibonacci\((n)\)

\[
\begin{align*}
\text{if } n &= 1 \text{ or } n = 2 \text{ return } (1); \\
\text{else} \\
M[1] &= 1, \ M[2] = 1 \\
\text{for } i &= 3 \text{ to } n \text{ do} \\
M[i] &= M[i - 1] + M[i - 2] \\
\text{return } (M[n])
\end{align*}
\]

How fast is it?

\[
T_{total} = \max\{T_{if}, T_{else}\}
\]

where \(T_{if} = c_1\), \(T_{else} = c_2 + T_{for} = c_2 + d \times (n - 2)\)

\[
T_{total} \leq c_1
\]
Iterative-Fibonacci($n$)

if $n = 1$ or $n = 2$ return (1);
else
    $M[1] = 1$, $M[2] = 1$
    for $i = 3$ to $n$
        $M[i] = M[i - 1] + M[i - 2]$
    return ($M[n]$)

How fast is it?

$$T_{total} = \max\{T_{if}, T_{else}\}$$

where $T_{if} = c_1$, $T_{else} = c_2 + T_{for} = c_2 + d \times (n - 2)$

$$T_{total} \leq c_1 + c_2 + d(n - 2),$$
Iterative-Fibonacci($n$)

if $n = 1$ or $n = 2$ return (1);
else
  $M[1] = 1, \ M[2] = 1$
  for $i = 3$ to $n$ do
    $M[i] = M[i - 1] + M[i - 2]$
  return ($M[n]$)

How fast is it?

$$T_{total} = \max\{T_{if}, T_{else}\}$$

where $T_{if} = c_1$, $T_{else} = c_2 + T_{for} = c_2 + d \times (n - 2)$

$$T_{total} \leq c_1 + c_2 + d(n - 2), \text{ a linear function in } n$$
Part I. Foundations

Iterative-Fibonacci\((n)\)

\[
\begin{align*}
\text{if } n &= 1 \text{ or } n = 2 \text{ return (1);} \\
\text{else} & \quad M[1] = 1, M[2] = 1 \\
& \quad \text{for } i = 3 \text{ to } n \text{ do} \\
& \quad \quad M[i] = M[i - 1] + M[i - 2] \\
& \quad \text{return } (M[n])
\end{align*}
\]

How fast is it?

\[
T_{total} = \max\{T_{if}, T_{else}\}
\]

where \(T_{if} = c_1\), \(T_{else} = c_2 + T_{for} = c_2 + d \times (n - 2)\)

\[
T_{total} \leq c_1 + c_2 + d(n - 2), \text{ a linear function in } n
\]

Iterative-Fibonacci\((n)\) is a simple dynamic programming algorithm.
Part I. Foundations

Actually, all the algorithm \texttt{ITERATIVE-FIBONACCI}(n) does is:
Part I. Foundations

Actually, all the algorithm $\text{ITERATIVE-FIBONACCI}(n)$ does is:

To fill out a table of size $n$, with
Part I. Foundations

Actually, all the algorithm \texttt{ITERATIVE-FIBONACCI}(n) does is:

To fill out a table of size $n$, with

- each entry being filled out exactly once, and
Part I. Foundations

Actually, all the algorithm Iterative-Fibonacci\((n)\) does is:

To fill out a table of size \(n\), with

- each entry being filled out exactly once, and
- filling out an entry takes a constant, say \(c\) steps.
Part I. Foundations

Actually, all the algorithm \textsc{Iterative-Fibonacci}(n) does is:

To fill out a table of size $n$, with

- each entry being filled out exactly once, and
- filling out an entry takes a constant, say $c$ steps.

So the total time \textsc{Iterative-Fibonacci}(n) uses is
Actually, all the algorithm Iterative-Fibonacci\((n)\) does is:

To fill out a table of size \(n\), with

- each entry being filled out exactly once, and
- filling out an entry takes a constant, say \(c\) steps.

So the total time Iterative-Fibonacci\((n)\) uses is

\[
T(n) = c \times n
\]
Chapter 1. The role of algorithms in computing
Chapter 1. The role of algorithms in computing

What is an Algorithm: a well-defined, finite procedure that takes an input and produces an output.
Chapter 1. The Role of Algorithms in Computing

Chapter 1. The role of algorithms in computing

What is an Algorithm: a well-defined, finite procedure that takes an input and produces an output.

Example 2: An algorithm skeleton;

Algorithm Maximum;

\[ \text{Input: list } X = \{a_1, \cdots, a_n\}; \]

\[ \text{Body that is a series of instructions; } \]

\[ \text{Output: } y, \text{ the maximum of } a_1, \cdots, a_n. \]
Chapter 1. The Role of Algorithms in Computing

Alternatively, an algorithm specifies a finite process to compute a function or a relation.
Alternatively, an algorithm specifies a finite process to compute a function or a relation.

e.g., algorithm \textsc{Maximum} computes the following function:

\[ f_{\text{max}}(X) = y, \text{ where } \forall a \in X, y \geq a, \]
Alternatively, an algorithm specifies a finite process to compute a function or a relation.

e.g., algorithm MAXIMUM computes the following function:

\[ f_{\text{max}}(X) = y, \text{ where } \forall a \in X, y \geq a, \]

For some problems, the functions computed are predicates, i.e., output \( y \in \{\text{TRUE, FALSE}\} \)
Chapter 1. The Role of Algorithms in Computing

Algorithms as a technology to resolve efficiency issues
Algorithms as a technology to resolve efficiency issues

Efficient use of computer resources such as time and space is necessary.
Chapter 1. The Role of Algorithms in Computing

Algorithms as a technology to resolve efficiency issues

Efficient use of computer resources such as time and space is necessary.

Two typical situations:
Chapter 1. The Role of Algorithms in Computing

Algorithms as a technology to resolve efficiency issues

Efficient use of computer resources such as time and space is necessary.

Two typical situations:

- very large input data for “easy” problems;
Chapter 1. The Role of Algorithms in Computing

Algorithms as a technology to resolve efficiency issues

Efficient use of computer resources such as time and space is necessary.

Two typical situations:

- very large input data for “easy” problems;
- moderately large input data for “hard” problems.
Chapter 2. Getting Started

The Sorting Problem

Input: \(n\) numbers \(\langle a_1, \cdots, a_n \rangle\);

Output: a reordering \(\langle a'_1, \cdots, a'_n \rangle\) of the input such that \(a'_1 \leq a'_2 \leq \cdots \leq a'_n\).

Insertion Sort

idea: an iterative process to produce a new list such that at each iteration, the new list consists of two sublists,

- a sorted sublist followed by an unsorted sublist,
- the leftmost number of the unsorted is being inserted into the sorted.

As the process goes, the sorted sublist gets longer, the unsorted sublist gets shorter, until the unsorted becomes empty.
Chapter 2. Getting Started

Chapter 2. Getting started

The Sorting Problem

\textbf{Input}: \(n\) numbers \(\langle a_1, \cdots, a_n \rangle\);
Chapter 2. Getting Started

Chapter 2. Getting started

The Sorting Problem

**INPUT:** \( n \) numbers \( \langle a_1, \cdots, a_n \rangle \);

**OUTPUT:** a reordering \( \langle a'_1, \cdots, a'_n \rangle \) of the input such that

\[
a'_1 \leq a'_2 \leq \cdots \leq a'_n.
\]
Chapter 2. Getting Started

Chapter 2. Getting started

The Sorting Problem

Input: \( n \) numbers \( \langle a_1, \ldots, a_n \rangle \);

Output: a reordering \( \langle a'_1, \ldots, a'_n \rangle \) of the input such that
\[ a'_1 \leq a'_2 \leq \cdots \leq a'_n. \]

Insertion Sort

Idea: an iterative process to produce a new list such that
Chapter 2. Getting Started

The Sorting Problem

**Input:** $n$ numbers $\langle a_1, \cdots, a_n \rangle$;

**Output:** a reordering $\langle a'_1, \cdots, a'_n \rangle$ of the input such that

$$a'_1 \leq a'_2 \leq \cdots \leq a'_n.$$ 

**Insertion Sort**

**Idea:** an iterative process to produce a new list such that

at each iteration, the new list consists of two sublists,
Chapter 2. Getting Started

The Sorting Problem

**Input:** $n$ numbers $\langle a_1, \cdots, a_n \rangle$;

**Output:** a reordering $\langle a'_1, \cdots, a'_n \rangle$ of the input such that

\[ a'_1 \leq a'_2 \leq \cdots \leq a'_n. \]

**Insertion Sort**

**Idea:** an iterative process to produce a new list such that

at each iteration, the new list consists of two sublists,

- a sorted sublist followed by an unsorted sublist, and
The Sorting Problem

**Input:** \( n \) numbers \( \langle a_1, \cdots, a_n \rangle \);

**Output:** a reordering \( \langle a'_1, \cdots, a'_n \rangle \) of the input such that
\[
a'_1 \leq a'_2 \leq \cdots \leq a'_n.
\]

### Insertion Sort

**Idea:** an iterative process to produce a new list such that
at each iteration, the new list consists of two sublists,
- a *sorted sublist* followed by an *unsorted sublist*, and
- the leftmost number of the *unsorted* is being inserted into the *sorted*. 
Chapter 2. Getting Started

The Sorting Problem

**Input:** \( n \) numbers \( \langle a_1, \cdots, a_n \rangle \);

**Output:** a reordering \( \langle a'_1, \cdots, a'_n \rangle \) of the input such that
\[
a'_1 \leq a'_2 \leq \cdots \leq a'_n.
\]

**Insertion Sort**

**Idea:** an iterative process to produce a new list such that

- at each iteration, the new list consists of two sublists,
  - a **sorted sublist** followed by an **unsorted sublist**, and
  - the leftmost number of the **unsorted** is being inserted into the **sorted**.

As the process goes, the **sorted sublist** gets longer, the **unsorted sublist** gets shorter, until the **unsorted** becomes empty.
Chapter 2. Getting Started

Algorithm \textsc{Insertion-Sort}(A)
Algorithm \textsc{Insertion-Sort}(A)

1. \hspace{1em} for $j = 2$ to $\text{length}[A]$ do
Algorithm \textsc{Insertion-Sort}(A)

1. \textbf{for} $j = 2$ \textbf{to} \textit{length}[A] \textbf{do}
2. \hspace{1em} \textit{key} = A[j]

Analysis of the algorithm:
• (correctness proof): to show that the algorithm is as desired;
• (efficiency proof): to show a guaranteed efficiency of the algorithm
Algorithm \textsc{Insertion-Sort}(A)

1. \textbf{for } $j = 2$ \textbf{to } length[$A$] \textbf{ do}
2. \hspace{1em} key = $A[j]$
3. \hspace{1em} \{\text{Insert } A[j] \text{ into sorted } A[1..j - 1]\}
Algorithm \textsc{Insertion-Sort}(A)

1. \textbf{for} $j = 2$ to $\text{length}[A]$ \textbf{do}
2. \hspace{1em} key = $A[j]$
3. \hspace{1em} \{Insert $A[j]$ into sorted $A[1..j - 1]$\}
4. \hspace{1em} $i = j - 1$

Analysis of the algorithm:
- (correctness proof): to show that the algorithm is as desired;
- (efficiency proof): to show a guaranteed efficiency of the algorithm.
Algorithm \textsc{Insertion-Sort}(A)

1. \textbf{for} $j = 2$ \textbf{to} \textit{length}[A] \textbf{do}
2. \hspace{1em} key = A[j]
3. \hspace{1em} \{\text{Insert } A[j] \text{ into sorted } A[1..j-1]\}
4. \hspace{1em} i = j - 1
5. \hspace{1em} \textbf{while} $i > 0$ \textbf{and} $A[i] > key$

\textbf{Analysis of the algorithm:}
- (correctness proof): to show that the algorithm is as desired;
- (efficiency proof): to show a guaranteed efficiency of the algorithm.
Algorithm \textsc{Insertion-Sort}(A)

1. for \( j = 2 \) to \( \text{length}[A] \) do
2. \hspace{1em} key = \( A[j] \)
3. \hspace{1em} \{Insert \( A[j] \) into sorted \( A[1..j - 1] \}\}
4. \hspace{1em} \( i = j - 1 \)
5. while \( i > 0 \) and \( A[i] > key \) do
6. \hspace{2em} \( A[i + 1] = A[i] \)
Chapter 2. Getting Started

Algorithm \textsc{Insertion-Sort}(A)

1. \textbf{for} $j = 2$ \textbf{to} $\text{length}[A]$ \textbf{do}
2. \hspace{1em} \textit{key} $=$ $A[j]$
3. \hspace{1em} \{\text{Insert } A[j] \text{ into sorted } A[1..j - 1]\}
4. \hspace{1em} $i = j - 1$
5. \textbf{while} $i > 0$ \textbf{and} $A[i] > \textit{key}$
6. \hspace{1em} \textbf{do} $A[i + 1] = A[i]$
7. \hspace{1em} $i = i - 1$

Analysis of the algorithm:
• (correctness proof): to show that the algorithm is as desired;
• (efficiency proof): to show a guaranteed efficiency of the algorithm
Chapter 2. Getting Started

Algorithm \textsc{Insertion-Sort}(A)

1 \hspace{1em} \textbf{for} \hspace{0.5em} j = 2 \hspace{0.5em} \textbf{to} \hspace{0.5em} \text{length}[A] \hspace{0.5em} \textbf{do}

2 \hspace{1em} \textit{key} = A[j]

3 \hspace{1em} \{\text{Insert } A[j] \text{ into sorted } A[1..j - 1]\}

4 \hspace{1em} i = j - 1

5 \hspace{1em} \textbf{while} \hspace{0.5em} i > 0 \hspace{0.5em} \text{and} \hspace{0.5em} A[i] > \textit{key} \hspace{0.5em} \textbf{do}

6 \hspace{1em} \hspace{1em} A[i + 1] = A[i]

7 \hspace{1em} \hspace{1em} i = i - 1

8 \hspace{1em} A[i + 1] = \textit{key}
Algorithm **INSERTION-SORT**(A)

1. **for** $j = 2$ **to** length[A] **do**
2. \hspace{1em} key = A[j]
3. \hspace{1em} \{Insert A[j] into sorted A[1..j−1]\}
4. \hspace{1em} i = j − 1
5. \hspace{1em} **while** i > 0 and A[i] > key **do**
6. \hspace{2em} A[i + 1] = A[i]
7. \hspace{2em} i = i − 1
8. A[i + 1] = key

Analysis of the algorithm:
Chapter 2. Getting Started

Algorithm \textsc{Insertion-Sort}(A)

1. for \( j = 2 \) to \( \text{length}[A] \) do
2. \hspace{1em} key = \( A[j] \)
3. \hspace{1em} \{Insert \( A[j] \) into sorted \( A[1..j-1] \}\}
4. \hspace{1em} i = j - 1
5. \hspace{1em} while \( i > 0 \) and \( A[i] > key \) do
6. \hspace{2em} \hspace{1em} \( A[i + 1] = A[i] \)
7. \hspace{2em} \hspace{1em} \( i = i - 1 \)
8. \hspace{1em} \hspace{1em} \( A[i + 1] = key \)

Analysis of the algorithm:

\begin{itemize}
\item (correctness proof): to show that the algorithm is as desired;
\end{itemize}
Chapter 2. Getting Started

Algorithm INSERTION-SORT(A)

1. \textbf{for} \( j = 2 \) to \( \text{length}[A] \) \textbf{do}
2. \hspace{1em} \text{key} = A[j]
3. \hspace{1em} \{} \text{Insert} \ A[j] \text{ into sorted} \ A[1..j - 1] \}\n4. \hspace{1em} i = j - 1
5. \hspace{1em} \textbf{while} \( i > 0 \) and \( A[i] > \text{key} \) \textbf{do}
6. \hspace{2em} \text{do} \( A[i + 1] = A[i] \)
7. \hspace{2em} \hspace{1em} i = i - 1
8. \hspace{2em} A[i + 1] = \text{key}

Analysis of the algorithm:

- (correctness proof): to show that the algorithm is as desired;
- (efficiency proof): to show a guaranteed efficiency of the algorithm
Chapter 2. Getting Started

Correctness proof: this is to prove
Chapter 2. Getting Started

Correctness proof: this is to prove

the pre-condition (condition for the input)
Chapter 2. Getting Started

Correctness proof: this is to prove

the pre-condition (condition for the input)

is transformed by the algorithm to
Chapter 2. Getting Started

Correctness proof: this is to prove

the pre-condition (condition for the input)

is transformed by the algorithm to

the post-condition (condition for the output)
Chapter 2. Getting Started

Correctness proof: this is to prove

the pre-condition (condition for the input)

is transformed by the algorithm to

the post-condition (condition for the output)

If the algorithm consists of sequential blocks of instructions, the task is to prove the correct transformation by each block.
Chapter 2. Getting Started

Correctness proof: this is to prove

the pre-condition (condition for the input)

is transformed by the algorithm to

the post-condition (condition for the output)

If the algorithm consists of sequential blocks of instructions,

the task is to prove the correct transformation by each block.

This means we need to prove that every sequential statement
in the algorithm transforms the given pre-condition to
the given post-condition.
The most difficult task is to do this for a loop statement.
The most difficult task is to do this for a loop statement. Finding loop invariant becomes necessary and sufficient.
Chapter 2. Getting Started

The most difficult task is to do this for a loop statement. Finding loop invariant becomes necessary and sufficient. In Insertion-Sort, the loop invariant is
Chapter 2. Getting Started

The most difficult task is to do this for a loop statement. Finding loop invariant becomes necessary and sufficient.

In Insertion-Sort, the loop invariant is

at each iteration, the sublist $A[1..j - 1]$ consists of the elements originally in the positions $[1..j-1]$ but in sorted order.
The most difficult task is to do this for a loop statement. Finding loop invariant becomes necessary and sufficient.

In Insertion-Sort, the loop invariant is

at each iteration, the sublist $A[1..j - 1]$ consists of the elements originally in the positions $[1..j-1]$ but in sorted order.

However, finding loop invariants is difficult!
Chapter 2. Getting Started

Efficiency analysis: This is to show that...
Chapter 2. Getting Started

Efficiency analysis: This is to show that

- For all cases of input, the needed computation resources for the algorithm.
Chapter 2. Getting Started

Efficiency analysis: This is to show that

- For all cases of input, the needed computation resources for the algorithm.
- resources can be CPU time and memory space used in the computation.
Chapter 2. Getting Started

Efficiency analysis: This is to show that

- For all cases of input, the needed computation resources for the algorithm.
- resources can be CPU time and memory space used in the computation.
- however, the unit measured is not real time or memory unit.
Chapter 2. Getting Started

Time of an algorithm $A(x)$

Input Instances $x$
Chapter 2. Getting Started

Time of an algorithm $A(x)$

Input Instances   worst case

$x$
Chapter 2. Getting Started

Upper bound for algorithm A, bounding all cases of instances
Resource measurement based on
Chapter 2. Getting Started

Resource measurement based on

- random-access machine (RAM)
Chapter 2. Getting Started

Resource measurement based on

- random-access machine (RAM)

- counting primitive operations: addition, substraction, floor, ceiling, multiplication, jump, memory movement,
Chapter 2. Getting Started

Resource measurement based on

- random-access machine (RAM)

- counting primitive operations: addition, subtraction, floor, ceiling, multiplication, jump, memory movement,

these operations differs in time by a constant multiplicative factor.
Chapter 2. Getting Started

Resource measurement based on

- random-access machine (RAM)

- counting primitive operations: addition, subtraction, floor, ceiling, multiplication, jump, memory movement,
  these operations differs in time by a constant multiplicative factor.

- speed between different machines: a constant multiplicative factor.
Analysis of Algorithm \texttt{INSERTION-SORT}(A)
Chapter 2. Getting Started

Analysis of

Algorithm \texttt{INSERTION-SORT}(A)

\begin{enumerate}
\item \texttt{for } $j = 2$ \texttt{to } length[$A$] \texttt{do}
\item \hspace{1em} key = $A[j]$
\item \hspace{1em} \{Insert $A[j]$ into sorted $A[1..j - 1]$\}
\item \hspace{1em} $i = j - 1$
\item \texttt{while } $i > 0$ \texttt{and } $A[i] > key$
\item \hspace{1em} \texttt{do } $A[i + 1] = A[i]$
\item \hspace{1em} \hspace{1em} $i = i - 1$
\item \hspace{1em} \hspace{1em} $A[i + 1] = key$
\end{enumerate}
Analysis of Algorithm \textsc{Insertion-Sort}(A)

1. \textbf{for} $j = 2$ to $\text{length}[A]$ \textbf{do}
2. \hspace{1em} $key = A[j]$
3. \hspace{1em} \{Insert $A[j]$ into sorted $A[1..j-1]$\}
4. \hspace{1em} $i = j - 1$
5. \textbf{while} $i > 0$ and $A[i] > key$
6. \hspace{1em} \textbf{do} $A[i+1] = A[i]$
7. \hspace{1em} \hspace{1em} $i = i - 1$
8. \hspace{1em} $A[i+1] = key$

Assume $t_j$ to be the number of times \textbf{while} is executed for every $j$.

$$T(n) = c_1n + c_2(n-1) + c_4(n-1) + c_5 \sum_{j=2}^{n} t_j + c_6 \sum_{j=2}^{n} (t_j - 1) + c_7 \sum_{j=2}^{n} (t_j - 1) + c_8(n-1)$$
Chapter 2. Getting Started

\[ T(n) = c_1 n + c_2 (n-1) + c_4 (n-1) + c_5 \sum_{j=2}^{n} t_j + c_6 \sum_{j=2}^{n} (t_j - 1) + c_7 \sum_{j=2}^{n} (t_j - 1) + c_8 (n-1) \]

For some constants \(a, b, c\), for example, \(a \geq c_5 + c_6 + c_7\), \(b \geq c_1 + c_2 + c_4 + c_8\).
Chapter 2. Getting Started

\[ T(n) = c_1 n + c_2 (n-1) + c_4 (n-1) + c_5 \sum_{j=2}^{n} t_j + c_6 \sum_{j=2}^{n} (t_j - 1) + c_7 \sum_{j=2}^{n} (t_j - 1) + c_8 (n-1) \]

\[ T(n) \leq a \sum_{j=2}^{n} t_j + bn + c \]

for some constants \( a, b, c, \)
Chapter 2. Getting Started

\[ T(n) = c_1 n + c_2 (n-1) + c_4 (n-1) + c_5 \sum_{j=2}^{n} t_j + c_6 \sum_{j=2}^{n} (t_j - 1) + c_7 \sum_{j=2}^{n} (t_j - 1) + c_8 (n-1) \]

\[ T(n) \leq a \sum_{j=2}^{n} t_j + b n + c \]

for some constants \( a, b, c \), for example, \( a \geq c_5 + c_6 + c_7 \), \( b \geq c_1 + c_2 + c_4 + c_8 \).
Chapter 2. Getting Started

\[ T(n) = c_1 n + c_2 (n - 1) + c_4 (n - 1) + c_5 \sum_{j=2}^{n} t_j + c_6 \sum_{j=2}^{n} (t_j - 1) + c_7 \sum_{j=2}^{n} (t_j - 1) + c_8 (n - 1) \]

\[ T(n) \leq a \sum_{j=2}^{n} t_j + b n + c \]

for some constants \(a, b, c\), for example, \(a \geq c_5 + c_6 + c_7\), \(b \geq c_1 + c_2 + c_4 + c_8\).

Because \(t_j = j\) in the worst case (e.g., list is reversely sorted).
Chapter 2. Getting Started

\[ T(n) = c_1 n + c_2 (n-1) + c_4 (n-1) + c_5 \sum_{j=2}^{n} t_j + c_6 \sum_{j=2}^{n} (t_j - 1) + c_7 \sum_{j=2}^{n} (t_j - 1) + c_8 (n-1) \]

\[ T(n) \leq a \sum_{j=2}^{n} t_j + bn + c \]

for some constants \( a, b, c \), for example, \( a \geq c_5 + c_6 + c_7 \), \( b \geq c_1 + c_2 + c_4 + c_8 \).

Because \( t_j = j \) in the worst case (e.g., list is reversely sorted).

\[ T(n) \leq a \frac{n}{2} (n + 1) + bn + c - a \]
Chapter 2. Getting Started

\[ T(n) = c_1 n + c_2 (n-1) + c_4 (n-1) + c_5 \sum_{j=2}^{n} t_j + c_6 \sum_{j=2}^{n} (t_j - 1) + c_7 \sum_{j=2}^{n} (t_j - 1) + c_8 (n-1) \]

\[ T(n) \leq a \sum_{j=2}^{n} t_j + bn + c \]

for some constants \( a, b, c \), for example, \( a \geq c_5 + c_6 + c_7 \), \( b \geq c_1 + c_2 + c_4 + c_8 \).

Because \( t_j = j \) in the worst case (e.g., list is reversely sorted).

\[ T(n) \leq a \frac{n}{2} (n + 1) + bn + c - a \leq xn^2 + yn + z \]

for some constants \( x, y, z \).
Chapter 2. Getting Started

So we have proved:

\[ T(n) \leq n^2 + yn + z \]

for some constants \( x, y, z \) \leq \text{means in all cases for which} \leq \text{holds;}

\( n^2 + yn + z \) is a complexity upper bound for \( T(n) \).
Chapter 2. Getting Started

So we have proved:

\[ T(n) \leq xn^2 + yn + z \]

for some constants \( x, y, z \)

\( \leq \) means in all cases for which \( \leq \) holds;
So we have proved:

\[ T(n) \leq xn^2 + yn + z \]

for some constants \( x, y, z \)

\( \leq \) means in all cases for which \( \leq \) holds;

\( xn^2 + yn + z \) is a complexity upper bound for \( T(n) \)
Chapter 2. Getting Started

Important complexity issues:

1. size of input $n$: the number of bits encoding input $x$, i.e., $n = |x|$. It is inaccurate for $n$ to represent the number of items in the input.

Consider to sort $4$ items $⟨x_1, x_2, x_3, x_4⟩$ of values in the scale of $2^N$, for some very large $N$.

- If $n$ is the number of items, $n = 4$, then any sorting algorithm would run in constant time.
- However, since $x_1, x_2, x_3, x_4$ are of very large values, a single comparison $x_1 \leq x_2$ would need a time proportional to $N$.

Hence, if $n = |⟨x_1, x_2, x_3, x_4⟩|$, then $n \approx N$.

To sort the 4 items, a constant number of comparisons is needed, each taking a time linear in $N$ (i.e., total time is linear in $n$).
Important complexity issues:

1. The size of input $n$: the number of bits encoding input $x$, i.e., $n = |x|$. It is inaccurate for $n$ to represent the number of items in the input.

Consider sorting 4 items $\langle x_1, x_2, x_3, x_4 \rangle$ of very large values in the scale of $2^N$, for some very large $N$.

- If $n$ is the number of items, $n = 4$, then any sorting algorithm would run in constant time.
- However, since $x_1, x_2, x_3, x_4$ are of very large values, a single comparison $x_1 \leq x_2$ would need a time proportional to $N$.

Hence, if $n = |\langle x_1, x_2, x_3, x_4 \rangle|$, then $n \approx N$.

To sort the 4 items, a constant number of comparisons is needed, each taking a time linear in $N$ (i.e., total time is linear in $n$).
Important complexity issues:

1. **size of input** $n$: the number of bits encoding input $x$, i.e., $n = |x|$.
Chapter 2. Getting Started

Important complexity issues:

1. **size of input** $n$: the number of bits encoding input $x$, i.e., $n = |x|$.

   It is inaccurate for $n$ to represent the number of items in the input.
Chapter 2. Getting Started

Important complexity issues:

1. **size of input** \( n \): the number of bits encoding input \( x \), i.e., \( n = |x| \).

   **It is inaccurate** for \( n \) to represent the number of items in the input.

Consider to sort 4 items \( \langle x_1, x_2, x_3, x_4 \rangle \) of values in the scale of \( 2^N \), for some very large \( N \).
Important complexity issues:

1. **size of input** $n$: the number of bits encoding input $x$, i.e., $n = |x|$.

   It is inaccurate for $n$ to represent the number of items in the input.

Consider to sort 4 items $\langle x_1, x_2, x_3, x_4 \rangle$ of values in the scale of $2^N$, for some very large $N$.

- If $n$ is the number of items, $n = 4$, then any sorting algorithm would run in **constant time**.
Chapter 2. Getting Started

Important complexity issues:

1. **size of input** $n$: the number of bits encoding input $x$, i.e., $n = |x|$.  

   It is inaccurate for $n$ to represent the number of items in the input.

Consider to sort 4 items $\langle x_1, x_2, x_3, x_4 \rangle$ of values in the scale of $2^N$, for some very large $N$.

- If $n$ is the number of items, $n = 4$, then any sorting algorithm would run in **constant time**.

- However, since $x_1, x_2, x_3, x_4$ are of very large values, a single comparison $x_1 \leq x_2$? would need a **time proportional to** $N$. 

Chapter 2. Getting Started

Important complexity issues:

1. **size of input** $n$: the number of bits encoding input $x$, i.e., $n = |x|$.

   It is inaccurate for $n$ to represent the number of items in the input.

Consider to sort 4 items $\langle x_1, x_2, x_3, x_4 \rangle$ of values in the scale of $2^N$, for some very large $N$.

- If $n$ is the number of items, $n = 4$, then any sorting algorithm would run in **constant time**.

- However, since $x_1, x_2, x_3, x_4$ are of very large values, a single comparison $x_1 \leq x_2$? would need a **time proportional to** $N$.

  Hence, if $n = |\langle x_1, x_2, x_3, x_4 \rangle|$, then $n \approx N$. 
Important complexity issues:

1. **size of input** $n$: the number of bits encoding input $x$, i.e., $n = |x|$. It is inaccurate for $n$ to represent the number of items in the input.

Consider to sort 4 items $\langle x_1, x_2, x_3, x_4 \rangle$ of values in the scale of $2^N$, for some very large $N$.

- If $n$ is the number of items, $n = 4$, then any sorting algorithm would run in **constant time**.

- However, since $x_1, x_2, x_3, x_4$ are of very large values, a single comparison $x_1 \leq x_2$? would need a **time proportional to $N$**.

Hence, if $n = |\langle x_1, x_2, x_3, x_4 \rangle|$, then $n \approx N$.

To sort the 4 items, a constant number of comparisons is needed, each taking a time linear in $N$ (i.e., **total time is linear in $n$**).
Chapter 2. Getting Started

2. Running time

$T(n)$: the number of primitive operations executed,
a function in $n$ worst-case running time: the running time upper bound for all inputs.

order of growth: $T(n) = an^2 + bn + c$ grows the same rate as $an^2$ (if $a > 0$).
2. Running time $T(n)$: the number of primitive operations executed, a function in $n$. 
2. Running time $T(n)$: the number of primitive operations executed, a function in $n$

   **worst-case running time**: the running time upper bound for all inputs.
2. Running time $T(n)$: the number of primitive operations executed, a function in $n$

worst-case running time: the running time upper bound for all inputs.

order of growth: $T(n) = an^2 + bn + c$ grows the same rate as $an^2$ (if $a > 0$).
Chapter 3. Growth of Functions

Big-O: set $O(n^2)$ contains all functions of growth rate $\leq cn^2$.

So for function $T(n) = an^2 + bn + c$, $T(n) \in O(n^2)$, but written as $T(n) = O(n^2)$.

In general, $O(g(n)) = \{ f(n) : \exists c > 0, k > 0$ such that $0 \leq f(n) \leq cg(n)$, for all $n \geq k \}$.
Big-O: set $O(n^2)$ contains all functions of growth rate $\leq cn^2$.

So for function $T(n) = an^2 + bn + c$, $T(n) \in O(n^2)$, but written as $T(n) = O(n^2)$.

In general, $O(g(n)) = \{ f(n) : \exists c > 0, k > 0 \text{ such that } 0 \leq f(n) \leq cg(n), \text{ for all } n \geq k \}$.
Chapter 3. Growth of Functions

**Big-O:** set $O(n^2)$ contains all functions of growth rate $\leq cn^2$. 
Chapter 3. Growth of Functions

Big-O: set $O(n^2)$ contains all functions of growth rate $\leq cn^2$.

So for function $T(n) = an^2 + bn + c$, $T(n) \in O(n^2)$, but written as

$$T(n) = O(n^2)$$
Chapter 3. Growth of Functions

**Big-O**: set $O(n^2)$ contains all functions of growth rate $\leq cn^2$.

So for function $T(n) = an^2 + bn + c$, $T(n) \in O(n^2)$, but written as

$$T(n) = O(n^2)$$

In general,

$$O(g(n)) = \{ f(n) : \exists c > 0, k > 0 \text{ such that } 0 \leq f(n) \leq cg(n), \text{ for all } n \geq k \}$$
Chapter 3. Growth of Functions

**Big-O**: set $O(n^2)$ contains all functions of growth rate $\leq cn^2$.

So for function $T(n) = an^2 + bn + c$, $T(n) \in O(n^2)$, but written as

$$T(n) = O(n^2)$$

In general,

$$O(g(n)) = \{f(n) : \exists c > 0, k > 0 \text{ such that } 0 \leq f(n) \leq cg(n), \text{ for all } n \geq k\}$$
Chapter 3. Growth of Functions

For example: the following functions are all of the order of $O(n^2)$:

(1) $3n^2$

(2) $5n^2 + 6n \log_2 n + 90$

(3) $0.001n^2 - 200n - 5000$

(4) $3n \log_2 n^2 + 6n$

(5) $\sqrt{n} - 20 \log_2 n + 56$

(6) $\log_2 n + 56$

(7) $345$

But the following are not:

(8) $3n^2 \log_2 n - 400n$

(9) $n^2$.001
Chapter 3. Growth of Functions

For example: the following functions are all of the order of $O(n^2)$:

1. $3n^2$
2. $5.3n^2 + 6n\log_2 n + 90$
3. $0.001n^2 - 200n - 5000$
4. $3n\log_2 n^2 + 6n$
5. $\sqrt{n} - 20\log_2 n$
6. $\log_2 n + 56$
7. $345$
For example: the following functions are all of the order of $O(n^2)$:

1. $3n^2$
2. $5.3n^2 + 6n \log_2 n + 90$
3. $0.001n^2 - 200n - 5000$
4. $3n \log_2 n + 6n$
5. $\sqrt{n} - 20 \log_2 n$
6. $\log_2 n + 56$
7. $345$
For example: the following functions are all of the order of $O(n^2)$:

(1) $3n^2$
(2) $5.3n^2 + 6n \log_2 n + 90$
(3) $0.001n^2 - 200n - 5000$
(4) $3n \log_2^n + 6n$
(5) $\sqrt{n} - 20 \log_2 n$
(6) $\log_2 n + 56$
(7) $345$

But the following are not:
Chapter 3. Growth of Functions

For example: the following functions are all of the order of $O(n^2)$:

(1) $3n^2$
(2) $5.3n^2 + 6n \log_2 n + 90$
(3) $0.001n^2 - 200n - 5000$
(4) $3n \log_2^n + 6n$
(5) $\sqrt{n} - 20 \log_2 n$
(6) $\log_2 n + 56$
(7) $345$

But the following are not:

(8) $3n^2 \log_2 n - 400n$
(9) $n^{2.001}$
Example: binary search algorithm

```
Algorithm Binary Search (A, p, r, key)
1. if p > r return (NULL)
2. else
3. q = ⌊(p + r) / 2⌋
4. if A[q] = key return (k)
5. else
6. if A[q] > key Binary Search (A, p, q−1, key)
7. else Binary Search (A, q+1, r, key)
```

Let \( T(n) \) be the worst case time complexity for Binary Search, where parameter \( n = r - p + 1 \).

The time \( T(n) \) can be upper-bounded with the recurrence, \( T(n) \leq T(n/2) + c \) and \( T(0) \leq c \) where \( c > 0 \) is a constant.

Note: we can also set base case \( T(1) \leq c \).
Chapter 4. Solving Recurrences

Example: binary search algorithm

Algorithm $\text{Binary Search}(A, p, r, key)$

1. if $p > r$ return (NULL)
2. else
3. $q = \floor{\frac{p+r}{2}}$
4. if $A[q] = key$ return ($k$)
5. else
6. if $A[q] > key$ $\text{Binary Search} (A, p, q - 1, key)$
7. else
8. $\text{Binary Search} (A, q + 1, r, key)$
Algorithm Binary Search \((A, p, r, key)\)

1. \(\text{if } p > r \text{ return (NULL)}\)
2. \(\text{else}\)
3. \(q = \left\lfloor \frac{p + r}{2} \right\rfloor\)
4. \(\text{if } A[q] = key \text{ return } (k)\)
5. \(\text{else}\)
6. \(\text{if } A[q] > key \text{ Binary Search } (A, p, q - 1, key)\)
7. \(\text{else}\)
8. \(\text{Binary Search } (A, q + 1, r, key)\)

Let \(T(n)\) be the worst case time complexity for Binary Search, where parameter \(n = r - p + 1\).
Chapter 4. Solving Recurrences

Example: binary search algorithm

Algorithm **Binary Search** \(A, p, r, key\)

1. if \(p > r\) return (NULL)
2. else
3. \(q = \left\lfloor \frac{p+r}{2} \right\rfloor\)
4. if \(A[q] = key\) return \(k\)
5. else
6. if \(A[q] > key\) **Binary Search** \(A, p, q-1, key\)
7. else
8. **Binary Search** \(A, q+1, r, key\)

Let \(T(n)\) be the worst case time complexity for **Binary Search**, where parameter \(n = r - p + 1\).

The time \(T(n)\) can be upper-bounded with the recurrence,

\[T(n) \leq T\left(\frac{n}{2}\right) + c\] and \(T(0) \leq c\)

where \(c > 0\) is a constant.
Chapter 4. Solving Recurrences

Example: binary search algorithm

Algorithm \texttt{Binary Search}(A, p, r, key)

1. if $p > r$ return (NULL)
2. else
3. \hspace{1em} $q = \left\lfloor \frac{p + r}{2} \right\rfloor$
4. \hspace{1em} if $A[q] = key$ return ($k$)
5. \hspace{1em} else
6. \hspace{2em} if $A[q] > key$ \texttt{Binary Search} (A, $p$, $q - 1$, key)
7. \hspace{2em} else
8. \hspace{3em} \texttt{Binary Search} (A, $q + 1$, $r$, key)

Let $T(n)$ be the worst case time complexity for \texttt{Binary Search}, where parameter $n = r - p + 1$.

The time $T(n)$ can be upper-bounded with the recurrence,

$$T(n) \leq T\left(\frac{n}{2}\right) + c$$

where $c > 0$ is a constant.

Note: we can also set base case $T(1) \leq c$. 
We have recurrence

\[ T(n) \leq T(n/2) + c \]

and

\[ T(1) \leq c \]

for the Binary Search algorithm.

The recurrence can be resolved by unfolding the recursive terms. The process to unfold the recurrence \( T(n) \) can be:

1. simple, straightforward
2. induction based (called substitution method)
3. graph based (called recursive tree method).
Chapter 4. Solving Recurrences

We have recurrence

\[ T(n) \leq T\left(\frac{n}{2}\right) + c \text{ and } T(1) \leq c \]

for the Binary Search algorithm.
We have recurrence

\[ T(n) \leq T\left(\frac{n}{2}\right) + c \text{ and } T(1) \leq c \]

for the Binary Search algorithm.

The recurrence can be resolved by unfolding the recursive terms. The process to unfold the recurrence \( T(n) \) can be:
We have recurrence
\[ T(n) \leq T\left(\frac{n}{2}\right) + c \] 
and \( T(1) \leq c \)

for the **Binary Search** algorithm.

The recurrence can be resolved by unfolding the recursive terms. The process to unfold the recurrence \( T(n) \) can be:

1. simple, straightforward
We have recurrence

\[ T(n) \leq T\left(\frac{n}{2}\right) + c \text{ and } T(1) \leq c \]

for the **Binary Search** algorithm.

The recurrence can be resolved by unfolding the recursive terms. The process to unfold the recurrence \( T(n) \) can be:

1. simple, straightforward
2. induction based (called **substitution method**)

\[ T(n) \leq T\left(\frac{n}{2}\right) + c \text{ and } T(1) \leq c \]

for the **Binary Search** algorithm.

The recurrence can be resolved by unfolding the recursive terms. The process to unfold the recurrence \( T(n) \) can be:

1. simple, straightforward
2. induction based (called **substitution method**)

\[ T(n) \leq T\left(\frac{n}{2}\right) + c \text{ and } T(1) \leq c \]
Chapter 4. Solving Recurrences

We have recurrence

\[ T(n) \leq T\left(\frac{n}{2}\right) + c \quad \text{and} \quad T(1) \leq c \]

for the Binary Search algorithm.

The recurrence can be resolved by unfolding the recursive terms. The process to unfold the recurrence \( T(n) \) can be:

1. simple, straightforward
2. induction based (called substitution method)
3. graph based (called recursive tree method).
Chapter 4. Solving Recurrences

1. Simple, straightforward unfolding:

\[ T(n) \leq T(n^2) + c \]

\[ T(n^2) \leq T(n^4) + c \]

\[ T(n^{2^h}) \leq T(n^{2^{h+1}}) + c \]

where \( n^{2^{h+1}} = 1 \),

\[ T(n) \leq T(n^{2^h}) + (h+1)c \]

there are \( h+1 \) inequalities

\[ T(n) \leq c + c \log_2 n = O(\log_2 n) \]

proved this last equation!
1. Simple, straightforward unfolding:
1. Simple, straightforward unfolding:

use \( T(n) \leq T\left(\frac{n}{2}\right) + c \) as a template;
Chapter 4. Solving Recurrences

1. Simple, straightforward unfolding:

\[ T(n) \leq T\left(\frac{n}{2}\right) + c \]

use \( T(n) \leq T\left(\frac{n}{2}\right) + c \) as a template;

\( T(n) \leq T\left(\frac{n}{2}\right) + c \)
Chapter 4. Solving Recurrences

1. Simple, straightforward unfolding:

   use $T(n) \leq T\left(\frac{n}{2}\right) + c$ as a template;

   $T(n) \leq T\left(\frac{n}{2}\right) + c$
   $T\left(\frac{n}{2}\right) \leq T\left(\frac{n}{2^2}\right) + c$
Chapter 4. Solving Recurrences

1. Simple, straightforward unfolding:

   use $T(n) \leq T\left(\frac{n}{2}\right) + c$ as a template;

   $T(n) \leq T\left(\frac{n}{2}\right) + c$
   $T\left(\frac{n}{2}\right) \leq T\left(\frac{n}{2^2}\right) + c$
   $T\left(\frac{n}{2^2}\right) \leq T\left(\frac{n}{2^3}\right) + c$
Chapter 4. Solving Recurrences

1. Simple, straightforward unfolding: use $T(n) \leq T\left(\frac{n}{2}\right) + c$ as a template;

   $T(n) \leq T\left(\frac{n}{2}\right) + c$
   $T\left(\frac{n}{2}\right) \leq T\left(\frac{n}{2^2}\right) + c$
   $T\left(\frac{n}{2^2}\right) \leq T\left(\frac{n}{2^3}\right) + c$

   .......

   $T\left(\frac{n}{2^h}\right) \leq T\left(\frac{n}{2^{h+1}}\right) + c$

   where $n \geq 2$, $h \geq 1$. There are $h + 1$ inequalities

   $T(n) \leq T\left(\frac{n}{2^{h+1}}\right) + (h + 1)c$

   and $T(n) = O(\log_2 n)$.
Chapter 4. Solving Recurrences

1. Simple, straightforward unfolding:

use $T(n) \leq T\left(\frac{n}{2}\right) + c$ as a template;

\[
\begin{align*}
T(n) &\leq T\left(\frac{n}{2}\right) + c \\
T\left(\frac{n}{2}\right) &\leq T\left(\frac{n}{2^2}\right) + c \\
T\left(\frac{n}{2^2}\right) &\leq T\left(\frac{n}{2^3}\right) + c \\
\cdots &
\end{align*}
\]

$T\left(\frac{n}{2^h}\right) \leq T\left(\frac{n}{2^{h+1}}\right) + c$ \hspace{1cm} \text{where} \hspace{1cm} \frac{n}{2^{h+1}} = 1, 2^{h+1} = n, h + 1 = \log_2 n$
1. Simple, straightforward unfolding:

use $T(n) \leq T\left(\frac{n}{2}\right) + c$ as a template;

$T(n) \leq T\left(\frac{n}{2}\right) + c$

$T\left(\frac{n}{2}\right) \leq T\left(\frac{n}{2^2}\right) + c$

$T\left(\frac{n}{2^2}\right) \leq T\left(\frac{n}{2^3}\right) + c$

\[\cdots\cdots\]

$T\left(\frac{n}{2^h}\right) \leq T\left(\frac{n}{2^{h+1}}\right) + c$ \hspace{1cm} \text{where} \hspace{0.5cm} \frac{n}{2^{h+1}} = 1, \hspace{0.5cm} 2^{h+1} = n, \hspace{0.5cm} h + 1 = \log_2 n$
1. Simple, straightforward unfolding:

use $T(n) \leq T(\frac{n}{2}) + c$ as a template;

\[
\begin{align*}
T(n) & \leq T(\frac{n}{2}) + c \\
T(\frac{n}{2}) & \leq T(\frac{n}{2^2}) + c \\
T(\frac{n}{2^2}) & \leq T(\frac{n}{2^3}) + c \\
\vdots \\
T(\frac{n}{2^h}) & \leq T(\frac{n}{2^{h+1}}) + c
\end{align*}
\]

where $\frac{n}{2^{h+1}} = 1$, $2^{h+1} = n$, $h + 1 = \log_2 n$

\[
T(n) \leq T(\frac{n}{2^{h+1}}) + (h + 1)c
\]

there are $h + 1$ inequalities
Chapter 4. Solving Recurrences

1. Simple, straightforward unfolding:

use $T(n) \leq T\left(\frac{n}{2}\right) + c$ as a template;

$$
T(n) \leq T\left(\frac{n}{2}\right) + c \\
T\left(\frac{n}{2}\right) \leq T\left(\frac{n}{2^2}\right) + c \\
T\left(\frac{n}{2^2}\right) \leq T\left(\frac{n}{2^3}\right) + c \\
\ldots \ldots \\
T\left(\frac{n}{2^h}\right) \leq T\left(\frac{n}{2^{h+1}}\right) + c \quad \text{where} \quad \frac{n}{2^{h+1}} = 1, \ 2^{h+1} = n, \ h + 1 = \log_2 n
$$

$\therefore$

$$
T(n) \leq T\left(\frac{n}{2^{h+1}}\right) + (h + 1)c \\
\leq c + c \log_2 n
$$
Chapter 4. Solving Recurrences

1. Simple, straightforward unfolding:

use \( T(n) \leq T\left(\frac{n}{2}\right) + c \) as a template;

\[
\begin{align*}
T(n) & \leq T\left(\frac{n}{2}\right) + c \\
T\left(\frac{n}{2}\right) & \leq T\left(\frac{n}{2^2}\right) + c \\
T\left(\frac{n}{2^2}\right) & \leq T\left(\frac{n}{2^3}\right) + c \\
& \cdots \\
T\left(\frac{n}{2^h}\right) & \leq T\left(\frac{n}{2^{h+1}}\right) + c
\end{align*}
\]

where \( \frac{n}{2^{h+1}} = 1 \), \( 2^{h+1} = n \), \( h + 1 = \log_2 n \)

there are \( h + 1 \) inequalities

\[
\begin{align*}
T(n) & \leq T\left(\frac{n}{2^{h+1}}\right) + (h + 1)c \\
& \leq c + c \log_2 n \\
& = O(\log_2 n)
\end{align*}
\]
1. Simple, straightforward unfolding:

use \( T(n) \leq T\left(\frac{n}{2}\right) + c \) as a template;

\[
\begin{align*}
T(n) & \leq T\left(\frac{n}{2}\right) + c \\
T\left(\frac{n}{2}\right) & \leq T\left(\frac{n}{2^2}\right) + c \\
T\left(\frac{n}{2^2}\right) & \leq T\left(\frac{n}{2^3}\right) + c \\
& \vdots \\
T\left(\frac{n}{2^h}\right) & \leq T\left(\frac{n}{2^{h+1}}\right) + c
\end{align*}
\]

where \( \frac{n}{2^{h+1}} = 1 \), \( 2^{h+1} = n \), \( h + 1 = \log_2 n \)

\[
T(n) \leq T\left(\frac{n}{2^{h+1}}\right) + (h + 1)c \quad \text{there are } h + 1 \ \text{inequalities}
\]

\[
\leq c + c \log_2 n
\]

\[
= O(\log_2 n) \quad \text{proved this last equation!}
\]
Chapter 4. Solving Recurrences

Another example:

Algorithm Merge Sort ($A, p, r$)

1. if $p < r$
2. then $q = \lfloor \frac{p + r}{2} \rfloor$
3. Merge Sort ($A, p, q$)
4. Merge Sort ($A, q + 1, r$)
5. Merging2Lists ($A, p, q, r$)

Analysis of the algorithm.

• Assume $n = r - p + 1$, a power of 2; also assume $T(n)$ is time for Merge Sort ($A, p, r$).

• $t_1, t_2 = c$
• $t_3 = t_4 = T(n/2)$
• $t_5 \leq n$ (why?)

$T(n) = t_1, t_2 + t_3 + t_4 + t_5 \leq 2T(n/2) + n + c$

base case: $T(1) = c$. 
Chapter 4. Solving Recurrences

Another example:

### Algorithm Merge Sort \((A, p, r)\)

1. If \(p < r\)
2. Then \(q = \lfloor \frac{p + r}{2} \rfloor\)
3. Merge Sort \((A, p, q)\)
4. Merge Sort \((A, q + 1, r)\)
5. Merging2Lists \((A, p, q, r)\)

**Analysis of the algorithm.**

- Assume \(n = r - p + 1\), a power of 2; also assume \(T(n)\) is time for Merge Sort \((A, p, r)\).

- \(t_1, t_2 = c \cdot t_3 = t_4 = T(n/2)\)

- \(t_5 \leq n\) (why?)

\[ T(n) = t_1 + t_2 + t_3 + t_4 + t_5 \leq 2T(n/2) + n + c \]

**Base case:** \(T(1) = c\).
Another example:

Algorithm MERGE SORT($A, p, r$)

1. if $p < r$
2. then $q = \left\lceil \frac{p+r}{2} \right\rceil$
3. MERGE SORT($A, p, q$)
4. MERGE SORT($A, q + 1, r$)
5. MERGING2LISTS($A, p, q, r$)
Another example:

Algorithm $\text{MERGE\Sort}(A, p, r)$

1. if $p < r$
2. then $q = \left\lceil \frac{p+r}{2} \right\rceil$
3. $\text{MERGE\Sort}(A, p, q)$
4. $\text{MERGE\Sort}(A, q+1, r)$
5. $\text{MERGING2LISTS}(A.p, q, r)$

Analysis of the algorithm.

- Assume $n = r - p + 1$, a power of 2;
- also assume $T(n)$ is time for $\text{MERGE\Sort}(A, p, r)$. Then
Chapter 4. Solving Recurrences

Another example:

Algorithm \textsc{Merge Sort}(A, p, r)

1. \textbf{if} \, p < r
2. \textbf{then} \, q = \left\lfloor \frac{p+r}{2} \right\rfloor
3. \textsc{Merge Sort}(A, p, q)
4. \textsc{Merge Sort}(A, q + 1, r)
5. \textsc{Merging2Lists}(A.p, q, r)

Analysis of the algorithm.

- Assume \( n = r - p + 1 \), a power of 2;
  also assume \( T(n) \) is time for \textsc{Merge Sort}(A, p, r). Then

- \( t_{1,2} = c \)
Chapter 4. Solving Recurrences

Another example:

Algorithm $\text{MERGE\ SORT}(A, p, r)$

1. if $p < r$
2. then $q = \lfloor \frac{p+r}{2} \rfloor$
3. $\text{MERGE\ SORT}(A, p, q)$
4. $\text{MERGE\ SORT}(A, q + 1, r)$
5. $\text{MERGING2LISTS}(A, p, q, r)$

Analysis of the algorithm.

- Assume $n = r - p + 1$, a power of 2;
  also assume $T(n)$ is time for $\text{MERGE\ SORT}(A, p, r)$. Then

- $t_{1,2} = c$
- $t_3 = t_4 = T\left(\frac{n}{2}\right)$
Chapter 4. Solving Recurrences

Another example:

Algorithm **MERGE SORT**(*A, p, r*)

1. \(\text{if } p < r\)
2. \(\text{then } q = \left\lfloor \frac{p+r}{2} \right\rfloor\)
3. \(\text{MERGE SORT}(A, p, q)\)
4. \(\text{MERGE SORT}(A, q + 1, r)\)
5. \(\text{MERGING2LISTS}(A, p, q, r)\)

Analysis of the algorithm.

- Assume \(n = r - p + 1\), a power of 2; also assume \(T(n)\) is time for **MERGE SORT**(*A, p, r*). Then
  - \(t_{1,2} = c\)
  - \(t_3 = t_4 = T \left( \frac{n}{2} \right)\)
  - \(t_5 \leq n\) (why?)
Chapter 4. Solving Recurrences

Another example:

Algorithm \textsc{Merge Sort}(A, p, r)

1. \textbf{if} \( p < r \)
2. \textbf{then} \( q = \lceil \frac{p+r}{2} \rceil \)
3. \textsc{Merge Sort}(A, p, q)
4. \textsc{Merge Sort}(A, q + 1, r)
5. \textsc{Merging2Lists}(A, p, q, r)

Analysis of the algorithm.

- Assume \( n = r - p + 1 \), a power of 2;
  also assume \( T(n) \) is time for \textsc{Merge Sort}(A, p, r). Then

- \( t_{1,2} = c \)
- \( t_3 = t_4 = T(\frac{n}{2}) \)
- \( t_5 \leq n \) (why?)

\[
T(n) = t_{1,2} + t_3 + t_4 + t_5 \leq 2T(\frac{n}{2}) + n + c
\]
Chapter 4. Solving Recurrences

Another example:

Algorithm \textsc{Merge Sort}(A, p, r)

1. \textbf{if} \; p < r
2. \textbf{then} \; q = \left\lfloor \frac{p+r}{2} \right\rfloor
3. \textsc{Merge Sort}(A, p, q)
4. \textsc{Merge Sort}(A, q + 1, r)
5. \textsc{Merging2Lists}(A, p, q, r)

Analysis of the algorithm.

- Assume \( n = r - p + 1 \), a power of 2;
  also assume \( T(n) \) is time for \textsc{Merge Sort}(A, p, r). Then

- \( t_{1,2} = c \)
- \( t_3 = t_4 = T\left(\frac{n}{2}\right) \)
- \( t_5 \leq n \) \textbf{(why?)}

\[
T(n) = t_{1,2} + t_3 + t_4 + t_5 \leq 2T\left(\frac{n}{2}\right) + n + c
\]

base case: \( T(1) = c \).
Chapter 4. Solving Recurrences

Solve recurrence

\[ T(n) \leq 2T(n^2) + n + c \]

with base case

\[ T(1) = c \]

with a simple method:

\[ T(n) \leq 2T(n^2) + n + c \]

\[ 2T(n^2) \leq 2T(n^4) + 2n^2 + 2c \]

\[ 3T(n^2) \leq 2T(n^4) + 2n^2 + 2c \]

\[ \vdots \]

\[ hT(n^{2h}) \leq 2T(n^{2h+1}) + 2n^{2h} + 2hc \]

where

\[ n^{2h+1} = 1 \]
Chapter 4. Solving Recurrences

Solve recurrence

\[ T(n) \leq 2T\left(\frac{n}{2}\right) + n + c \quad \text{with base case } T(1) = c \]

with a simple method:
Chapter 4. Solving Recurrences

Solve recurrence

\[ T(n) \leq 2T\left(\frac{n}{2}\right) + n + c \quad \text{with base case } T(1) = c \]

with a simple method:

\[ T(n) \leq 2T\left(\frac{n}{2}\right) + n + c \]
Chapter 4. Solving Recurrences

Solve recurrence

\[ T(n) \leq 2T\left(\frac{n}{2}\right) + n + c \]

with base case \( T(1) = c \)

with a simple method:

\[ T(n) \leq 2T\left(\frac{n}{2}\right) + n + c \]
\[ T\left(\frac{n}{2}\right) \leq 2T\left(\frac{n}{2^2}\right) + \frac{n}{2} + c \]
Chapter 4. Solving Recurrences

Solve recurrence

\[ T(n) \leq 2T\left(\frac{n}{2}\right) + n + c \]

with base case \( T(1) = c \)

with a simple method:

\[
\begin{align*}
T(n) &\leq 2T\left(\frac{n}{2}\right) + n + c \\
T\left(\frac{n}{2}\right) &\leq 2T\left(\frac{n}{2^2}\right) + \frac{n}{2} + c \\
T\left(\frac{n}{2^2}\right) &\leq 2T\left(\frac{n}{2^3}\right) + \frac{n}{2^2} + c
\end{align*}
\]
Solve recurrence

\[ T(n) \leq 2T\left(\frac{n}{2}\right) + n + c \] with base case \( T(1) = c \)

with a simple method:

\[
\begin{align*}
T(n) & \leq 2T\left(\frac{n}{2}\right) + n + c \\
T\left(\frac{n}{2}\right) & \leq 2T\left(\frac{n}{2^2}\right) + \frac{n}{2} + c \\
T\left(\frac{n}{2^2}\right) & \leq 2T\left(\frac{n}{2^3}\right) + \frac{n}{2^2} + c \\
& \vdots
\end{align*}
\]
Solve recurrence

\[ T(n) \leq 2T\left(\frac{n}{2}\right) + n + c \]  

with base case \( T(1) = c \)

with a simple method:

\[ T(n) \leq 2T\left(\frac{n}{2}\right) + n + c \]
\[ T\left(\frac{n}{2}\right) \leq 2T\left(\frac{n}{2^2}\right) + \frac{n}{2} + c \]
\[ T\left(\frac{n}{2^2}\right) \leq 2T\left(\frac{n}{2^3}\right) + \frac{n}{2^2} + c \]
\[ \vdots \]
\[ T\left(\frac{n}{2^h}\right) \leq 2T\left(\frac{n}{2^{h+1}}\right) + \frac{n}{2^h} + c \]

where \( \frac{n}{2^{h+1}} = 1 \)
Chapter 4. Solving Recurrences

Solve recurrence

\[ T(n) \leq 2T\left(\frac{n}{2}\right) + n + c \] with base case \( T(1) = c \)

with a simple method:

\[
\begin{align*}
T(n) & \leq 2T\left(\frac{n}{2}\right) + n + c \\
T\left(\frac{n}{2}\right) & \leq 2T\left(\frac{n}{2^2}\right) + \frac{n}{2} + c \\
T\left(\frac{n}{2^2}\right) & \leq 2T\left(\frac{n}{2^3}\right) + \frac{n}{2^2} + c \\
& \vdots \\
T\left(\frac{n}{2^h}\right) & \leq 2T\left(\frac{n}{2^{h+1}}\right) + \frac{n}{2^h} + c \quad \text{where } \frac{n}{2^{h+1}} = 1
\end{align*}
\]

multiplying \(2, 2^2, \ldots\) to the second, third, \(\ldots\) inequalities, respectively,
Chapter 4. Solving Recurrences

Solve recurrence

\[ T(n) \leq 2T\left(\frac{n}{2}\right) + n + c \quad \text{with base case } T(1) = c \]

with a simple method:

- \[ T(n) \leq 2T\left(\frac{n}{2}\right) + n + c \]
- \[ T\left(\frac{n}{2}\right) \leq 2T\left(\frac{n}{2^2}\right) + \frac{n}{2} + c \]
- \[ T\left(\frac{n}{2^2}\right) \leq 2T\left(\frac{n}{2^3}\right) + \frac{n}{2^2} + c \]
- \[ \cdots \]
- \[ T\left(\frac{n}{2^h}\right) \leq 2T\left(\frac{n}{2^{h+1}}\right) + \frac{n}{2^h} + c \quad \text{where } \frac{n}{2^{h+1}} = 1 \]

multiplying 2, 2\(^2\), \ldots to the second, third, \ldots inequalities, respectively,

\[ T(n) \leq 2T\left(\frac{n}{2}\right) + n + c \]
Chapter 4. Solving Recurrences

Solve recurrence

\[ T(n) \leq 2T\left(\frac{n}{2}\right) + n + c \] with base case \( T(1) = c \)

with a simple method:

\[
\begin{align*}
T(n) &\leq 2T\left(\frac{n}{2}\right) + n + c \\
T\left(\frac{n}{2}\right) &\leq 2T\left(\frac{n}{4}\right) + \frac{n}{2} + c \\
T\left(\frac{n}{4}\right) &\leq 2T\left(\frac{n}{8}\right) + \frac{n}{4} + c \\
&\vdots \\
T\left(\frac{n}{2^h}\right) &\leq 2T\left(\frac{n}{2^{h+1}}\right) + \frac{n}{2^h} + c \quad \text{where } \frac{n}{2^{h+1}} = 1
\end{align*}
\]

multiplying \(2, 2^2, \ldots\) to the second, third, \ldots inequalities, respectively,

\[
\begin{align*}
T(n) &\leq 2T\left(\frac{n}{2}\right) + n + c \\
2T\left(\frac{n}{2}\right) &\leq 2^2T\left(\frac{n}{2^2}\right) + 2 \times \frac{n}{2} + 2c
\end{align*}
\]
Chapter 4. Solving Recurrences

Solve recurrence

\[ T(n) \leq 2T\left(\frac{n}{2}\right) + n + c \quad \text{with base case } T(1) = c \]

with a simple method:

\[
\begin{align*}
T(n) &\leq 2T\left(\frac{n}{2}\right) + n + c \\
T\left(\frac{n}{2}\right) &\leq 2T\left(\frac{n}{2^2}\right) + \frac{n}{2} + c \\
T\left(\frac{n}{2^2}\right) &\leq 2T\left(\frac{n}{2^3}\right) + \frac{n}{2^2} + c \\
\dotsc \\
T\left(\frac{n}{2^h}\right) &\leq 2T\left(\frac{n}{2^{h+1}}\right) + \frac{n}{2^h} + c \quad \text{where } \frac{n}{2^{h+1}} = 1
\end{align*}
\]

multiplying \(2, 2^2, \dotsc\) to the second, third, \(\dotsc\) inequalities, respectively,

\[
\begin{align*}
T(n) &\leq 2T\left(\frac{n}{2}\right) + n + c \\
2T\left(\frac{n}{2}\right) &\leq 2^2T\left(\frac{n}{2^2}\right) + 2 \times \frac{n}{2} + 2c \\
2^2T\left(\frac{n}{2^2}\right) &\leq 2^3T\left(\frac{n}{2^3}\right) + 2^2 \times \frac{n}{2^2} + 2^2c
\end{align*}
\]
Chapter 4. Solving Recurrences

Solve recurrence

\[ T(n) \leq 2T\left(\frac{n}{2}\right) + n + c \quad \text{with base case } T(1) = c \]

with a simple method:

\[
\begin{align*}
T(n) & \leq 2T\left(\frac{n}{2}\right) + n + c \\
T\left(\frac{n}{2}\right) & \leq 2T\left(\frac{n}{2^2}\right) + \frac{n}{2} + c \\
T\left(\frac{n}{2^2}\right) & \leq 2T\left(\frac{n}{2^3}\right) + \frac{n}{2^2} + c \\
& \quad \ldots \\
T\left(\frac{n}{2^h}\right) & \leq 2T\left(\frac{n}{2^{h+1}}\right) + \frac{n}{2^h} + c \quad \text{where } \frac{n}{2^{h+1}} = 1
\end{align*}
\]

multiplying 2, 2^2, \ldots to the second, third, \ldots inequalities, respectively,

\[
\begin{align*}
T(n) & \leq 2T\left(\frac{n}{2}\right) + n + c \\
2T\left(\frac{n}{2}\right) & \leq 2^2T\left(\frac{n}{2^2}\right) + 2 \times \frac{n}{2} + 2c \\
2^2T\left(\frac{n}{2^2}\right) & \leq 2^3T\left(\frac{n}{2^3}\right) + 2^2 \times \frac{n}{2^2} + 2^2c \\
& \quad \ldots
\end{align*}
\]
Chapter 4. Solving Recurrences

Solve recurrence

\[ T(n) \leq 2T\left(\frac{n}{2}\right) + n + c \] with base case \( T(1) = c \)

with a simple method:

\[
\begin{align*}
T(n) & \leq 2T\left(\frac{n}{2}\right) + n + c \\
T\left(\frac{n}{2}\right) & \leq 2T\left(\frac{n}{2^2}\right) + \frac{n}{2} + c \\
T\left(\frac{n}{2^2}\right) & \leq 2T\left(\frac{n}{2^3}\right) + \frac{n}{2^2} + c \\
& \vdots \\
T\left(\frac{n}{2^h}\right) & \leq 2T\left(\frac{n}{2^{h+1}}\right) + \frac{n}{2^h} + c \quad \text{where } \frac{n}{2^{h+1}} = 1
\end{align*}
\]

multiplying \(2, 2^2, \ldots\) to the second, third, \ldots inequalities, respectively,

\[
\begin{align*}
T(n) & \leq 2T\left(\frac{n}{2}\right) + n + c \\
2T\left(\frac{n}{2}\right) & \leq 2^2 T\left(\frac{n}{2^2}\right) + 2 \times \frac{n}{2} + 2c \\
2^2 T\left(\frac{n}{2^2}\right) & \leq 2^3 T\left(\frac{n}{2^3}\right) + 2^2 \times \frac{n}{2^2} + 2^2 c \\
& \vdots \\
2^h T\left(\frac{n}{2^h}\right) & \leq 2^{h+1} T\left(\frac{n}{2^{h+1}}\right) + 2^h \times \frac{n}{2^h} + 2^h c \quad \text{where } \frac{n}{2^{h+1}} = 1
\end{align*}
\]
Chapter 4. Solving Recurrences

Solve recurrence

\[ T(n) \leq 2T\left(\frac{n}{2}\right) + n + c \]  
with base case \( T(1) = c \)

with a simple method:

\[
\begin{align*}
T(n) & \leq 2T\left(\frac{n}{2}\right) + n + c \\
T\left(\frac{n}{2}\right) & \leq 2T\left(\frac{n}{2^2}\right) + \frac{n}{2} + c \\
T\left(\frac{n}{2^2}\right) & \leq 2T\left(\frac{n}{2^3}\right) + \frac{n}{2^2} + c \\
\vdots \\
T\left(\frac{n}{2^h}\right) & \leq 2T\left(\frac{n}{2^{h+1}}\right) + \frac{n}{2^h} + c \quad \text{where } \frac{n}{2^{h+1}} = 1
\end{align*}
\]

multiplying \( 2, 2^2, \ldots \) to the second, third, \ldots inequalities, respectively,

\[
\begin{align*}
T(n) & \leq 2T\left(\frac{n}{2}\right) + n + c \\
2T\left(\frac{n}{2}\right) & \leq 2^2T\left(\frac{n}{2^2}\right) + 2 \times \frac{n}{2} + 2c \\
2^2T\left(\frac{n}{2^2}\right) & \leq 2^3T\left(\frac{n}{2^3}\right) + 2^2 \times \frac{n}{2^2} + 2^2c \\
\vdots \\
2^hT\left(\frac{n}{2^h}\right) & \leq 2^{h+1}T\left(\frac{n}{2^{h+1}}\right) + 2^h \times \frac{n}{2^h} + 2^hc \quad \text{where } \frac{n}{2^{h+1}} = 1
\end{align*}
\]
Solve recurrence

\[ T(n) \leq 2T\left(\frac{n}{2}\right) + n + c \text{ with base case } T(1) = c \]

with a simple method:

\[
\begin{align*}
T(n) &\leq 2T\left(\frac{n}{2}\right) + n + c \\
T\left(\frac{n}{2}\right) &\leq 2T\left(\frac{n}{2^2}\right) + \frac{n}{2} + c \\
T\left(\frac{n}{2^2}\right) &\leq 2T\left(\frac{n}{2^3}\right) + \frac{n}{2^2} + c \\
\vdots \\
T\left(\frac{n}{2^h}\right) &\leq 2T\left(\frac{n}{2^{h+1}}\right) + \frac{n}{2^h} + c \quad \text{where } \frac{n}{2^{h+1}} = 1
\end{align*}
\]

multiplying \(2, 2^2, \ldots\) to the second, third, \ldots inequalities, respectively,

\[
\begin{align*}
T(n) &\leq 2T\left(\frac{n}{2}\right) + n + c \\
2T\left(\frac{n}{2}\right) &\leq 2^2T\left(\frac{n}{2^2}\right) + 2 \times \frac{n}{2} + 2c \\
2^2T\left(\frac{n}{2^2}\right) &\leq 2^3T\left(\frac{n}{2^3}\right) + 2^2 \times \frac{n}{2^2} + 2^2c \\
\vdots \\
2^hT\left(\frac{n}{2^h}\right) &\leq 2^{h+1}T\left(\frac{n}{2^{h+1}}\right) + 2^h \times \frac{n}{2^h} + 2^hc \quad \text{where } \frac{n}{2^{h+1}} = 1
\end{align*}
\]

\[
T(n) \leq 2^{h+1}T\left(\frac{n}{2^{h+1}}\right) + (h + 1)n + c \sum_{i=0}^{h} 2^i
\]
Chapter 4. Solving Recurrences

\[ T(n) \leq 2h + 1 \]

\[ T\left( n^{2h+1} \right) + (h+1)n + ch \sum_{i=0}^{2i} \]

With \( n^{2h+1} = 1 \), we have \( n = 2h+1 \) or \( h+1 = \log_2 n \).

\[ T(n) \leq 2h + 1 \]

\[ T(1) + n \log_2 n + c(2h+1-1) = cn + n \log_2 n + c(n-1) = O(n \log_2 n) \]

We need to prove the last equality, i.e., find constants \( a \) and \( k \) such that

\[ n \log_2 n + 2cn - c \leq an \log_2 n \]

when \( n > k \).

Choose \( a = 2 \). Then to make (1) holds, we need \( \log_2 n > 2c \). So \( k = 2^{2c} \) suffices.

That is, \( n \log_2 n + 2cn - c \leq 2n \log_2 n \) when \( n > k = 2^{2c} \).

So \( T(n) = O(n \log_2 n) \).
Chapter 4. Solving Recurrences

\[ T(n) \leq 2^{h+1} T\left(\frac{n}{2^{h+1}}\right) + (h + 1)n + c \sum_{i=0}^{h} 2^i \]
Chapter 4. Solving Recurrences

\[ T(n) \leq 2^{h+1}T\left(\frac{n}{2^{h+1}}\right) + (h + 1)n + c \sum_{i=0}^{h} 2^i \]

With \( \frac{n}{2^{h+1}} = 1 \), we have \( n = 2^{h+1} \) or \( h + 1 = \log_2 n \)
Chapter 4. Solving Recurrences

\[ T(n) \leq 2^{h+1} T\left(\frac{n}{2^{h+1}}\right) + (h + 1)n + c \sum_{i=0}^{h} 2^i \]

With \( \frac{n}{2^{h+1}} = 1 \), we have \( n = 2^{h+1} \) or \( h + 1 = \log_2 n \)

\[ T(n) \leq 2^{h+1} T(1) + n \log_2 n + c(2^{h+1} - 1) \]
Chapter 4. Solving Recurrences

\[ T(n) \leq 2^{h+1} T\left(\frac{n}{2^{h+1}}\right) + (h + 1)n + c \sum_{i=0}^{h} 2^i \]

With \( \frac{n}{2^{h+1}} = 1 \), we have \( n = 2^{h+1} \) or \( h + 1 = \log_2 n \)

\[ T(n) \leq 2^{h+1} T(1) + n \log_2 n + c(2^{h+1} - 1) \]
\[ = cn + n \log_2 n + c(n - 1) \]
Chapter 4. Solving Recurrences

\[ T(n) \leq 2^{h+1} T\left(\frac{n}{2^{h+1}}\right) + (h+1)n + c \sum_{i=0}^{h} 2^i \]

With \( \frac{n}{2^{h+1}} = 1 \), we have \( n = 2^{h+1} \) or \( h + 1 = \log_2 n \)

\[ T(n) \leq 2^{h+1} T(1) + n \log_2 n + c(2^{h+1} - 1) \]
\[ = cn + n \log_2 n + c(n - 1) \]
\[ = n \log_2 n + 2cn - c \]
Chapter 4. Solving Recurrences

\[ T(n) \leq 2^{h+1} T(\frac{n}{2^{h+1}}) + (h + 1)n + c \sum_{i=0}^{h} 2^i \]

With \( \frac{n}{2^{h+1}} = 1 \), we have \( n = 2^{h+1} \) or \( h + 1 = \log_2 n \)

\[ T(n) \leq 2^{h+1} T(1) + n \log_2 n + c(2^{h+1} - 1) \]
\[ = cn + n \log_2 n + c(n - 1) \]
\[ = n \log_2 n + 2cn - c \]
\[ = O(n \log_2 n) \]
Chapter 4. Solving Recurrences

\[
T(n) \leq 2^{h+1} T\left(\frac{n}{2^h+1}\right) + (h + 1)n + c \sum_{i=0}^{h} 2^i
\]

With \( \frac{n}{2^h+1} = 1 \), we have \( n = 2^{h+1} \) or \( h + 1 = \log_2 n \)

\[
T(n) \leq 2^{h+1} T(1) + n \log_2 n + c(2^{h+1} - 1)
= cn + n \log_2 n + c(n - 1)
= n \log_2 n + 2cn - c
= O(n \log_2 n)
\]

We need to prove the last equality, i.e., find constants \( a \) and \( k \) such that

\[
n \log_2 n + 2cn - c \leq an \log_2 n \quad (1)
\]

when \( n > k \).
Chapter 4. Solving Recurrences

\[ T(n) \leq 2^{h+1} T\left(\frac{n}{2^{h+1}}\right) + (h+1)n + c \sum_{i=0}^{h} 2^i \]

With \( \frac{n}{2^{h+1}} = 1 \), we have \( n = 2^{h+1} \) or \( h + 1 = \log_2 n \)

\[ T(n) \leq 2^{h+1} T(1) + n \log_2 n + c(2^{h+1} - 1) \]
\[ = cn + n \log_2 n + c(n - 1) \]
\[ = n \log_2 n + 2cn - c \]
\[ = O(n \log_2 n) \]

We need to prove the last equality, i.e., find constants \( a \) and \( k \) such that

\[ n \log_2 n + 2cn - c \leq an \log_2 n \quad (1) \]

when \( n > k \).

Choose \( a = 2 \). Then to make (1) holds, we need \( \log_2 n > 2c \). So \( k = 2^{2c} \) suffices.
Chapter 4. Solving Recurrences

\[ T(n) \leq 2^{h+1} T\left(\frac{n}{2^{h+1}}\right) + (h + 1)n + c \sum_{i=0}^{h} 2^i \]

With \( \frac{n}{2^{h+1}} = 1 \), we have \( n = 2^{h+1} \) or \( h + 1 = \log_2 n \)

\[ T(n) \leq 2^{h+1} T(1) + n \log_2 n + c(2^{h+1} - 1) \]
\[ = cn + n \log_2 n + c(n - 1) \]
\[ = n \log_2 n + 2cn - c \]
\[ = O(n \log_2 n) \]

We need to prove the last equality, i.e., find constants \( a \) and \( k \) such that

\[ n \log_2 n + 2cn - c \leq an \log_2 n \quad (1) \]

when \( n > k \).

Choose \( a = 2 \). Then to make \( (1) \) holds, we need \( \log_2 n > 2c \). So \( k = 2^{2c} \) suffices.

That is, \( n \log_2 n + 2cn - c \leq 2n \log_2 n \) when \( n > k = 2^{2c} \).
Chapter 4. Solving Recurrences

\[ T(n) \leq 2^{h+1} T\left(\frac{n}{2^{h+1}}\right) + (h+1)n + c \sum_{i=0}^{h} 2^i \]

With \( \frac{n}{2^{h+1}} = 1 \), we have \( n = 2^{h+1} \) or \( h + 1 = \log_2 n \)

\[ T(n) \leq 2^{h+1} T(1) + n \log_2 n + c(2^{h+1} - 1) \]
\[ = cn + n \log_2 n + c(n - 1) \]
\[ = n \log_2 n + 2cn - c \]
\[ = O(n \log_2 n) \]

We need to prove the last equality, i.e., find constants \( a \) and \( k \) such that

\[ n \log_2 n + 2cn - c \leq an \log_2 n \quad (1) \]

when \( n > k \).

Choose \( a = 2 \). Then to make (1) holds, we need \( \log_2 n > 2c \). So \( k = 2^{2c} \) suffices.

That is, \( n \log_2 n + 2cn - c \leq 2n \log_2 n \) when \( n > k = 2^{2c} \).

So

\[ T(n) = O(n \log_2 n) \]
Chapter 4. Solving Recurrences

When \( n \) is not a power of 2

- choose \( m \) \( n \) such that \( m \) \( n \) is a power of 2 and the smallest such that \( n \leq m \) \( n \);
- \( T(n) \leq T(m \) \( n \), why?
  - assume \( T \) to be monotonic;
  - use the analysis we just did, \( T(m \) \( n = O(m \) \( n \log_2 m \) \( n \);
    - that is, \( \exists c, k, T(m \) \( n \) \( \leq cm \) \( n \log_2 m \) \( n \) when \( m \) \( n \geq k \);
  - but \( m \) \( n < 2n \), why?
    - because \( m \) \( n \frac{1}{2} < n \);
- So \( T(n) \leq T(m \) \( n \) \( \leq cm \) \( n \log_2 m \) \( n \) \( \leq 2cn \log_2 (2n) \) \( \leq 2cn \log_2 n \) \( = c' \) \( n \log_2 n \), here \( c' = 4c \), when \( m \) \( n \geq k \) \( (\geq 4) \);
- therefore, \( T(n) = O(n \log_2 n) \).
When $n$ is not a power of 2
When $n$ is not a power of 2

- choose $m_n$ such that $m_n$ is a power of 2
When $n$ is not a power of 2

- choose $m_n$ such that $m_n$ is a power of 2 and the smallest such that $n \leq m_n$.
When $n$ is not a power of 2

- choose $m_n$ such that $m_n$ is a power of 2 and the smallest such that $n \leq m_n$;
- $T(n) \leq T(m_n)$, why?
When $n$ is not a power of 2

- choose $m_n$ such that $m_n$ is a power of 2 and the smallest such that $n \leq m_n$;

- $T(n) \leq T(m_n)$, why? assume $T$ to be monotonic;
Chapter 4. Solving Recurrences

When $n$ is not a power of 2

• choose $m_n$ such that $m_n$ is a power of 2 and the smallest such that $n \leq m_n$;

• $T(n) \leq T(m_n)$, why? assume $T$ to be monotonic;

• use the analysis we just did, $T(m_n) = O(m_n \log_2 m_n)$;
When $n$ is not a power of 2

- choose $m_n$ such that $m_n$ is a power of 2 and the smallest such that $n \leq m_n$;

- $T(n) \leq T(m_n)$, why? assume $T$ to be monotonic;

- use the analysis we just did, $T(m_n) = O(m_n \log_2 m_n)$; that is, $\exists c, k, T(m_n) \leq cm_n \log_2 m_n$ when $m_n \geq k$;
Chapter 4. Solving Recurrences

When $n$ is not a power of 2

- choose $m_n$ such that $m_n$ is a power of 2 and the smallest such that $n \leq m_n$;
- $T(n) \leq T(m_n)$, why? assume $T$ to be monotonic;
- use the analysis we just did, $T(m_n) = O(m_n \log_2 m_n)$; that is, $\exists c, k, T(m_n) \leq cm_n \log_2 m_n$ when $m_n \geq k$;
- but $m_n < 2n$, why?
When $n$ is not a power of 2

- choose $m_n$ such that $m_n$ is a power of 2 and the smallest such that $n \leq m_n$;
- $T(n) \leq T(m_n)$, why? assume $T$ to be monotonic;
- use the analysis we just did, $T(m_n) = O(m_n \log_2 m_n)$; that is, $\exists c, k, T(m_n) \leq cm_n \log_2 m_n$ when $m_n \geq k$;
- but $m_n < 2n$, why? because $\frac{m_n}{2} < n$;
Chapter 4. Solving Recurrences

When $n$ is not a power of 2

- choose $m_n$ such that $m_n$ is a power of 2 and the smallest such that $n \leq m_n$;
- $T(n) \leq T(m_n)$, why? assume $T$ to be monotonic;
- use the analysis we just did, $T(m_n) = O(m_n \log_2 m_n)$;
  that is, $\exists c, k, T(m_n) \leq cm_n \log_2 m_n$ when $m_n \geq k$;
- but $m_n < 2n$, why? because $\frac{m_n}{2} < n$;
- So $T(n)$
When $n$ is not a power of 2

- choose $m_n$ such that $m_n$ is a power of 2 and the smallest such that $n \leq m_n$;
- $T(n) \leq T(m_n)$, why? assume $T$ to be monotonic;
- use the analysis we just did, $T(m_n) = O(m_n \log_2 m_n)$; that is, $\exists c, k, T(m_n) \leq cm_n \log_2 m_n$ when $m_n \geq k$;
- but $m_n < 2n$, why? because $\frac{m_n}{2} < n$;
- So $T(n) \leq T(m_n)$
Chapter 4. Solving Recurrences

When $n$ is not a power of 2

- choose $m_n$ such that $m_n$ is a power of 2 and the smallest such that $n \leq m_n$;
- $T(n) \leq T(m_n)$, why? assume $T$ to be monotonic;
- use the analysis we just did, $T(m_n) = O(m_n \log_2 m_n)$; that is, $\exists c, k, T(m_n) \leq cm_n \log_2 m_n$ when $m_n \geq k$;
- but $m_n < 2n$, why? because $\frac{m_n}{2} < n$;
- So $T(n) \leq T(m_n) \leq cm_n \log_2 m_n$
When $n$ is not a power of 2

- choose $m_n$ such that $m_n$ is a power of 2 and the smallest such that $n \leq m_n$;
- $T(n) \leq T(m_n)$, why? assume $T$ to be monotonic;
- use the analysis we just did, $T(m_n) = O(m_n \log_2 m_n)$; that is, $\exists c, k$, $T(m_n) \leq cm_n \log_2 m_n$ when $m_n \geq k$;
- but $m_n < 2n$, why? because $\frac{m_n}{2} < n$;
- So $T(n) \leq T(m_n) \leq cm_n \log_2 m_n \leq 2cn \log_2(2n)$
When \( n \) is not a power of 2

- choose \( m_n \) such that \( m_n \) is a power of 2 and the smallest such that \( n \leq m_n \);

- \( T(n) \leq T(m_n) \), why? assume \( T \) to be monotonic;

- use the analysis we just did, \( T(m_n) = O(m_n \log_2 m_n) \);
  that is, \( \exists c, k, T(m_n) \leq cm_n \log_2 m_n \) when \( m_n \geq k \);

- but \( m_n < 2n \), why? because \( \frac{m_n}{2} < n \);

- So \( T(n) \leq T(m_n) \leq cm_n \log_2 m_n \leq 2cn \log_2 (2n) \leq 2cn \log_2 n^2 \).
When $n$ is not a power of 2

- choose $m_n$ such that $m_n$ is a power of 2 and the smallest such that $n \leq m_n$;

- $T(n) \leq T(m_n)$, why? assume $T$ to be monotonic;

- use the analysis we just did, $T(m_n) = O(m_n \log_2 m_n)$; that is, $\exists c, k, T(m_n) \leq cm_n \log_2 m_n$ when $m_n \geq k$;

- but $m_n < 2n$, why? because $\frac{m_n}{2} < n$;

- So $T(n) \leq T(m_n) \leq cm_n \log_2 m_n \leq 2cn \log_2(2n) \leq 2cn \log_2 n^2 = 4cn \log_2 n =$
When $n$ is not a power of 2

- choose $m_n$ such that $m_n$ is a power of 2 and the smallest such that $n \leq m_n$;
- $T(n) \leq T(m_n)$, why? assume $T$ to be monotonic;
- use the analysis we just did, $T(m_n) = O(m_n \log_2 m_n)$;
  that is, $\exists c, k, T(m_n) \leq cm_n \log_2 m_n$ when $m_n \geq k$;
- but $m_n < 2n$, why? because $\frac{m_n}{2} < n$;
- So $T(n) \leq T(m_n) \leq cm_n \log_2 m_n \leq 2cn \log_2 (2n) \leq 2cn \log_2 n^2$
  $= 4cn \log_2 n = c'n \log_2 n$, here $c' = 4c$. 

---

Chapter 4. Solving Recurrences
Chapter 4. Solving Recurrences

When $n$ is not a power of 2

- choose $m_n$ such that $m_n$ is a power of 2 and the smallest such that $n \leq m_n$;

- $T(n) \leq T(m_n)$, why? assume $T$ to be monotonic;

- use the analysis we just did, $T(m_n) = O(m_n \log_2 m_n)$; that is, $\exists c, k$, $T(m_n) \leq cm_n \log_2 m_n$ when $m_n \geq k$;

- but $m_n < 2n$, why? because $\frac{m_n}{2} < n$;

- So $T(n) \leq T(m_n) \leq cm_n \log_2 m_n \leq 2cn \log_2 (2n) \leq 2cn \log_2 n^2 = 4cn \log_2 n = c'n \log_2 n$, here $c' = 4c$, when $m_n \geq k(\geq 4)$,
Chapter 4. Solving Recurrences

When $n$ is not a power of 2

- choose $m_n$ such that $m_n$ is a power of 2 and the smallest such that $n \leq m_n$;

- $T(n) \leq T(m_n)$, why? assume $T$ to be monotonic;

- use the analysis we just did, $T(m_n) = O(m_n \log_2 m_n)$; that is, $\exists c, k, T(m_n) \leq cm_n \log_2 m_n$ when $m_n \geq k$;

- but $m_n < 2n$, why? because $\frac{m_n}{2} < n$;

- So $T(n) \leq T(m_n) \leq cm_n \log_2 m_n \leq 2cn \log_2 (2n) \leq 2cn \log_2 n^2$
  $= 4cn \log_2 n = c'n \log_2 n$, here $c' = 4c$, when $m_n \geq k(\geq 4)$, i.e., when $n \geq \lceil \frac{k}{2} \rceil(\geq 2)$;
Chapter 4. Solving Recurrences

When $n$ is not a power of 2

- choose $m_n$ such that $m_n$ is a power of 2 and the smallest such that $n \leq m_n$;
- $T(n) \leq T(m_n)$, why? assume $T$ to be monotonic;
- use the analysis we just did, $T(m_n) = O(m_n \log_2 m_n)$; that is, $\exists c, k, T(m_n) \leq cm_n \log_2 m_n$ when $m_n \geq k$;
- but $m_n < 2n$, why? because $\frac{m_n}{2} < n$;
- So $T(n) \leq T(m_n) \leq cm_n \log_2 m_n \leq 2cn \log_2 (2n) \leq 2cn \log_2 n^2$ = $4cn \log_2 n = c' n \log_2 n$, here $c' = 4c$, when $m_n \geq k (\geq 4)$, i.e., when $n \geq \lceil \frac{k}{2} \rceil (\geq 2)$;
- therefore, $T(n) = O(n \log_2 n)$. 
Chapter 4. Solving Recurrences

Methods for solving recurrences
Chapter 4. Solving Recurrences

Methods for solving recurrences

1. Substitution method (based on math induction)
Chapter 4. Solving Recurrences

Methods for solving recurrences

1. Substitution method (based on math induction)

First we recall the principle of the math induction:

To prove a property \( P(n) \) for every natural number \( n \geq 1 \), it suffices to prove

- \( P(1) \) holds;
Chapter 4. Solving Recurrences

Methods for solving recurrences

1. Substitution method (based on math induction)

First we recall the principle of the math induction:

To prove a property $P(n)$ for every natural number $n \geq 1$, it suffices to prove

- $P(1)$ holds;
- for every $k \geq 1$, if $P(k)$ holds, then $P(k + 1)$ holds.
Chapter 4. Solving Recurrences

Methods for solving recurrences

1. Substitution method (based on math induction)

First we recall the principle of the math induction:

To prove a property $P(n)$ for every natural number $n \geq 1$, it suffices to prove

- $P(1)$ holds;
- for every $k \geq 1$, if $P(k)$ holds, then $P(k + 1)$ holds.

"The principle for dominos to fall".
Chapter 4. Solving Recurrences

Math induction comes with different forms or variants
Chapter 4. Solving Recurrences

Math induction comes with different forms or variants

- the base case can be for any integer, e.g., $P(3)$ instead of $P(1)$;
Math induction comes with different forms or variants

- the base case can be for any integer, e.g., \( P(3) \) instead of \( P(1) \);
- the statement to prove: \( P(k) \rightarrow P(k + 1) \) has the variant:
Math induction comes with different forms or variants

- the base case can be for any integer, e.g., \( P(3) \) instead of \( P(1) \);
- the statement to prove: \( P(k) \rightarrow P(k + 1) \) has the variant:

\[
P(1) \land P(2) \land \cdots \land P(k) \rightarrow P(k + 1)
\]
Math induction comes with different forms or variants

- the base case can be for any integer, e.g., $P(3)$ instead of $P(1)$;
- the statement to prove: $P(k) \rightarrow P(k + 1)$ has the variant:

\[ P(1) \land P(2) \land \cdots \land P(k) \rightarrow P(k + 1) \]

- other variants: e.g., $P(k) \rightarrow P(2k)$
Chapter 4. Solving Recurrences

Math induction comes with different forms or variants

- the base case can be for any integer, e.g., $P(3)$ instead of $P(1)$;
- the statement to prove: $P(k) \rightarrow P(k+1)$ has the variant:
  \[ P(1) \land P(2) \land \cdots \land P(k) \rightarrow P(k+1) \]
- other variants: e.g., $P(k) \rightarrow P(2k)$
  but we need to make sure all $n$’s beyond the base cases are covered.
Math induction comes with different forms or variants

- the base case can be for any integer, e.g., $P(3)$ instead of $P(1)$;
- the statement to prove: $P(k) \rightarrow P(k + 1)$ has the variant:

$$P(1) \land P(2) \land \cdots \land P(k) \rightarrow P(k + 1)$$

- other variants: e.g., $P(k) \rightarrow P(2k)$
  but we need to make sure all $n$’s beyond the base cases are covered.

$$P(k) \rightarrow P(2k) \land P(2k + 1)$$
Chapter 4. Solving Recurrences

**Theorem:** Arithmetic sequence of first $n$ terms

$$1 + 2 + 3 + \cdots + n = \frac{n}{2}(n + 1)$$
Chapter 4. Solving Recurrences

**Theorem:** Arithmetic sequence of first $n$ terms

$$1 + 2 + 3 + \cdots + n = \frac{n}{2}(n + 1)$$

**Proof:** (use math induction)
Chapter 4. Solving Recurrences

**Theorem:** Arithmetic sequence of first \( n \) terms

\[
1 + 2 + 3 + \cdots + n = \frac{n}{2}(n + 1)
\]

**Proof:** (use math induction)

step 1: base case: \( n = 1 \), left = 1, right = \( \frac{1}{2}(1 + 1) = 1 \);
step 2: assumption:

\[
1 + 2 + 3 + \cdots + n - 1 = \frac{n-1}{2}(n - 1 + 1)
\]
Theorem: Arithmetic sequence of first $n$ terms

$$1 + 2 + 3 + \cdots + n = \frac{n}{2}(n + 1)$$

Proof: (use math induction)

step 1: base case: $n = 1$, left = 1, right = $\frac{1}{2}(1 + 1) = 1$;

step 2: assumption:

$$1 + 2 + 3 + \cdots + n - 1 = \frac{n - 1}{2}(n - 1 + 1) \text{ which is } = \frac{n - 1}{2}n$$

step 3: induction:

$$1 + 2 + 3 + \cdots + n$$
Chapter 4. Solving Recurrences

**Theorem**: Arithmetic sequence of first $n$ terms

$$1 + 2 + 3 + \cdots + n = \frac{n}{2}(n + 1)$$

**Proof**: (use math induction)

step 1: base case: $n = 1$, left = 1, right = $\frac{1}{2}(1 + 1) = 1$;
step 2: assumption:

$$1 + 2 + 3 + \cdots + n - 1 = \frac{n - 1}{2}(n - 1 + 1) \text{ which is } = \frac{n - 1}{2}n$$

step 3: induction:

$$1 + 2 + 3 + \cdots + n = (1 + 2 + 3 + \cdots + n - 1) + n$$
Theorem: Arithmetic sequence of first $n$ terms

$$1 + 2 + 3 + \cdots + n = \frac{n}{2} (n + 1)$$

Proof: (use math induction)

step 1: base case: $n = 1$, left $= 1$, right $= \frac{1}{2} (1 + 1) = 1$;
step 2: assumption:

$$1 + 2 + 3 + \cdots + n - 1 = \frac{n - 1}{2} (n - 1 + 1) \text{ which is } = \frac{n - 1}{2} n$$

step 3: induction:

$$1 + 2 + 3 + \cdots + n = (1 + 2 + 3 + \cdots + n - 1) + n = \frac{n - 1}{2} n + n$$
Chapter 4. Solving Recurrences

Theorem: Arithmetic sequence of first $n$ terms

$$1 + 2 + 3 + \cdots + n = \frac{n}{2}(n + 1)$$

Proof: (use math induction)

step 1: base case: $n = 1$, left = 1, right = $\frac{1}{2}(1 + 1) = 1$;

step 2: assumption:

$$1 + 2 + 3 + \cdots + n - 1 = \frac{n - 1}{2}(n - 1 + 1) \text{ which is } = \frac{n - 1}{2}n$$

step 3: induction:

$$1 + 2 + 3 + \cdots + n = (1 + 2 + 3 + \cdots + n - 1) + n = \frac{n - 1}{2}n + n = \frac{n}{2}(n + 1)$$
Chapter 4. Solving Recurrences

**Theorem:** Arithmetic sequence of first $n$ terms

$$1 + 2 + 3 + \cdots + n = \frac{n}{2}(n + 1)$$

**Proof:** (use math induction)

step 1: base case: $n = 1$, left = 1, right = $\frac{1}{2}(1 + 1) = 1$;

step 2: assumption:

$$1 + 2 + 3 + \cdots + n - 1 = \frac{n - 1}{2}(n - 1 + 1) \text{ which is } \frac{n - 1}{2}n$$

step 3: induction:

$$1 + 2 + 3 + \cdots + n = (1 + 2 + 3 + \cdots + n - 1) + n = \frac{n - 1}{2}n + n = \frac{n}{2}(n + 1)$$

So

$$1 + 2 + 3 + \cdots + n = \frac{n}{2}(n + 1)$$
Algorithm Merge Sort \((A, p, r)\)

1. \textbf{if} \(p < r\)
2. \textbf{then} \(q = \left\lfloor \frac{p+r}{2} \right\rfloor\)
3. \textbf{Merge Sort} \((A, p, q)\)
4. \textbf{Merge Sort} \((A, q + 1, r)\)
5. \textbf{Merging2Lists} \((A.p, q, r)\)
Algorithm MERGE\_SORT(A, p, r)

1. if $p < r$
2. then $q = \lfloor \frac{p+r}{2} \rfloor$
3. MERGE\_SORT(A, p, q)
4. MERGE\_SORT(A, q + 1, r)
5. MERGING2LISTS(A.p, q, r)

$T(n) \leq 2T(\frac{n}{2}) + n + c$ with base case $T(1) = c$
Chapter 4. Solving Recurrences

Algorithm `MERGE SORT(A, p, r)`

1. \textbf{if} $p < r$
2. \hspace{1em} \textbf{then} $q = \lfloor \frac{p+r}{2} \rfloor$
3. \hspace{1em} `MERGE SORT(A, p, q)`
4. \hspace{1em} `MERGE SORT(A, q + 1, r)`
5. \hspace{1em} `MERGING2LISTS(A, p, q, r)`

\[ T(n) \leq 2T\left(\frac{n}{2}\right) + n + c \quad \text{with base case} \quad T(1) = c \]
MergeSort has the time complexity:

\[ T(n) \leq 2T\left(\frac{n}{2}\right) + n + c \quad \text{with base case} \quad T(1) = c \]

Use \textbf{substitution method} to prove that \( T(n) = O(n \log_2 n) \).
MergeSort has the time complexity:

$$T(n) \leq 2T\left(\frac{n}{2}\right) + n + c \quad \text{with base case } T(1) = c$$

Use substitution method to prove that $T(n) = O(n \log_2 n)$.

Proof. We claim that there are constants $a, k$ such that

$$T(n) \leq an \log_2 n \quad \text{when } n \geq k \quad (2)$$
MergeSort has the time complexity:

\[ T(n) \leq 2T\left(\frac{n}{2}\right) + n + c \quad \text{with base case } T(1) = c \]

Use substitution method to prove that \( T(n) = O(n \log_2 n) \).

Proof. We claim that there are constants \( a, k \) such that

\[ T(n) \leq an \log_2 n \quad \text{when } n \geq k \quad (2) \]

step 1. base case: when \( n = 1 \), \( T(1) = c \),
Chapter 4. Solving Recurrences

**MergeSort** has the time complexity:

\[ T(n) \leq 2T\left(\frac{n}{2}\right) + n + c \quad \text{with base case } T(1) = c \]

Use **substitution method** to prove that \( T(n) = O(n \log_2 n) \).

**Proof.** We claim that there are constants \( a, k \) such that

\[ T(n) \leq an \log_2 n \quad \text{when } n \geq k \quad (2) \]

step 1. base case: when \( n = 1 \), \( T(1) = c \), to make \( T(1) \leq an \log_2 n \),
Chapter 4. Solving Recurrences

MergeSort has the time complexity:

\[ T(n) \leq 2T\left(\frac{n}{2}\right) + n + c \quad \text{with base case } T(1) = c \]

Use substitution method to prove that \( T(n) = O(n \log_2 n) \).

**Proof.** We claim that there are constants \( a, k \) such that

\[ T(n) \leq an \log_2 n \quad \text{when } n \geq k \quad (2) \]

step 1. base case: when \( n = 1 \), \( T(1) = c \), to make \( T(1) \leq an \log_2 n \), we need \( n \geq k = 2 \), **So we need to use \( T(2) \) as base case.**
MergeSort has the time complexity:

\[ T(n) \leq 2T\left(\frac{n}{2}\right) + n + c \quad \text{with base case } T(1) = c \]

Use substitution method to prove that \( T(n) = O(n \log_2 n) \).

**Proof.** We claim that there are constants \( a, k \) such that

\[ T(n) \leq an \log_2 n \quad \text{when } n \geq k \quad (2) \]

step 1. base case: when \( n = 1 \), \( T(1) = c \), to make \( T(1) \leq an \log_2 n \),
we need \( n \geq k = 2 \), **So we need to use \( T(2) \) as base case.**

from the recurrence, we known

\[ T(2) \leq 2T(1) + 2 + c = 3c + 2 \leq an \log_2 n \]

when we choose \( a = 3c + 2 \);
Chapter 4. Solving Recurrences

**MergeSort** has the time complexity:

\[ T(n) \leq 2T\left(\frac{n}{2}\right) + n + c \quad \text{with base case } T(1) = c \]

Use substitution method to prove that \( T(n) = O(n \log_2 n) \).

**Proof.** We claim that there are constants \( a, k \) such that

\[ T(n) \leq an \log_2 n \quad \text{when } n \geq k \quad (2) \]

step 1. base case: when \( n = 1 \), \( T(1) = c \), to make \( T(1) \leq an \log_2 n \),
we need \( n \geq k = 2 \), **So we need to use** \( T(2) \) **as base case.**

from the recurrence, we known

\[ T(2) \leq 2T(1) + 2 + c = 3c + 2 \leq an \log_2 n \]

when **we choose** \( a = 3c + 2; \)
Chapter 4. Solving Recurrences

step 2. assumption: for \( \frac{n}{2} \), the claim is true,
step 2. assumption: for $\frac{n}{2}$, the claim is true, i.e.,

$$T\left(\frac{n}{2}\right) \leq a \frac{n}{2} \log_2 \frac{n}{2}, \quad \text{when } n \geq 4$$
step 2. assumption: for $\frac{n}{2}$, the claim is true, i.e.,

$$T\left(\frac{n}{2}\right) \leq a \frac{n}{2} \log_2 \frac{n}{2}, \quad \text{when } n \geq 4$$

step 3. induction:

$$T(n)$$
Chapter 4. Solving Recurrences

step 2. assumption: for \( \frac{n}{2} \), the claim is true, i.e.,

\[
T\left(\frac{n}{2}\right) \leq a \frac{n}{2} \log_2 \frac{n}{2}, \text{ when } n \geq 4
\]

step 3. induction:

\[
T(n) \leq 2T\left(\frac{n}{2}\right) + n + c
\]
Chapter 4. Solving Recurrences

step 2. assumption: for \( \frac{n}{2} \), the claim is true, i.e.,

\[
T\left(\frac{n}{2}\right) \leq a \frac{n}{2} \log_2 \frac{n}{2}, \quad \text{when } n \geq 4
\]

step 3. induction:

\[
T(n) \leq 2T\left(\frac{n}{2}\right) + n + c = 2T\left(\frac{n}{2}\right) + n + c
\]
step 2. assumption: for $\frac{n}{2}$, the claim is true, i.e.,

$$T\left(\frac{n}{2}\right) \leq a \frac{n}{2} \log_2 \frac{n}{2}, \text{ when } n \geq 4$$

step 3. induction:

$$T(n) \leq 2T\left(\frac{n}{2}\right) + n + c = 2T\left(\frac{n}{2}\right) + n + c \leq 2a \frac{n}{2} \log_2 \frac{n}{2} + n + c$$
step 2. assumption: for $\frac{n}{2}$, the claim is true, i.e.,

$$T\left(\frac{n}{2}\right) \leq a \frac{n}{2} \log_2 \frac{n}{2}, \text{ when } n \geq 4$$

step 3. induction:

$$T(n) \leq 2T\left(\frac{n}{2}\right) + n + c = 2T\left(\frac{n}{2}\right) + n + c \leq 2a \frac{n}{2} \log_2 \frac{n}{2} + n + c$$

$$= an(\log_2 n - \log_2 2) + n + c$$
Chapter 4. Solving Recurrences

step 2. assumption: for \( \frac{n}{2} \), the claim is true, i.e.,

\[
T\left(\frac{n}{2}\right) \leq a \frac{n}{2} \log_2 \frac{n}{2}, \text{ when } n \geq 4
\]

step 3. induction:

\[
T(n) \leq 2T\left(\frac{n}{2}\right) + n + c = 2T\left(\frac{n}{2}\right) + n + c \leq 2a \frac{n}{2} \log_2 \frac{n}{2} + n + c
\]

\[
= an(\log_2 n - \log_2 2) + n + c = an \log_2 n - an + n + c
\]
Chapter 4. Solving Recurrences

step 2. assumption: for \( \frac{n}{2} \), the claim is true, i.e.,

\[
T\left(\frac{n}{2}\right) \leq a \frac{n}{2} \log_2 \frac{n}{2}, \text{ when } n \geq 4
\]

step 3. induction:

\[
T(n) \leq 2T\left(\frac{n}{2}\right) + n + c = 2T\left(\frac{n}{2}\right) + n + c \leq 2a \frac{n}{2} \log_2 \frac{n}{2} + n + c
\]

\[
= an(\log_2 n - \log_2 2) + n + c = an \log_2 n - an + n + c = an \log_2 n - (3c + 2)n + n + c
\]
step 2. assumption: for $n \geq 2$, the claim is true, i.e.,

$$T\left(\frac{n}{2}\right) \leq a \frac{n}{2} \log_2 \frac{n}{2}, \quad \text{when } n \geq 4$$

step 3. induction:

$$T(n) \leq 2T\left(\frac{n}{2}\right) + n + c = 2T\left(\frac{n}{2}\right) + n + c \leq 2a \frac{n}{2} \log_2 \frac{n}{2} + n + c$$

$$= an (\log_2 n - \log_2 2) + n + c = an \log_2 n - an + n + c = an \log_2 n - (3c + 2)n + n + c$$

$$= an \log_2 n - c3n - 2n + n + c$$
Chapter 4. Solving Recurrences

step 2. assumption: for $\frac{n}{2}$, the claim is true, i.e.,

$$T\left(\frac{n}{2}\right) \leq a \frac{n}{2} \log_2 \frac{n}{2}, \text{ when } n \geq 4$$

step 3. induction:

$$T(n) \leq 2T\left(\frac{n}{2}\right) + n + c = 2T\left(\frac{n}{2}\right) + n + c \leq 2a \frac{n}{2} \log_2 \frac{n}{2} + n + c$$

$$= an \left(\log_2 n - \log_2 2\right) + n + c = an \log_2 n - an + n + c = an \log_2 n -(3c+2)n + n + c$$

$$= an \log_2 n - c3n - 2n + n + c = an \log_2 n + c(1 - 3n) - n$$
step 2. assumption: for $\frac{n}{2}$, the claim is true, i.e.,

$$T\left(\frac{n}{2}\right) \leq a \frac{n}{2} \log_2 \frac{n}{2}, \text{ when } n \geq 4$$

step 3. induction:

$$T(n) \leq 2T\left(\frac{n}{2}\right) + n + c = 2T\left(\frac{n}{2}\right) + n + c \leq 2a \frac{n}{2} \log_2 \frac{n}{2} + n + c$$

$$= an(\log_2 n - \log_2 2) + n + c = an \log_2 n - an + n + c = an \log_2 n - (3c + 2)n + n + c$$

$$= an \log_2 n - c3n - 2n + n + c = an \log_2 n + c(1 - 3n) - n \leq an \log_2 n$$

because $c(1 - 3n) - n < 0$
Chapter 4. Solving Recurrences

step 2. assumption: for $\frac{n}{2}$, the claim is true, i.e.,

$$T\left(\frac{n}{2}\right) \leq a\frac{n}{2} \log_2 \frac{n}{2}, \text{ when } n \geq 4$$

step 3. induction:

$$T(n) \leq 2T\left(\frac{n}{2}\right) + n + c = 2T\left(\frac{n}{2}\right) + n + c \leq 2a\frac{n}{2} \log_2 \frac{n}{2} + n + c$$

$$= an(\log_2 n - \log_2 2) + n + c = an \log_2 n - an + n + c = an \log_2 n - (3c + 2)n + n + c$$

$$= an \log_2 n - c3n - 2n + n + c = an \log_2 n + c(1 - 3n) - n \leq an \log_2 n$$

because $c(1 - 3n) - n < 0$
Chapter 4. Solving Recurrences

A little review on logarithm functions:

\[
\begin{align*}
\log_a n + \log_a m &= \log_a (nm); \\
\log_a n^b &= b \log_a n, \text{ especially } \log_a 1^n &= -\log_a n; \\
a^{\log_a n} &= n; \\
\log_a n &= \log_b n \log_a b = \frac{1}{\log_b a} \log_b n.
\end{align*}
\]
A little review on logarithm functions:

- $\log_a n + \log_a m = \log_a nm$;
Chapter 4. Solving Recurrences

A little review on logarithm functions:

- \( \log_a n + \log_a m = \log_a nm; \)
- \( \log_a n^b = b \log_a n, \) especially \( \log_a \frac{1}{n} = -\log_a n; \)
Chapter 4. Solving Recurrences

A little review on logarithm functions:

- $\log_a n + \log_a m = \log_a (nm)$;
- $\log_a n^b = b \log_a n$, especially $\log_a \frac{1}{n} = -\log_a n$;
- $a^{\log_a n} = n$. 

![Graph of $y = 2^x$ and $y = \log_2 x$](image)
Chapter 4. Solving Recurrences

A little review on logarithm functions:

- \( \log_a n + \log_a m = \log_a nm \);
- \( \log_a n^b = b \log_a n \), especially \( \log_a \frac{1}{n} = -\log_a n \);
- \( a^{\log_a n} = n \);
- \( \log_a n = \frac{\log_b n}{\log_b a} = \frac{1}{\log_b a} \log_b n \);
Chapter 4. Solving Recurrences

A little review on logarithm functions:

- \( \log_a n + \log_a m = \log_a nm \);
- \( \log_a n^b = b \log_a n \), especially \( \log_a \frac{1}{n} = -\log_a n \);
- \( a^{\log_a n} = n \);
- \( \log_a n = \frac{\log_b n}{\log_b a} = \frac{1}{\log_b a} \log_b n \);
- \( \log_a^m n = (\log_a n)^m \neq \log_a n^m \).
Chapter 4. Solving Recurrences

Solving recurrence with the substitution method (guess then verify)

\[ T(n) = \frac{3}{2} T(\lfloor \frac{2n}{3} \rfloor) + n, \text{ where } T(1) = 2 \]
Chapter 4. Solving Recurrences

Solving recurrence with the substitution method (guess then verify)

\[ T(n) = \frac{3}{2} T(\lfloor \frac{2n}{3} \rfloor) + n, \text{ where } T(1) = 2 \]

Guess \( T(n) \leq cn \log_2 n \), for some constant \( c \) to be determined later.
Chapter 4. Solving Recurrences

Solving recurrence with the substitution method (guess then verify)

\[ T(n) = \frac{3}{2} T(\lfloor \frac{2n}{3} \rfloor) + n, \text{ where } T(1) = 2 \]

Guess \( T(n) \leq cn \log_2 n \), for some constant \( c \) to be determined later.
Verify with induction:
Chapter 4. Solving Recurrences

Solving recurrence with the **substitution method** (guess then verify)

\[ T(n) = \frac{3}{2} T\left(\lceil \frac{2n}{3} \rceil \right) + n, \quad \text{where } T(1) = 2 \]

Guess \( T(n) \leq cn \log_2 n \), for some constant \( c \) to be determined later. 
**Verify with induction:**

Step 1, base case: \( T(1) = 2 \leq cn \log_2 n = 0 \) does not hold for the guessed inequality.
Chapter 4. Solving Recurrences

Solving recurrence with the substitution method (guess then verify)

\[ T(n) = \frac{3}{2} T(\lfloor \frac{2n}{3} \rfloor) + n, \text{ where } T(1) = 2 \]

Guess \( T(n) \leq cn \log_2 n \), for some constant \( c \) to be determined later.
Verify with induction:

Step 1, base case: \( T(1) = 2 \leq cn \log_2 n = 0 \) does not hold for the guessed inequality.

Instead, we choose \( T(2) \) to be the base case. From the recurrence, we have

\[ T(2) = \frac{3}{2} T(\lfloor \frac{2n}{3} \rfloor) + n \]
Chapter 4. Solving Recurrences

Solving recurrence with the substitution method (guess then verify)

\[ T(n) = \frac{3}{2} T(\left\lfloor \frac{2n}{3} \right\rfloor) + n, \text{ where } T(1) = 2 \]

Guess \( T(n) \leq cn \log_2 n \), for some constant \( c \) to be determined later.

Verify with induction:

Step 1, base case: \( T(1) = 2 \leq cn \log_2 n = 0 \) does not hold for the guessed inequality.

Instead, we choose \( T(2) \) to be the base case. From the recurrence, we have

\[ T(2) = \frac{3}{2} T(\left\lfloor \frac{4}{3} \right\rfloor) + 2 = \frac{3}{2} T(1) + 2 \]
Chapter 4. Solving Recurrences

Solving recurrence with the substitution method (guess then verify)

\[ T(n) = \frac{3}{2} T(\lfloor \frac{2n}{3} \rfloor) + n, \text{ where } T(1) = 2 \]

Guess \( T(n) \leq cn \log_2 n \), for some constant \( c \) to be determined later.

Verify with induction:

Step 1, base case: \( T(1) = 2 \leq cn \log_2 n = 0 \) does not hold for the guessed inequality. Instead, we choose \( T(2) \) to be the base case. From the recurrence, we have

\[ T(2) = \frac{3}{2} T(\lfloor \frac{2 \cdot 2}{3} \rfloor) + 2 = \frac{3}{2} T(\lfloor \frac{4}{3} \rfloor) + 2 = \frac{3}{2} T(1) + 2 = 5 \]

We make \( T(2) = 5 \leq cn \log_2 n \) holds for \( c = 3 \), and when \( n \geq k = 2 \),
Chapter 4. Solving Recurrences

Solving recurrence with the **substitution method** (guess then verify)

\[ T(n) = \frac{3}{2}T(\lfloor \frac{2n}{3} \rfloor) + n, \text{ where } T(1) = 2 \]

**Guess** \( T(n) \leq cn \log_2 n \), for some constant \( c \) to be determined later.

**Verify with induction:**

Step 1, base case: \( T(1) = 2 \leq cn \log_2 n = 0 \) does not hold for the guessed inequality.

Instead, we choose \( T(2) \) to be the base case. From the recurrence, we have

\[ T(2) = \frac{3}{2}T(\lfloor \frac{2n}{3} \rfloor) + n = \frac{3}{2}T(\lfloor \frac{4}{3} \rfloor) + 2 = \frac{3}{2}T(1) + 2 = 5 \]

We make \( T(2) = 5 \leq cn \log_2 n \) holds for \( c = 3 \), and when \( n \geq k = 2 \),
Chapter 4. Solving Recurrences

Solving recurrence with the substitution method (guess then verify)

\[ T(n) = \frac{3}{2} T(\lfloor \frac{2n}{3} \rfloor) + n, \quad \text{where } T(1) = 2 \]

**Guess** \( T(n) \leq cn \log_2 n \), for some constant \( c \) to be determined later.

**Verify with induction:**

Step 1, base case: \( T(1) = 2 \leq cn \log_2 n = 0 \) does not hold for the guessed inequality.

Instead, we choose \( T(2) \) to be the base case. From the recurrence, we have

\[
T(2) = \frac{3}{2} T(\lfloor \frac{2 \times 2}{3} \rfloor) + 2 = 3 \frac{3}{2} T(1) + 2 = 5
\]

We make \( T(2) = 5 \leq cn \log_2 n \) holds for \( c = 3 \), and when \( n \geq k = 2 \),

Step 2, assumption: assume the guessed upper bound holds for \( \lfloor \frac{2n}{3} \rfloor \), i.e.,

\[
T(\lfloor \frac{2n}{3} \rfloor) \leq c \lfloor \frac{2n}{3} \rfloor \log_2 \lfloor \frac{2n}{3} \rfloor
\]
Chapter 4. Solving Recurrences

Solving recurrence with the substitution method (guess then verify)

\[ T(n) = \frac{3}{2} T\left(\left\lfloor \frac{2n}{3} \right\rfloor \right) + n, \text{ where } T(1) = 2 \]

Guess \( T(n) \leq cn \log_2 n \), for some constant \( c \) to be determined later.

Verify with induction:

Step 1, base case: \( T(1) = 2 \leq cn \log_2 n = 0 \) does not hold for the guessed inequality.

Instead, we choose \( T(2) \) to be the base case. From the recurrence, we have

\[ T(2) = \frac{3}{2} T\left(\left\lfloor \frac{2n}{3} \right\rfloor \right) + n = \frac{3}{2} T\left(\left\lfloor \frac{4}{3} \right\rfloor \right) + 2 = \frac{3}{2} T(1) + 2 = 5 \]

We make \( T(2) = 5 \leq cn \log_2 n \) holds for \( c = 3 \), and when \( n \geq k = 2 \),

Step 2, assumption: assume the guessed upper bound holds for \( \left\lfloor \frac{2n}{3} \right\rfloor \), i.e.,

\[ T\left(\left\lfloor \frac{2n}{3} \right\rfloor \right) \leq c\left(\frac{2n}{3}\right) \log_2 \left(\frac{2n}{3}\right) \]

Step 3, substitute it in the recurrence, we get

\[ T(n) = \frac{3}{2} T\left(\left\lfloor \frac{2n}{3} \right\rfloor \right) + n \]
Chapter 4. Solving Recurrences

Solving recurrence with the **substitution method** (guess then verify)

\[
T(n) = \frac{3}{2} T(\lfloor \frac{2n}{3} \rfloor) + n, \text{ where } T(1) = 2
\]

**Guess** \(T(n) \leq cn \log_2 n\), for some constant \(c\) to be determined later.

**Verify with induction:**

Step 1, base case: \(T(1) = 2 \leq cn \log_2 n = 0\) does not hold for the guessed inequality.

Instead, we choose \(T(2)\) to be the base case. From the recurrence, we have

\[
T(2) = \frac{3}{2} T(\lfloor \frac{2n}{3} \rfloor) + n = \frac{3}{2} T(\lfloor \frac{4}{3} \rfloor) + 2 = \frac{3}{2} T(1) + 2 = 5
\]

We make \(T(2) = 5 \leq cn \log_2 n\) holds for \(c = 3\), and when \(n \geq k = 2\),

Step 2, assumption: assume the guessed upper bound holds for \(\lfloor \frac{2n}{3} \rfloor\), i.e.,

\[
T(\lfloor \frac{2n}{3} \rfloor) \leq c \lfloor \frac{2n}{3} \rfloor \log_2 \lfloor \frac{2n}{3} \rfloor
\]

Step 3, substitute it in the recurrence, we get

\[
T(n) = \frac{3}{2} T(\lfloor \frac{2n}{3} \rfloor) + n \leq \frac{3}{2} (c \lfloor \frac{2n}{3} \rfloor \log_2 \lfloor \frac{2n}{3} \rfloor) + n
\]
Step 3, substitute it in the recurrence, we get

\[ T(n) = \frac{3}{2} T\left(\left\lfloor \frac{2n}{3} \right\rfloor \right) + n \]
Step 3, substitute it in the recurrence, we get

\[ T(n) = \frac{3}{2} T(\lfloor \frac{2n}{3} \rfloor) + n \leq \frac{3}{2} (c \lfloor \frac{2n}{3} \rfloor \log_2 \lfloor \frac{2n}{3} \rfloor) + n \]
Step 3, substitute it in the recurrence, we get

\[ T(n) = \frac{3}{2} T(\lfloor \frac{2n}{3} \rfloor) + n \leq \frac{3}{2} \left( c \lfloor \frac{2n}{3} \rfloor \log_2 \left( \frac{2n}{3} \right) \right) + n \]

\[ \leq \frac{3}{2} \left( c \frac{2n}{3} \log_2 \frac{2n}{3} \right) + n \]
Step 3, substitute it in the recurrence, we get

\[
T(n) = \frac{3}{2} T\left(\left\lfloor \frac{2n}{3} \right\rfloor \right) + n \leq \frac{3}{2} \left( c \left\lfloor \frac{2n}{3} \right\rfloor \log_2 \left\lfloor \frac{2n}{3} \right\rfloor \right) + n
\]

\[
\leq \frac{3}{2} \left( c \frac{2n}{3} \log_2 \frac{2n}{3} \right) + n \leq cn \left( \log_2 n + \log_2 \frac{2}{3} \right) + n
\]
Step 3, substitute it in the recurrence, we get

\[ T(n) = \frac{3}{2} T\left(\lfloor \frac{2n}{3} \rfloor \right) + n \leq \frac{3}{2} \left( c \lfloor \frac{2n}{3} \rfloor \log_2 \lfloor \frac{2n}{3} \rfloor \right) + n \]

\[ \leq \frac{3}{2} \left( c \frac{2n}{3} \log_2 \frac{2n}{3} \right) + n \leq cn(\log_2 n + \log_2 \frac{2}{3}) + n \]

\[ \leq cn(\log_2 n - \log_2 \frac{3}{2}) + n \]
Step 3, substitute it in the recurrence, we get

\[
T(n) = \frac{3}{2} T\left( \left\lfloor \frac{2n}{3} \right\rfloor \right) + n \leq \frac{3}{2} \left( c \left\lfloor \frac{2n}{3} \right\rfloor \log_2 \left\lfloor \frac{2n}{3} \right\rfloor \right) + n
\]

\[
\leq \frac{3}{2} \left( c \frac{2n}{3} \log_2 \frac{2n}{3} \right) + n \leq cn(\log_2 n + \log_2 \frac{2}{3}) + n
\]

\[
\leq cn(\log_2 n - \log_2 \frac{3}{2}) + n = cn \log_2 n - cn \log_2 \frac{3}{2} + n
\]
Step 3, substitute it in the recurrence, we get

\[ T(n) = \frac{3}{2} \, T(\lfloor \frac{2n}{3} \rfloor) + n \leq \frac{3}{2} \left( c \lfloor \frac{2n}{3} \rfloor \log_2 \lfloor \frac{2n}{3} \rfloor \right) + n \]

\[ \leq \frac{3}{2} \left( c \frac{2n}{3} \log_2 \frac{2n}{3} \right) + n \leq cn(\log_2 n + \log_2 \frac{2}{3}) + n \]

\[ \leq cn(\log_2 n – \log_2 \frac{3}{2}) + n = cn \log_2 n – cn \log_2 \frac{3}{2} + n \]

Since \( \log_2 \frac{3}{2} > 0.5 \),
Chapter 4. Solving Recurrences

Step 3, substitute it in the recurrence, we get

\[ T(n) = \frac{3}{2} T(\lfloor \frac{2n}{3} \rfloor) + n \leq \frac{3}{2} \left( c \lfloor \frac{2n}{3} \rfloor \log_2 \lfloor \frac{2n}{3} \rfloor \right) + n \]

\[ \leq \frac{3}{2} \left( c \frac{2n}{3} \log_2 \frac{2n}{3} \right) + n \leq cn(\log_2 n + \log_2 \frac{2}{3}) + n \]

\[ \leq cn(\log_2 n - \log_2 \frac{3}{2}) + n = cn \log_2 n - cn \log_2 \frac{3}{2} + n \]

Since \( \log_2 \frac{3}{2} > 0.5 \), \(-cn \log_2 \frac{3}{2} + n < 0 \) when \( c \geq 2 \).
Chapter 4. Solving Recurrences

Step 3, substitute it in the recurrence, we get

\[ T(n) = \frac{3}{2} T(\lfloor \frac{2n}{3} \rfloor) + n \leq \frac{3}{2} (c \lfloor \frac{2n}{3} \rfloor \log_2 \lfloor \frac{2n}{3} \rfloor) + n \]

\[ \leq \frac{3}{2} (c \frac{2n}{3} \log_2 \frac{2n}{3}) + n \leq cn(\log_2 n + \log_2 \frac{2}{3}) + n \]

\[ \leq cn(\log_2 n - \log_2 \frac{3}{2}) + n = cn \log_2 n - cn \log_2 \frac{3}{2} + n \]

Since \( \log_2 \frac{3}{2} > 0.5 \), \(-cn \log_2 \frac{3}{2} + n < 0 \) when \( c \geq 2 \).

But we already know \( c = 3 \) from the base case proof,
Step 3, substitute it in the recurrence, we get

\[ T(n) = \frac{3}{2} T(\lfloor \frac{2n}{3} \rfloor) + n \leq \frac{3}{2} (c \lfloor \frac{2n}{3} \rfloor \log_2 \lfloor \frac{2n}{3} \rfloor) + n \]

\[ \leq \frac{3}{2} (c \frac{2n}{3} \log_2 \frac{2n}{3}) + n \leq cn(\log_2 n + \log_2 \frac{2}{3}) + n \]

\[ \leq cn(\log_2 n - \log_2 \frac{3}{2}) + n = cn \log_2 n - cn \log_2 \frac{3}{2} + n \]

Since \( \log_2 \frac{3}{2} > 0.5 \), \(-cn \log_2 \frac{3}{2} + n < 0 \) when \( c \geq 2 \).

But we already know \( c = 3 \) from the base case proof, we have

\[ T(n) \leq cn \log_2 n - cn \log_2 \frac{3}{2} + n \]
Chapter 4. Solving Recurrences

Step 3, substitute it in the recurrence, we get

\[ T(n) = \frac{3}{2} T\left(\left\lfloor \frac{2n}{3} \right\rfloor \right) + n \leq \frac{3}{2} \left( c \left\lfloor \frac{2n}{3} \right\rfloor \log_2 \left\lfloor \frac{2n}{3} \right\rfloor \right) + n \]

\[ \leq \frac{3}{2} \left( c \frac{2n}{3} \log_2 \frac{2n}{3} \right) + n \leq cn \left( \log_2 n + \log_2 \frac{2}{3} \right) + n \]

\[ \leq cn \left( \log_2 n - \log_2 \frac{3}{2} \right) + n = cn \log_2 n - cn \log_2 \frac{3}{2} + n \]

Since \( \log_2 \frac{3}{2} > 0.5 \), \( -cn \log_2 \frac{3}{2} + n < 0 \) when \( c \geq 2 \).

But we already know \( c = 3 \) from the base case proof, we have

\[ T(n) \leq cn \log_2 n - cn \log_2 \frac{3}{2} + n \leq cn \log_2 n \]
Chapter 4. Solving Recurrences

2. Changing variables

Example:

\[ T(n) = 2T(\sqrt{n}) + \log_2 n \]

Define \( m = \log_2 n \), i.e., \( n = 2^m \)

\[ T(2^m) = 2T(2^{m/2}) + m \]

rename the function:

\[ S(m) = T(2^m) \]

\[ S(m) = 2S(m/2) + m \]

solve it, we have

\[ S(m) = O(m \log m) \]

so

\[ T(n) = T(2^m) = O(m \log m) = O(\log n \log \log n) \]
2. Changing variables

Example: \( T(n) = 2T(\sqrt{n}) + \log_2 n \)
Chapter 4. Solving Recurrences

2. Changing variables

Example: \( T(n) = 2T(\sqrt{n}) + \log_2 n \)

Define \( m = \log_2 n \),
2. Changing variables

Example: \( T(n) = 2T(\sqrt{n}) + \log_2 n \)

Define \( m = \log_2 n \), i.e., \( n = 2^m \)
2. Changing variables

Example: \( T(n) = 2T(\sqrt{n}) + \log_2 n \)

Define \( m = \log_2 n \), i.e., \( n = 2^m \)

\( T(2^m) = 2T(2^{m/2}) + m \)
2. Changing variables

Example: \( T(n) = 2T(\sqrt{n}) + \log_2 n \)

Define \( m = \log_2 n \), i.e., \( n = 2^m \)

\( T(2^m) = 2T(2^{m/2}) + m \)

rename the function: \( S(m) = T(2^m) \)
2. Changing variables

Example: \( T(n) = 2T(\sqrt{n}) + \log_2 n \)

Define \( m = \log_2 n \), i.e., \( n = 2^m \)

\( T(2^m) = 2T(2^{m/2}) + m \)

rename the function: \( S(m) = T(2^m) \)

\( S(m) = 2S(m/2) + m \)
2. Changing variables

Example: $T(n) = 2T(\sqrt{n}) + \log_2 n$

Define $m = \log_2 n$, i.e., $n = 2^m$

$T(2^m) = 2T(2^{m/2}) + m$

rename the function: $S(m) = T(2^m)$

$S(m) = 2S(m/2) + m$

solve it, we have $S(m) = O(m \log m)$
2. Changing variables

Example: \( T(n) = 2T(\sqrt{n}) + \log_2 n \)

Define \( m = \log_2 n \), i.e., \( n = 2^m \)

\( T(2^m) = 2T(2^{m/2}) + m \)

rename the function: \( S(m) = T(2^m) \)

\( S(m) = 2S(m/2) + m \)

solve it, we have \( S(m) = O(m \log m) \)

so \( T(n) = T(2^m) = O(m \log m) = O(\log n \log \log n) \).
Chapter 4. Solving Recurrences

3. Recursive tree method
Chapter 4. Solving Recurrences

3. Recursive tree method

By unfolding the recurrence to make a recursive-tree.
3. Recursive tree method

By unfolding the recurrence to make a recursive-tree.

(1) $T(n)$ is a tree with non-recursive terms as the root and recursive terms as its children.
Chapter 4. Solving Recurrences

3. Recursive tree method

By unfolding the recurrence to make a recursive-tree.

(1) $T(n)$ is a tree with non-recursive terms as the root and recursive terms as its children.

(2) for each child, replace it with then non-recursive terms and produce children that are then recursive terms.
3. Recursive tree method

By unfolding the recurrence to make a recursive-tree.

(1) $T(n)$ is a tree with non-recursive terms as the root and recursive terms as its children.

(2) for each child, replace it with then non-recursive terms and produce children that are then recursive terms

(3) repeat (2), expand the tree until all children are the base case.
Chapter 4. Solving Recurrences

Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$
Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$
Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

\begin{align*}
    l_0: & & T(n) \\
    l_1: & & T(n/4) \\
    & & T(n/4) \\
    & & T(n/4)
\end{align*}
Chapter 4. Solving Recurrences

Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

\[
\begin{align*}
 l_0: \\
 l_1: & \quad T(n/4) & T(n) & T(n/4) & T(n/4) & n^2
\end{align*}
\]
Chapter 4. Solving Recurrences

Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

$l_0$: $T(n)$ $n^2$

$l_1$: $T(n/4)$ $T(n/4)$ $T(n/4)$ $n^2$
Chapter 4. Solving Recurrences

Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

\[ l_0: \quad T(n) \]
\[ l_1: \quad T(n/4) \quad T(n/4) \quad T(n/4) \]
\[ l_2: \quad T(\frac{n}{4^2}) \quad T(\frac{n}{4^2}) \quad T(\frac{n}{4^2}) \quad T(\frac{n}{4^2}) \quad T(\frac{n}{4^2}) \quad T(\frac{n}{4^2}) \quad T(\frac{n}{4^2}) \quad n^2 \]
Chapter 4. Solving Recurrences

Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

$l_0$: 

$l_1$: 

$l_2$: $T(n/4)$ $T(n/4)$ $T(n/4)$ $T(n/4)$ $T(n/4)$ $T(n/4)$ $T(n/4)$ $T(n/4)$ $T(n/4)$ $T(n/4)$ $n^2$ $3(\frac{n}{4})^2$
Chapter 4. Solving Recurrences

Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

\[
\begin{align*}
  l_0: & & T(n) \\
  l_1: & & T(n/4) \\
  l_2: & & T(n/4) & T(n/4) & T(n/4) \\
  l_3: & & \ldots & \ldots & \ldots & \ldots & 3(n/4)^2
\end{align*}
\]
Chapter 4. Solving Recurrences

Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

$l_0$: $T(n)$

$l_1$: $T(n/4)$ $T(n/4)$

$l_2$: $T(n/4^2)$ $T(n/4^2)$ $T(n/4^2)$ $T(n/4^2)$ $T(n/4^2)$ $T(n/4^2)$ $T(n/4^2)$ $n^2$ $3(n/4)^2$ $3^2(n/4^2)^2$

$l_3$: .......
Chapter 4. Solving Recurrences

Example \( T(n) = 3T(n/4) + n^2 \), with base case \( T(1) = 1 \)

\[
\begin{align*}
&l_0: & T(n) \\
&l_1: & T(n/4) & T(n/4) \\
&l_2: & T(n/4) & T(n/4) & T(n/4) \\
&l_3: & \ldots \ldots \\
&l_4: & \ldots \ldots \\
\end{align*}
\]
Chapter 4. Solving Recurrences

Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

$l_0$: $T(n)$

$l_1$: $T(n/4)$ $T(n/4)$ $T(n/4)$ $T(n/4)$ $T(n/4)$ $T(n/4)$ $T(n/4)$ $n^2$

$l_2$: $T(n/4^2)$ $T(n/4^2)$ $T(n/4^2)$ $T(n/4^2)$ $T(n/4^2)$ $T(n/4^2)$ $T(n/4^2)$ $3(n/4)^2$

$l_3$: ......

$l_4$: ......
Chapter 4. Solving Recurrences

Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

$l_0$: 

$l_1$: $T(n/4)$ $T(n/4)$ $T(n)$

$l_2$: $T(n/4)$ $T(n/4)$ $T(n/4)$ $T(n/4)$ $T(n/4)$ $T(n/4)$

$l_3$: ......

$l_4$: ......

$l_5$: ......
Chapter 4. Solving Recurrences

Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

$l_0$: 
$l_1$: $T(n/4)$ $T(n/4)$ $T(n/4)$ $n^2$
$l_2$: $T(n/4^2)$ $T(n/4^2)$ $T(n/4^2)$ $T(n/4^2)$ $T(n/4^2)$ $T(n/4^2)$ $3(n/4)^2$
$l_3$: ....
$l_4$: ....
$l_5$: ....
$l_{m-1}$: ....
Chapter 4. Solving Recurrences

Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

$l_0$: 

$l_1$: 

$l_2$: $T(n/4) T(n/4) T(n/4)$ 

$l_3$: 

$l_4$: 

$l_5$: 

$l_m-1$: 

$l_m$:
Chapter 4. Solving Recurrences

Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

$l_0:\quad T(n)$

$l_1:\quad T(n/4)$

$l_2:\quad T(n/4)\ T(n/4)$

$l_3:\quad \dotso$

$l_4:\quad \dotso$

$l_5:\quad \dotso$

$l_{m-1}:\quad \dotso$

$l_m:\quad T(1), T(1), T(1), T(1), T(1), \dotso, T(1)$
Chapter 4. Solving Recurrences

Example  $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

$l_0$:  

$l_1$: 

$l_2$:  

$l_3$: 

$l_4$: 

$l_5$: 

$l_{m-1}$: 

$l_m$: $T(1), T(1), T(1), T(1), T(1), \ldots, T(1)$

where $\frac{n}{4^m} = 1$, i.e., $m = \log_4 n$. 

\[\begin{array}{cccc}
T(n) & T(n/4) & T(n/4) & n^2 \\
T(n/4) & T(n/4) & T(n/4) & 3\left(\frac{n}{4}\right)^2 \\
T(n/4) & T(n/4) & T(n/4) & \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \\
T(1) & T(1) & T(1) & \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \\
\end{array}\]
Chapter 4. Solving Recurrences

Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

<table>
<thead>
<tr>
<th>Level</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>$l_0$</td>
<td>$T(n)$</td>
</tr>
<tr>
<td>$l_1$</td>
<td>$T(n/4)$, $T(n/4)$, $T(n/4)$</td>
</tr>
<tr>
<td>$l_2$</td>
<td>$T(n/4)$, $T(n/4)$, $T(n/4)$, $T(n/4)$, $T(n/4)$, $T(n/4)$, $T(n/4)$, $n^2$</td>
</tr>
<tr>
<td>$l_3$</td>
<td>$\ldots$</td>
</tr>
<tr>
<td>$l_4$</td>
<td>$\ldots$</td>
</tr>
<tr>
<td>$l_5$</td>
<td>$\ldots$</td>
</tr>
<tr>
<td>$l_{m-1}$</td>
<td>$\ldots$</td>
</tr>
<tr>
<td>$l_m$</td>
<td>$T(1), T(1), T(1), T(1), T(1), \ldots, T(1)$</td>
</tr>
</tbody>
</table>

where $\frac{n}{4^m} = 1$, i.e., $m = \log_4 n$.

Then $T(n)$ is the sum

$$T(n) = n^2[1 + 3\left(\frac{1}{4}\right)^2 + 3^2\left(\frac{1}{4^2}\right)^2 + 3^3\left(\frac{1}{4^3}\right)^2 + \cdots + 3^{m-1}\left(\frac{1}{4^{m-1}}\right)^2] + 3^m T(1)$$
Chapter 4. Solving Recurrences

Example \( T(n) = 3T(n/4) + n^2 \), with base case \( T(1) = 1 \)

\[
\begin{align*}
l_0: & \quad T(n) \\
l_1: & \quad T(n/4) \\
l_2: & \quad T(n/4^2) T(n/4^2) \\
l_3: & \quad \ldots \\
l_4: & \quad \ldots \\
l_5: & \quad \ldots \\
l_{m-1}: & \quad \ldots \\
l_m: & \quad T(1), T(1), T(1), T(1), T(1), \ldots, T(1)
\end{align*}
\]

where \( \frac{n}{4^m} = 1 \), i.e., \( m = \log_4 n \).

Then \( T(n) \) is the sum

\[
T(n) = n^2[1 + 3\left(\frac{1}{4}\right)^2 + 3^2\left(\frac{1}{4^2}\right)^2 + 3^3\left(\frac{1}{4^3}\right)^2 + \cdots + 3^{m-1}\left(\frac{1}{4^{m-1}}\right)^2] + 3^m T(1)
\]

\[
T(n) = n^2[1 + 3\left(\frac{1}{4}\right)^2 + 3^2\left(\frac{1}{4^2}\right)^2 + 3^3\left(\frac{1}{4^3}\right)^2 + \cdots + 3^{m-1}\left(\frac{1}{4^{m-1}}\right)^2] + 3^m \times 1
\]
Chapter 4. Solving Recurrences

Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

\[
\begin{align*}
l_0: & & T(n) & \\
l_1: & & T(n/4) & & T(n/4) & \\
l_2: & & T(n/4^2) & T(n/4^2) & T(n/4^2) & T(n/4^2) & T(n/4^2) & \cdots & n^2 \\
l_3: & & \cdots & & \cdots & & \cdots & & \cdots & & \cdots \\
l_4: & & \cdots & & \cdots & & \cdots & & \cdots & & \cdots \\
l_5: & & \cdots & & \cdots & & \cdots & & \cdots & & \cdots \\
l_{m-1}: & & \cdots & & \cdots & & \cdots & & \cdots & & \cdots \\
l_m: & & T(1), T(1), T(1), T(1), T(1), \ldots, T(1)
\end{align*}
\]

where $\frac{n}{4^m} = 1$, i.e., $m = \log_4 n$.

Then $T(n)$ is the sum

\[
T(n) = n^2[1 + 3\left(\frac{1}{4}\right)^2 + 3^2\left(\frac{1}{4^2}\right)^2 + 3^3\left(\frac{1}{4^3}\right)^2 + \cdots + 3^{m-1}\left(\frac{1}{4^{m-1}}\right)^2] + 3^m T(1) = n^2[1 + 3\left(\frac{1}{4}\right)^2 + 3^2\left(\frac{1}{4^2}\right)^2 + 3^3\left(\frac{1}{4^3}\right)^2 + \cdots + 3^{m-1}\left(\frac{1}{4^{m-1}}\right)^2] + 3^m \times 1 = n^2[1 + 3\left(\frac{1}{4}\right)^2 + 3^2\left(\frac{1}{4^2}\right)^2 + 3^3\left(\frac{1}{4^3}\right)^2 + \cdots + 3^{m-1}\left(\frac{1}{4^{m-1}}\right)^2] + 3^m \left(\frac{n}{4^m}\right)^2
\]
Chapter 4. Solving Recurrences

Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

\[
\begin{align*}
&l_0: \quad T(n) \\
&l_1: \quad T(n/4) \\
&l_2: \quad T(n/4) T(n/4) T(n/4) \\
&l_3: \quad \ldots \ldots \\
&l_4: \quad \ldots \ldots \\
&l_5: \quad \ldots \ldots \\
&l_{m-1}: \quad \ldots \ldots \\
&l_m: \quad T(1), T(1), T(1), T(1), T(1), \ldots, T(1)
\end{align*}
\]

where $\frac{n}{4^m} = 1$, i.e., $m = \log_4 n$.

Then $T(n)$ is the sum

\[
T(n) = n^2[1 + 3(\frac{1}{4})^2 + 3^2(\frac{1}{4^2})^2 + 3^3(\frac{1}{4^3})^2 + \ldots + 3^{m-1}(\frac{1}{4^{m-1}})^2] + 3^m T(1)
\]

\[
T(n) = n^2[1 + 3(\frac{1}{4})^2 + 3^2(\frac{1}{4^2})^2 + 3^3(\frac{1}{4^3})^2 + \ldots + 3^{m-1}(\frac{1}{4^{m-1}})^2] + 3^m \times 1
\]

\[
= n^2[1 + 3(\frac{1}{4})^2 + 3^2(\frac{1}{4^2})^2 + 3^3(\frac{1}{4^3})^2 + \ldots + 3^{m-1}(\frac{1}{4^{m-1}})^2] + 3^m (\frac{n}{4^m})^2
\]

\[
= n^2[1 + \frac{3}{16} + (\frac{3}{16})^2 + (\frac{3}{16})^3 + \ldots + (\frac{3}{16})^m]
\]
Chapter 4. Solving Recurrences

Example \( T(n) = 3T(n/4) + n^2 \), with base case \( T(1) = 1 \)

\[
\begin{align*}
\text{l}_0: & \quad T(n) \\
\text{l}_1: & \quad T(n/4) \quad T(n/4) \\
\text{l}_2: & \quad T(n/4^2) T(n/4^2) T(n/4^2) T(n/4^2) T(n/4^2) T(n/4^2) T(n/4^2) T(n/4^2) n^2 \\
\text{l}_3: & \quad \ddots \\
\text{l}_4: & \quad \ddots \\
\text{l}_5: & \quad \ddots \\
\text{l}_{m-1}: & \quad \ddots \\
\text{l}_m: & \quad T(1), T(1), T(1), T(1), T(1), \ldots, T(1)
\end{align*}
\]

where \( \frac{n}{4^m} = 1 \), i.e., \( m = \log_4 n \).

Then \( T(n) \) is the sum

\[
\begin{align*}
T(n) &= n^2[1 + 3\left(\frac{1}{4}\right)^2 + 3^2\left(\frac{1}{4^2}\right)^2 + 3^3\left(\frac{1}{4^3}\right)^2 + \cdots + 3^{m-1}\left(\frac{1}{4^{m-1}}\right)^2] + 3^m T(1) \\
T(n) &= n^2[1 + 3\left(\frac{1}{4}\right)^2 + 3^2\left(\frac{1}{4^2}\right)^2 + 3^3\left(\frac{1}{4^3}\right)^2 + \cdots + 3^{m-1}\left(\frac{1}{4^{m-1}}\right)^2] + 3^m \times 1 \\
&= n^2[1 + 3\left(\frac{1}{4}\right)^2 + 3^2\left(\frac{1}{4^2}\right)^2 + 3^3\left(\frac{1}{4^3}\right)^2 + \cdots + 3^{m-1}\left(\frac{1}{4^{m-1}}\right)^2] + 3^m \left(\frac{n}{4^m}\right)^2 \\
&= n^2[1 + \frac{3}{16} + \left(\frac{3}{16}\right)^2 + \left(\frac{3}{16}\right)^3 + \cdots + \left(\frac{3}{16}\right)^m] \\
&= n^2\left(\frac{1 - \left(\frac{3}{16}\right)^{m+1}}{1 - \frac{3}{16}}\right)
\end{align*}
\]
Chapter 4. Solving Recurrences

Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

\[
\begin{align*}
\text{l}_0: & & T(n) \\
\text{l}_1: & & T(n/4) \\
\text{l}_2: & & T(n/4) T(n/4) T(n/4) \\
\text{l}_3: & & \ldots \ldots \\
\text{l}_4: & & \ldots \ldots \\
\text{l}_5: & & \ldots \ldots \\
\text{l}_{m-1}: & & \ldots \ldots \\
\text{l}_m: & & T(1), T(1), T(1), T(1), T(1), \ldots, T(1)
\end{align*}
\]

where $\frac{n}{4^m} = 1$, i.e., $m = \log_4 n$.

Then $T(n)$ is the sum

\[
T(n) = n^2[1 + 3 \left(\frac{1}{4}\right)^2 + 3^2 \left(\frac{1}{4}\right)^2 + 3^3 \left(\frac{1}{4}\right)^2 + \cdots + 3^{m-1} \left(\frac{1}{4^{m-1}}\right)^2] + 3^m T(1)
\]

\[
= n^2[1 + 3 \left(\frac{1}{4}\right)^2 + 3^2 \left(\frac{1}{4}\right)^2 + 3^3 \left(\frac{1}{4}\right)^2 + \cdots + 3^{m-1} \left(\frac{1}{4^{m-1}}\right)^2] + 3^m \times 1
\]

\[
= n^2[1 + 3 \left(\frac{1}{4}\right)^2 + 3^2 \left(\frac{1}{4}\right)^2 + 3^3 \left(\frac{1}{4}\right)^2 + \cdots + 3^{m-1} \left(\frac{1}{4^{m-1}}\right)^2] + 3^m \left(\frac{n}{4^m}\right)^2
\]

\[
= n^2[1 + \frac{3}{16} + \left(\frac{3}{16}\right)^2 + \left(\frac{3}{16}\right)^3 + \cdots + \left(\frac{3}{16}\right)^m] \\
= n^2 \left[\frac{1 - \left(\frac{3}{16}\right)^{m+1}}{1 - \frac{3}{16}}\right] \\
\leq n^2 \left(\frac{1}{1 - \frac{3}{16}}\right)
\]
Chapter 4. Solving Recurrences

Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

\[ l_0: \quad T(n) \\
 l_1: \quad T(n/4) \quad T(n/4) \\
 l_2: \quad T(n/4^2) \quad T(n/4^2) \quad T(n/4^2) \quad T(n/4^2) \quad T(n/4^2) \quad T(n/4^2) \quad n^2 \\
 l_3: \quad \ldots \\
 l_4: \quad \ldots \\
 l_5: \quad \ldots \\
 l_{m-1}: \quad \ldots \\
 l_m: \quad T(1), T(1), T(1), T(1), T(1), \ldots, T(1) \\
\]

where $\frac{n}{4^m} = 1$, i.e., $m = \log_4 n$.

Then $T(n)$ is the sum

\[
T(n) = n^2[1 + 3\left(\frac{1}{4}\right)^2 + 3^2\left(\frac{1}{4^2}\right)^2 + 3^3\left(\frac{1}{4^3}\right)^2 + \cdots + 3^{m-1}\left(\frac{1}{4^{m-1}}\right)^2] + 3mT(1) \\
T(n) = n^2[1 + 3\left(\frac{1}{4}\right)^2 + 3^2\left(\frac{1}{4^2}\right)^2 + 3^3\left(\frac{1}{4^3}\right)^2 + \cdots + 3^{m-1}\left(\frac{1}{4^{m-1}}\right)^2] + 3m \times 1 \\
= n^2[1 + 3\left(\frac{1}{4}\right)^2 + 3^2\left(\frac{1}{4^2}\right)^2 + 3^3\left(\frac{1}{4^3}\right)^2 + \cdots + 3^{m-1}\left(\frac{1}{4^{m-1}}\right)^2] + 3^m \left(\frac{n}{4^m}\right)^2 \\
= n^2[1 + \frac{3}{16} + \left(\frac{3}{16}\right)^2 + \left(\frac{3}{16}\right)^3 + \cdots + \left(\frac{3}{16}\right)^m] \\
= n^2\left(\frac{\frac{1}{1-\frac{3}{16}}^{m+1}}{1-\frac{3}{16}}\right) \\
\leq n^2\left(\frac{1}{1-\frac{3}{16}}\right) \\
= \frac{16}{13} n^2
\]

for all $n > 0$. 
Chapter 4. Solving Recurrences

Recursive tree notation in the textbook:

\[ T(n) = 3T(n/4) + n^2, \text{ with base case } T(1) = 1 \]

\( l_0: \)

\[ n^2 \]

\( l_m: \ldots \]

\( l_{m-1}: \ldots \]

\( l_1: \ldots \]

\( l_0: \ldots \]

\( l_{m-1}: T(1), T(1), T(1), T(1), T(1), \ldots, 3m \text{ nodes of } T(1) \)
Chapter 4. Solving Recurrences

Recursive tree notation in the textbook:

\[ T(n) = 3T(n/4) + n^2, \text{ with base case } T(1) = 1 \]

\[ l_0: \quad n^2 \]
\[ l_1: \quad (n/4)^2 \quad (n/4)^2 \quad (n/4)^2 \]
Chapter 4. Solving Recurrences

Recursive tree notation in the textbook:

\[ T(n) = 3T(n/4) + n^2, \text{ with base case } T(1) = 1 \]

\[ l_0: \]

\[ l_1: \]

\[ l_2: \]

\[ l_m: \text{... } \]

\[ l_m - 1: \text{... } \]

\[ l_m: T(1), T(1), T(1), T(1), T(1), \ldots, T(1) \]
Chapter 4. Solving Recurrences

Recursive tree notation in the textbook:

\[ T(n) = 3T(n/4) + n^2, \text{ with base case } T(1) = 1 \]

\[ l_0: \]
\[ l_1: \quad (n/4)^2 \quad (n/4)^2 \quad (n/4)^2 \quad (n/4)^2 \quad (n/4)^2 \quad (n/4)^2 \]
\[ l_2: \quad (n/4^2)^2 \quad (n/4^2)^2 \quad (n/4^2)^2 \quad (n/4^2)^2 \quad (n/4^2)^2 \quad (n/4^2)^2 \]
\[ l_3: \ldots \ldots \]
Chapter 4. Solving Recurrences

Recursive tree notation in the textbook:

\[ T(n) = 3T(n/4) + n^2, \text{ with base case } T(1) = 1 \]

\begin{align*}
  l_0: & \quad n^2 \\
  l_1: & \quad \left(\frac{n}{4}\right)^2 \left(\frac{n}{4}\right)^2 \left(\frac{n}{4}\right)^2 \quad \left(\frac{n}{4}\right)^2 \left(\frac{n}{4}\right)^2 \left(\frac{n}{4}\right)^2 \\
  l_2: & \quad \left(\frac{n}{4^2}\right)^2 \left(\frac{n}{4^2}\right)^2 \left(\frac{n}{4^2}\right)^2 \quad \left(\frac{n}{4^2}\right)^2 \left(\frac{n}{4^2}\right)^2 \left(\frac{n}{4^2}\right)^2 \\
  l_3: & \quad \ldots \ldots \\
  l_4: & \quad \ldots \ldots \\
\end{align*}
Chapter 4. Solving Recurrences

Recursive tree notation in the textbook:

\[ T(n) = 3T(n/4) + n^2, \text{ with base case } T(1) = 1 \]

\[
\begin{align*}
l_0: & & n^2 \\
l_1: & & (n/4)^2 \quad (n/4)^2 \\
l_2: & & (n/4)^2 \quad (n/4)^2 \quad (n/4)^2 \quad (n/4)^2 \\
l_3: & & \ldots \ldots \quad \ldots \ldots \quad \ldots \ldots \quad \ldots \ldots \quad \ldots \ldots \quad \ldots \ldots \quad \ldots \ldots \quad \ldots \ldots \quad \ldots \ldots \\
l_4: & & \ldots \ldots \quad \ldots \ldots \quad \ldots \ldots \quad \ldots \ldots \quad \ldots \ldots \quad \ldots \ldots \quad \ldots \ldots \quad \ldots \ldots \quad \ldots \ldots \\
l_{m-1}: & & \ldots \ldots \quad \ldots \ldots \quad \ldots \ldots \quad \ldots \ldots \quad \ldots \ldots \quad \ldots \ldots \quad \ldots \ldots \quad \ldots \ldots \quad \ldots \ldots \\
\end{align*}
\]

\[ 3^m \text{ nodes of } \left(\frac{n}{4^3}\right)^2 \]
Chapter 4. Solving Recurrences

Recursive tree notation in the textbook:

\[ T(n) = 3T(n/4) + n^2, \text{ with base case } T(1) = 1 \]

\[ l_0: \quad n^2 \]
\[ l_1: \quad \frac{n}{4^2} \quad \frac{(n/4)^2}{4} \quad \frac{(n/4)^2}{4} \quad \frac{(n/4)^2}{4} \]
\[ l_2: \quad \left(\frac{n}{4^2}\right)^2 \quad \left(\frac{n}{4^2}\right)^2 \quad \left(\frac{n}{4^2}\right)^2 \quad \left(\frac{n}{4^2}\right)^2 \quad \left(\frac{n}{4^2}\right)^2 \quad \left(\frac{n}{4^2}\right)^2 \quad \left(\frac{n}{4^2}\right)^2 \quad \left(\frac{n}{4^2}\right)^2 \quad \left(\frac{n}{4^2}\right)^2 \]
\[ l_3: \quad \ldots \quad 3^3 \text{ nodes of } \left(\frac{n}{4^3}\right)^2 \]
\[ l_4: \quad \ldots \quad 3^{m-1} \text{ of } \left(\frac{n}{4^{m-1}}\right)^2 \]
\[ l_{m-1}: \quad \ldots \]
Chapter 4. Solving Recurrences

Recursive tree notation in the textbook:

\[ T(n) = 3T(n/4) + n^2, \text{ with base case } T(1) = 1 \]

- \( l_0: \) \( n^2 \)
- \( l_1: \) \( (n/4)^2 \) \( (n/4)^2 \) \( (n/4)^2 \) \( (n/4)^2 \)
- \( l_2: \) \( (n/4^2)^2 \) \( (n/4^2)^2 \) \( (n/4^2)^2 \) \( (n/4^2)^2 \) \( (n/4^2)^2 \) \( (n/4^2)^2 \)
- \( l_3: \ldots \ldots \)
- \( l_4: \ldots \ldots \)
- \( l_{m-1}: \ldots \ldots \)
- \( l_m: T(1), T(1), T(1), T(1), T(1), \ldots, \)

3\(^m\) nodes of \( T(1) \)

3\(^{m-1}\) of \( (n/4^{m-1})^2 \)

3\(^3\) nodes of \( (n/4^3)^2 \)
Recursive tree notation in the textbook:

\[ T(n) = 3T(n/4) + n^2, \text{ with base case } T(1) = 1 \]

\[ \begin{align*}
    l_0: & \quad n^2 \\
    l_1: & \quad (n/4)^2, (n/4)^2, (n/4)^2 \\
    l_2: & \quad \left(\frac{n}{4^2}\right)^2, \left(\frac{n}{4^2}\right)^2, \left(\frac{n}{4^2}\right)^2, \left(\frac{n}{4^2}\right)^2 \\
    l_3: & \quad \ldots \\
    l_4: & \quad \ldots \\
    l_{m-1}: & \quad \ldots \\
    l_m: & \quad T(1), T(1), T(1), T(1), T(1), \ldots,
\end{align*} \]

\[ \begin{align*}
    & 3^3 \text{ nodes of } \left(\frac{n}{4^3}\right)^2 \\
    & 3^{m-1} \text{ of } \left(\frac{n}{4^{m-1}}\right)^2 \\
    & 3^m \text{ nodes of } T(1)
\end{align*} \]
Chapter 5. Probabilistic Analysis of Algorithms

Chapter 5. Probabilistic analysis and randomized algorithms
Chapter 5. Probabilistic Analysis of Algorithms

Chapter 5. Probabilistic analysis and randomized algorithms

- Estimate efficiency of algorithms on a majority of inputs, not all inputs;
Chapter 5. Probabilistic Analysis of Algorithms

Chapter 5. Probabilistic analysis and randomized algorithms

• Estimate efficiency of algorithms on a majority of inputs, not all inputs;
• Performance is “average” cases, not the worst case;
Chapter 5. Probabilistic Analysis of Algorithms

Chapter 5. Probabilistic analysis and randomized algorithms

- Estimate efficiency of algorithms on a majority of inputs, not all inputs;
- Performance is “average” cases, not the worst case;
- With assumption that input data are in a probabilistic distribution
Chapter 5. Probabilistic Analysis of Algorithms

Chapter 5. Probabilistic analysis and randomized algorithms

• Estimate efficiency of algorithms on a majority of inputs, not all inputs;
• Performance is “average” cases, not the worst case;
• With assumption that input data are in a probabilistic distribution
• Close relationship with randomized algorithms
A good example: **QUICK SORT** algorithm [Hoare’1959]
A good example: Quick Sort algorithm [Hoare’1959]

- It has the worst case time $T_{wc}(n) \geq an^2$ for some constant $a > 0$. 
A good example: **Quick Sort** algorithm [Hoare’1959]

- It has the worst case time $T_{wc}(n) \geq an^2$ for some constant $a > 0$.
- It has the average case time $T_{ac}(n) \leq bn \log_2 n$ for a constant $b > 0$. 
A good example: **Quick Sort** algorithm [Hoare’1959]

- It has the worst case time $T_{wc}(n) \geq an^2$ for some constant $a > 0$.
- It has the average case time $T_{ac}(n) \leq bn \log_2 n$ for a constant $b > 0$. In other word, it is efficient in most cases;
A good example: **Quick Sort** algorithm [Hoare’1959]

- It has the worst case time \( T_{wc}(n) \geq an^2 \) for some constant \( a > 0 \).
- It has the average case time \( T_{ac}(n) \leq bn \log_2 n \) for a constant \( b > 0 \).

In other word, it is efficient in most cases;
assumption: the input data are of the uniform distribution.

Alternatively, we can enforce the desired distribution by using randomness (tossing coins) in the algorithms.
Chapter 5. Probabilistic Analysis of Algorithms

A good example: Quick Sort algorithm [Hoare’1959]

- It has the worst case time $T_{wc}(n) \geq an^2$ for some constant $a > 0$.
- It has the average case time $T_{ac}(n) \leq bn \log_2 n$ for a constant $b > 0$.

  in other word, it is efficient in most cases;
  assumption: the input data are of the uniform distribution.

- But usually the input data do not satisfy uniform distribution.
Chapter 5. Probabilistic Analysis of Algorithms

A good example: **Quick Sort** algorithm [Hoare’1959]

- It has the worst case time $T_{wc}(n) \geq an^2$ for some constant $a > 0$.
- It has the average case time $T_{ac}(n) \leq bn \log_2 n$ for a constant $b > 0$.

  In other word, it is efficient in most cases; assumption: the input data are of the uniform distribution.

- But usually the input data do not satisfy uniform distribution.

  Alternatively, we can enforce the desired distribution by using randomness (tossing coins) in the algorithms.
Chapter 5. Probabilistic Analysis of Algorithms

A good example: Quick Sort algorithm [Hoare’1959]

- It has the worst case time \( T_{wc}(n) \geq an^2 \) for some constant \( a > 0 \).
- It has the average case time \( T_{ac}(n) \leq bn\log_2 n \) for a constant \( b > 0 \).

In other word, it is efficient in most cases; assumption: the input data are of the uniform distribution.

- But usually the input data do not satisfy uniform distribution.

Alternatively, we can enforce the desired distribution by using randomness (tossing coins) in the algorithms.

That is, we use randomized algorithms.
Chapter 5. Probabilistic Analysis of Algorithms

In both probabilistic analysis of deterministic algorithms and analysis of randomized algorithms
- actions in the algorithm are considered random events
In both probabilistic analysis of deterministic algorithms and analysis of randomized algorithms:

- actions in the algorithm are considered random events;
- such random events are driven by random data.

You can compute the expected time (i.e., averaged time).
Chapter 5. Probabilistic Analysis of Algorithms

In both probabilistic analysis of deterministic algorithms and analysis of randomized algorithms

- actions in the algorithm are considered random events
- such random events are driven by random data which are either input data or randomly tossed coins
In both probabilistic analysis of deterministic algorithms and analysis of randomized algorithms

- actions in the algorithm are considered random events
- such random events are driven by random data which are either input data or randomly tossed coins
- So the running time come with a probability
In both probabilistic analysis of deterministic algorithms and analysis of randomized algorithms

- actions in the algorithm are considered random events
- such random events are driven by random data which are either input data or randomly tossed coins
- So the running time come with a probability you can compute the expected time (i.e., averaged time)
Chapter 5. Probabilistic Analysis of Algorithms

In both probabilistic analysis of deterministic algorithms and analysis of randomized algorithms

- actions in the algorithm are considered random events
- such random events are driven by random data which are either input data or randomly tossed coins
- So the running time come with a probability you can compute the expected time (i.e., averaged time)
There are two types of randomized algorithms:

- Las Vegas algorithms

Accuracy 75% can be improved to 99.99% with multiple trials.

Las Vegas algorithms is as powerful as Monte Carlo algorithms, if not more.
Chapter 5. Probabilistic Analysis of Algorithms

There are two types of randomized algorithms:

- Las Vegas algorithms
  - always gives answer correctly;

- Monte Carlo algorithms
  - on 'NO' instances, 100% accuracy; \( \text{Prob}(\text{to answer 'NO' on 'NO' instance}) = 1 \)
  - on 'YES' instances, \( \geq 75\% \) accuracy; \( \text{Prob}(\text{to answer 'YES' on 'YES' instance}) \geq 0.75 \)

Accuracy 75% can be improved to 99.99% with multiple trials.

Las Vegas algorithms is as powerful as Monte Carlo algorithms, if not more.
Chapter 5. Probabilistic Analysis of Algorithms

There are two types of randomized algorithms:

- Las Vegas algorithms
  - always gives answer correctly;
  - running time comes with a probability distribution

- Monte Carlo algorithms
  - on 'NO' instances, $100\%$ accuracy; $\text{Prob}(\text{to answer 'NO' on 'NO' instance}) = 1$
  - on 'YES' instances, $\geq 75\%$ accuracy; $\text{Prob}(\text{to answer 'YES' on 'YES' instance}) \geq 0.75$

Accuracy $75\%$ can be improved to $99.99\%$ with multiple trials.

Las Vegas algorithms is as powerful as Monte Carlo algorithms, if not more.
Chapter 5. Probabilistic Analysis of Algorithms

There are two types of randomized algorithms:

- **Las Vegas algorithms**
  - always gives answer correctly;
  - running time comes with a probability distribution
  e.g., for QUICKSORT, \( \text{Prob}(T(n) \leq cn \log n) \geq 0.75 \)

- **Monte Carlo algorithms**
  - on ‘NO’ instances, 100\% accuracy;
  \( \text{Prob}(\text{to answer ‘NO’ on ‘NO’ instance}) = 1 \)
  - on ‘YES’ instances, \( \geq 75\% \) accuracy;
  \( \text{Prob}(\text{to answer ‘YES’ on ‘YES’ instance}) \geq 0.75 \)

Accuracy 75\% can be improved to 99.99\% with multiple trials.

Las Vegas algorithms is as powerful as Monte Carlo algorithms, if not more.
Chapter 5. Probabilistic Analysis of Algorithms

There are two types of randomized algorithms:

- **Las Vegas algorithms**
  - always gives answer correctly;
  - running time comes with a probability distribution
    e.g., for QUICKSORT, $\text{Prob}(T(n) \leq cn \log n) \geq 0.75$

- **Monte Carlo algorithms**
  - on 'NO' instances, 100% accuracy; $\text{Prob}(\text{to answer 'NO' on 'NO' instance}) = 1$
  - on 'YES' instances, $\geq 75\%$ accuracy; $\text{Prob}(\text{to answer 'YES' on 'YES' instance}) \geq 0.75$

Accuracy 75% can be improved to 99.99% with multiple trials.

- Las Vegas algorithms is as powerful as Monte Carlo algorithms, if not more.
Chapter 5. Probabilistic Analysis of Algorithms

There are two types of randomized algorithms:

- **Las Vegas algorithms**
  - always gives answer correctly;
  - running time comes with a probability distribution
    e.g., for QUICKSORT, \( \text{Prob}(T(n) \leq cn \log n) \geq 0.75 \)

- **Monte Carlo algorithms**
  - on 'NO' instances, 100% accuracy;
  - on 'YES' instances, \( \geq 75\% \) accuracy;
    \( \text{Prob}(\text{to answer 'YES' on 'YES' instance}) \geq 0.75 \)

Accuracy 75% can be improved to 99.9% with multiple trials.

- Las Vegas algorithms is as powerful as Monte Carlo algorithms, if not more.
Chapter 5. Probabilistic Analysis of Algorithms

There are two types of randomized algorithms:

- **Las Vegas algorithms**
  - always gives answer correctly;
  - running time comes with a probability distribution
    e.g., for QUICKSORT, $\text{Prob}(T(n) \leq cn \log n) \geq 0.75$

- **Monte Carlo algorithms**
  - on 'NO' instances, 100% accuracy;
    $\text{Prob}(\text{to answer 'NO' on 'NO' instance}) = 1$

  Accuracy 75% can be improved to 99.9% with multiple trials.

- Las Vegas algorithms is as powerful as Monte Carlo algorithms, if not more.
There are two types of randomized algorithms:

- **Las Vegas algorithms**
  - always gives answer correctly;
  - running time comes with a probability distribution
e.g., for **QUICKSORT**, \( \text{Prob}(T(n) \leq cn \log n) \geq 0.75 \)

- **Monte Carlo algorithms**
  - on 'NO' instances, 100% accuracy;
    \( \text{Prob}(\text{to answer 'NO' on 'NO' instance}) = 1 \)
  - on 'YES' instances, \( \geq 75\% \) accuracy;
Chapter 5. Probabilistic Analysis of Algorithms

There are two types of randomized algorithms:

- **Las Vegas algorithms**
  - always gives answer correctly;
  - running time comes with a probability distribution
    
    e.g., for **QUICKSORT**, \( \text{Prob}(T(n) \leq cn \log n) \geq 0.75 \)

- **Monte Carlo algorithms**
  - on 'NO' instances, 100% accuracy;
    \( \text{Prob}(\text{to answer 'NO' on 'NO' instance}) = 1 \)
  - on 'YES' instances, \( \geq 75\% \) accuracy;
    \( \text{Prob}(\text{to answer 'YES' on 'YES' instance}) \geq 0.75 \)

Accuracy 75% can be improved to 99.99% with multiple trials.

Las Vegas algorithms is as powerful as Monte Carlo algorithms, if not more.
There are two types of randomized algorithms:

- **Las Vegas algorithms**
  - always gives answer correctly;
  - running time comes with a probability distribution
    e.g., for QUICKSORT, $\text{Prob}(T(n) \leq cn \log n) \geq 0.75$

- **Monte Carlo algorithms**
  - on 'NO' instances, 100% accuracy;
    $\text{Prob}(\text{to answer 'NO' on 'NO' instance}) = 1$
  - on 'YES' instances, $\geq 75\%$ accuracy;
    $\text{Prob}(\text{to answer 'YES' on 'YES' instance}) \geq 0.75$

Accuracy 75% can be improved to 99.99% with multiple trials.

Las Vegas algorithms is as powerful as Monte Carlo algorithms, if not more.
There are two types of randomized algorithms:

- **Las Vegas algorithms**
  - always gives answer correctly;
  - running time comes with a probability distribution
    e.g., for QUICKSORT, \( \text{Prob}(T(n) \leq cn \log n) \geq 0.75 \)

- **Monte Carlo algorithms**
  - on 'NO' instances, 100% accuracy;
    \( \text{Prob}(\text{to answer 'NO' on 'NO' instance}) = 1 \)
  - on 'YES' instances, \( \geq 75\% \) accuracy;
    \( \text{Prob}(\text{to answer 'YES' on 'YES' instance}) \geq 0.75 \)

  Accuracy 75% can be improved to 99.99% with multiple trials.

- **Las Vegas algorithms** is as powerful as Monte Carlo algorithms, if not more.