Part II Sorting and Order Statistics
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- Chapter 6. Heapsort, the use of priority queue
- Chapter 7. Quicksort, probabilistic analysis, randomized algorithms
- Chapter 8. Sorting in linear time, lower bounds
- Chapter 9. Medians and order statistics
Chapter 6. Heapsort

Chapter 6. Heapsort and the use of priority queue
Chapter 6. Heapsort

Chapter 6. Heapsort and the use of priority queue
Chapter 6. Heapsort

Chapter 6. Heapsort and the use of priority queue

- key(parent) ≥ key(leftChild), key(rightChild);

![Diagram of a binary heap with key values and parent-child relationships.]
Chapter 6. Heapsort

Chapter 6. Heapsort and the use of priority queue

- \( \text{key(parent)} \geq \text{key(leftChild)}, \text{key(rightChild)} \);
- relationships are modeled with a complete binary tree.
Chapter 6. Heapsort and the use of priority queue

- key(parent) ≥ key(leftChild), key(rightChild);
- relationships are modeled with a complete binary tree
- can be stored in arrays (indexes begin with 0),
Chapter 6. Heapsort

Chapter 6. Heapsort and the use of priority queue

• key(parent) ≥ key(leftChild), key(rightChild);
• relationships are modeled with a complete binary tree
• can be stored in arrays (indexes begin with 0),
  index(leftChild) = 2 × index(parent) + 1
Chapter 6. Heapsort and the use of priority queue

- key(parent) ≥ key(leftChild), key(rightChild);
- relationships are modeled with a complete binary tree
- can be stored in arrays (indexes begin with 0),
  index(leftChild) = 2 × index(parent) + 1
  index(rightChild) = 2 × index(parent) + 2
The heap sort algorithm consists of subroutines:
Chapter 6. Heapsort

The heap sort algorithm consists of subroutines:

- **Build-Max-Heap** \((A)\)
The heap sort algorithm consists of subroutines:

- **Build-Max-Heap**$(A)$
- **Max-Heapify**$(A, i)$
Chapter 6. Heapsort

The heap sort algorithm consists of subroutines:

- **Build-Max-Heap**\( (A) \)
- **Max-Heapify**\( (A, i) \)
- **HeapSort**\( (A) \)
Chapter 6. Heapsort

The heap sort algorithm consists of subroutines:

- **Build-Max-Heap** \((A)\)
- **Max-Heapify** \((A, i)\)
- **HeapSort** \((A)\)

heaps as priority queues
Chapter 6. Heapsort

The heap sort algorithm consists of subroutines:

- `Build-Max-Heap(A)`
- `Max-Heapify(A, i)`
- `HeapSort(A)`

Heaps as priority queues

- `Heap-Maximum(A)`
Chapter 6. Heapsort

The heap sort algorithm consists of subroutines:

- **Build-Max-Heap** \((A)\)
- **Max-Heapify** \((A, i)\)
- **HeapSort** \((A)\)

heaps as priority queues

- **Heap-Maximum** \((A)\)
- **Heap-Extract-Max** \((A)\)
The heap sort algorithm consists of subroutines:

- \texttt{Build-Max-Heap}(A)
- \texttt{Max-Heapify}(A, i)
- \texttt{HeapSort}(A)

Heaps as priority queues

- \texttt{Heap-Maximum}(A)
- \texttt{Heap-Extract-Max}(A)
- \texttt{Heap-Increase-Key}(A, I, key)
Chapter 6. Heapsort

The heap sort algorithm consists of subroutines:

- **Build-Max-Heap**($A$)
- **Max-Heapify**($A, i$)
- **HeapSort**($A$)

Heaps as priority queues

- **Heap-Maximum**($A$)
- **Heap-Extract-Max**($A$)
- **Heap-Increase-Key**($A, I, key$)
- **Max-Heap-Insert**($A, key$)
Chapter 6. Heapsort

Algorithm HEAPSORT(A)

T_{BMH}(n) = \lceil n/2 \rceil - 1 \sum_{i=0}^{n/2-1} c_2 T_{MH}(n,i)

Subroutine Build-Max-Heap (A)

T_{BMH}(n) = \lceil n/2 \rceil - 1 \sum_{i=0}^{n/2-1} c_2 T_{MH}(n,i)
Chapter 6. Heapsort

Algorithm $\text{HeapSort}(A)$

1. $\text{Build-Max-Heap}(A)$
Algorithm \textsc{HeapSort}(A)

1. \textbf{Build-Max-Heap}(A)
2. \textbf{for} \( i = \text{length}[A] - 1 \) \textbf{downto} 1 \{ indexes begin from 0\}
Algorithm \texttt{HeapSort}(A)

1. \texttt{Build-Max-Heap}(A)
2. \texttt{for} \( i = \text{length}[A] - 1 \) \texttt{downto} 1 \hspace{1em} \{ indexes begin from 0\}
3. exchange \( A[0] \leftrightarrow A[i] \)
Chapter 6. Heapsort

Algorithm HeapSort(A)

1. Build-Max-Heap(A)
2. for $i = \text{length}[A] - 1$ downto 1 \{ indexes begin from 0\}
3. exchange $A[0] \leftrightarrow A[i]$
4. $\text{heapsize}[A] = \text{heapsize}[A] - 1$

Subroutine Build-Max-Heap(A)

1. $\text{heapsize}[A] = \text{length}[A]$
2. for $i = \lceil \frac{1}{2} \text{length}[A] \rceil - 1$ downto 0 \{ indexes begin from 0\}
3. Max-Heapify(A, i)
Algorithm `HeapSort(A)`

1. `Build-Max-Heap(A)`
2. for \( i = \text{length}[A] - 1 \) downto 1 \{ indexes begin from 0\}
3. exchange \( A[0] \leftrightarrow A[i] \)
4. \( \text{heapsize}[A] = \text{heapsize}[A] - 1 \)
5. `Max-Heapify(A, 0)`
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Algorithm **HEAPSORT**(\(A\))

1. **BUILD-MAX-HEAP**(\(A\))
2. for \(i = \text{length}[A] - 1\) **downto** 1 { indexes begin from 0}
3. exchange \(A[0] \leftrightarrow A[i]\)
4. \(\text{heapsize}[A] = \text{heapsize}[A] - 1\)
5. **MAX-HEAPIFY**(\(A, 0\))

\[ T_{HS}(n) = c_1 + T_{BMH}(n) + (n - 1)T_{MH}(n, 0), \text{ where } n = |A| \]
Chapter 6. Heapsort

Algorithm \texttt{HEAPSORT}(A)

1. \texttt{BUILD-MAX-HEAP}(A)
2. \texttt{for} $i = \text{length}[A] - 1$ \texttt{downto} 1 \quad \{ \text{indexes begin from 0}\}
3. exchange $A[0] \leftrightarrow A[i]$
4. $\text{heapsize}[A] = \text{heapsize}[A] - 1$
5. \texttt{MAX-HEAPIFY}(A, 0)

\[
T_{HS}(n) = c_1 + T_{BMH}(n) + (n - 1)T_{MH}(n, 0), \text{ where } n = |A|
\]

Subroutine \texttt{BUILD-MAX-HEAP}(A)
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Algorithm \textsc{HeapSort}(A)

1. \textsc{Build-Max-Heap}(A)
2. \textbf{for} $i = \text{length}[A] - 1$ \textbf{downto} 1 \{ indexes begin from 0\}
3. exchange $A[0] \leftrightarrow A[i]$
4. $\text{heapsize}[A] = \text{heapsize}[A] - 1$
5. \textsc{Max-Heapify}(A, 0)

$T_{HS}(n) = c_1 + T_{BMH}(n) + (n - 1)T_{MH}(n, 0)$, where $n = |A|$

Subroutine \textsc{Build-Max-Heap}(A)

1. $\text{heapsize}[A] = \text{length}[A]$
Chapter 6. Heapsort

Algorithm \texttt{HEAPSort}(A)

1. \texttt{BUILD-MAX-HEAP}(A)
2. \texttt{for } i = \texttt{length}[A] - 1 \texttt{ downto } 1 \quad \{ \text{indexes begin from 0}\}
3. \quad \text{exchange } A[0] \leftrightarrow A[i]
4. \quad \texttt{heapsize}[A] = \texttt{heapsize}[A] - 1
5. \quad \texttt{MAX-HEAPIFY}(A, 0)

\[ T_{HS}(n) = c_1 + T_{BMH}(n) + (n - 1)T_{MH}(n, 0) \], where \( n = |A| \)

Subroutine \texttt{BUILD-MAX-HEAP}(A)

1. \texttt{heapsize}[A] = \texttt{length}[A]
2. \texttt{for } i = \lfloor \frac{1}{2}\texttt{length}[A] \rfloor - 1 \texttt{ downto } 0 \quad \{ \text{indexes begin from 0}\}
Chapter 6. Heapsort

Algorithm \textsc{HeapSort}(A)

1. \textbf{Build-Max-Heap}(A)
2. \textbf{for} \( i = \text{length}[A] - 1 \) \textbf{downto} 1 \{ indexes begin from 0\}
3. exchange \( A[0] \leftarrow A[i] \)
4. \( \text{heapsize}[A] = \text{heapsize}[A] - 1 \)
5. \textsc{Max-Heapify}(A, 0)

\[ T_{HS}(n) = c_1 + T_{BMH}(n) + (n - 1)T_{MH}(n, 0) \text{, where } n = \lvert A \rvert \]

Subroutine \textbf{Build-Max-Heap}(A)

1. \( \text{heapsize}[A] = \text{length}[A] \)
2. \textbf{for} \( i = \lfloor \frac{1}{2}\text{length}[A] \rfloor - 1 \) \textbf{downto} 0 \{ indexes begin from 0\}
3. \textsc{Max-Heapify}(A, i)
Chapter 6. Heapsort

Algorithm **HEAPSORT**(*A*)

1. **BUILD-MAX-HEAP**(*A*)
2. for *i* = length[*A*] − 1 downto 1 { indexes begin from 0}
3. exchange *A*[0] ↔ *A*[i]
4. heapsize[*A*] = heapsize[*A*] − 1
5. **MAX-HEAPIFY**(*A*, 0)

\[ T_{HS}(n) = c_1 + T_{BMH}(n) + (n - 1)T_{MH}(n, 0), \text{ where } n = |A| \]

Subroutine **BUILD-MAX-HEAP**(*A*)

1. heapsize[*A*] = length[*A*]
2. for *i* = ⌊\frac{1}{2}\text{length}[A]\rfloor − 1 downto 0 { indexes begin from 0}
3. **MAX-HEAPIFY**(*A*, *i*)

\[ T_{BMH}(n) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor - 1} c_2 T_{MH}(n, i) \]
Chapter 6. Heapsort

Subroutine MAX-HEAPIFY(A, i)

1. \( l = 2 \times i + 1 \)
2. \( r = 2 \times i + 2 \)
3. if \((l \leq \text{heapsize}[A])\) and \((A[l] > A[i])\) then
4. \( \text{largest} = l \)
5. else \( \text{largest} = i \)
6. if \((r \leq \text{heapsize}[A])\) and \((A[r] > A[\text{largest}])\) then
7. \( \text{largest} = r \)
8. if \(\text{largest} \neq i\) then
9. Exchange \(A[i] \leftrightarrow A[\text{largest}]\)
10. Max-Heapify \((A, \text{largest})\)

\( T_{MH}(n, i) \leq c_3 + T_{MH}(n, 2i + 1) \) because
\( T_{MH}(n, i) = c_3 + T_{MH}(n, 2i + 1) \), or
\( T_{MH}(n, i) \leq c_4 \log_2 n \), for all \( i = 0, 1, \ldots, n - 1 \). (Prove it!)

\( T_{BMH}(n) = \lfloor n/2 \rfloor - 1 \sum_{i=0}^{n-1} c_2 T_{MH}(n, i) \leq c_4 n^2 \log_2 n \)

\( T_{HS}(n) = c_1 + T_{BMH}(n) + (n - 1)c_2 T_{MH}(n, 0) \leq c_4 n^2 \log_2 n + (n - 1)c_2 c_4 \log_2 n = O(n \log n) \)
Chapter 6. Heapsort

Subroutine MAX-HEAPIFY($A, i$)

1. $l = 2 \times i + 1$
Chapter 6. Heapsort

Subroutine MAX-HEAPIFY\((A, i)\)

1. \(l = 2 \times i + 1\)
2. \(r = 2 \times i + 2\)
Chapter 6. Heapsort

Subroutine MAX-HEAPIFY(A, i)

1. \( l = 2 \times i + 1 \)
2. \( r = 2 \times i + 2 \)
3. \( \text{if } (l \leq \text{heapsize}[A]) \text{ and } (A[l] > A[i]) \)
Chapter 6. Heapsort

Subroutine MAX-HEAPIFY($A, i$)

1. $l = 2 \times i + 1$
2. $r = 2 \times i + 2$
3. if ($l \leq \text{heapsize}[A]$) and ($A[l] > A[i]$)
4. then $\text{largest} = l$
Chapter 6. Heapsort

Subroutine MAX-HEAPIFY($A, i$)

1. $l = 2 \times i + 1$
2. $r = 2 \times i + 2$
3. if ($l \leq \text{heapsize}[A]$) and ($A[l] > A[i]$)
   then $largest = l$
4. else $largest = i$

$M_{max} (n, i) \leq c_3 + M_{max} (n, 2i + 1)$

Because $M_{max} (n, i) = c_3 + M_{max} (n, 2i + 1)$,
or $= c_3 + M_{max} (n, 2i + 2)$

$M_{max} (n, i) \leq c_4 \log_2 n$, for all $i = 0$, $1$, $\ldots$, $n - 1$.
(Prove it!)

$B_{max} (n) = \lfloor \frac{n}{2} \rfloor - 1 \sum_{i=0}^{n-1} c_2 M_{max} (n, i)$

$\leq c_4 n^2 \log_2 n$

$H_{sort} (n) = c_1 + B_{max} (n) + (n - 1) c_2 M_{max} (n, 0)$

$\leq c_4 n^2 \log_2 n + (n - 1) c_2 c_4 \log_2 n$

$= O(n \log n)$
Chapter 6. Heapsort

Subroutine **MAX-HEAPIFY**(*A*, *i*)

1. \( l = 2 \times i + 1 \)
2. \( r = 2 \times i + 2 \)
3. **if** \((l \leq \text{heapsize}[A]) \text{ and } (A[l] > A[i])\)**
4. **then** \( \text{largest} = l \)
5. **else** \( \text{largest} = i \)
6. **if** \((r \leq \text{heapsize}[A]) \text{ and } (A[r] > A[\text{largest}])\)**
Chapter 6. Heapsort

Subroutine MAX-HEAPIFY($A, i$)

1. $l = 2 \times i + 1$
2. $r = 2 \times i + 2$
3. if ($l \leq \text{heapsize}[A]$) and ($A[l] > A[i]$)
   then largest = $l$
4. else largest = $i$
5. if ($r \leq \text{heapsize}[A]$) and ($A[r] > A[largest]$)
   then largest = $r$
Chapter 6. Heapsort

Subroutine MAX-HEAPIFY($A, i$)

1. $l = 2 \times i + 1$
2. $r = 2 \times i + 2$
3. if ($l \leq \text{heapsize}[A]$) and ($A[l] > A[i]$)
4. then largest = $l$
5. else largest = $i$
6. if ($r \leq \text{heapsize}[A]$) and ($A[r] > A[\text{largest}]$)
7. then largest = $r$
8. if largest $\neq i$
Chapter 6. Heapsort

Subroutine MAX-HEAPIFY(A, i)

1. \( l = 2 \times i + 1 \)
2. \( r = 2 \times i + 2 \)
3. if \( (l \leq \text{heapsize}[A]) \) and \( (A[l] > A[i]) \) then \( \text{largest} = l \)
4. \( \text{else largest} = i \)
5. if \( (r \leq \text{heapsize}[A]) \) and \( (A[r] > A[\text{largest}]) \) then \( \text{largest} = r \)
6. if \( \text{largest} \neq i \) then exchange \( A[i] \leftrightarrow A[\text{largest}] \)

\[ T_{BMH}(n) = \lceil \frac{n}{2} \rceil - 1 \sum_{i=0}^{n-1} c_2 T_{MH}(n, i) \leq c_4 n^2 \log_2 n \]

\[ T_{HS}(n) = c_1 + T_{BMH}(n) + (n - 1) c_2 T_{MH}(n, 0) \leq c_4 n^2 \log_2 n + (n - 1) c_2 c_4 \log_2 n = O(n \log n) \]
Chapter 6. Heapsort

Subroutine MAX-HEAPIFY(A, i)

1. \( l = 2 \times i + 1 \)
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3. if \( (l \leq \text{heapsize}[A]) \text{ and } (A[l] > A[i]) \) then \( \text{largest} = l \)
4. else \( \text{largest} = i \)
5. if \( (r \leq \text{heapsize}[A]) \text{ and } (A[r] > A[\text{largest}]) \) then \( \text{largest} = r \)
6. if \( \text{largest} \neq i \) then exchange \( A[i] \leftrightarrow A[\text{largest}] \)
10. MAX-HEAPIFY(A, \text{largest})
Chapter 6. Heapsort

Subroutine MAX-HEAPIFY(A, i)

1. \( l = 2 \times i + 1 \)
2. \( r = 2 \times i + 2 \)
3. if \( (l \leq \text{heapsize}[A]) \) and \( (A[l] > A[i]) \)
4. then \( \text{largest} = l \)
5. else \( \text{largest} = i \)
6. if \( (r \leq \text{heapsize}[A]) \) and \( (A[r] > A[\text{largest}]) \)
7. then \( \text{largest} = r \)
8. if \( \text{largest} \neq i \)
9. then exchange \( A[i] \leftrightarrow A[\text{largest}] \)
10. MAX-HEAPIFY(A, largest)

\[ T_{MH}(n, i) \leq c_3 + T_{MH}(n, 2i + 1) \quad \text{Because } T_{MH}(n, i) = c_3 + T_{MH}(n, 2i + 1), \text{ or } \]
\[ = c_3 + T_{MH}(n, 2i + 2) \]
Chapter 6. Heapsort

Subroutine MAX-HEAPIFY$(A, i)$

1. \( l = 2 \times i + 1 \)
2. \( r = 2 \times i + 2 \)
3. \textbf{if} \ (l \leq \text{heapsize}[A]) \textbf{and} (A[l] > A[i]) \textbf{then} largest = l
4. \textbf{else} largest = i
5. \textbf{if} \ (r \leq \text{heapsize}[A]) \textbf{and} (A[r] > A[\text{largest}]) \textbf{then} largest = r
6. \textbf{if} largest \neq i \textbf{then} exchange \ A[i] \leftrightarrow A[\text{largest}]\text{MAX-HEAPIFY}(A, largest)$

\[ T_{MH}(n, i) \leq c_3 + T_{MH}(n, 2i + 1) \quad \text{Because} \quad T_{MH}(n, i) = c_3 + T_{MH}(n, 2i + 1), \text{or} \]
\[ = c_3 + T_{MH}(n, 2i + 2) \]

\[ T_{MH}(n, i) \leq c_4 \log_2 n, \text{ for all } i = 0, 1, \ldots, n - 1. \quad \text{(Prove it!)} \]
Chapter 6. Heapsort

Subroutine MAX-HEAPIFY$(A, i)$

1. \( l = 2 \times i + 1 \)
2. \( r = 2 \times i + 2 \)
3. if \((l \leq \text{heapsize}[A]) \text{ and } (A[l] > A[i])\)
   then largest = \( l \)
4. else largest = \( i \)
5. if \((r \leq \text{heapsize}[A]) \text{ and } (A[r] > A[\text{largest}])\)
   then largest = \( r \)
6. if largest \( \neq i \)
   then exchange \( A[i] \leftrightarrow A[\text{largest}] \)
7. MAX-HEAPIFY$(A, \text{largest})$

\[ T_{MH}(n, i) \leq c_3 + T_{MH}(n, 2i + 1) \quad \text{Because } T_{MH}(n, i) = c_3 + T_{MH}(n, 2i + 1), \text{ or} \]
\[ = c_3 + T_{MH}(n, 2i + 2) \]

\[ T_{MH}(n, i) \leq c_4 \log_2 n, \text{ for all } i = 0, 1, \ldots, n - 1. \text{ (Prove it!)} \]

\[ T_{BMH}(n) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor - 1} c_2 T_{MH}(n, i) \]
Chapter 6. Heapsort

Subroutine MAX-HEAPIFY($A, i$)

1. $l = 2 \times i + 1$
2. $r = 2 \times i + 2$
3. if ($l \leq \text{heapsize}[A]$) and ($A[l] > A[i]$)
4. then largest = $l$
5. else largest = $i$
6. if ($r \leq \text{heapsize}[A]$) and ($A[r] > A[\text{largest}]$)
7. then largest = $r$
8. if largest $\neq i$
9. then exchange $A[i] \leftrightarrow A[\text{largest}]$
10. MAX-HEAPIFY($A, \text{largest}$)

$T_{MH}(n, i) \leq c_3 + T_{MH}(n, 2i + 1)$  
Because $T_{MH}(n, i) = c_3 + T_{MH}(n, 2i + 1)$, or
$= c_3 + T_{MH}(n, 2i + 2)$

$T_{MH}(n, i) \leq c_4 \log_2 n$, for all $i = 0, 1, \ldots, n - 1$. (Prove it!)

$T_{BMH}(n) = \sum_{i=0}^{[n/2]-1} c_2 T_{MH}(n, i) \leq c_4 \frac{n}{2} \log_2 n$
Chapter 6. Heapsort

Subroutine $\text{MAX-HEAPIFY}(A, i)$

1. $l = 2 \times i + 1$
2. $r = 2 \times i + 2$
3. if $(l \leq \text{heapsize}[A])$ and $(A[l] > A[i])$

4. then $\text{largest} = l$
5. else $\text{largest} = i$
6. if $(r \leq \text{heapsize}[A])$ and $(A[r] > A[\text{largest}])$
7. then $\text{largest} = r$
8. if $\text{largest} \neq i$
9. then exchange $A[i] \leftrightarrow A[\text{largest}]$
10. $\text{MAX-HEAPIFY}(A, \text{largest})$

$T_{MH}(n, i) \leq c_3 + T_{MH}(n, 2i + 1)$  
Because $T_{MH}(n, i) = c_3 + T_{MH}(n, 2i + 1)$, or 
$= c_3 + T_{MH}(n, 2i + 2)$

$T_{MH}(n, i) \leq c_4 \log_2 n$, for all $i = 0, 1, \ldots, n - 1$. (Prove it!)

$T_{BMH}(n) = \sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor - 1} c_2 T_{MH}(n, i) \leq c_4 \frac{n}{2} \log_2 n$

$T_{HS}(n) = c_1 + T_{BMH}(n) + (n - 1)c_2 T_{MH}(n, 0)$
Chapter 6. Heapsort

Subroutine MAX-HEAPIFY(A, i)

1. \( l = 2 \times i + 1 \)
2. \( r = 2 \times i + 2 \)
3. if \( (l \leq \text{heapsize}[A]) \) and \( (A[l] > A[i]) \)
   then \( \text{largest} = l \)
4. else \( \text{largest} = i \)
5. if \( (r \leq \text{heapsize}[A]) \) and \( (A[r] > A[\text{largest}]) \)
   then \( \text{largest} = r \)
6. if \( \text{largest} \neq i \)
   then exchange \( A[i] \leftrightarrow A[\text{largest}] \)
7. MAX-HEAPIFY(A, largest)

\[ T_{MH}(n, i) \leq c_3 + T_{MH}(n, 2i + 1) \quad \text{Because } T_{MH}(n, i) = c_3 + T_{MH}(n, 2i + 1), \text{ or } \\
= c_3 + T_{MH}(n, 2i + 2) \]

\[ T_{MH}(n, i) \leq c_4 \log_2 n, \text{ for all } i = 0, 1, \ldots, n - 1. \quad \text{(Prove it!)} \]

\[ T_{BMH}(n) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor - 1} c_2 T_{MH}(n, i) \leq c_4 \frac{n}{2} \log_2 n \]

\[ T_{HS}(n) = c_1 + T_{BMH}(n) + (n - 1)c_2 T_{MH}(n, 0) \leq c_4 \frac{n}{2} \log_2 n + (n - 1)c_2 c_4 \log_2 n \]
Chapter 6. Heapsort

Subroutine Max-Heapify($A, i$)

1. $l = 2 \times i + 1$
2. $r = 2 \times i + 2$
3. if ($l \leq \text{heapsize}[A]$) and ($A[l] > A[i]$) then
   largest = $l$
4. else largest = $i$
5. if ($r \leq \text{heapsize}[A]$) and ($A[r] > A[\text{largest}]$) then
   largest = $r$
6. if largest $\neq i$
7. then exchange $A[i] \leftrightarrow A[\text{largest}]$
8. MAX-HEAPIFY($A, \text{largest}$)

$T_{MH}(n, i) \leq c_3 + T_{MH}(n, 2i + 1)$  
Because $T_{MH}(n, i) = c_3 + T_{MH}(n, 2i + 1)$, or  
$= c_3 + T_{MH}(n, 2i + 2)$

$T_{MH}(n, i) \leq c_4 \log_2 n$, for all $i = 0, 1, \ldots, n - 1$. (Prove it!)

$T_{BMH}(n) = \sum_{i=0}^{\lfloor n/2 \rfloor - 1} c_2 T_{MH}(n, i) \leq c_4 \frac{n}{2} \log_2 n$

$T_{HS}(n) = c_1 + T_{BMH}(n) + (n - 1)c_2 T_{MH}(n, 0) \leq$
$c_4 \frac{n}{2} \log_2 n + (n - 1)c_2c_4 \log_2 n = O(n \log n)$
Chapter 6. Heapsort

Operations on heaps:

Function Heap-Maximum(A):
1. return (A[1])

Function Heap-Extract-Max(A):
1. if heapsize[A] < 1 then return "heap underflow"
2. max = A[1]
4. heapsize[A] = heapsize[A] - 1
5. Max-Heapify(A, 1)
6. return (max)

Function Heap-Increase-Key(A, i, key):
1. if key < A[i] then return "new key is smaller than current key"
2. A[i] = key
3. while i > 1 and A[PARENT[i]] < A[i] exchange
5. i = PARENT[i]

Function Max-Heap-Insert(A, key):
1. heapsize[A] = heapsize[A] + 1
2. A[heapsize[A]] = −∞
3. Heap-Increase-Key(A, heapsize[A], key)
Chapter 6. Heapsort

Operations on heaps:

Function $\text{Heap-Maximum}(A)$
1. return $(A[1])$

Function $\text{Heap-Extract-Max}(A)$
1. if $\text{heapsize}[A] < 1$
2. then return ("heap underflow")
3. $\text{max} = A[1]$
5. $\text{heapsize}[A] = \text{heapsize}[A] - 1$
6. $\text{Max-Heapify}(A, 1)$
7. return $(\text{max})$

Function $\text{Heap-Increase-Key}(A, i, \text{key})$
1. if $\text{key} < A[i]$
2. then return ("new key is smaller than current key")
3. $A[i] = \text{key}$
4. while $i > 1$ and $A[\text{Parent}[i]] < A[i]$
5. exchange $A[i] \leftarrow A[\text{Parent}[i]]$
6. $i = \text{Parent}[i]$

Function $\text{Max-Heap-Insert}(A, \text{key})$
1. $\text{heapsize}[A] = \text{heapsize}[A] + 1$
2. $A[\text{heapsize}[A]] = -\infty$
3. $\text{Heap-Increase-Key}(A, \text{heapsize}[A], \text{key})$
Chapter 6. Heapsort

Operations on heaps:

Function **Heap-Maximum**($A$) obtain the maximum

Function **Heap-Extract-Max**($A$) obtain and remove the maximum
1. **if** $heapsize[A] < 1$
2. **then return** ("heap underflow")
3. $max = A[1]$
5. $heapsize[A] = heapsize[A] - 1$
6. **Max-Heapify**($A, 1$)
7. **return** ($max$)
Chapter 6. Heapsort

Operations on heaps:

Function **HEAP-MAXIMUM**(A) obtain the maximum
1. return (A[1])

Function **HEAP-EXTRACT-MAX**(A) obtain and remove the maximum
1. if heapsize[A] < 1
2. then return ("heap underflow")
3. max = A[1]
5. heapsize[A] = heapsize[A] - 1
6. **MAX-HEAPIFY**(A, 1)
7. return (max)

Function **HEAP-INCREASE-KEY**(A, i, key) replace a key with a larger value
1. if key < A[i]
2. then return ("new key is smaller than current key")
3. A[i] = key
4. while i > 1 and A[PARENT[i]] < A[i]
6. i = PARENT[i]
Chapter 6. Heapsort

Operations on heaps:

Function **Heap-Maximum**(A)  
1. **return** (A[1])  
   
Function **Heap-Extract-Max**(A)  
1. **if** heapsize[A] < 1  
2. **then return** ("heap underflow")  
3. max = A[1]  
5. heapsize[A] = heapsize[A] − 1  
6. **Max-Heapify**(A, 1)  
7. **return** (max)

Function **Heap-Increase-Key**(A, i, key)  
1. **if** key < A[i]  
2. **then return** ("new key is smaller than current key")  
3. A[i] = key  
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6. i = PARENT[i]

Function **Max-Heap-Insert**(A, key)  
1. heapsize[A] = heapsize[A] + 1  
2. A[heapsize[A]] = −∞  
3. **Heap-Increase-Key**(A, heapsize[A], key)
Chapter 7. Quicksort

Idea of the Quicksort: divide-and-conquer

- divide: re-organize list $A[p, r]$ into two sublists $A[p, q-1]$ and $A[q+1, r]$ based on pivot $A[q]$ such that
  - (a) $A[i] \leq A[q]$ for all $i = p, \ldots, q-1$
  - (b) $A[i] \geq A[q]$ for all $i = q+1, \ldots, r$

Chapter 7. Quicksort

Chapter 7. Quicksort and randomized algorithms

Idea of the Quicksort: divide-and-conquer
Chapter 7. Quicksort

Chapter 7. Quicksort and randomized algorithms

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\begin{align*}
(a) & \quad A[i] \leq A[q] \quad \text{for all } i = p, \ldots, q - 1 \\
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\end{align*}
Chapter 7. Quicksort

Chapter 7. Quicksort and randomized algorithms

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Chapter 7. Quicksort and randomized algorithms

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Chapter 7. Quicksort

Chapter 7. Quicksort and randomized algorithms

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  (b) $A[i] \geq A[q]$ for all $i = q + 1, \cdots, r$

Chapter 7. Quicksort

Algorithm Quicksort \( (A, p, r) \)

1. if \( p < r \)
2. then 
   \( q = \text{Partition}(A, p, r) \)
3. \( \text{QuickSort}(A, p, q-1) \)
4. \( \text{QuickSort}(A, q+1, r) \)

How the pivot \( A[q] \) is identified is crucial to the performance of Quicksort.

- Assume \( A[q] \) partitions list \( A,p,r \) even, then 
  \( T(n) \leq 2T(n/2) + cn = O(n \log_2 n) \)
- Assume \( A[q] \) partitions the list 20% vs 80%, then 
  \( T(n) \leq T(n/5) + T(4n/5) + cn = O(n \log_2 n) \)
- Assume \( A[q] \) partitions the list 1% vs 99%, then 
  \( T(n) \leq T(n/100) + T(99n/100) + cn = O(n \log_2 n) \)
Chapter 7. Quicksort

Chapter 7. Quicksort and Randomized algorithms

Algorithm QUICKSORT \((A, p, r)\)
Algorithm \textsc{QuickSort} \((A, p, r)\)

1. \textbf{if} \( p < r \)
Chapter 7. Quicksort

Chapter 7. Quicksort and Randomized algorithms

Algorithm \texttt{QUICKSORT} \((A, p, r)\)
1. \textbf{if} \(p < r\)
2. \textbf{then} \(q = \text{PARTITION}(A, p, r)\)
Chapter 7. Quicksort

Chapter 7. Quicksort and Randomized algorithms

Algorithm \textsc{QuickSort} \((A, p, r)\)
\begin{enumerate}
\item \textbf{if} \(p < r\)
\item \textbf{then} \(q = \text{Partition}(A, p, r)\)
\item \textbf{QuickSort} \((A, p, q - 1)\)
\end{enumerate}

How the pivot \(A[q]\) is identified is crucial to the performance of Quicksort.

- Assume \(A[q]\) partitions list \(A[p, r]\) evenly, then
  \[T(n) \leq 2T\left(\frac{n}{2}\right) + cn = O(n \log_2 n)\]

- Assume \(A[q]\) partitions the list 20% vs 80%, then
  \[T(n) \leq T\left(\frac{5n}{2}\right) + T\left(\frac{4n}{5}\right) + cn = O(n \log_2 n)\]

- Assume \(A[q]\) partitions the list 1% vs 99%, then
  \[T(n) \leq T\left(\frac{100n}{99}\right) + T\left(\frac{99n}{100}\right) + cn = O(n \log_2 n)\]
Chapter 7. Quicksort

Chapter 7. Quicksort and Randomized algorithms

Algorithm QuickSort (A, p, r)
1. if p < r
2. then q = Partition(A, p, r)
3. QuickSort (A, p, q − 1)
4. QuickSort (A, q + 1, r)

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- Assume \( A[q] \) partitions list \( A,p,r \) evenly, then
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  \[ T(n) \leq T(n5) + T(4n5) + cn = O(n \log_2 n) \]

- Assume \( A[q] \) partitions the list 1% vs 99%, then
  \[ T(n) \leq T(100n) + T(99n100) + cn = O(n \log_2 n) \]
Chapter 7. Quicksort

Chapter 7. Quicksort and Randomized algorithms

Algorithm QUICKSORT \((A, p, r)\)
1. \textbf{if} \(p < r\)
2. \textbf{then} \(q = \text{PARTITION}(A, p, r)\)
3. QUICKSORT \((A, p, q - 1)\)
4. QUICKSORT \((A, q + 1, r)\)

How the pivot \(A[q]\) is identified is crucial to the performance of Quicksort.
Chapter 7. Quicksort and Randomized algorithms

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  \[T(n) \leq 2T(n/2) + cn = O(n \log_2 n)\]
Chapter 7. Quicksort

Chapter 7. Quicksort and Randomized algorithms

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Chapter 7. Quicksort

Chapter 7. Quicksort and Randomized algorithms

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Chapter 7. Quicksort

Chapter 7. Quicksort and Randomized algorithms

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How can we identify such a pivot?
Chapter 7. Quicksort

quicksort(A, p, q-1) quicksort(A, q+1, r)
Chapter 7. Quicksort

\[ \text{PARTITION}(A, p, r) \]
1. \( x \leftarrow A[r] \)
2. \( i \leftarrow p - 1 \)
3. for \( j \leftarrow p \) to \( r - 1 \)
   4. do if \( A[j] \leq x \)
      5. then \( i \leftarrow i + 1 \)
      6. exchange \( A[i] \leftrightarrow A[j] \)
6. exchange \( A[i + 1] \leftrightarrow A[r] \)
8. return \( i + 1 \)
Partition may not guarantee to partition the list to two fractions of sizes $\epsilon n : (1 - \epsilon)n$, for a constant $\epsilon > 0$. 
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- skewed situation like $1 : n - 1$ partition may happen, resulting in running time $\geq cn^2$. 
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Chapter 7. Quicksort

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- that is, the cases other than the skewed ones occur much more often.
**Partition** may not guarantee to partition the list to two fractions of sizes $\epsilon n : (1 - \epsilon)n$, for a constant $\epsilon > 0$.

- skewed situation like $1 : n - 1$ partition may happen, resulting in running time $\geq cn^2$.

- however, chances for skewed cases like above are very small.

- that is, the cases other than the skewed ones occur much more often. So the idea of Quicksort may work well on a majority of data.
Chapter 7. Quicksort

Assume that the equal likely chance for every number to be in the last position, what is the chance to partition the list into

\[ x\% \text{ vs } (100 - x)\% \]

fragments, for \(10 \leq x \leq 90\)?
Chapter 7. Quicksort

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fragments, for \(10 \leq x \leq 90\)?

The chance is \(= 80\%\)
Chapter 7. Quicksort

What running time would it be if 10:90 partition is always guaranteed?
Chapter 7. Quicksort

What running time would it be if 10:90 partition is always guaranteed?

\[ T(n) \leq T(n/10) + T(9n/10) + cn \]
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Using the recursive-tree method (in the book notation), we have
Chapter 7. Quicksort

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\[ l_0: \quad cn \]

\[ cn \]
Chapter 7. Quicksort

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\[
\begin{align*}
l_0: & \quad cn \\
l_1: & \quad cn/10 \quad \quad \quad \quad 9cn/10 \\
\end{align*}
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Chapter 7. Quicksort

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Using the recursive-tree method (in the book notation), we have

\[
\begin{align*}
  l_0: & \quad & cn \\
  l_1: & \quad & \frac{cn}{10} & \quad \frac{9cn}{10} \\
  l_2: & \quad & \frac{cn}{10^2} & \frac{9cn}{10^2} & \frac{9cn}{10^2} & \frac{9^2cn}{10^2} \\
\end{align*}
\]
Chapter 7. Quicksort

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\[ l_1: \quad cn/10 \quad 9cn/10 \]
\[ l_2: \quad cn/10^2 \quad 9cn/10^2 \quad 9^2cn/10^2 \]

\[ \ldots \]

where \( c'/\log_2 10/9 \)
What running time would it be if 10:90 partition is always guaranteed?

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- \( l_1: \) \( cn/10 \) \( 9cn/10 \)
- \( l_2: \) \( cn/10^2 \) \( 9cn/10^2 \) \( 9^2cn/10^2 \)
- \( l_h: \) \( cn/10^h \) \( \ldots \) \( c9^h n/10^h \)
Chapter 7. Quicksort

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\[ T(n) \leq T(n/10) + T(9n/10) + cn \]

Using the recursive-tree method (in the book notation), we have

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\[ l_2: \quad \text{cn/10^2} \quad \text{9cn/10^2} \quad \text{9^2cn/10^2} \quad \text{cn} \]

\[ l_h: \quad \text{cn/10^h} \quad \ldots \quad \text{c9^h n/10^h} \quad \text{cn} \]

\[ \ldots \quad \ldots \]
Chapter 7. Quicksort

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\begin{align*}
l_0: & \quad \text{cn} \\
l_1: & \quad \frac{cn}{10}, \frac{9cn}{10} \\
l_2: & \quad \frac{cn}{10^2}, \frac{9cn}{10^2}, \frac{9^2cn}{10^2} \\
l_h: & \quad \frac{cn}{10^h}, \ldots, \frac{c9^h n}{10^h} \\
l_k: & \quad \ldots \quad \frac{c9^k n}{10^k} \leq cn
\end{align*}
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Chapter 7. Quicksort

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  \vdots & \quad \vdots \quad \vdots \\
  l_h: & \quad cn/10^h \quad 9^h n/10^h \quad cn \\
  l_k: & \quad \vdots \quad \vdots \quad c9^k n/10^k \quad \leq cn
\end{align*}
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where \( (\frac{1}{10})^h n = 1 \), i.e., \( h = \log_{10} n \)
Chapter 7. Quicksort

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  l_h: & \quad cn/10^h \quad \ldots \ldots \quad c9^hn/10^h \quad cn \\
  l_k: & \quad \ldots \ldots \quad c9^kn/10^k \quad \leq cn
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Chapter 7. Quicksort

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  l_2: & \quad cn/10^2, 9cn/10^2 \\
  \vdots & \quad \vdots \\
  l_h: & \quad cn/10^h, 9cn/10^h \\
  \vdots & \quad \vdots \\
  l_k: & \quad cn/10^k, \quad \leq cn \\
\end{align*}

where \((\frac{1}{10})^h n = 1\), i.e., \(h = \log_{10} n\)

\((\frac{9}{10})^k n = 1\), i.e., \(k = \log_{\frac{10}{9}} n\)

\[ cn \log_{10} n \leq T(n) \leq cn \log_{\frac{10}{9}} n \]
Chapter 7. Quicksort

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  \vdots & \quad \vdots \\
  l_h: & \quad cn/10^h \quad \cdots \quad c9^h n/10^h \quad cn \\
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\end{align*}
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where \((\frac{1}{10})^h n = 1\), i.e., \(h = \log_{10} n\)

\((\frac{9}{10})^k n = 1\), i.e., \(k = \log_{\frac{10}{9}} n\)

\[ cn \log_{10} n \leq T(n) \leq cn \log_{\frac{10}{9}} n \]

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Chapter 7. Quicksort

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\[ l_2: \quad cn/10^2 \quad 9cn/10^2 \quad 9cn/10^2 \quad 9^2cn/10^2 \quad \text{cn} \]
\[ l_h: \quad cn/10^h \quad \ldots \quad \text{cn} \quad 9^h n/10^h \]
\[ l_k: \quad \ldots \quad \text{cn} \quad 9^k n/10^k \quad \leq \text{cn} \]

where \((\frac{1}{10})^h n = 1\), i.e., \(h = \log_{10} n\)

\((\frac{9}{10})^k n = 1\), i.e., \(k = \log_{\frac{9}{10}} n\)

\[ cn \log_{10} n \leq T(n) \leq cn \log_{\frac{9}{10}} n \]

\[ T(n) \leq cn \log_{\frac{10}{9}} n = cn \frac{\log_2 n}{\log_2 \frac{10}{9}} \]
Chapter 7. Quicksort

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l_2: & \quad cn/10^2, 9cn/10^2, cn/10^2, 9^2cn/10^2 \\
l_h: & \quad cn/10^h, \ldots, c9^h n/10^h \\
l_k: & \quad \ldots, c9^k n/10^k \leq cn
\end{align*}

where \( \left(\frac{1}{10}\right)^h n = 1 \), i.e., \( h = \log_{10} n \)

\( \left(\frac{9}{10}\right)^k n = 1 \), i.e., \( k = \log_{10} \frac{9}{n} \)

\[ cn \log_{10} n \leq T(n) \leq cn \log_{10} \frac{9}{n} \]

\[ T(n) \leq cn \log_{10} \frac{9}{n} = cn \frac{\log_2 n}{\log_2 \frac{10}{9}} = c'n \log_2 n \]
Chapter 7. Quicksort

What running time would it be if 10:90 partition is always guaranteed?

\[ T(n) \leq T(n/10) + T(9n/10) + cn \]

Using the recursive-tree method (in the book notation), we have

\[
\begin{align*}
  l_0: & \quad cn \\
  l_1: & \quad cn/10 \\
  l_2: & \quad cn/10^2 \\
  l_h: & \quad cn/10^h \\
  l_k: & \quad \ldots \ldots \\
  l_h: & \quad c9^h n/10^h \\
  l_k: & \quad \ldots \ldots \\
\end{align*}
\]

where \((\frac{1}{10})^h n = 1\), i.e., \(h = \log_{10} n\)

\[ (\frac{9}{10})^k n = 1, \text{ i.e., } k = \log_{\frac{9}{10}} n \]

\[ cn \log_{10} n \leq T(n) \leq cn \log_{\frac{9}{10}} n \]

\[ T(n) \leq cn \log_{\frac{9}{10}} n = cn \frac{\log_2 n}{\log_2 \frac{10}{9}} = c' n \log_2 n = O(n \log_2 n) \]
Chapter 7. Quicksort

What running time would it be if 10:90 partition is always guaranteed?

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\begin{align*}
  l_0: & \quad cn \\  l_1: & \quad cn/10 \\  l_2: & \quad cn/10^2, 9cn/10^2, 9cn/10 \\  l_h: & \quad cn/10^h, \ldots, c9^h n/10^h \\  l_k: & \quad \vdots, c9^k n/10^k \leq cn
\end{align*}
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\[ T(n) \leq cn \log_{\frac{9}{10}} n = cn \frac{\log_2 n}{\log_2 \frac{9}{10}} = c' n \log_2 n = O(n \log_2 n) \]

where \(c' = c / \log_2 \frac{10}{9}\)
Chapter 7. Quicksort

Instead of analyzing \textsc{QuickSort} (with uniformly distributed for input) we design a randomized version of the algorithm and analyze it.
Chapter 7. Quicksort

Instead of analyzing \textsc{QuickSort} (with uniformly distributed for input) we design a randomized version of the algorithm and analyze it.

Algorithm \textsc{Randomized-Partition}(A, p, r)
1. $i = \text{random}(p, r)$
2. exchange $A[r] \leftrightarrow A[i]$
3. return \text{\textsc{Partition}}($A, p, r$)
Instead of analyzing \textsc{QuickSort} (with uniformly distributed for input) we design a randomized version of the algorithm and analyze it.

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Algorithm \texttt{Randomized QuickSort} (A, p, r)
Instead of analyzing QuickSort (with uniformly distributed for input) we design a randomized version of the algorithm and analyze it.

Algorithm **RANDOMIZED-PARTITION** $(A, p, r)$
1. $i = random(p, r)$
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3. **return** $(\text{PARTITION}(A, p, r))$

Algorithm **RANDOMIZED QUICKSORT** $(A, p, r)$
1. if $p < r$

---

**Chapter 7. Quicksort**
Instead of analyzing QuickSort (with uniformly distributed for input) we design a randomized version of the algorithm and analyze it.

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1. \(i = \text{random}(p, r)\)
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Algorithm **Randomized QuickSort** \((A, p, r)\)
1. **if** \(p < r\)
2. **then** \(q = \text{Randomized-Partition}(A, p, r)\)
Chapter 7. Quicksort

Instead of analyzing QuickSort (with uniformly distributed for input) we design a randomized version of the algorithm and analyze it.

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Algorithm Randomized QuickSort \((A, p, r)\)
1. if \(p < r\)
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3. Randomized QuickSort \((A, p, q - 1)\)
Chapter 7. Quicksort

Instead of analyzing \texttt{QuickSort} (with uniformly distributed for input) we design a randomized version of the algorithm and analyze it.

Algorithm \texttt{RANDOMIZED-PARTITION}(A, p, r)
1. \hspace{1cm} $i = \text{random}(p, r)$
2. \hspace{1cm} exchange $A[r] \leftrightarrow A[i]$
3. \hspace{1cm} \textbf{return} $(\texttt{PARTITION}(A, p, r))$

Algorithm \texttt{RANDOMIZED QUICKSORT} $(A, p, r)$
1. \hspace{1cm} \textbf{if} $p < r$
2. \hspace{1cm} \textbf{then} $q = \texttt{RANDOMIZED-PARTITION}(A, p, r)$
3. \hspace{1cm} \texttt{RANDOMIZED QUICKSORT} $(A, p, q - 1)$
4. \hspace{1cm} \texttt{RANDOMIZED QUICKSORT} $(A, q + 1, r)$
Chapter 7. Quicksort

Up to this point, you should have known:

1. the details of QuickSort algorithm, especially Partition;
2. Why it runs $O(n \log n)$ on uniformly distributed data, intuitively;
3. the connection between
   (a) requiring prob distribution in the input data;
   (b) randomized algorithms;
Chapter 7. Quicksort

**Analysis of Randomized-QuickSort**
Chapter 7. Quicksort

Analysis of **Randomized-QuickSort**

- count the expected number of comparisons between $x_i$ and $x_j$;

Observation 1: $x_i$ is compared with $x_j$ only when either is a pivot;

Observation 2: $x_i$ is compared with $x_j$ at most once;

- define random variable $X_{i,j} \in \{0, 1\}$, such that $X_{i,j} = 1$ iff a comparison between $x_i$ and $x_j$ occurs;

- let $X = \sum_{i<j} X_{i,j}$, total number of comparisons;

- the expected number of comparisons is $E(X) = \sum_{i<j} E(X_{i,j}) = \sum_{i<j} P(X_{i,j} = 1)$ by linearity of expectations.
Chapter 7. Quicksort

Analysis of **RANDOMIZED-QUICKSORT**
- count the expected number of comparisons between \( x_i \) and \( x_j \);

Observation 1: \( x_i \) is compared with \( x_j \) only when either is a pivot;
Analysis of Randomized-QuickSort

• count the expected number of comparisons between \( x_i \) and \( x_j \);

Observation 1: \( x_i \) is compared with \( x_j \) only when either is a pivot;

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Chapter 7. Quicksort

Analysis of **Randomized-QuickSort**

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Chapter 7. Quicksort

Analysis of Randomized-QuickSort

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$$X_{i,j} = 1 \text{ iff a comparison between } x_i \text{ and } x_j \text{ occurs}$$
Chapter 7. Quicksort

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\[
X_{i,j} = 1 \text{ iff a comparison between } x_i \text{ and } x_j \text{ occurs}
\]

• let \( X = \sum_{i<j} X_{i,j} \), total number of comparisons
Chapter 7. Quicksort

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- the **expected** number of comparisons is

  \[ E(X) = E\left(\sum_{i<j} X_{i,j}\right) \]
Chapter 7. Quicksort

Analysis of Randomized-QuickSort

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• the expected number of comparisons is

$$E(X) = E\left(\sum_{i<j} X_{i,j}\right) = \sum_{i<j} E(X_{i,j}) = \ldots$$
Chapter 7. Quicksort

Analysis of Randomized-QuickSort

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$$E(X) = E\left( \sum_{i<j} X_{i,j} \right) = \sum_{i<j} E(X_{i,j}) = \sum_{i<j} P(X_{i,j} = 1)$$

by linearity of expectations.
Chapter 7. Quicksort

Analysis of `RANDOMIZED-QUICKSORT` (cont.)

\[
E(X) = E\left(\sum_{i<j} X_{i,j}\right) = \sum_{i<j} E(X_{i,j}) = \sum_{i<j} \text{Prob}(X_{i,j} = 1)
\]
Analysis of \textsc{Randomized-QuickSort} (cont.)

\[ E(X) = E\left(\sum_{i<j} X_{i,j}\right) = \sum_{i<j} E(X_{i,j}) = \sum_{i<j} \text{Prob}(X_{i,j} = 1) \]

\( X_{i,j} = 1 \), i.e., comparison between \( x_i \) and \( x_j \) occurs only when
Chapter 7. Quicksort

Analysis of \texttt{RANDOMIZED-QUICKSORT} (cont.)

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\(X_{i,j} = 1\), i.e., comparison between \(x_i\) and \(x_j\) occurs only when

(1) \(x_i, x_j\) are in the same sublist \(L\);
Analysis of **RANDOMIZED-QUICKSORT** (cont.)

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Chapter 7. Quicksort

Analysis of Randomized-Quicksort (cont.)

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1. \( x_i, x_j \) are in the same sublist \( L \);
2. either is chosen to be the pivot;

\[ P(X_{i,j} = 1) = 2 \frac{1}{|L|}, \text{ where } |L| \text{ is the size of the sublist. why?} \]
Chapter 7. Quicksort

Analysis of RANDOMIZED-QUICKSORT (cont.)

\[ E(X) = E(\sum_{i<j} X_{i,j}) = \sum_{i<j} E(X_{i,j}) = \sum_{i<j} \text{Prob}(X_{i,j} = 1) \]

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but we do not know the size of the sublist \( L \)!
Chapter 7. Quicksort

Analysis of RANDOMIZED-QUICKSORT (cont.)

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however, if \( x_i, x_j \) are so indexed in the final sorted list,
Chapter 7. Quicksort

Analysis of **RANDOMIZED-QUICKSORT** (cont.)

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E(X) = E\left(\sum_{i<j} X_{i,j}\right) = \sum_{i<j} E(X_{i,j}) = \sum_{i<j} \text{Prob}(X_{i,j} = 1)
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\(X_{i,j} = 1\), i.e., comparison between \(x_i\) and \(x_j\) occurs only when

1. \(x_i, x_j\) are in the same sublist \(L\);
2. either is chosen to be the pivot;

\[
P(X_{i,j} = 1) = 2 \frac{1}{|L|}, \text{ where } |L| \text{ is the size of the sublist. why?}
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but we do not know the size of the sublist \(L\)!

however, if \(x_i, x_j\) are so indexed in the final sorted list, then
Chapter 7. Quicksort

Analysis of **RANDOMIZED-QUICKSORT** (cont.)

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E(X) = E(\sum_{i<j} X_{i,j}) = \sum_{i<j} E(X_{i,j}) = \sum_{i<j} \text{Prob}(X_{i,j} = 1)
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\[
P(X_{i,j} = 1) = 2 \frac{1}{|L|}, \text{ where } |L| \text{ is the size of the sublist. why?}
\]

but we do not know the size of the sublist \(L\)!

however, if \(x_i, x_j\) are so indexed in the final sorted list, then

- size of the sublist (which \(x_i, x_j\) belongs to)
- \(|L| \geq (j - i + 1)\)
Chapter 7. Quicksort

Analysis of **Randomized-QuickSort** (cont.)

\[
E(X) = E\left(\sum_{i<j} X_{i,j}\right) = \sum_{i<j} E(X_{i,j}) = \sum_{i<j} \text{Prob}(X_{i,j} = 1)
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\(X_{i,j} = 1\), i.e., comparison between \(x_i\) and \(x_j\) occurs only when

1. \(x_i, x_j\) are in the same sublist \(L\);
2. either is chosen to be the pivot;

\[P(X_{i,j} = 1) = 2 \frac{1}{|L|},\] where \(|L|\) is the size of the sublist. **why?**

but we do not know the size of the sublist \(L\)!

however, if \(x_i, x_j\) are so indexed in the final sorted list, then

- size of the sublist (which \(x_i, x_j\) belongs to)
  \[|L| \geq (j - i + 1)\]

So \(P(X_{i,j} = 1) \leq 2 \frac{1}{|L|} \leq 2 \frac{1}{j - i + 1}\)
Chapter 7. Quicksort

original unsorted list

\[
\begin{array}{c}
5 \quad 23 \quad 10
\end{array}
\]

sublist L containing elements 5 and 10
10 is a pivot

\[
\begin{array}{c}
5 \quad 10
\end{array}
\]

\[|L|\]

L has to contain elements between 5 and 10
i.e., L has to contain elements 6, 7, 8, 9
\[|L| \geq j - i + 1 = 10 - 5 + 1 = 6\]

final sorted list

\[
\begin{array}{c}
1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10
\end{array}
\]

\[
\begin{array}{c}
x_5 \quad x_{10}
\end{array}
\]
Analysis of \texttt{RANDOMIZED-QUICKSORT} (cont.)

\[ E(X) = E\left(\sum_{i<j} X_{i,j}\right) = \sum_{i<j} E(X_{i,j}) = \sum_{i<j} \text{Prob}(X_{i,j} = 1) \]

\[ \leq \sum_{i<j} 2 \frac{1}{j - i + 1} \]

for some constant \( c > 0 \).

So \( E(X) = O(n \log_2 n) \).
Analysis of \texttt{RANDOMIZED-QUICKSORT} (cont.)

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E(X) = E\left(\sum_{i<j} X_{i,j}\right) = \sum_{i<j} E(X_{i,j}) = \sum_{i<j} \text{Prob}(X_{i,j} = 1)
\]

\[
\leq \sum_{i<j} 2^{\frac{1}{j-i+1}} = \sum_{i=1}^{n-1} \sum_{j=2}^{n} 2^{\frac{1}{j-i+1}}
\]

for some constant \(c > 0\).

So \(E(X) = \mathcal{O}(n \log_2 n)\).
Analysis of \texttt{RANDOMIZED-QUICKSORT} (cont.)

\begin{align*}
E(X) &= E\left(\sum_{i<j} X_{i,j}\right) = \sum_{i<j} E(X_{i,j}) = \sum_{i<j} \text{Prob}(X_{i,j} = 1) \\
&\leq \sum_{i<j} 2 \frac{1}{j-i+1} = \sum_{i=1}^{n-1} \sum_{j=2}^{n} 2 \frac{1}{j-i+1} \\
&\leq \sum_{i=1}^{n-1} 2 \sum_{k=1}^{n-1} \frac{1}{k+1} \\
&\leq \sum_{i=1}^{n-1} 2 \sum_{k=1}^{n-1} \frac{1}{k+1} \\
&\leq cn \log_2 n
\end{align*}

for some constant $c > 0$.

So $E(X) = O(n \log_2 n)$. 
Analysis of **Randomized-QuickSort** (cont.)

\[
E(X) = E\left(\sum_{i<j} X_{i,j}\right) = \sum_{i<j} E(X_{i,j}) = \sum_{i<j} \text{Prob}(X_{i,j} = 1)
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\[
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\]

\[
\leq \sum_{i=1}^{n-1} 2 \sum_{k=1}^{n-1} \frac{1}{k + 1} \leq \sum_{i=1}^{n-1} c \log_2 n
\]
Analysis of **RANDOMIZED-QUICKSORT** (cont.)

\[ E(X) = E\left( \sum_{i<j} X_{i,j} \right) = \sum_{i<j} E(X_{i,j}) = \sum_{i<j} \text{Prob}(X_{i,j} = 1) \]

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Chapter 7. Quicksort

Analysis of Randomized-QuickSort (cont.)

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for some constant \( c > 0 \).
Chapter 7. Quicksort

Analysis of \textsc{Randomized-QuickSort} (cont.)

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for some constant \( c > 0 \).

So \( E(X) = O(n \log_2 n) \).
Chapter 7. Quicksort

O(n log n) Sorting Algorithms

Graph showing the performance of Heap, Merge, and Quick sort algorithms.
Chapter 8. Lower Bounds and Sorting in Linear Time

We have used Big-$O$ for upper bounds. We need another notation for lower bounds.

Define $\Omega(g(n))$ be the set of functions that have growth rates not slower than $cg(n)$ for any given constant $c > 0$.

$\Omega(g(n)) = \{ f(n) : \exists c > 0, k > 0$ such that $f(n) \geq cg(n)$ for all $n \geq k \}$

• e.g., $\Omega(n \log n)$ includes the following functions: $14n \log n, \frac{1}{100}n \log n, n^2, n^3 \log n, 3^7002n, n!$, ...

Proof techniques for Big-$\Omega$ are similar to those for Big-$O$. 
Chapter 8. Lower Bounds and Sorting in Linear Time

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• We have used Big-$O$ for upper bounds.

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Chapter 8. Lower Bounds and Sorting in Linear Time

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Chapter 8. Lower Bounds and Sorting in Linear Time

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- We need another notation for lower bounds.

Define $\Omega(g(n))$ be the set of functions that have growth rates not slower than $cg(n)$ for any given constant $c > 0$.

$$\Omega(g(n)) = \{f(n) : \exists c > 0, k > 0 \text{ such that } f(n) \geq cg(n) \text{ for all } n \geq k\}$$

- e.g., $\Omega(n \log n)$ includes the following functions:

  $$14n \log n, \frac{1}{100}n \log n, n^2, n^3 \log n, \frac{3}{700}2^n, n!, \ldots$$
Chapter 8. Lower Bounds and Sorting in Linear Time

- We have used Big-$O$ for upper bounds.
- We need another notation for lower bounds.

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- Proof techniques for Big-$\Omega$ are similar to those for Big-$O$. 
Upper bound of an algorithm is a time sufficient (i.e., enough) for the algorithm to solve all instances. We make sure an upper bound should cover all instances; e.g., MergeSort has upper bound $O(n \log n)$. Is it correct to say MergeSort has upper bound $O(n^2)$? It is correct for two reasons: (1) since $cn \log n$ is sufficient, so is $cn^2$. (2) $O(n^2)$ contains all functions that $O(n \log n)$ contains. Is it correct to say MergeSort has upper bound $O(n)$? 
Chapter 8. Lower Bounds and Sorting in Linear Time

Upper bound of an algorithm
Upper bound of an algorithm
A time sufficient (i.e., enough) for the algorithm to solve all instances.
Upper bound of an algorithm

A time sufficient (i.e., enough) for the algorithm to solve all instances.

We make sure an upper bound should cover all instances;
Chapter 8. Lower Bounds and Sorting in Linear Time

Upper bound of **an algorithm**

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**Is it correct to say `MERGE_SORT` has upper bound $O(n^2)$?**
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Upper bound of \textit{an algorithm}

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e.g., \textsc{MergeSort} has upper bound $O(n \log n)$.

\textbf{Is it correct to say \textsc{MergeSort} has upper bound $O(n^2)$?}

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Is it correct to say MergeSort has upper bound $O(n)$?
Chapter 8. Lower Bounds and Sorting in Linear Time

Lower bound of an algorithm is a time necessary (i.e., needed) for the algorithm to solve all instances. \( l(n) \) is a lower bound – if some (generic) instance requires time \( l(n) \) or more to be solved by the algorithm.

For example, MergeSort has lower bound \( \Omega(n \log n) \).

Is it correct to say that MergeSort has lower bound \( \Omega(n^2) \)?

It is correct for two reasons:

1. Since \( cn \log n \) is necessary, so is \( cn \).
2. \( \Omega(n) \) contains all functions that \( \Omega(n \log n) \) contains.
Chapter 8. Lower Bounds and Sorting in Linear Time

Lower bound of an algorithm

Is it correct to say MergeSort has lower bound $\Omega(n \log n)$?

It is correct for two reasons:

1. since $cn \log n$ is necessary, so is $cn$.
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Is it correct to say MergeSort has lower bound $\Omega(n^2)$?
Chapter 8. Lower Bounds and Sorting in Linear Time

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Chapter 8. Lower Bounds and Sorting in Linear Time

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$l(n)$ is a lower bound – if some (generic) instance requires time $l(n)$ or more to be solved by the algorithm.
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Chapter 8. Lower Bounds and Sorting in Linear Time

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Chapter 8. Lower Bounds and Sorting in Linear Time

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Is it correct to say **MergeSort** has lower bound \( \Omega(n^2) \)?
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The best known upper bound for MergeSort is $O(n \log n)$, coinciding with the best known lower bound $\Omega(n \log n)$; both bounds are tight (i.e., optimal). Thus complexity is denoted with $\theta(n \log n)$, meaning both $O(n \log n)$ and $\Omega(n \log n)$.

We may not be so lucky for some other algorithms.
• The best known upper bound for MERGESORT is $O(n \log n)$,
Chapter 8. Lower Bounds and Sorting in Linear Time

- The best known upper bound for \texttt{MERGESORT} is $O(n \log n)$, coinciding with the best known lower bound $\Omega(n \log n)$;
• The best known upper bound for \textsc{MergeSort} is $O(n \log n)$, coinciding with the best known lower bound $\Omega(n \log n)$;

• Both bounds are tight (i.e., optimal). Thus complexity is denoted with $\theta(n \log n)$, meaning both $O(n \log n)$ and $\Omega(n \log n)$. 
The best known upper bound for MergeSort is $O(n \log n)$, coinciding with the best known lower bound $\Omega(n \log n)$;

Both bounds are tight (i.e., optimal). Thus complexity is denoted with $\theta(n \log n)$, meaning both $O(n \log n)$ and $\Omega(n \log n)$.

We may not be so lucky for some other algorithms.
Chapter 8. Lower Bounds and Sorting in Linear Time

Rec-Fibonacci($n$)

if $n = 1$ or $n = 2$, return (1);
else
    $T_1 =$ Rec-Fibonacci($n - 1$);
    $T_2 =$ Rec-Fibonacci($n - 2$);
return ($T_1 + T_2$);

Derive an upper bound:

$T(n) = c + T(n - 1) + T(n - 2)$, with $T(1) = T(2) = c \leq c + 2T(n - 1) \leq 2c + 2T(n - 2) \leq \ldots \leq (n - 2)c + 2n - 2T(2) \leq (n - 2)c + c^2 n - 2 = O(2^n)$. 
Chapter 8. Lower Bounds and Sorting in Linear Time

\textbf{Rec-Fibonacci}(n)

\textbf{if} $n = 1$ or $n = 2$, \textbf{return} (1);  
\textbf{else}  
$T_1 = \text{Rec-Fibonacci}(n - 1)$;  
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\textbf{return} ($T_1 + T_2$);

Derive an upper bound:

\[ T(n) = c + T(n - 1) + T(n - 2), \quad T(1) = T(2) = c \leq c + 2T(n - 1) \leq 2c + 2T(n - 2) \ldots \leq (n - 2)c + 2T(2) = O(2^n). \]
Chapter 8. Lower Bounds and Sorting in Linear Time

**Rec-Fibonacci**

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\text{if } n = 1 \text{ or } n = 2, \text{ return } (1); \\
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T(n) = c + T(n - 1) + T(n - 2), \text{ with } T(1) = T(2) = c
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Chapter 8. Lower Bounds and Sorting in Linear Time

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Derive an upper bound:

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$\leq c + 2T(n - 1)$
Chapter 8. Lower Bounds and Sorting in Linear Time

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$\leq c + 2T(n - 1)$

$\leq 2c + 2^2T(n - 2)$

$= O(2^n)$.
Chapter 8. Lower Bounds and Sorting in Linear Time

**Rec-Fibonacci**

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if \( n = 1 \) or \( n = 2 \), return \((1)\);
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```

Derive an upper bound:

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T(n) = c + T(n - 1) + T(n - 2), \text{ with } T(1) = T(2) = c \\
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Chapter 8. Lower Bounds and Sorting in Linear Time

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T(n) = c + T(n - 1) + T(n - 2), \text{ with } T(1) = T(2) = c
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\ldots
\leq (n - 2)c + 2^{n-2}T(2)
\]
Chapter 8. Lower Bounds and Sorting in Linear Time

Rec-Fibonacci\((n)\)

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\text{if } n = 1 \text{ or } n = 2, \text{ return } (1);
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\]

\[
\ldots
\]

\[
\leq (n - 2)c + 2^{n-2}T(2)
\]

\[
\leq (n - 2)c + c2^{n-2}
\]
Chapter 8. Lower Bounds and Sorting in Linear Time

Rec-Fibonacci(n)

if $n = 1$ or $n = 2$, return $(1)$;
else

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$T_2 = \text{Rec-Fibonacci}(n - 2)$;
return $(T_1 + T_2)$;

Derive an upper bound:

$T(n) = c + T(n - 1) + T(n - 2)$, with $T(1) = T(2) = c$

$\leq c + 2T(n - 1)$

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$\ldots$

$\leq (n - 2)c + 2^{n-2}T(2)$

$\leq (n - 2)c + c2^{n-2}$

$= O(2^n)$. 
Chapter 8. Lower Bounds and Sorting in Linear Time

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    if \ n = 1 \ or \ n = 2, \ \textbf{return} \ (1);
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\end{verbatim}
Chapter 8. Lower Bounds and Sorting in Linear Time

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Chapter 8. Lower Bounds and Sorting in Linear Time

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```

Derive a lower bound:

\[ T(n) = c + T(n - 1) + T(n - 2), \text{ with } T(1) = T(2) = c \]

\[ T(n) \geq c + 2T(n - 2), \quad \text{for } n \geq 3 \]

\[ T(n) \geq 2c + 2T(n - 4), \quad \text{for } n \geq 5 \]

\[ T(n) \geq \left( \frac{2}{1} \right)^{\frac{n}{2}} \cdot \left( \frac{1}{2} \right)^{n/2}, \quad \text{for } n \geq 5 \]

\[ T(n) \geq \Omega(2^{\frac{n}{2}}), \quad \text{for } n \geq 5 \]
Chapter 8. Lower Bounds and Sorting in Linear Time

**Rec-Fibonacci**

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\text{if } n = 1 \text{ or } n = 2, \text{ return } (1);
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\text{return } (T_1 + T_2);

Derive a lower bound:

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T(n) = c + T(n - 1) + T(n - 2), \text{ with } T(1) = T(2) = c
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Rec-Fibonacci(n)

if $n = 1$ or $n = 2$, return (1);
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    $T_1 = \text{Rec-Fibonacci}(n - 1)$;
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Derive a lower bound:
$T(n) = c + T(n - 1) + T(n - 2)$, with $T(1) = T(2) = c$
\[
\geq c + 2T(n - 2)
\geq 2c + 2^2T(n - 4)
\]
Chapter 8. Lower Bounds and Sorting in Linear Time

\textbf{Rec-Fibonacci}(n)

\textbf{if} \ n = 1 \textbf{ or } n = 2, \textbf{return} \ (1); 
\textbf{else}
\hspace{1cm} T_1 = \textbf{Rec-Fibonacci}(n - 1);
\hspace{1cm} T_2 = \textbf{Rec-Fibonacci}(n - 2);
\hspace{1cm} \textbf{return} \ (T_1 + T_2);

Derive a lower bound:
\[ T(n) = c + T(n - 1) + T(n - 2), \text{ with } T(1) = T(2) = c \]
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Chapter 8. Lower Bounds and Sorting in Linear Time

Rec-Fibonacci($n$)

if $n = 1$ or $n = 2$, return (1);
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Derive a lower bound:
$T(n) = c + T(n - 1) + T(n - 2)$, with $T(1) = T(2) = c$
$\geq c + 2T(n - 2)$
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\ldots
$\geq \frac{(n-2)}{2}c + 2^{\frac{n-2}{2}}T(2)$
Rec-Fibonacci\( (n) \)

\[
\text{if } n = 1 \text{ or } n = 2, \text{ return } (1); \\
\text{else} \\
\quad T_1 = \text{Rec-Fibonacci}(n - 1); \\
\quad T_2 = \text{Rec-Fibonacci}(n - 2); \\
\quad \text{return } (T_1 + T_2);
\]

Derive a lower bound:
\[
T(n) = c + T(n - 1) + T(n - 2), \text{ with } T(1) = T(2) = c \\
\geq c + 2T(n - 2) \\
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\ldots \\
\geq \frac{(n-2)c}{2} + 2^\frac{n-2}{2}T(2) \\
\geq \frac{(n-2)c}{2} + c2^\frac{n-2}{2}
\]
Chapter 8. Lower Bounds and Sorting in Linear Time

Rec-Fibonacci\((n)\)

\[
\begin{align*}
\text{if } n = 1 \text{ or } n = 2, & \quad \text{return (1);} \\
\text{else} & \\
T_1 & = \text{Rec-Fibonacci}(n - 1); \\
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\text{return } (T_1 + T_2); \\
\end{align*}
\]

Derive a lower bound:
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T(n) = c + T(n - 1) + T(n - 2), \text{ with } T(1) = T(2) = c \\
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\vdots \\
\geq \frac{(n-2)}{2}c + 2^{\frac{n-2}{2}}T(2) \\
\geq \frac{(n-2)}{2}c + c2^{\frac{n-2}{2}} \\
= \Omega(2^{\frac{n}{2}}).
\]
Chapter 8. Lower Bounds and Sorting in Linear Time

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= \Omega(2^{\frac{n}{2}}).
\]

Compare to the upper bound we have derived \( O(2^n) \),


Chapter 8. Lower Bounds and Sorting in Linear Time

Rec-Fibonacci\( (n) \)

\[
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= \Omega(2^{\frac{n}{2}}).
\]

Compare to the upper bound we have derived \( O(2^n) \), not tight!
Upper bound of a problem
A time sufficient (i.e., enough) to solve all instances of the problem. To derive an upper bound, we can resort to algorithms solving the problem; an upper bound is of such an algorithm is also an upper bound for the problem. e.g., \(O(n^2)\) is an upper bound for Sorting (why?) \(O(n \log n)\) is also an upper bound for Sorting (why?) One important task in algorithm research: to design algorithms achieving better upper bounds (small time complexity)
Chapter 8. Lower Bounds and Sorting in Linear Time

Upper bound of a problem
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**e.g.,** $O(n^2)$ is an upper bound for **Sorting**
Upper bound of a problem

A time sufficient (i.e., enough) to solve all instances of the problem.

To derive an upper bound, we can resort to algorithms solving the problem; an upper bound is of such an algorithm is also an upper bound for the problem.

e.g., $O(n^2)$ is an upper bound for Sorting (why?)
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One important task in algorithm research: to design algorithms achieving better upper bounds (small time complexity)
Chapter 8. Lower Bounds and Sorting in Linear Time

Lower bound of a problem. A time necessary (i.e., needed) for all instances in the problem to be solved. Can we use an algorithm lower bound for the problem lower bound? e.g., consider the Sorting problem. Insertion Sort has lower bound $\Omega(n^2)$ (why?), can we say the Sorting problem has lower bound $\Omega(n^2)$? No! because MergeSort has upper bound $O(n \log n)$. Likewise, we cannot say the Sorting has lower bound $\Omega(n \log n)$. Statement "problem Sorting has lower bound $\Omega(n \log n)$" $\iff$ statement "there is no algorithm running faster than time $cn \log n$".
Chapter 8. Lower Bounds and Sorting in Linear Time

Lower bound of a problem
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Chapter 8. Lower Bounds and Sorting in Linear Time

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Statement “problem **Sorting** has lower bound $\Omega(n \log n)$”
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• To derive a lower bound for a **problem**, we **cannot** examine an infinite number of algorithms!

• Lower bounds can only be derived mathematically, but not from existing algorithms.
Chapter 8. Lower Bounds and Sorting in Linear Time

Deriving a lower bound for sorting

with decision tree as algorithm/computation model

Claim 1: total number of leaves is $\geq n!$.

Claim 2: the height of the tree at least $\geq \log n!$.

(The minimum of heights of all such trees!)
Chapter 8. Lower Bounds and Sorting in Linear Time

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**Theorem**: Sorting needs $\Omega(n \log n)$ comparisons on comparison-based computation models.
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Prove.
The longest path from the root to a leaf is $\Omega(\log n!)$. I.e.,
the number of comparisons needed in the worst case is $\Omega(\log n!)$.  

$n! = n(n-1)(n-2) \cdots (n-n) \geq \left(\frac{n}{2}\right)^{n/2} \cdot 2^{n/2-1} \geq \frac{n^{n/2}}{2^{n/2}}$ or by Stirling's formula:

$n! = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n (1 + O(1/n))$.

$\Omega(\log(\log n)) = \Omega(n \log n)$
**Theorem**: Sorting needs $\Omega(n \log n)$ comparisons on comparison-based computation models.

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The longest path from the root to a leaf is $\Omega(\log n!)$. I.e., the number of comparisons needed in the worst case is $\Omega(\log n!)$.

$$n! = n(n - 1)(n - 2) \cdots (n - \frac{n}{2})(n - \frac{n}{2} - 1) \cdots 2 \times 1$$

$$\geq \left(\frac{n}{2}\right)^{\frac{n}{2}} \times 2^{\frac{n}{2}} - 1 \geq \frac{1}{2} n^{\frac{n}{2}}$$

or by Stirling’s formula:

$$n! = \sqrt{2\pi n} (n/e)^n (1 + O(1/n))$$

$$\Omega(\log(n!)) = \Omega(n \log n)$$
Chapter 8. Lower Bounds and Sorting in Linear Time

Sorting algorithms with worst case linear time
Chapter 8. Lower Bounds and Sorting in Linear Time

Sorting algorithms with worst case linear time

- count sort
- radix sort
- bucket sort
Count sort

Algorithm Counting-Sort(A, B, k)
1. A contains n integers;
k is the max
2. C[i] = 0
3. for j = 1 to length[A]
5. {C[i] contains the number of elements whose values = i}
6. for i = 1 to k
7. C[i] = C[i] + C[i−1]
8. {C[i] contains the number of elements whose values ≤ i}
9. for j = length[A] downto 1

Example: A: 2 5 3 0 2 3 0 3, k = 5, C: 2 0 2 3 0 1

analysis: T(n) = O(k + n)
Count sort

Algorithm COUNTING-SORT \((A, B, k)\) \{\(A\) contains \(n\) integers; \(k\) is the max\}
Chapter 8. Lower Bounds and Sorting in Linear Time

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1. \(\textbf{for} \ i = 0 \ \textbf{to} \ k\)
2. \(C[i] = 0\)
Chapter 8. Lower Bounds and Sorting in Linear Time

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1. for \(i = 0\) to \(k\)
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3. for \(j = 1\) to \(\text{length}[A]\)
Chapter 8. Lower Bounds and Sorting in Linear Time

**Count sort**

Algorithm `COUNTING-SORT (A, B, k)` \{A contains n integers; k is the max\}

1. \textbf{for} $i = 0$ \textbf{to} $k$
2. \hspace{1em} $C[i] = 0$
3. \textbf{for} $j = 1$ \textbf{to} `length[A]`
4. \hspace{1em} $C[A[j]] = C[A[j]] + 1$

Example: $A: 2 \ 5 \ 3 \ 0 \ 2 \ 3 \ 0 \ 3$, $k = 5$, $C: 2 \ 0 \ 2 \ 3 \ 0 \ 1$

analysis: $T(n) = O(k + n)$
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1. \textbf{for} \(i = 0\) \textbf{to} \(k\)
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Chapter 8. Lower Bounds and Sorting in Linear Time

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8. \(\{C[i] \text{ contains the number of elements whose values } \leq i\}\)
Count sort

Algorithm COUNTING-SORT \((A, B, k)\) \{\(A\) contains \(n\) integers; \(k\) is the max\}

1. \textbf{for} \(i = 0\) \textbf{to} \(k\)
2. \hspace{0.5cm} \(C[i] = 0\)
3. \textbf{for} \(j = 1\) \textbf{to} length\([A]\)
4. \hspace{0.5cm} \(C[A[j]] = C[A[j]] + 1\)
5. \hspace{0.5cm} \(\{C[i] \text{ contains the number of elements whose values } = i\}\)
6. \textbf{for} \(i = 1\) \textbf{to} \(k\)
7. \hspace{0.5cm} \(C[i] = C[i] + C[i - 1]\)
8. \hspace{0.5cm} \(\{C[i] \text{ contains the number of elements whose values } \leq i\}\)
9. \textbf{for} \(j = \text{length}[A]\) \textbf{downto} 1

Example: \(A: 2\ 5\ 3\ 0\ 2\ 3\ 0\ 3\), \(k = 5\), \(C: 2\ 0\ 2\ 3\ 0\ 1\)

\(T(n) = O(k + n)\)
Count sort

Algorithm COUNTING-SORT (A, B, k) \{ A contains n integers; k is the max \}
1. \textbf{for} \ i = 0 \ \textbf{to} \ k \\
2. \hspace{2em} \texttt{C}[i] = 0 \\
3. \textbf{for} \ j = 1 \ \textbf{to} \ \text{length}[A] \\
4. \hspace{2em} \texttt{C}[A[j]] = \texttt{C}[A[j]] + 1 \\
5. \hspace{2em} \{\texttt{C}[i] \text{ contains the number of elements whose values } = i\} \\
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10. \hspace{2em} \texttt{B}[\texttt{C}[A[j]]] = A[j] \\

Example: A: 2 5 3 0 2 3 0 3, \ k = 5, \ C: 2 0 2 3 0 1 \\
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Chapter 8. Lower Bounds and Sorting in Linear Time

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9. for \(j = \text{length}[A]\) downto 1
10. \(B[C[A[j]]] = A[j]\)
11. \(C[A[j]] = C[A[j]] - 1\)

Example: \(A: 2\ 5\ 3\ 0\ 2\ 3\ 0\ 3,\ k = 5,\ C: 2\ 0\ 2\ 3\ 0\ 1\)

analysis: \(T(n) = O(k + n)\)
Chapter 8. Lower Bounds and Sorting in Linear Time

Count sort

Algorithm \text{COUNTING-SORT} (A, B, k) \quad \{A contains n integers; k is the max\}
1. \textbf{for} \ i = 0 \ \textbf{to} \ k
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Chapter 8. Lower Bounds and Sorting in Linear Time

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Example: \(A: 2\ 5\ 3\ 0\ 2\ 3\ 0\ 3, \ k = 5,\ C: 2\ 0\ 2\ 3\ 0\ 1\)
Chapter 8. Lower Bounds and Sorting in Linear Time

Count sort

Algorithm Counting-Sort \((A, B, k)\) \{\(A\) contains \(n\) integers; \(k\) is the max\}
1. \textbf{for} \(i = 0\) \textbf{to} \(k\)
2. \hspace{1em} \(C[i] = 0\)
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Example: \(A: 2 \ 5 \ 3 \ 0 \ 2 \ 3 \ 0 \ 3\), \(k = 5\), \(C: 2 \ 0 \ 2 \ 3 \ 0 \ 1\)

analysis:
Chapter 8. Lower Bounds and Sorting in Linear Time

Count sort

Algorithm COUNTING-SORT \((A, B, k)\) \(\{A \text{ contains } n \text{ integers}; k \text{ is the max}\}\)

1. \textbf{for } \(i = 0 \text{ to } k\)
2. \hspace{10pt} \(C[i] = 0\)
3. \textbf{for } \(j = 1 \text{ to } \text{length}[A]\)
4. \hspace{10pt} \(C[A[j]] = C[A[j]] + 1\)
5. \{\(C[i] \text{ contains the number of elements whose values } = i\}\)
6. \textbf{for } \(i = 1 \text{ to } k\)
7. \hspace{10pt} \(C[i] = C[i] + C[i-1]\)
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Example: \(A: 2 \ 5 \ 3 \ 0 \ 2 \ 3 \ 0 \ 3\), \(k = 5\), \(C: 2 \ 0 \ 2 \ 3 \ 0 \ 1\)

analysis: \(T(n) = O(k + n)\)
Radix Sort:

Algorithm Radix-Sort \((A,d)\)

1. for \(i = 1\) to \(d\)
2. sort \(A\) on the \(i\)th digit

Lemma. Given \(n\) \(b\)-bit binary numbers and any positive \(r \leq b\).

Radix-Sort uses \(\Theta(\lceil b/r \rceil (n + 2r))\) time.
Chapter 8. Lower Bounds and Sorting in Linear Time

Radix Sort:

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<th>329</th>
</tr>
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<tbody>
<tr>
<td>457</td>
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<td>657</td>
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<td>839</td>
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<tr>
<td>720</td>
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<td>457</td>
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<tr>
<td>355</td>
<td>839</td>
<td>657</td>
<td>839</td>
</tr>
</tbody>
</table>
Chapter 8. Lower Bounds and Sorting in Linear Time

Radix Sort:

329  720  720  329
457  355  329  355
657  436  436  436
839  457  839  457
436  657  355  657
720  329  457  720
355  839  657  839

Algorithm Radix-Sort($A, d$)
1. for $i = 1$ to $d$
Chapter 8. Lower Bounds and Sorting in Linear Time

Radix Sort:

329  720  720  329  
457  355  329  355  
657  436  436  436  
839  457  839  457  
436  657  355  657  
720  329  457  720  
355  839  657  839  

Algorithm Radix-Sort \((A, d)\)
1. \(\text{for } i = 1 \text{ to } d\)
2. \(\text{sort } A \text{ on the } i\text{th digit}\)
Chapter 8. Lower Bounds and Sorting in Linear Time

Radix Sort:

329 720 720 329
457 355 329 355
657 436 436 436
839 457 839 457
436 657 355 657
720 329 457 720
355 839 657 839

Algorithm \textsc{Radix-Sort}(A, d)
1. \textbf{for} $i = 1$ \textbf{to} $d$
2. \hspace{0.5cm} \textbf{sort} $A$ on the $i$th digit

Lemma. Given $n$ $b$-bit binary numbers and any positive $r \leq b$. \textsc{Radix-Sort} uses $\Theta([b/r](n + 2^r))$ time.
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Proof. Each $b$-digit binary number can be regarded as $[b/r]$ $r$-digit binary numbers. These $r$-digit binary numbers are of integer values in the range of $\{0, 1, \ldots, 2^r - 1\}$. 
Lemma. Given $n$ $b$-bit binary numbers and any positive $r \leq b$. Radix-Sort uses $\Theta(\lceil b/r \rceil (n + 2^r))$ time.

Proof. Each $b$-digit binary number can be regarded as $\lceil b/r \rceil$ $r$-digit binary numbers. These $r$-digit binary numbers are of integer values in the range of $\{0, 1, \ldots, 2^r - 1\}$. Run Radix-Sort on the original binary numbers assumed to be $\lceil b/r \rceil$ columns.
Lemma. Given $n$ $b$-bit binary numbers and any positive $r \leq b$. Radix-Sort uses $\Theta([b/r](n + 2^r))$ time.

Proof. Each $b$-digit binary number can be regarded as $[b/r]$ $r$-digit binary numbers. These $r$-digit binary numbers are of integer values in the range of $\{0, 1, \ldots, 2^r - 1\}$.

Run Radix-Sort on the original binary numbers assumed to be $[b/r]$ columns.

For every column, sorting by Counting-Sort with $2^r - 1$ being the maximum.
Lemma. Given $n$ $b$-bit binary numbers and any positive $r \leq b$. Radix-Sort uses $\Theta(\lceil b/r \rceil (n + 2^r))$ time.

Proof. Each $b$-digit binary number can be regarded as $\lceil b/r \rceil$ $r$-digit binary numbers. These $r$-digit binary numbers are of integer values in the range of $\{0, 1, \ldots, 2^r - 1\}$.

Run Radix-Sort on the original binary numbers assumed to be $\lceil b/r \rceil$ columns.

For every column, sorting by Counting-Sort with $2^r - 1$ being the maximum.

The total time is $O(\lceil b/r \rceil (n + 2^r))$, where $(n + 2^r)$ is time for Counting-Sort.
Lemma. Given \( n \) \( b \)-bit binary numbers and any positive \( r \leq b \). Radix-Sort uses \( \Theta(\lceil b/r \rceil (n + 2^r)) \) time.

Proof. Each \( b \)-digit binary number can be regarded as \( \lceil b/r \rceil \) \( r \)-digit binary numbers. These \( r \)-digit binary numbers are of integer values in the range of \( \{0, 1, \ldots, 2^r - 1\} \).

Run Radix-Sort on the original binary numbers assumed to be \( \lceil b/r \rceil \) columns.

For every column, sorting by Counting-Sort with \( 2^r - 1 \) being the maximum.

The total time is \( O(\lceil b/r \rceil (n + 2^r)) \), where \( (n + 2^r) \) is time for Counting-Sort.

Since all steps in the two algorithms are mandatory, the total time is also \( \Omega(\lceil b/r \rceil (n + 2^r)) \), thus \( \Theta(\lceil b/r \rceil (n + 2^r)) \).
Lemma. Given \( n \) \( b \)-bit binary numbers and any positive \( r \leq b \). Radix-Sort uses \( \Theta(\lceil b/r \rceil (n + 2^r)) \) time.

Proof. Each \( b \)-digit binary number can be regarded as \( \lceil b/r \rceil \) \( r \)-digit binary numbers. These \( r \)-digit binary numbers are of integer values in the range of \( \{0, 1, \ldots, 2^r - 1\} \).

Run Radix-Sort on the original binary numbers assumed to be \( \lceil b/r \rceil \) columns.

For every column, sorting by Counting-Sort with \( 2^r - 1 \) being the maximum.

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Since all steps in the two algorithms are mandatory, the total time is also \( \Omega(\lceil b/r \rceil (n + 2^r)) \), thus \( \Theta(\lceil b/r \rceil (n + 2^r)) \).

Once \( b \) and \( n \) are given, we can choose \( r \) to minimize the quantity \( \lceil b/r \rceil (n + 2^r) \).
Chapter 8. Lower Bounds and Sorting in Linear Time

Bucket Sort (assuming uniform distribution of inputs)
Chapter 8. Lower Bounds and Sorting in Linear Time

**Bucket Sort** (assuming uniform distribution of inputs)

**Algorithm** \texttt{Bucket-Sort}(A)
Bucket Sort (assuming uniform distribution of inputs)

Algorithm Bucket-Sort(A)
1. $n = length[A]$
Bucket Sort (assuming uniform distribution of inputs)

Algorithm Bucket-Sort($A$)
1. $n = length[A]$
2. for $i = 1$ to $n$
Chapter 8. Lower Bounds and Sorting in Linear Time

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Algorithm **Bucket-Sort**($A$)
1. $n = length[A]$
2. for $i = 1$ to $n$
3. insert $A[i]$ into list $B[\lfloor nA[i] \rfloor]$

A: .78 .17 .39 .26 .72 .94 .21 .12 .23 .68

B: 0 / 1 → .12 2 → .17 3 → .21 4 → .23 5 → .26 6 → .39 7 → .68 8 → .72 9 → .78 10 → .94
Bucket Sort (assuming uniform distribution of inputs)

Algorithm Bucket-Sort(A)
1. \( n = length[A] \)
2. \( \text{for } i = 1 \text{ to } n \)
3. \( \text{insert } A[i] \text{ into list } B[\lfloor nA[i] \rfloor] \)
4. \( \text{for } i = 0 \text{ to } n - 1 \)
Chapter 8. Lower Bounds and Sorting in Linear Time

**Bucket Sort** (assuming uniform distribution of inputs)

Algorithm **Bucket-Sort** \( (A) \)
1. \( n = length[A] \)
2. \( \text{for } i = 1 \text{ to } n \)
3.     insert \( A[i] \) into list \( B[\lfloor nA[i] \rfloor] \)
4. \( \text{for } i = 0 \text{ to } n - 1 \)
5.     sort list \( B[i] \) with Insertion Sort
Chapter 8. Lower Bounds and Sorting in Linear Time

**Bucket Sort** (assuming uniform distribution of inputs)

Algorithm **Bucket-Sort**($A$)

1. $n = \text{length}[A]$
2. for $i = 1$ to $n$
3. \hspace{1em} insert $A[i]$ into list $B[\lfloor nA[i] \rfloor]$
4. for $i = 0$ to $n - 1$
5. \hspace{1em} sort list $B[i]$ with **Insertion Sort**
6. concatenate the lists $B[0], B[1], ..., B[n - 1]$
Chapter 8. Lower Bounds and Sorting in Linear Time

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A: 0.78 0.17 0.39 0.26 0.72 0.94 0.21 0.12 0.23 0.68

B: 0 /
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A: .78 .17 .39 .26 .72 .94 .21 .12 .23 .68

B: 0 /
1 \rightarrow .12 \rightarrow .17
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B: 0 /
1 → .12 → .17
2 → .21 → .23 → .26
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A: \( .78 \ .17 \ .39 \ .26 \ .72 \ .94 \ .21 \ .12 \ .23 \ .68 \)

B: \( 0 / \)
\( 1 \rightarrow .12 \rightarrow .17 \)
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\( 3 \rightarrow .39 \)
Chapter 8. Lower Bounds and Sorting in Linear Time

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A: \( .78 \ .17 \ .39 \ .26 \ .72 \ .94 \ .21 \ .12 \ .23 \ .68 \)

B: \( 0 / \)
  \( 1 \rightarrow .12 \rightarrow .17 \)
  \( 2 \rightarrow .21 \rightarrow .23 \rightarrow .26 \)
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  \( 4 / \)
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B: 0 /
1 → .12 → .17
2 → .21 → .23 → .26
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5 /
Chapter 8. Lower Bounds and Sorting in Linear Time

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1 \(\rightarrow\) .12 \(\rightarrow\) .17
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\begin{itemize}
\item A: .78 .17 .39 .26 .72 .94 .21 .12 .23 .68
\item B: 0 /
\hspace{1em} 1 → .12 → .17
\hspace{1em} 2 → .21 → .23 → .26
\hspace{1em} 3 → .39
\hspace{1em} 4 /
\hspace{1em} 5 /
\hspace{1em} 6 → .68
\hspace{1em} 7 → .72 → .78
\end{itemize}
Chapter 8. Lower Bounds and Sorting in Linear Time

**Bucket Sort** (assuming uniform distribution of inputs)

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A: .78 .17 .39 .26 .72 .94 .21 .12 .23 .68

B: 0 /
   1 → .12 → .17
   2 → .21 → .23 → .26
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   8 /
Bucket Sort (assuming uniform distribution of inputs)

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1 \( \rightarrow .12 \rightarrow .17 \)
2 \( \rightarrow .21 \rightarrow .23 \rightarrow .26 \)
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8 /
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Bucket Sort (assuming uniform distribution of inputs)

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B: \( 0 / \)
  1 → .12 → .17
  2 → .21 → .23 → .26
  3 → .39
  4 /
  5 /
  6 → .68
  7 → .72 → .78
  8 /
  9 → .94
Chapter 9. Medians and Order Statistics

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• find the maximum: linear time
Chapter 9. Medians and Order Statistics

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- find the maximum: linear time
- find the minimum: linear time
Chapter 9. Medians and Order Statistics

• find the maximum: linear time
• find the minimum: linear time
• find the median (i.e., the \( \frac{n}{2} \)th smallest element)?
Chapter 9. Medians and Order Statistics

Chapter 9. Medians and order statistics

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  the problem has upper bound $O(n \log_2 n)$. 
Chapter 9. Medians and Order Statistics

• find the maximum: linear time
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Chapter 9. Medians and Order Statistics

- find the maximum: linear time
- find the minimum: linear time
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  the problem has upper bound $O(n \log_2 n)$. why?

Can we do better?
Chapter 9. Medians and Order Statistics

Selection problem

Input: a list $A$ of elements, an integer $i$;

Output: the $i$th smallest element in $A$;

There are algorithms solving it in linear time.

Two types of algorithms:

- Selection in expected linear time (but worst case $\Theta(n^2)$)
- Selection in worst case linear time
Chapter 9. Medians and Order Statistics

Selection problem
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Chapter 9. Medians and Order Statistics

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Chapter 9. Medians and Order Statistics

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Two types of algorithms:

- Selection in *expected* linear time (but worst case $\Theta(n^2)$)
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Chapter 9. Medians and Order Statistics

Selection in *expected* linear time
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**Input:** a list \( A \) of elements, an integer \( i \);
Chapter 9. Medians and Order Statistics

Selection in *expected* linear time

**Input:** a list $A$ of elements, an integer $i$;
**Output:** the $i$th smallest element in $A$;
Chapter 9. Medians and Order Statistics

Selection in *expected* linear time

**Input:** a list $A$ of elements, an integer $i$;

**Output:** the $i$th smallest element in $A$;

Idea of the algorithm:

- randomly identify a pivot $x$ and partition the list $A$ into two sublists $A_l$ and $A_u$; assume the **rank** of $x$ is $k$;
Chapter 9. Medians and Order Statistics

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- if $i = k$, done, return $(x)$;
Selection in *expected* linear time

**Input:** a list $A$ of elements, an integer $i$;

**Output:** the $i$th smallest element in $A$;

Idea of the algorithm:

- randomly identify a pivot $x$ and partition the list $A$ into two sublists $A_l$ and $A_u$; assume the rank of $x$ is $k$;
- if $i = k$, done, return ($x$);
- else if $k > i$, recursively do for $A_l$ with $i$;
Selection in expected linear time

**Input:** a list $A$ of elements, an integer $i$;  
**Output:** the $i$th smallest element in $A$;

Idea of the algorithm:

- randomly identify a pivot $x$ and partition the list $A$ into two sublists $A_l$ and $A_u$;  
  assume the rank of $x$ is $k$;
- if $i = k$, done, return $(x)$;
- else if $k > i$, recursively do for $A_l$ with $i$;
  else recursively do for $A_u$ with $i - k$;
Chapter 9. Medians and Order Statistics

Algorithm RANDOMIZED-SELECT \((A, p, r, i)\)
Algorithm RANDOMIZED-SELECT (A, p, r, i)
1. if \( p = r \)
Algorithm RANDOMIZED-SELECT \((A, p, r, i)\)

1. \textbf{if} \(p = r\)
2. \textbf{return} \((A[p])\)
Chapter 9. Medians and Order Statistics

Algorithm RANDOMIZED-SELECT \((A, p, r, i)\)
1. \textbf{if} \(p = r\)
2. \textbf{return} \((A[p])\)
3. \(q = \text{RANDOMIZED PARTITION} \((A, p, r)\)\)
Algorithm RANDOMIZED-SELECT \((A, p, r, i)\)

1. if \(p = r\)
2. return \((A[p])\)
3. \(q = \text{RANDOMIZED PARTITION}(A, p, r)\)
4. \(k = q - p + 1\)
Algorithm Randomized-Select \((A, p, r, i)\)
1. \textbf{if} \(p = r\)
2. \hspace{1em} \textbf{return} \((A[p])\)
3. \quad \(q = \text{Randomized Partition} \((A, p, r)\)\)
4. \quad \(k = q - p + 1\)
5. \textbf{if} \(i = k\)

If pivots always partition lists into \(n^r\), for some \(r > 1\),
the time \(T(n)\) would have the recurrence
\[ T(n) \leq \max\{T(\frac{n}{r}), T(\frac{n}{r} - 1)\} + cn \]
assuming \(r \geq 2\),
\[ T(n) \leq cn(\frac{n}{r} - 1) + cn(\frac{n}{r} - 1)^2 + ... + cn(\frac{n}{r} - 1)^m = O(n) \]
where \((\frac{n}{r} - 1)^m = 1\), \(m = \log_r n\).
Algorithm \textsc{Randomized-Select} \hspace{1pt} (A, p, r, i)
\begin{enumerate}
  \item \textbf{if} \hspace{1pt} p = r
  \item \hspace{1pt} \textbf{return} \hspace{1pt} (A[p])
  \item \hspace{1pt} q = \textsc{Randomized Partition} \hspace{1pt} (A, p, r)
  \item \hspace{1pt} k = q - p + 1
  \item \textbf{if} \hspace{1pt} i = k
  \item \hspace{1pt} \textbf{return} \hspace{1pt} (A[q])
\end{enumerate}
Algorithm Randomized-Select \((A, p, r, i)\)

1. \textbf{if} \(p = r\)
2. \hspace{1em} \textbf{return} \((A[p])\)
3. \(q = \text{Randomized Partition} (A, p, r)\)
4. \(k = q - p + 1\)
5. \textbf{if} \(i = k\)
6. \hspace{1em} \textbf{return} \((A[q])\)
7. \textbf{else if} \(i < k\)
Algorithm `RANDOMIZED-SELECT (A, p, r, i)`
1. if \( p = r \)
2. return \( A[p] \)
3. \( q = \text{RANDOMIZED PARTITION} (A, p, r) \)
4. \( k = q - p + 1 \)
5. if \( i = k \)
6. return \( A[q] \)
7. else if \( i < k \)
8. return \( \text{RANDOMIZED-SELECT} (A, p, q - 1, i) \)
Algorithm Randomized-Select\( (A, p, r, i) \)
1. \textbf{if} \( p = r \)
2. \hspace{1em} \textbf{return} \( (A[p]) \)
3. \( q = \text{Randomized Partition} \( (A, p, r) \) \)
4. \( k = q - p + 1 \)
5. \textbf{if} \( i = k \)
6. \hspace{1em} \textbf{return} \( (A[q]) \)
7. \textbf{else if} \( i < k \)
8. \hspace{2em} \textbf{return} \( \text{Randomized-Select} \( (A, p, q - 1, i) \) \)
9. \hspace{2em} \textbf{else return} \( \text{Randomized-Select} \( (A, q + 1, r, i - k) \) \)
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Algorithm \textsc{Randomized-Select} \((A, p, r, i)\)

1. \textbf{if} \(p = r\)
2. \hspace{1em} \textbf{return} \((A[p])\)
3. \(q = \textsc{Randomized Partition} \((A, p, r)\)\)
4. \(k = q - p + 1\)
5. \textbf{if} \(i = k\)
6. \hspace{1em} \textbf{return} \((A[q])\)
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If pivots always partition lists into \(\frac{n}{r} : \frac{r-1}{r} n\), for some \(r > 1\),
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If pivots always partition lists into \(\frac{n}{r} : \frac{r - 1}{r} n\), for some \(r > 1\), time \(T(n)\) would have the recurrence

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\]

assuming \(r \geq 2\),

\[
T(n) \leq cn\left(\frac{r-1}{r}\right) + cn\left(\frac{r-1}{r}\right)^2 + cn\left(\frac{r-1}{r}\right)^3 + \ldots cn\left(\frac{r-1}{r}\right)^m = O(n)
\]

where \(\left(\frac{r-1}{r}\right)^m n = 1\), \(m = \log_{\frac{r-1}{r}} n\)
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Performance analysis

The worst case: running time $\Theta(n^2)$.

Average case: $E[T(n)]$ • on sublist $A[p..r]$, assume $n = r - p + 1$;
• the algorithm identifies a pivot and recursively computes on sublist $A[p..q]$ (or $A[q+1..r]$);
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- so the expected time $E[T(n)]$ needs to include the average time of recursion on the case when sublist $A[p..q]$ possibly has lengths $k = 0, 1, 2, \ldots, n - 1$
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• thus the expected time $E[T(n)]$ is computed as

$$E[T(n)] \leq \sum_{k=1}^{n} \frac{1}{n} \cdot E[T(\max\{k-1, n-k\})] + an, \text{ for some constant } a > 0$$
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because $\max\{k - 1, n - k\} = k - 1$ if $k > n/2$ and $\max\{k - 1, n - k\} = n - k$ if $k \leq n/2$

$$E[T(n)] \leq \frac{2}{n} \sum_{k=n/2}^{n-1} E[T(k)] + an$$
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We conclude that

\[ E[T(n)] \leq \frac{2}{n} \sum_{k=2}^{n-1} k = \frac{n}{2} \]

Theorem.

\[ E[T(n)] = O(n). \]

Proof (by substitution method).

We will prove that

\[ E[T(n)] \leq cn \quad \text{for some } c > 0. \]

• Base case: \( n = 2 \), we will decide later;

• Assumption: for all \( k \leq n - 1 \),

\[ E[T(k)] \leq ck; \]

• Induction:

\[ E[T(k)] \leq \frac{2}{n} \sum_{k=2}^{n-1} k = \frac{n}{2} E[T(k)] + an \leq \frac{2}{n} \sum_{k=2}^{n-1} k = \frac{n}{2} ck + an = 2c \left( \frac{n}{2} - \frac{1}{2} \sum_{k=1}^{n-1} k \right) + an = \cdots = 3cn/4 + c/2 + an \leq cn \quad \text{when } (cn/4 - c/2 - an) \geq 0. \]

That is when

\[ n \geq \frac{2c}{c-4a}. \]
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We conclude that $E[T(n)] \leq \frac{2}{n} \sum_{k=n/2}^{n-1} E[T(k)] + an$
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**Theorem.** $E[T(n)] = O(n)$. 

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We will prove that $E[T(n)] \leq cn$ for some $c > 0$. 

- **Base case:** $n = ?$, we will decide later; 
- **Assumption:** for all $k \leq n-1$, $E[T(k)] \leq ck$; 
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  \[
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  \]
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  \]
  \[
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  \]
  \[
  = \frac{3c}{4} n - \frac{c}{2} a n + an 
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- **Base case:** $T(n) \leq cn$, for $n < \frac{2c}{c - 4a}$, 

How to prove?
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We conclude that $E[T(n)] \leq 2/n\sum_{k=n/2}^{n-1} E[T(k)] + an$

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- **Base case:** \( T(n) \leq cn \), for \( n < 2c/(c - 4a) \),
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- Base case: \( T(n) \leq cn \), for \( n < \frac{2c}{(c - 4a)} \), How to prove?
Chapter 9. Medians and Order Statistics

Selection in worst case linear time

Input: set $S$ of $n$ elements and $i$;
Output: the $i$th smallest element in $S$;

Main idea:
• find a pivot $x$ to partition the list $S$ into two sublists $S_1$ and $S_2$, such that $\forall y \in S_1 \ y < x$ and $\forall z \in S_2 \ z > x$;
• both $S_1$ and $S_2$ are guaranteed only a fraction of $S$;
• the $i$th smallest element is either $x$, or in $S_1$ or in $S_2$ (but not both);
• in either of the latter two cases, the algorithm is applied recursively.

$T(n) \leq T(\beta n) + cn$ where $0 < \beta < 1$, such that $T(n) \leq cn + c\beta n + c\beta^2 n + \ldots \leq cn + c\beta n + c\beta^2 n + \ldots \leq c\frac{1}{1-\beta}n = O(n)$.
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$T(n) \leq T(\beta n) + cn$ where $0 < \beta < 1$, such that $T(n) \leq cn + c\beta n + c\beta^2 n + \ldots + c\beta^m n \leq c n(1 - \beta) = O(n)$
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Selection in worst case linear time

**INPUT:** set $S$ of $n$ elements and $i$;
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Main idea:

- find a pivot $x$ to partition the list $S$ into two sublists $S_1$ and $S_2$, such that $\forall y \in S_1 \; y < x$ and $\forall z \in S_2 \; z > x$
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T(n) \leq T(\beta n) + cn \quad \text{where} \quad 0 < \beta < 1,
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How to find such a pivot?
• the very selection algorithm is recursively called for finding the pivot
• the size of the sublist to find the pivot is also a fraction \( \alpha n \) of the original list \( S \), \( |S| = n \);
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3. recursively call \textsc{Select} \((M, \lceil n/10 \rceil)\);
   let the result be \(x\) and let the rank of \(x\) be \(k\) in \(S\)
4. if \(i = k\) return \((x)\)
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   such that \(\forall y \in S_1 \ y < x\) and \(\forall z \in S_2 \ z > x\)
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Note: the number of elements $\leq x$ is at least:

$$|S_1| \geq 3\left\lceil \frac{n}{5} \right\rceil^2 \geq \frac{3n}{10} \implies |S_2| < n - \frac{3n}{10} = \frac{7n}{10}$$

Similarly, the number of elements $\geq x$ is at least:

$$|S_2| \geq 3\left\lceil \frac{n}{5} \right\rceil^2 - 2 \geq \frac{3n}{10} - 6 \geq \frac{3n}{10} \implies |S_1| < n - \frac{3n}{10} + 6 = \frac{7n}{10} + 6$$

So a time upper bound for \textit{Select} is $T(n) = T_{\text{mom}} + T_{\text{sub}} + O(n)$ when $n \geq 140$. 
Note: the number of elements \( \leq x \) is at least:

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$$T(n) \leq T(\lceil n/5 \rceil) + T(\lceil 7n/10 + 6 \rceil) + O(n)$$

When $n \geq 140$
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Summary of Algorithm Analysis Scenarios

• analyzing time $T(n)$ of the algorithm
• obtain an expression $T(n) = \ldots$
• guess an upper (or lower) bound (e.g., $T(n) = O(\ )$)
• prove the correctness of the bound.

For example, given Insertion Sort:

• we first analyzed the algorithm and obtained
  $T(n) = c_1 n + c_2 (n - 1) + c_4 (n - 1) + c_5 n \sum_{j=2}^{n} t_j + c_6 n \sum_{j=2}^{n} (t_j - 1) + c_7 n \sum_{j=2}^{n} (t_j - 1) + c_8 (n - 1)$

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- analyzing time $T(n)$ of the algorithm
- obtain an expression $T(n) = \ldots$
- guess an upper (or lower) bound (e.g., $T(n) = O(\cdot)$)
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For example, given Insertion Sort:

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$$T(n) = c_1 n + c_2 (n - 1) + c_4 (n - 1) + c_5 \sum_{j=2}^{n} t_j + c_6 \sum_{j=2}^{n} (t_j - 1) + c_7 \sum_{j=2}^{n} (t_j - 1) + c_8 (n - 1)$$
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- and finally proved that it was indeed the case.
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For recursive algorithms

For example, given the Binary Search algorithm,

- we first analyze the time $T(n)$ of the algorithm and obtained a recurrence for $T(n)$:
  \[ T(n) \leq T(\lfloor n/2 \rfloor) + c \]

- we guess upper bound $T(n) = O(\log_2 n)$, i.e., $T(n) \leq c \log_2 n$;

- we prove the guessed bound.

  (1) we can use the recursive tree method by unfolding the time function; or
  (2) we can use the substitution method by the principle of induction.

But we need the recurrence to apply induction.

Using the recurrence: $T(n) \leq T(\lfloor n/2 \rfloor) + c$ to prove $T(n) \leq c \log_2 n$.

See previous lecture notes.
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