Part II Sorting and Order Statistics
Part II Sorting and Order Statistics

- Chapter 6. Heapsort, the use of priority queue
- Chapter 7. Quicksort, probabilistic analysis, randomized algorithms
- Chapter 8. Sorting in linear time, lower bounds
- Chapter 9. Medians and order statistics
Chapter 6. Heapsort

Chapter 6. Heapsort and the use of priority queue
Chapter 6. Heapsort

Chapter 6. Heapsort and the use of priority queue
Chapter 6. Heapsort

Chapter 6. Heapsort and the use of priority queue

- $\text{key}(\text{parent}) \geq \text{key}(\text{leftChild}), \text{key}(\text{rightChild})$;
Chapter 6. Heapsort

Chapter 6. Heapsort and the use of priority queue

- key(parent) \(\geq\) key(leftChild), key(rightChild);
- relationships are modeled with a complete binary tree
Chapter 6. Heapsort

Chapter 6. Heapsort and the use of priority queue

- key(parent) ≥ key(leftChild), key(rightChild);
- relationships are modeled with a complete binary tree;
- can be stored in arrays (indexes begin with 0),
Chapter 6. Heapsort

Chapter 6. Heapsort and the use of priority queue

- key(parent) ≥ key(leftChild), key(rightChild);
- relationships are modeled with a complete binary tree;
- can be stored in arrays (indexes begin with 0),
  \[ \text{index(leftChild)} = 2 \times \text{index(parent)} + 1 \]
Chapter 6. Heapsort and the use of priority queue

• key(parent) ≥ key(leftChild), key(rightChild);
• relationships are modeled with a complete binary tree
• can be stored in arrays (indexes begin with 0),
  \[ \text{index(leftChild)} = 2 \times \text{index(parent)} + 1 \]
  \[ \text{index(rightChild)} = 2 \times \text{index(parent)} + 2 \]
Chapter 6. Heapsort

The heap sort algorithm consists of subroutines:
Chapter 6. Heapsort

The heap sort algorithm consists of subroutines:

- **Build-Max-Heap** \((A)\)
Chapter 6. Heapsort

The heap sort algorithm consists of subroutines:

- **Build-Max-Heap**(A)
- **Max-Heapify**(A, i)
Chapter 6. Heapsort

The heap sort algorithm consists of subroutines:

- **Build-Max-Heap**$(A)$
- **Max-Heapify**$(A, i)$
- **HeapSort**$(A)$
Chapter 6. Heapsort

The heap sort algorithm consists of subroutines:

- **Build-Max-Heap** \( A \)
- **Max-Heapify** \( A, i \)
- **HeapSort** \( A \)

heaps as priority queues
Chapter 6. Heapsort

The heap sort algorithm consists of subroutines:

- **Build-Max-Heap**(A)
- **Max-Heapify**(A, i)
- **HeapSort**(A)

heaps as priority queues

- **Heap-Maximum**(A)
Chapter 6. Heapsort

The heap sort algorithm consists of subroutines:

- **Build-Max-Heap**(A)
- **Max-Heapify**(A, i)
- **HeapSort**(A)

heaps as priority queues

- **Heap-Maximum**(A)
- **Heap-Extract-Max**(A)
Chapter 6. Heapsort

The heap sort algorithm consists of subroutines:

- **Build-Max-Heap** \((A)\)
- **Max-Heapify** \((A, i)\)
- **HeapSort** \((A)\)

heaps as priority queues

- **Heap-Maximum** \((A)\)
- **Heap-Extract-Max** \((A)\)
- **Heap-Increase-Key** \((A, I, key)\)
The heap sort algorithm consists of subroutines:

- **Build-Max-Heap**($A$)
- **Max-Heapify**($A, i$)
- **HeapSort**($A$)

heaps as priority queues

- **Heap-Maximum**($A$)
- **Heap-Extract-Max**($A$)
- **Heap-Increase-Key**($A, I, key$)
- **Max-Heap-Insert**($A, key$)
Algorithm \texttt{HEAPSORT}(A)
Algorithm HeapSort\((A)\)

1. Build-Max-Heap\((A)\)
Chapter 6. Heapsort

Algorithm HeapSort(A)
1. Build-Max-Heap(A)
2. for $i = \text{length}[A] - 1$ downto 1 \{ indexes begin from 0\}

T_{HS}(n) = c_1 + T_{BMH}(n) + (n - 1)T_{MH}(n, 0), where $n = |A|$
Algorithm **HEAPSORT**(A)

1. **BUILD-MAX-HEAP**(A)
2. for \( i = \text{length}[A] - 1 \) downto 1 \{ indexes begin from 0\}
3. exchange \( A[0] \leftrightarrow A[i] \)

**T**\(_{HS}\)\((n)\) = \( c_1 + T_{BMH}(n) + (n-1)T_{MH}(n,0) \), where \( n = |A| \)
Chapter 6. Heapsort

Algorithm \texttt{HEAPSORT}(A)

1. \texttt{BUILD-MAX-HEAP}(A)
2. \texttt{for} $i = \text{length}[A] - 1$ \texttt{downto} 1 \hspace{1em} \{ indexes begin from 0\}
3. exchange $A[0] \leftrightarrow A[i]$
4. $\text{heapsize}[A] = \text{heapsize}[A] - 1$
Chapter 6. Heapsort

Algorithm \textsc{HeapSort}(A)

1. \textsc{Build-Max-Heap}(A)
2. \textbf{for} $i = \text{length}[A] - 1$ \textbf{downto} 1 \hspace{1em} \{ \text{indexes begin from 0} \}
3. exchange $A[0] \leftrightarrow A[i]$
4. $\text{heapsize}[A] = \text{heapsize}[A] - 1$
5. \textsc{Max-Heapify}(A, 0)
Chapter 6. Heapsort

Algorithm HeapSort(A)

1. Build-Max-Heap(A)
2. for $i = \text{length}[A] - 1$ downto 1 { indexes begin from 0}
3. exchange $A[0] \leftrightarrow A[i]$
4. $\text{heapsize}[A] = \text{heapsize}[A] - 1$
5. Max-Heapify(A, 0)

$T_{HS}(n) = c_1 + T_{BMH}(n) + (n - 1)T_{MH}(n, 0)$, where $n = |A|$
Chapter 6. Heapsort

Algorithm \textsc{HeapSort}(A)

1. \textbf{Build-Max-Heap}(A)
2. \textbf{for} $i = \text{length}[A] - 1$ \textbf{downto} 1 \quad \{ \text{indexes begin from 0} \}
3. \quad \text{exchange } A[0] \leftrightarrow A[i]
4. \quad \text{heapsize}[A] = \text{heapsize}[A] - 1
5. \quad \textbf{Max-Heapify}(A, 0)

\[ T_{HS}(n) = c_1 + T_{BMH}(n) + (n - 1)T_{MH}(n, 0), \text{ where } n = |A| \]

Subroutine \textbf{Build-Max-Heap}(A)
Chapter 6. Heapsort

Algorithm \textsc{HeapSort}(A)

1. \textsc{Build-Max-Heap}(A)
2. \textbf{for} \( i = \text{length}[A] - 1 \) \textbf{downto} 1 \hspace{1em} \{ \text{indexes begin from 0} \}
3. \text{exchange} \( A[0] \leftrightarrow A[i] \)
4. \( \text{heapsize}[A] = \text{heapsize}[A] - 1 \)
5. \textsc{Max-Heapify}(A, 0)

\( T_{HS}(n) = c_1 + T_{BMH}(n) + (n - 1)T_{MH}(n, 0) \), where \( n = |A| \)

Subroutine \textsc{Build-Max-Heap}(A)

1. \( \text{heapsize}[A] = \text{length}[A] \)
Chapter 6. Heapsort

Algorithm HeapSort(A)

1. Build-Max-Heap(A)
2. \textbf{for} i = length[A] − 1 \textbf{downto} 1 \quad \{ \text{indexes begin from 0} \}
3. exchange A[0] \leftrightarrow A[i]
4. heapsize[A] = heapsize[A] − 1
5. Max-Heapify(A, 0)

\[T_{HS}(n) = c_1 + T_{BMH}(n) + (n - 1)T_{MH}(n, 0), \text{ where } n = |A|\]

Subroutine Build-Max-Heap(A)

1. heapsize[A] = length[A]
2. \textbf{for} i = \left\lfloor \frac{1}{2} \text{length}[A] \right\rfloor − 1 \textbf{downto} 0 \quad \{ \text{indexes begin from 0} \}
Chapter 6. Heapsort

Algorithm \texttt{Heapsort}(A)

1. \texttt{Build-Max-Heap}(A)
2. \texttt{for} $i = \text{length}[A] - 1$ \texttt{downto} 1 \quad \{ \text{indexes begin from 0}\}
3. \quad \text{exchange } A[0] \leftrightarrow A[i]
4. \quad \text{heapsize}[A] = \text{heapsize}[A] - 1
5. \quad \text{Max-Heapify}(A, 0)

$T_{HS}(n) = c_1 + T_{BMH}(n) + (n - 1)T_{MH}(n, 0)$, where $n = |A|$

Subroutine \texttt{Build-Max-Heap}(A)

1. \text{heapsize}[A] = \text{length}[A]
2. \texttt{for} $i = \lceil \frac{1}{2}\text{length}[A] \rceil - 1$ \texttt{downto} 0 \quad \{ \text{indexes begin from 0}\}
3. \quad \text{Max-Heapify}(A, i)$
Chapter 6. Heapsort

Algorithm \texttt{Heapsort}(A)

1. \texttt{Build-Max-Heap}(A)
2. \texttt{for} \ \texttt{i} = \texttt{length}[A] - 1 \ \texttt{downto} \ 1 \quad \{ \text{indexes begin from 0} \}
3. exchange \ A[0] \longleftrightarrow \ A[i]
4. \ \texttt{heapsize}[A] = \texttt{heapsize}[A] - 1
5. \ \texttt{Max-Heapify}(A, 0)

\[ T_{HS}(n) = c_1 + T_{BMH}(n) + (n - 1)T_{MH}(n, 0) , \text{ where } n = |A| \]

Subroutine \texttt{Build-Max-Heap}(A)

1. \ \texttt{heapsize}[A] = \texttt{length}[A]
2. \texttt{for} \ \texttt{i} = \lfloor \frac{1}{2}\texttt{length}[A] \rfloor - 1 \ \texttt{downto} \ 0 \quad \{ \text{indexes begin from 0} \}
3. \ \texttt{Max-Heapify}(A, i)

\[ T_{BMH}(n) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor - 1} c_2 T_{MH}(n, i) \]
Chapter 6. Heapsort

Subroutine MAX-HEAPIFY($A, i$)

1. $l = 2 \times i + 1$
2. $r = 2 \times i + 2$
3. if ($l \leq \text{heapsize} \{ A \}$) and ($A[l] > A[i]$) then
4.     $\text{largest} = l$
5. else
6.     $\text{largest} = i$
7. if ($r \leq \text{heapsize} \{ A \}$) and ($A[r] > A[\text{largest}]$) then
8.     $\text{largest} = r$
9. if $\text{largest} \neq i$ then
10.     $A[i] \leftrightarrow A[\text{largest}]$
11.     Max-HEAPIFY($A, \text{largest}$)

$T_{MH}(n, i) \leq c_3 + T_{MH}(n, 2i + 1)$

Because

$T_{MH}(n, i) = c_3 + T_{MH}(n, 2i + 1)$, or

$T_{MH}(n, i) \leq c_4 \log_2 n$, for all $i = 0, 1, \ldots, n - 1$. (Prove it!)

$T_{BMH}(n) = \lfloor n/2 \rfloor - 1 \sum_{i=0}^{n-1} c_2 T_{MH}(n, i)$

$\leq c_4 n^2 \log_2 n$

$T_{HS}(n) = c_1 + T_{BMH}(n) + (n - 1) c_2 T_{MH}(n, 0)$

$\leq c_4 n^2 \log_2 n + (n - 1) c_2 c_4 \log_2 n = O(n \log n)$
Chapter 6. Heapsort

Subroutine MAX-HEAPIFY(A, i)

1. $l = 2 \times i + 1$
Chapter 6. Heapsort

Subroutine \texttt{MAX-HEAPIFY}(A, i)

1. \hspace{1em} l = 2 \times i + 1
2. \hspace{1em} r = 2 \times i + 2
Chapter 6. Heapsort

Subroutine MAX-HEAPIFY$(A, i)$

1. $l = 2 \times i + 1$
2. $r = 2 \times i + 2$
3. if $(l \leq \text{heapsize}[A]) \text{ and } (A[l] > A[i])$

T_MH$(n,i) \leq c_3 + T_MH(n,2i+1)$

Because $T_MH(n,i) = c_3 + T_MH(n,2i+1)$, or $= c_3 + T_MH(n,2i+2)$

$T_MH(n,i) \leq c_4 \log_2 n$, for all $i = 0, 1, ..., n-1$. (Prove it!)

$T_{BMH}(n) = \lfloor \frac{n}{2} \rfloor - 1 \sum_{i=0}^{c_2 T_{MH}(n,i)}$}

$\leq c_4 n^2 \log_2 n$

$T_{HS}(n) = c_1 + T_{BMH}(n) + (n-1)c_2 T_{MH}(n,0)$

$\leq c_4 n^2 \log_2 n + (n-1)c_2 c_4 \log_2 n = O(n \log n)$
Chapter 6. Heapsort

Subroutine \textsc{Max-Heapify}(A, i)

1. \( l = 2 \times i + 1 \)
2. \( r = 2 \times i + 2 \)
3. \textbf{if} \ (l \leq \text{heapsize}[A]) \ \textbf{and} \ (A[l] > A[i])
4. \quad \textbf{then} \ largest = l
Chapter 6. Heapsort

Subroutine MAX-HEAPIFY($A, i$)

1. $l = 2 \times i + 1$
2. $r = 2 \times i + 2$
3. if ($l \leq \text{heapsize}[A]$) and ($A[l] > A[i]$)
4. then $\text{largest} = l$
5. else $\text{largest} = i$
6. if ($r \leq \text{heapsize}[A]$) and ($A[r] > A[\text{largest}]$)
7. then $\text{largest} = r$
8. if $\text{largest} \neq i$
9. then exchange $A[i] \leftrightarrow A[\text{largest}]
10. \text{Max-Heapify}(A, \text{largest})$

$T_{MH}(n,i) \leq c_3 + T_{MH}(n, 2i + 1)$

Because

$T_{MH}(n,i) = c_3 + T_{MH}(n, 2i + 1),$ or

$T_{MH}(n,i) \leq c_4 \log_2 n,$ for all $i = 0, 1, \ldots, n - 1$. (Prove it!)

$T_{BMH}(n) = \lfloor n/2 \rfloor - 1 \sum_{i=0}^{n} c_2 T_{MH}(n,i) \leq c_4 n^2 \log_2 n$

$T_{HS}(n) = c_1 + T_{BMH}(n) + (n-1) c_2 T_{MH}(n,0) \leq c_4 n^2 \log_2 n + (n-1) c_2 c_4 \log_2 n = O(n \log n)$
Chapter 6. Heapsort

Subroutine MAX-HEAPIFY(A, i)

1. \( l = 2 \times i + 1 \)
2. \( r = 2 \times i + 2 \)
3. \( \text{if } (l \leq \text{heapsize}[A]) \text{ and } (A[l] > A[i]) \) then \( \text{largest} = l \)
4. \( \text{else } \text{largest} = i \)
5. \( \text{if } (r \leq \text{heapsize}[A]) \text{ and } (A[r] > A[\text{largest}]) \)
Chapter 6. Heapsort

Subroutine MAX-HEAPIFY($A, i$)

1. $l = 2 \times i + 1$
2. $r = 2 \times i + 2$
3. if ($l \leq \text{heapsize}[A]$) and ($A[l] > A[i]$)
   then $\text{largest} = l$
4. else $\text{largest} = i$
6. if ($r \leq \text{heapsize}[A]$) and ($A[r] > A[\text{largest}]$)
   then $\text{largest} = r$
7. exchange $A[i] \leftarrow A[\text{largest}]$
8. MAX-HEAPIFY($A, \text{largest}$)

$T_{MH}(n, i) \leq c_3 + T_{MH}(n, 2i + 1)$

Because $T_{MH}(n, i) = c_3 + T_{MH}(n, 2i + 1)$, or $= c_3 + T_{MH}(n, 2i + 2)$

$T_{MH}(n, i) \leq c_4 \log_2 n$, for all $i = 0, 1, \ldots, n - 1$. (Prove it!)

$T_{BMH}(n) = \left\lfloor \frac{n}{2} \right\rfloor - 1 \sum_{i=0}^{c_2} T_{MH}(n, i)$

$\leq c_4 n^2 \log_2 n$

$T_{HSS}(n) = c_1 + T_{BMH}(n) + (n - 1)c_2 T_{MH}(n, 0)$

$\leq c_4 n^2 \log_2 n + (n - 1)c_2 c_4 \log_2 n = O(n \log n)$
Chapter 6. Heapsort

Subroutine \textsc{Max-Heapify}(A, i)

1. \[ l = 2 \times i + 1 \]
2. \[ r = 2 \times i + 2 \]
3. \textbf{if} \ (l \leq \text{heapsize}[A]) \textbf{and} \ (A[l] > A[i]) \textbf{then} \ largest = l
4. \textbf{else} \ largest = i
5. \textbf{if} \ (r \leq \text{heapsize}[A]) \textbf{and} \ (A[r] > A[largest]) \textbf{then} \ largest = r
6. \textbf{if} \ largest \neq i

\[ T_{\text{MH}}(n, i) \leq c_3 + T_{\text{MH}}(n, 2i + 1) \]

Because \[ T_{\text{MH}}(n, i) = c_3 + T_{\text{MH}}(n, 2i + 1) \], or \[ T_{\text{MH}}(n, i) \leq c_4 \log_2 n \], for all \[ i = 0, 1, \ldots, n-1 \]. (Prove it!)

\[ T_{\text{BMH}}(n) = \left\lfloor \frac{n}{2} \right\rfloor - 1 \sum_{i=0}^{n} c_2 T_{\text{MH}}(n, i) \leq c_4 n^2 \log_2 n \]

\[ T_{\text{HS}}(n) = c_1 + T_{\text{BMH}}(n) + (n - 1) c_2 T_{\text{MH}}(n, 0) \leq c_4 n^2 \log_2 n + (n - 1) c_2 c_4 \log_2 n = O(n \log n) \]
Chapter 6. Heapsort

Subroutine MAX-HEAPIFY($A, i$)

1. $l = 2 \times i + 1$
2. $r = 2 \times i + 2$
3. if ($l \leq \text{heapsize}[A]$) and ($A[l] > A[i]$) then
   largest = $l$
4. else
5. largest = $i$
6. if ($r \leq \text{heapsize}[A]$) and ($A[r] > A[\text{largest}]$) then
   largest = $r$
7. if largest $\neq i$
8. then exchange $A[i] \leftrightarrow A[\text{largest}]$
Chapter 6. Heapsort

Subroutine `MAX-HEAPIFY(A, i)`

1. \( l = 2 \times i + 1 \)
2. \( r = 2 \times i + 2 \)
3. \( \text{if} \ (l \leq \text{heapsize}[A]) \text{ and } (A[l] > A[i]) \)
4. \( \text{then} \ largest = l \)
5. \( \text{else} \ largest = i \)
6. \( \text{if} \ (r \leq \text{heapsize}[A]) \text{ and } (A[r] > A[largest]) \)
7. \( \text{then} \ largest = r \)
8. \( \text{if} \ largest \neq i \)
9. \( \text{then} \ exchange \ A[i] \leftrightarrow A[largest] \)
10. \( \text{MAX-HEAPIFY}(A, largest) \)
Chapter 6. Heapsort

Subroutine MAX-HEAPIFY(A, i)

1. \( l = 2 \times i + 1 \)
2. \( r = 2 \times i + 2 \)
3. if \( (l \leq \text{heapsize}[A]) \) and \( (A[l] > A[i]) \)
4. then \( \text{largest} = l \)
5. else \( \text{largest} = i \)
6. if \( (r \leq \text{heapsize}[A]) \) and \( (A[r] > A[\text{largest}]) \)
7. then \( \text{largest} = r \)
8. if \( \text{largest} \neq i \)
9. then exchange \( A[i] \leftarrow A[\text{largest}] \)
10. MAX-HEAPIFY(A, largest)

\[ T_{MH}(n, i) \leq c_3 + T_{MH}(n, 2i + 1) \] Because \( T_{MH}(n, i) = c_3 + T_{MH}(n, 2i + 1) \), or \[ = c_3 + T_{MH}(n, 2i + 2) \]
Chapter 6. Heapsort

Subroutine MAX-HEAPIFY($A, i$)

1. $l = 2 \times i + 1$
2. $r = 2 \times i + 2$
3. if ($l \leq \text{heapsize} \left[ A \right]$) and ($A[l] > A[i]$)
   then largest = $l$
4. else largest = $i$
5. if ($r \leq \text{heapsize} \left[ A \right]$) and ($A[r] > A[\text{largest}]$)
   then largest = $r$
6. if largest $\neq i$
   then exchange $A[i] \leftrightarrow A[\text{largest}]$
7. MAX-HEAPIFY($A$, largest)

$T_{MH}(n, i) \leq c_3 + T_{MH}(n, 2i + 1)$  

Because $T_{MH}(n, i) = c_3 + T_{MH}(n, 2i + 1)$, or

$= c_3 + T_{MH}(n, 2i + 2)$

$T_{MH}(n, i) \leq c_4 \log_2 n$, for all $i = 0, 1, \ldots, n - 1$. (Prove it!)
Chapter 6. Heapsort

Subroutine \textsc{Max-Heapify}(A, i)

1. \( l = 2 \times i + 1 \)
2. \( r = 2 \times i + 2 \)
3. \textbf{if} \ (l \leq \text{heapsize}[A]) \ \textbf{and} \ (A[l] > A[i])
   \textbf{then} \ largest = l
4. \textbf{else} largest = i
5. \textbf{if} \ (r \leq \text{heapsize}[A]) \ \textbf{and} \ (A[r] > A[largest])
   \textbf{then} largest = r
6. \textbf{if} \ largest \neq i
   \textbf{then} exchange \ A[i] \leftarrow A[largest]
7. \textsc{Max-Heapify}(A, largest)

\( T_{MH}(n, i) \leq c_3 + T_{MH}(n, 2i + 1) \quad \text{Because} \ T_{MH}(n, i) = c_3 + T_{MH}(n, 2i + 1), \text{or} \)
\( = c_3 + T_{MH}(n, 2i + 2) \)

\( T_{MH}(n, i) \leq c_4 \log_2 n, \text{ for all } i = 0, 1, \ldots, n - 1. \quad \text{(Prove it!)} \)

\( T_{BMH}(n) = \sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor - 1} c_2 T_{MH}(n, i) \)
Chapter 6. Heapsort

Subroutine MAX-HEAPIFY(A, i)

1. \( l = 2 \times i + 1 \)
2. \( r = 2 \times i + 2 \)
3. \textbf{if} \ (l \leq \text{heapsize}[A]) \ \textbf{and} \ (A[l] > A[i])
4. \hspace{1em} \textbf{then} \ largest = l
5. \hspace{1em} \textbf{else} \ largest = i
6. \hspace{1em} \textbf{if} \ (r \leq \text{heapsize}[A]) \ \textbf{and} \ (A[r] > A[largest])
7. \hspace{2em} \textbf{then} \ largest = r
8. \hspace{1em} \textbf{if} \ largest \neq i
9. \hspace{2em} \textbf{then} \ exchange \ A[i] \leftrightarrow A[largest]
10. \hspace{1em} \textbf{MAX-HEAPIFY}(A, largest)

\( T_{MH}(n, i) \leq c_3 + T_{MH}(n, 2i + 1) \) \quad \text{Because} \quad T_{MH}(n, i) = c_3 + T_{MH}(n, 2i + 1), \text{ or} \quad = c_3 + T_{MH}(n, 2i + 2)

\( T_{MH}(n, i) \leq c_4 \log_2 n, \text{ for all } i = 0, 1, \ldots, n - 1. \) \quad \text{(Prove it!)}

\( T_{BMH}(n) = \sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor - 1} c_2 T_{MH}(n, i) \leq c_4 \frac{n}{2} \log_2 n \)
Chapter 6. Heapsort

Subroutine MAX-HEAPIFY\( (A, i) \)

1. \( l = 2 \times i + 1 \)
2. \( r = 2 \times i + 2 \)
3. If \( (l \leq \text{heapsize}[A]) \) and \( (A[l] > A[i]) \)
   4. then largest = l
   5. else largest = i
6. If \( (r \leq \text{heapsize}[A]) \) and \( (A[r] > A[\text{largest}]) \)
   7. then largest = r
8. If largest \( \neq i \)
   9. then exchange \( A[i] \leftarrow A[\text{largest}] \)
10. MAX-HEAPIFY\( (A, \text{largest}) \)

\[ T_{MH}(n, i) \leq c_3 + T_{MH}(n, 2i + 1) \quad \text{Because } T_{MH}(n, i) = c_3 + T_{MH}(n, 2i + 1), \text{ or} \]
\[ = c_3 + T_{MH}(n, 2i + 2) \]

\[ T_{MH}(n, i) \leq c_4 \log_2 n, \text{ for all } i = 0, 1, \ldots, n - 1. \quad \text{(Prove it!)} \]

\[ T_{BMH}(n) = \sum_{i=0}^{\lfloor n/2 \rfloor - 1} c_2 T_{MH}(n, i) \leq c_4 \frac{n}{2} \log_2 n \]

\[ T_{HS}(n) = c_1 + T_{BMH}(n) + (n - 1)c_2 T_{MH}(n, 0) \]
Chapter 6. Heapsort

Subroutine MAX-HEAPIFY($A, i$)

1. $l = 2 \times i + 1$
2. $r = 2 \times i + 2$
3. if ($l \leq \text{heapsize}[A]$) and ($A[l] > A[i]$) then largest $= l$
4. else largest $= i$
5. if ($r \leq \text{heapsize}[A]$) and ($A[r] > A[\text{largest}]$) then largest $= r$
6. if largest $\neq i$ then exchange $A[i] \leftrightarrow A[\text{largest}]$
7. MAX-HEAPIFY($A, \text{largest}$)

$T_{MH}(n, i) \leq c_3 + T_{MH}(n, 2i + 1)$  
Because $T_{MH}(n, i) = c_3 + T_{MH}(n, 2i + 1)$, or $= c_3 + T_{MH}(n, 2i + 2)$

$T_{MH}(n, i) \leq c_4 \log_2 n$, for all $i = 0, 1, \ldots, n - 1$. (Prove it!)

$T_{BMH}(n) = \sum_{i=0}^{\left\lceil \frac{n}{2} \right\rceil - 1} c_2 T_{MH}(n, i) \leq c_4 \frac{n}{2} \log_2 n$

$T_{HS}(n) = c_1 + T_{BMH}(n) + (n - 1)c_2 T_{MH}(n, 0) \leq c_4 \frac{n}{2} \log_2 n + (n - 1)c_2 c_4 \log_2 n$
Chapter 6. Heapsort

Subroutine MAX-HEAPIFY($A, i$)

1. $l = 2 \times i + 1$
2. $r = 2 \times i + 2$
3. if $(l \leq \text{heapsize}[A])$ and $(A[l] > A[i])$
   then largest = $l$
4. else largest = $i$
5. if $(r \leq \text{heapsize}[A])$ and $(A[r] > A[\text{largest}])$
   then largest = $r$
6. if largest $\neq i$
   then exchange $A[i] \leftrightarrow A[\text{largest}]
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$T_{MH}(n, i) \leq c_3 + T_{MH}(n, 2i + 1)$  Because $T_{MH}(n, i) = c_3 + T_{MH}(n, 2i + 1)$, or $= c_3 + T_{MH}(n, 2i + 2)$

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$T_{BMH}(n) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor - 1} c_2 T_{MH}(n, i) \leq c_4 \frac{n}{2} \log_2 n$

$T_{HS}(n) = c_1 + T_{BMH}(n) + (n - 1)c_2 T_{MH}(n, 0) \leq c_4 \frac{n}{2} \log_2 n + (n - 1)c_2 c_4 \log_2 n = O(n \log n)$
Chapter 6. Heapsort

Operations on heaps:

Function Heap-Maximum \(A\) obtain the maximum
1. \(\text{return} (A[1])\)

Function Heap-Extract-Max \(A\) obtain and remove the maximum
1. if heapsize \([A]\) < 1
2. then \(\text{return} \text{"heap underflow"}\)
3. \(\text{max} = A[1]\)
5. \(\text{heapsize}[A] = \text{heapsize}[A] - 1\)
6. Max-Heapify \((A, 1)\)
7. \(\text{return} (\text{max})\)

Function Heap-Increase-Key \(A, i, key\) replace a key with a larger value
1. if key < \(A[i]\)
2. then \(\text{return} \text{"new key is smaller than current key"}\)
3. \(A[i] = \text{key}\)
4. while \(i > 1\) and \(A[\text{PARENT}[i]] < A[i]\)
5. exchange \(A[i] \leftarrow A[\text{PARENT}[i]]\)
6. \(i = \text{PARENT}[i]\)

Function Max-Heap-Insert \(A, key\) insert a new key to heap
1. \(\text{heapsize}[A] = \text{heapsize}[A] + 1\)
2. \(A[\text{heapsize}[A]] = -\infty\)
3. Heap-Increase-Key \((A, \text{heapsize}[A], \text{key})\)
Chapter 6. Heapsort

Operations on heaps:

Function \texttt{Heap-Maximum}(A)
1. \textbf{return} \texttt{(A[1])}

\begin{algorithm}
\textbf{Function Heap-Extract-Max}(A)
1. \textbf{if} heapsize[A] < 1
\textbf{then return} \texttt{"heap underflow"}
2. \textbf{max} = A[1]
4. heapsize[A] = heapsize[A] - 1
5. Max-Heapify(A, 1)
6. \textbf{return} \texttt{(max)}
\end{algorithm}

\begin{algorithm}
\textbf{Function Heap-Increase-Key}(A, i, key)
1. \textbf{if} key < A[i]
\textbf{then return} \texttt{"new key is smaller than current key"}
2. A[i] = key
3. \textbf{while} i > 1 and A[PARENT[i]] < A[i]
4. \textbf{exchange} A[i] \leftrightarrow A[PARENT[i]]
5. i = PARENT[i]
\end{algorithm}

\begin{algorithm}
\textbf{Function Max-Heap-Insert}(A, key)
1. heapsize[A] = heapsize[A] + 1
2. A[heapsize[A]] = -\infty
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Operations on heaps:

Function $\text{HEAP-MAXIMUM}(A)$

Function $\text{HEAP-EXTRACT-MAX}(A)$
1. \textbf{if} $\text{heapsize}[A] < 1$
2. \textbf{then return} ("heap underflow")
3. $\text{max} = A[1]$
5. $\text{heapsize}[A] = \text{heapsize}[A] - 1$
6. $\text{Max-Heapify}(A, 1)$
7. \textbf{return} ($\text{max}$)
Chapter 6. Heapsort

Operations on heaps:

**Function** `Heap-Maximum(A)`  
1. return `A[1]`  
   - **obtain the maximum**

**Function** `Heap-Extract-Max(A)`  
1. if `heapsize[A] < 1`  
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3. `max = A[1]`  
5. `heapsize[A] = heapsize[A] - 1`  
6. `Max-Heapify(A, 1)`  
7. return `max`  
   - **obtain and remove the maximum**

**Function** `Heap-Increase-Key(A, i, key)`  
1. if `key < A[i]`  
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3. `A[i] = key`  
4. while `i > 1 and A[PARENT[i]] < A[i]`  
6. `i = PARENT[i]`  
   - **replace a key with a larger value**
Chapter 6. Heapsort

Operations on heaps:

Function \textsc{Heap-Maximum}(A)
1. \textbf{return} \((A[1])\)

Function \textsc{Heap-Extract-Max}(A)
1. \textbf{if} heapsize\([A] < 1\)
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5. \text{heapsize}[A] = heapsize\([A] - 1\)
6. \textsc{Max-Heapify}(A, 1)
7. \textbf{return} \((\text{max})\)

Function \textsc{Heap-Increase-Key}(A, \textit{i}, \textit{key})
1. \textbf{if} key < A[\textit{i}]
2. \textbf{then return} ("new key is smaller than current key")
3. \(A[\textit{i}] = \text{key}\)
4. \textbf{while} \textit{i} > 1 \textbf{and} A[\text{Parent}[\textit{i}]] < A[\textit{i}]
5. \textbf{exchange} \(A[\textit{i}] \leftrightarrow A[\text{Parent}[\textit{i}]]\)
6. \(\textit{i} = \text{Parent}[\textit{i}]\)

Function \textsc{Max-Heap-Insert}(A, \textit{key})
1. \text{heapsize}[A] = heapsize\([A] + 1\)
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Chapter 7. Quicksort

Chapter 7. Quicksort and randomized algorithms
Chapter 7. Quicksort

Chapter 7. Quicksort and randomized algorithms

Idea of the Quicksort: **divide-and-conquer**
Chapter 7. Quicksort

Idea of the Quicksort: divide-and-conquer

- divide: re-organize list $A[p, r]$ into two sublists $A[p, q - 1]$ and $A[q + 1, r]$ based on pivot $A[q]$, such that

  (a) $A[i] \leq A[q]$ for all $i = p, \ldots, q - 1$

  (b) $A[i] \geq A[q]$ for all $i = q + 1, \ldots, r$
Chapter 7. Quicksort

Chapter 7. Quicksort and randomized algorithms

Idea of the Quicksort: divide-and-conquer

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Chapter 7. Quicksort

Chapter 7. Quicksort and randomized algorithms

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Chapter 7. Quicksort

Chapter 7. Quicksort and randomized algorithms

Idea of the Quicksort: divide-and-conquer

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  (b) \( A[i] \geq A[q] \) for all \( i = q + 1, \ldots, r \)

Chapter 7. Quicksort

Chapter 7. Quicksort and Randomized algorithms

Algorithm Quicksort \((A, p, r)\)

1. if \(p < r\)
2. then \(q = \text{Partition}\(A, p, r\)\)
3. \(\text{QuickSort}\(A, p, q - 1\)\)
4. \(\text{QuickSort}\(A, q + 1, r\)\)

How the pivot \(A[q]\) is identified is crucial to the performance of Quicksort.

- Assume \(A[q]\) partitions list \(A, p, r\) evenly, then
  \[T(n) \leq 2T(n/2) + cn = O(n \log_2 n)\]

- Assume \(A[q]\) partitions the list 20% vs 80%, then
  \[T(n) \leq T(0.2n) + T(0.8n) + cn = O(n \log_2 n)\]

- Assume \(A[q]\) partitions the list 1% vs 99%, then
  \[T(n) \leq T(0.01n) + T(0.99n) + cn = O(n \log_2 n)\]
Chapter 7. Quicksort

Chapter 7. Quicksort and Randomized algorithms

Algorithm $\text{QUICKSORT} (A, p, r)$
Chapter 7. Quicksort

Chapter 7. Quicksort and Randomized algorithms

Algorithm QUICKSORT \((A, p, r)\)
1. \textbf{if } \(p < r\)
Chapter 7. Quicksort

Chapter 7. Quicksort and Randomized algorithms

Algorithm \textsc{QuickSort} \((A, p, r)\)

1. if \(p < r\)
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Chapter 7. Quicksort and Randomized algorithms

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- Assume \(A[q]\) partitions the list 1% vs 99%, then \(T(n) \leq T(n_{100}) + T(99n_{100}) + cn = O(n \log_2 n)\)
Algorithm **QUICKSORT** \((A, p, r)\)
1. **if** \(p < r\)
2. **then** \(q = \text{PARTITION}(A, p, r)\)
3. **QUICKSORT** \((A, p, q - 1)\)
4. **QUICKSORT** \((A, q + 1, r)\)
Chapter 7. Quicksort and Randomized algorithms

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Chapter 7. Quicksort

Chapter 7. Quicksort and Randomized algorithms

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Chapter 7. Quicksort

Chapter 7. Quicksort and Randomized algorithms

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Chapter 7. Quicksort

Chapter 7. Quicksort and Randomized algorithms

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- Assume \(A[q]\) partitions the list 1\% vs 99\%, then
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Chapter 7. Quicksort

Chapter 7. Quicksort and Randomized algorithms

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- Assume \(A[q]\) partitions the list 1\% vs 99\%, then
  \[ T(n) \leq T(n/100) + T(99n/100) + cn = O(n \log_2 n) \]

How can we identify such a pivot?
Chapter 7. Quicksort

\[
\begin{array}{cccccc}
\text{i} & \text{p} & \text{j} & \text{r} \\
2 & 8 & 7 & 1 & 3 & 5 & 6 & 4 \\
\text{p} & \text{i} & \text{j} & \text{r} \\
2 & 8 & 7 & 1 & 3 & 5 & 6 & 4 \\
\text{p} & \text{i} & \text{j} \\
2 & 8 & 7 & 1 & 3 & 5 & 6 & 4 \\
\text{p} & \text{i} & \text{j} \\
2 & 1 & 7 & 8 & 3 & 5 & 6 & 4 \\
\text{p} & \text{i} & \text{j} \\
2 & 1 & 3 & 8 & 7 & 5 & 6 & 4 \\
\text{p} & \text{i} & \text{j} \\
2 & 1 & 3 & 8 & 7 & 5 & 6 & 4 \\
\text{p} & \text{i} & \text{j} \\
2 & 1 & 3 & 8 & 7 & 5 & 6 & 4 \\
\text{p} & \text{i} & \text{j} \\
2 & 1 & 3 & 4 & 7 & 5 & 6 & 8
\end{array}
\]

\text{quicksort}(A, p, q-1) \quad \text{quicksort}(A, q+1, r)
Chapter 7. Quicksort

PARTITION\((A, p, r)\)
1 \(x \leftarrow A[r]\)
2 \(i \leftarrow p - 1\)
3 \(\text{for } j \leftarrow p \text{ to } r - 1\)
4 \(\text{do if } A[j] \leq x\)
5 \(\text{then } i \leftarrow i + 1\)
6 \(\text{exchange } A[i] \leftrightarrow A[j]\)
7 \(\text{exchange } A[i + 1] \leftrightarrow A[r]\)
8 \(\text{return } i + 1\)
Partition may not guarantee to partition the list to two fractions of sizes $\epsilon n : (1 - \epsilon)n$, for a constant $\epsilon > 0$. 
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- skewed situation like $1 : n - 1$ partition may happen, resulting in running time $\geq cn^2$. 
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- that is, the cases other than the skewed ones occur much more often.
Chapter 7. Quicksort

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- skewed situation like $1 : n - 1$ partition may happen, resulting in running time $\geq cn^2$.
- however, chances for skewed cases like above are very small.
- that is, the cases other than the skewed ones occur much more often.

So the idea of Quicksort may work well on a majority of data.
Chapter 7. Quicksort

Assume that the equal likely chance for every number to be in the last position, what is the chance to partition the list into

\[ x\% \text{ vs } (100 - x)\% \]

fragments, for \(10 \leq x \leq 90\)?
Chapter 7. Quicksort

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\[ x\% \text{ vs } (100 - x)\% \]

fragments, for \(10 \leq x \leq 90\)?

The chance is \(= 80\%\)
Chapter 7. Quicksort

What running time would it be if 10:90 partition is always guaranteed?
Chapter 7. Quicksort

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\[ T(n) \leq T(n/10) + T(9n/10) + cn \]
Chapter 7. Quicksort

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Using the recursive-tree method (in the book notation), we have
Chapter 7. Quicksort

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Using the recursive-tree method (in the book notation), we have

\[ l_0: \quad cn \]
\[ l_h: \quad c \left( \frac{n}{10} \right)^h \quad \text{for } h \geq 1 \]
\[ l_k: \quad c \left( \frac{9}{10} \right)^k \quad \text{for } k \geq 1 \]

where \( c' = c / \log_{10} 9 \).
Chapter 7. Quicksort

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\[ l_1: \quad cn/10 \quad 9cn/10 \]
Chapter 7. Quicksort

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\[
\begin{align*}
&l_0: & cn & & \quad \text{cn} \\
&l_1: & cn/10 & & 9cn/10 & & \quad \text{cn} \\
&l_2: & cn/10^2 & 9cn/10^2 & 9cn/10^2 & 9^2cn/10^2 & & \quad \text{cn}
\end{align*}
\]
Chapter 7. Quicksort

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       & & & & & & \cdots
\end{align*} \]
Chapter 7. Quicksort

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\[ l_2: \quad cn/10^2 \quad 9cn/10^2 \quad 9^2cn/10^2 \quad cn \]
\[ l_h: \quad cn/10^h \quad \ldots \quad c9^h n/10^h \quad cn \]
Chapter 7. Quicksort

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l_1: & \quad cn/10 \\
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l_h: & \quad cn/10^h \\
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l_h: & \quad cn/10^h \\
\end{align*}
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Chapter 7. Quicksort

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  l_h: & & cn/10^h & \cdots & c9^h n/10^h & cn \\
  l_k: & & \cdots & c9^k n/10^k & \leq cn
\end{align*}
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Chapter 7. Quicksort

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  l_h: & \quad cn/10^h \quad \cdots \quad c9^h n/10^h \quad cn \\
  l_k: & \quad \cdots \quad c9^k n/10^k \quad \leq cn
\end{align*}
\]

where \( (\frac{1}{10})^h n = 1 \), i.e., \( h = \log_{10} n \)
Chapter 7. Quicksort

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Chapter 7. Quicksort

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Chapter 7. Quicksort

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Chapter 7. Quicksort

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Chapter 7. Quicksort

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Instead of analyzing QuickSort (with uniformly distributed for input) we design a randomized version of the algorithm and analyze it.
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Algorithm \textsc{Randomized-Partition}(A, p, r)
1. \( i = \text{random}(p, r) \)
2. exchange \( A[r] \leftarrow A[i] \)
3. \textbf{return} (\textsc{Partition}(A, p, r))
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Algorithm \texttt{Randomized QuickSort} \( (A, p, r) \)
Chapter 7. Quicksort

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Instead of analyzing QuickSort (with uniformly distributed for input) we design a randomized version of the algorithm and analyze it.

Algorithm **RANDOMIZED-PARTITION** \((A, p, r)\)
1. \(i = \text{random}(p, r)\)
2. exchange \(A[r] \leftrightarrow A[i]\)
3. return (**PARTITION** \((A, p, r)\))

Algorithm **RANDOMIZED QUICKSORT** \((A, p, r)\)
1. if \(p < r\)
2. then \(q = \text{RANDOMIZED-PARTITION}(A, p, r)\)
Chapter 7. Quicksort

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Chapter 7. Quicksort

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4. randomized QuickSort \( (A, q + 1, r) \)
Chapter 7. Quicksort

Up to this point, you should have known:

1. the details of QuickSort algorithm, especially Partition;
2. Why it runs $O(n \log n)$ on uniformly distributed data, intuitively;
3. the connection between
   (a) requiring prob distribution in the input data;
   (b) randomized algorithms;
Analysis of **RANDOMIZED-QUICKSORT**
Chapter 7. Quicksort

Analysis of \texttt{RANDOMIZED-QUICKSORT}

- count the expected number of comparisons between $x_i$ and $x_j$;

Observation 1: $x_i$ is compared with $x_j$ only when either is a pivot;

Observation 2: $x_i$ is compared with $x_j$ at most once;

- define random variable $X_{i,j} \in \{0, 1\}$, such that $X_{i,j} = 1$ iff a comparison between $x_i$ and $x_j$ occurs;

- let $X = \sum_{i<j} X_{i,j}$, total number of comparisons;

- the expected number of comparisons is $E(X) = E(\sum_{i<j} X_{i,j}) = \sum_{i<j} E(X_{i,j}) = \sum_{i<j} P(X_{i,j} = 1)$ by linearity of expectations.
Chapter 7. Quicksort

Analysis of Randomized-QuickSort

- count the expected number of comparisons between $x_i$ and $x_j$;

Observation 1: $x_i$ is compared with $x_j$ only when either is a pivot;
Analysis of \textsc{Randomized-QuickSort}

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Chapter 7. Quicksort

Analysis of **RANDOMIZED-QUICKSORT**

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- let $X = \sum_{i<j} X_{i,j}$, total number of comparisons
Chapter 7. Quicksort

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Chapter 7. Quicksort

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  by linearity of expectations.
Chapter 7. Quicksort

Analysis of \textsc{Randomized-Quicksort} (cont.)

\[ E(X) = E\left(\sum_{i<j} X_{i,j}\right) = \sum_{i<j} E(X_{i,j}) = \sum_{i<j} \text{Prob}(X_{i,j} = 1) \]
Analysis of \texttt{RANDOMIZED-QUICKSORT} (cont.)

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Analysis of \textsc{Randomized-Quicksort} (cont.)

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\[ \text{Prob}(X_{i,j} = 1) \leq \frac{2}{|L|} \leq \frac{2}{j - i + 1} \]
Analysis of \textsc{Randomized-QuickSort} (cont.)

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\(P(X_{i,j} = 1) = 2 \frac{1}{|L|}\), where \(|L|\) is the size of the sublist. \text{\textit{why?}}
Chapter 7. Quicksort

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but we do not know the size of the sublist \( L \)!
Chapter 7. Quicksort

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Analysis of **RANDOMIZED-QUICKSORT** (cont.)

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Chapter 7. Quicksort

Analysis of \textsc{Randomized-Quicksort} (cont.)

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size of the sublist (which \( x_i, x_j \) belongs to)
\[ |L| \geq (j - i + 1) \]
Chapter 7. Quicksort

Analysis of RANDOMIZED-QUICKSORT (cont.)

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size of the sublist (which \(x_i, x_j\) belongs to)
\(|L| \geq (j - i + 1)\)

So \(P(X_{i,j} = 1) \leq 2 \frac{1}{|L|} \leq 2 \frac{1}{j-i+1}\)
Chapter 7. Quicksort

original unsorted list

5 23 10

sublist L containing elements 5 and 10
10 is a pivot

L has to contain elements between 5 and 10
i.e., L has to contain elements 6, 7, 8, 9
|L| ≥ j − i + 1 = 10 − 5 + 1 = 6

final sorted list

1 2 3 4 5 6 7 8 9 10

x₅ x₁₀
Chapter 7. Quicksort

Analysis of **RANDOMIZED-QUICKSORT** (cont.)

\[
E(X) = E(\sum_{i<j} X_{i,j}) = \sum_{i<j} E(X_{i,j}) = \sum_{i<j} Prob(X_{i,j} = 1)
\]

\[
\leq \sum_{i<j} 2 \frac{1}{j - i + 1}
\]

\[
\leq \sum_{i<j} c \log_2 n
\]

for some constant \(c > 0\).

\[
E(X) = O(n \log_2 n)
\]
Analysis of \texttt{RANDOMIZED-QUICKSORT} (cont.)

\[ E(X) = E(\sum_{i<j} X_{i,j}) = \sum_{i<j} E(X_{i,j}) = \sum_{i<j} \text{Prob}(X_{i,j} = 1) \]

\[ \leq \sum_{i<j} 2 \frac{1}{j-i+1} = \sum_{i=1}^{n-1} \sum_{j=2}^{n} 2 \frac{1}{j-i+1} \]
Analysis of **Randomized-Quicksort** (cont.)

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E(X) = E\left(\sum_{i<j} X_{i,j}\right) = \sum_{i<j} E(X_{i,j}) = \sum_{i<j} \text{Prob}(X_{i,j} = 1)
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\]

\[
\leq \sum_{i=1}^{n-1} 2 \sum_{k=1}^{n-1} \frac{1}{k + 1} \leq \]

for some constant \(c > 0\).

So \(E(X) = O(n \log_2 n)\).
Chapter 7. Quicksort

Analysis of Randomized-QuickSort (cont.)

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\leq \sum_{i=1}^{n-1} 2 \sum_{k=1}^{n-1} \frac{1}{k + 1} \leq \sum_{i=1}^{n-1} c \log_2 n \leq cn \log_2 n
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Analysis of \textsc{Randomized-QuickSort} (cont.)

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Chapter 7. Quicksort

Analysis of \texttt{RANDOMIZED-QUICKSORT} (cont.)

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for some constant \( c > 0 \).

So \( E(X) = O(n \log_2 n) \).
Chapter 7. Quicksort
Chapter 8. Lower Bounds and Sorting in Linear Time

• We have used Big-O for upper bounds.
• We need another notation for lower bounds.

Define \( \Omega(g(n)) \) be the set of functions that have growth rates not slower than \( cg(n) \) for any given constant \( c > 0 \).

\[ \Omega(g(n)) = \{ f(n) : \exists c > 0, k > 0 \text{ such that } f(n) \geq cg(n) \text{ for all } n \geq k \} \]

• e.g., \( \Omega(n \log n) \) includes the following functions:

\[ 14n \log n, \frac{1}{100}n \log n, n^2, n^3 \log n, \frac{3}{7002}n, n!, ... \]

• Proof techniques for Big-\( \Omega \) are similar to those for Big-O.
We have used Big-\(O\) for upper bounds.
Chapter 8. Lower Bounds and Sorting in Linear Time

• We have used Big-$O$ for upper bounds.
• We need another notation for lower bounds.
• We have used Big-$O$ for upper bounds.
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Define $\Omega(g(n))$ be the set of functions that have growth rates not slower than $cg(n)$ for any given constant $c > 0$. 

e.g., $\Omega(n \log n)$ includes the following functions: $14n \log n, 100n \log n, n^2, n^3 \log n, 7002n, n!$, ...
Chapter 8. Lower Bounds and Sorting in Linear Time

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We have used Big-$O$ for upper bounds.

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- Proof techniques for Big-$\Omega$ are similar to those for Big-$O$. 
Upper bound of an algorithm A time sufficient (i.e., enough) for the algorithm to solve all instances. We make sure an upper bound should cover all instances; e.g., MergeSort has upper bound $O(n \log n)$. Is it correct to say MergeSort has upper bound $O(n^2)$?

It is correct for two reasons:

1. Since $cn \log n$ is sufficient, so is $cn^2$.
2. $O(n^2)$ contains all functions that $O(n \log n)$ contains.

Is it correct to say MergeSort has upper bound $O(n)$?
Upper bound of an algorithm
Upper bound of an algorithm

A time sufficient (i.e., enough) for the algorithm to solve all instances.
Chapter 8. Lower Bounds and Sorting in Linear Time

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Upper bound of **an algorithm**

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Chapter 8. Lower Bounds and Sorting in Linear Time

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Upper bound of *an algorithm*

A time *sufficient* (i.e., *enough*) for the algorithm to solve all instances.

We make sure an upper bound should covers all instances;

- e.g., **MergeSort** has upper bound \( O(n \log n) \).

**Is it correct to say **MergeSort** has upper bound \( O(n^2) \)?**

It is correct for two reasons:

1. since \( cn \log n \) is sufficient, so is \( cn^2 \).
2. \( O(n^2) \) contains all functions that \( O(n \log n) \) contains.

**Is it correct to say **MergeSort** has upper bound \( O(n) \)?**
Lower bound of an algorithm is a time necessary (i.e., needed) for the algorithm to solve all instances. \(l(n)\) is a lower bound – if some (generic) instance requires time \(l(n)\) or more to be solved by the algorithm.

For example, MergerSort has lower bound \(Ω(n \log n)\).

Is it correct to say MergerSort has lower bound \(Ω(n)\)?

It is correct for two reasons:

1. Since \(cn \log n\) is necessary, so is \(cn\).
2. \(Ω(n)\) contains all functions that \(Ω(n \log n)\) contains.

Is it correct to say MergerSort has lower bound \(Ω(n^2)\)?
Chapter 8. Lower Bounds and Sorting in Linear Time

Lower bound of an algorithm

Is it correct to say MergeSort has lower bound $\Omega(n \log n)$?

It is correct for two reasons:

1. since $cn \log n$ is necessary, so is $cn$.
2. $\Omega(n)$ contains all functions that $\Omega(n \log n)$ contains.
Lower bound of an algorithm

A time necessary (i.e., needed) for the algorithm to solve all instances.
Lower bound of an algorithm

A time necessary (i.e., needed) for the algorithm to solve all instances.

$l(n)$ is a lower bound – if some (generic) instance requires time $l(n)$ or more to be solved by the algorithm.
Lower bound of an algorithm

A time necessary (i.e., needed) for the algorithm to solve all instances.

\( l(n) \) is a lower bound – if some (generic) instance requires time \( l(n) \) or more to be solved by the algorithm.

e.g., \texttt{MERGE\textsc{Sort}} has lower bound \( \Omega(n \log n) \)
Chapter 8. Lower Bounds and Sorting in Linear Time

**Lower bound of an algorithm**

A time necessary (i.e., needed) for the algorithm to solve all instances.

$l(n)$ is a lower bound – if some (generic) instance requires time $l(n)$ or more to be solved by the algorithm.

e.g., **MergeSort** has lower bound $\Omega(n \log n)$

Is it correct to say **MergeSort** has lower bound $\Omega(n)$?
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Lower bound of an algorithm

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Is it correct to say \texttt{MergeSort} has lower bound \( \Omega(n) \)?

It is correct for two reasons:

(1) since \( cn \log n \) is necessary, so is \( cn \).
Chapter 8. Lower Bounds and Sorting in Linear Time

**Lower bound of an algorithm**

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Is it correct to say Mergesort has lower bound \( \Omega(n) \)?

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Chapter 8. Lower Bounds and Sorting in Linear Time

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Chapter 8. Lower Bounds and Sorting in Linear Time

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(1) since \( cn \log n \) is necessary, so is \( cn \).

(2) \( \Omega(n) \) contains all functions that \( \Omega(n \log n) \) contains.

\textbf{Is it correct to say \textsc{MergeSort} has lower bound} \( \Omega(n^2) \)?
Chapter 8. Lower Bounds and Sorting in Linear Time

- The best known upper bound for MergeSort is $O(n \log n)$, coinciding with the best known lower bound $\Omega(n \log n)$;
- Both bounds are tight (i.e., optimal). Thus complexity is denoted with $\theta(n \log n)$, meaning both $O(n \log n)$ and $\Omega(n \log n)$.
- We may not be so lucky for some other algorithms.
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Chapter 8. Lower Bounds and Sorting in Linear Time

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Chapter 8. Lower Bounds and Sorting in Linear Time

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The best known upper bound for \texttt{MergeSort} is $O(n \log n)$, coinciding with the best known lower bound $\Omega(n \log n)$;

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We may not be so lucky for some other algorithms.
Chapter 8. Lower Bounds and Sorting in Linear Time

**Rec-Fibonacci**

If $n = 1$ or $n = 2$, return $1$;
Else

\[ T_1 = \text{Rec-Fibonacci}(n - 1); \]
\[ T_2 = \text{Rec-Fibonacci}(n - 2); \]
Return \((T_1 + T_2)\);
Chapter 8. Lower Bounds and Sorting in Linear Time

Rec-Fibonacci(n)

\[\text{if } n = 1 \text{ or } n = 2, \text{ return } (1);\]
\[\text{else}\]
\[T_1 = \text{Rec-Fibonacci}(n - 1);\]
\[T_2 = \text{Rec-Fibonacci}(n - 2);\]
\[\text{return } (T_1 + T_2);\]

Derive an upper bound:
Chapter 8. Lower Bounds and Sorting in Linear Time

\textbf{Rec-Fibonacci}(n)

if \( n = 1 \) or \( n = 2 \), return (1);
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    \( T_1 = \text{Rec-Fibonacci}(n - 1); \)
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    return \( (T_1 + T_2); \)

Derive an upper bound:
\( T(n) = c + T(n - 1) + T(n - 2) \), with \( T(1) = T(2) = c \)
Chapter 8. Lower Bounds and Sorting in Linear Time

**Rec-Fibonacci** ($n$)

if $n = 1$ or $n = 2$, return (1);
else
    $T_1 = \text{Rec-Fibonacci}(n - 1)$;
    $T_2 = \text{Rec-Fibonacci}(n - 2)$;
    return ($T_1 + T_2$);

Derive an upper bound:
$T(n) = c + T(n - 1) + T(n - 2)$, with $T(1) = T(2) = c$
$\leq c + 2T(n - 1)$
Chapter 8. Lower Bounds and Sorting in Linear Time

**Rec-Fibonacci** \( (n) \)

if \( n = 1 \) or \( n = 2 \), return \( (1) \);
else

\( T_1 = \text{Rec-Fibonacci}(n - 1); \)
\( T_2 = \text{Rec-Fibonacci}(n - 2); \)

return \( (T_1 + T_2); \)

Derive an upper bound:
\( T(n) = c + T(n - 1) + T(n - 2) \), with \( T(1) = T(2) = c \)
\[ \leq c + 2T(n - 1) \]
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Chapter 8. Lower Bounds and Sorting in Linear Time

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    $\ldots$
Chapter 8. Lower Bounds and Sorting in Linear Time

Rec-Fibonacci(n)

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$\ldots$
$\leq c + 2c + \ldots 2^{n-3}c + 2^{n-2}T(2)$
Chapter 8. Lower Bounds and Sorting in Linear Time

Rec-Fibonacci\((n)\)

\[
\begin{align*}
\text{if } n &= 1 \text{ or } n = 2, \text{ return } (1); \\
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T_1 &= \text{Rec-Fibonacci}(n - 1); \\
T_2 &= \text{Rec-Fibonacci}(n - 2); \\
\text{return } (T_1 + T_2); 
\end{align*}
\end{align*}
\]

Derive an upper bound:

\[
\begin{align*}
T(n) &= c + T(n - 1) + T(n - 2), \text{ with } T(1) = T(2) = c \\
&\leq c + 2T(n - 1) \\
&\leq c + 2c + 2^2T(n - 2) \\
&\ldots \\
&\leq c + 2c + \ldots 2^{n-3}c + 2^{n-2}T(2) \\
&= \frac{2^{n-2} - 1}{2 - 1}c + 2^{n-2}c
\end{align*}
\]
Chapter 8. Lower Bounds and Sorting in Linear Time

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Derive an upper bound:
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T(n) = c + T(n - 1) + T(n - 2), \text{ with } T(1) = T(2) = c
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\leq c + 2T(n - 1)
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\[
\leq c + 2c + 2^2T(n - 2)
\]
\[...
\]
\[
\leq c + 2c + \ldots 2^{n-3}c + 2^{n-2}T(2)
\]
\[
= \frac{2^{n-2} - 1}{2 - 1}c + 2^{n-2}c
\]
\[
= (2^{n-2} - 1)c + 2^{n-2}c
\]

\( \text{O}(2^n) \)
Chapter 8. Lower Bounds and Sorting in Linear Time

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\]

return \((T_1 + T_2)\);

Derive an upper bound:

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T(n) = c + T(n - 1) + T(n - 2), \text{ with } T(1) = T(2) = c
\]

\[
\leq c + 2T(n - 1)
\]

\[
\leq c + 2c + 2^2T(n - 2)
\]

\[
\cdots
\]

\[
\leq c + 2c + \ldots 2^{n-3}c + 2^{n-2}T(2)
\]

\[
= \frac{2^{n-2} - 1}{2-1}c + 2^{n-2}c
\]

\[
= (2^{n-2} - 1)c + 2^{n-2}c
\]

\[
= (2^{n-1} - 1)c
\]

\[
\text{\(= O(2^n)\).}
\]
Chapter 8. Lower Bounds and Sorting in Linear Time

\textbf{Rec-Fibonacci}(n)

\begin{verbatim}
if \( n = 1 \) or \( n = 2 \), return (1);
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\end{verbatim}

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\( T(n) = c + T(n - 1) + T(n - 2), \) with \( T(1) = T(2) = c \)
\begin{align*}
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&\leq c + 2c + 2^2T(n - 2) \\
&\ldots \\
&\leq c + 2c + \ldots 2^{n-3}c + 2^{n-2}T(2) \\
&= \frac{2^{n-2} - 1}{2-1}c + 2^{n-2}c \\
&= (2^{n-2} - 1)c + 2^{n-2}c \\
&= (2^{n-1} - 1)c \\
&= O(2^n).
\end{align*}
Chapter 8. Lower Bounds and Sorting in Linear Time

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\]
\[
T_2 = \text{Rec-Fibonacci}(n - 2);
\]
\[
\text{return } (T_1 + T_2);
\]
Chapter 8. Lower Bounds and Sorting in Linear Time

\textbf{Rec-Fibonacci}(n)

\textbf{if} \ n = 1 \ \textbf{or} \ n = 2, \ \textbf{return} \ (1); \\
\textbf{else} \\
\hspace{1em} T_1 = \textbf{Rec-Fibonacci}(n - 1); \\
\hspace{1em} T_2 = \textbf{Rec-Fibonacci}(n - 2); \\
\hspace{1em} \textbf{return} \ (T_1 + T_2);

Derive a lower bound:

\begin{align*}
T(n) &= c + T(n - 1) + T(n - 2), \text{ with } T(1) = T(2) = c \\
&\geq 2T(n - 2) \\
&\geq 2^2 T(n - 4) \\
&\vdots \\
&\geq 2^{n-2} T(2) \\
&= 2^{n-2} c \\
&\geq c 2^{\log_2 n} \\
&\geq c \frac{1}{2} n^{1.41} \\
&= \Omega(1.41^n).
\end{align*}

while the derived upper bound is $O(2^n)$, not tight!
Chapter 8. Lower Bounds and Sorting in Linear Time

**Rec-Fibonacci**

```plaintext
if n = 1 or n = 2, return 1;
else
    T_1 = Rec-Fibonacci(n - 1);
    T_2 = Rec-Fibonacci(n - 2);
    return (T_1 + T_2);
```

Derive a lower bound:

\[ T(n) = c + T(n - 1) + T(n - 2), \text{ with } T(1) = T(2) = c \]
Chapter 8. Lower Bounds and Sorting in Linear Time

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$\geq 2T(n - 2)$
Chapter 8. Lower Bounds and Sorting in Linear Time

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Chapter 8. Lower Bounds and Sorting in Linear Time

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Chapter 8. Lower Bounds and Sorting in Linear Time

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Chapter 8. Lower Bounds and Sorting in Linear Time

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Chapter 8. Lower Bounds and Sorting in Linear Time

Rec-Fibonacci\(n\)

\[
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\text{if } & n = 1 \text{ or } n = 2, \text{ return } (1); \\
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Derive a lower bound:
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T(n) = c + T(n - 1) + T(n - 2), \text{ with } T(1) = T(2) = c
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\[
\begin{align*}
& \geq 2T(n - 2) \\
& \geq 2^2 T(n - 4) \\
& \vdots \\
& \geq 2^{\frac{n-2}{2}} T(2) \\
& = 2^{\frac{n-2}{2}} c \\
& = 2^{\frac{n}{2}} 2^{-2} c
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Chapter 8. Lower Bounds and Sorting in Linear Time

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\]
\[
= 2^{\frac{n}{2}} 2^{-2} c
\]
\[
= c 2^{\frac{1}{2}} 2^{\frac{1}{2}} c
\]
\[
= \frac{c}{2} (2^{\frac{1}{2}})^n
\]
Chapter 8. Lower Bounds and Sorting in Linear Time

**Rec-Fibonacci**\( (n)\)

**if** \( n = 1 \) or \( n = 2 \), **return** \( (1)\);

**else**

\( T_1 = \text{Rec-Fibonacci}(n - 1); \)

\( T_2 = \text{Rec-Fibonacci}(n - 2); \)

**return** \( (T_1 + T_2); \)

Derive a lower bound:

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\[ \geq 2T(n - 2) \]

\[ \geq 2^2 T(n - 4) \]

\[ \ldots \]

\[ \geq 2^{\frac{n-2}{2}} T(2) \]

\[ = 2^{\frac{n-2}{2}} c \]

\[ = 2^{\frac{n}{2}} 2^{\frac{-2}{2}} c \]

\[ = \frac{c}{2} (2^{\frac{1}{2}})^n \]

\[ = \frac{c}{2} \sqrt{2}^n \]

while the derived upper bound is \( O(2^n) \), not tight!
Chapter 8. Lower Bounds and Sorting in Linear Time

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\[ = \frac{c}{2} \left( 2^{\frac{1}{2}} \right)^n \]

\[ = \frac{c}{2} \sqrt{2}^n \]

\[ \geq \frac{c}{2} 1.41^n \]
Chapter 8. Lower Bounds and Sorting in Linear Time

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if \( n = 1 \) or \( n = 2 \), return \((1)\);
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Derive a lower bound:
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\]
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Chapter 8. Lower Bounds and Sorting in Linear Time

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Chapter 8. Lower Bounds and Sorting in Linear Time

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Chapter 8. Lower Bounds and Sorting in Linear Time

Upper bound of a problem is a time sufficient (i.e., enough) to solve all instances of the problem. To derive an upper bound, we can resort to algorithms solving the problem; an upper bound on the runtime of such an algorithm is also an upper bound for the problem. For example, $O(n^2)$ is an upper bound for sorting (why?). $O(n \log n)$ is also an upper bound for sorting (why?). One important task in algorithm research: to design algorithms achieving better upper bounds (smaller time complexity).
Upper bound of a problem
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Chapter 8. Lower Bounds and Sorting in Linear Time

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Chapter 8. Lower Bounds and Sorting in Linear Time

Lower bound of a problem
A time necessary (i.e., needed) for all instances in the problem to be solved.

Can we use an algorithm lower bound for the problem lower bound? e.g., consider Sorting problem, Insertion Sort has lower bound $\Omega(n^2)$ (why?), Can we say Sorting problem has lower bound $\Omega(n^2)$? No! because MergeSort has upper bound $O(n \log n)$.

Likewise, we cannot say Sorting has lower bound $\Omega(n \log n)$.

Statement “problem Sorting has lower bound $\Omega(n \log n)$” $\iff$ statement “there is no algorithm running faster than time $cn \log n$.”
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Chapter 8. Lower Bounds and Sorting in Linear Time

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Chapter 8. Lower Bounds and Sorting in Linear Time

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Chapter 8. Lower Bounds and Sorting in Linear Time

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Chapter 8. Lower Bounds and Sorting in Linear Time

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Statement “problem **Sorting** has lower bound \( \Omega(n \log n) \)”

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• To derive a lower bound for a **problem**, we **cannot** examine an infinite number of algorithms!
Statement “problem **Sorting** has lower bound $\Omega(n \log n)$”

$\iff$

statement “there is no algorithm running faster than time $cn \log n$”.

• To derive a lower bound for a **problem**, we **cannot** examine an infinite number of algorithms!

• Lower bounds can only be derived mathematically, but not from existing algorithms.
Chapter 8. Lower Bounds and Sorting in Linear Time

Deriving a lower bound for sorting

with decision tree as algorithm/computation model

Claim 1: total number of leaves is $\geq n!$.

Claim 2: the height of the tree at least $\geq \log n!$. (The minimum of heights of all such trees!)

Chapter 8. Lower Bounds and Sorting in Linear Time

Deriving a lower bound for sorting

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- each internal node denotes \((x_i \leq x_j)\), with two outcomes

Claim 1: total number of leaves is \(\geq n!\).

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Chapter 8. Lower Bounds and Sorting in Linear Time

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• each path is for one permutation of generic list \((1, 2, \ldots, n)\)

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Chapter 8. Lower Bounds and Sorting in Linear Time

Deriving a lower bound for sorting with decision tree as algorithm/computation model

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Chapter 8. Lower Bounds and Sorting in Linear Time

**Theorem**: Sorting needs $\Omega(n \log n)$ comparisons on comparison-based computation models.
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Prove.
The longest path from the root to a leaf is $\Omega(\log n!)$. I.e., the number of comparisons needed in the worst case is $\Omega(\log n!)$. 

$n! = n(n-1)(n-2) \cdots (n-\frac{n}{2})(n-\frac{n}{2}-1) \cdots 2 \times 1 \geq \left(\frac{n}{2}\right)^{n/2} \times 2^{n/2-1} \geq \frac{1}{2} \left(\frac{n}{2}\right)^n = \Omega(n^{\log \log n})$
Theorem: Sorting needs $\Omega(n \log n)$ comparisons on comparison-based computation models.

Prove.
The longest path from the root to a leave is $\Omega(\log n!)$. I.e., the number of comparisons needed in the worst case is $\Omega(\log n!)$. 

\[
n! = n(n-1)(n-2)\cdots(n - \frac{n}{2})(n - \frac{n}{2} - 1)\cdots2 \times 1
\]

\[
\geq \left(\frac{n}{2}\right)^{\frac{n}{2}} \times 2^{\frac{n}{2} - 1} \geq \frac{1}{2}n^{\frac{n}{2}}
\]

or by Stirling’s formula:

\[
n! = \sqrt{2\pi n}(n/e)^n(1 + O(1/n))
\]

$\Omega(\log(n!)) = \Omega(n \log n)$
Chapter 8. Lower Bounds and Sorting in Linear Time

Sorting algorithms with worst case linear time
(To be covered after the next chapter)
Chapter 8. Lower Bounds and Sorting in Linear Time

Sorting algorithms with worst case linear time  
(To be covered after the next chapter)  

- count sort  
- radix sort  
- bucket sort
Count sort

Algorithm Counting-Sort(A, B, k)

1. for i = 0 to k
2. C[i] = 0
3. for j = 1 to length[A]
5. {C[i] contains the number of elements whose values = i}
6. for i = 1 to k
7. C[i] = C[i] + C[i−1]
8. {C[i] contains the number of elements whose values ≤ i}
9. for j = length[A] downto 1

Example: A: 2 5 3 0 2 3 0 3, k = 5
C: 2 0 2 3 0 1

analysis: T(n) = O(k + n)
Count sort

Algorithm COUNTING-SORT \((A, B, k)\) \(\{A \text{ contains } n \text{ integers; } k \text{ is the max}\}\)
Count sort

Algorithm COUNTING-SORT \((A, B, k)\) \{\(A\) contains \(n\) integers; \(k\) is the max\}

1. \textbf{for} \(i = 0\) \textbf{to} \(k\)
Count sort

Algorithm COUNTING-SORT \((A, B, k)\) \{\(A\) contains \(n\) integers; \(k\) is the max\}

1. \textbf{for} \(i = 0\) \textbf{to} \(k\)
2. \(C[i] = 0\)
Count sort

Algorithm COUNTING-SORT \((A, B, k)\) \{A contains \(n\) integers; \(k\) is the max\}

1. \hspace{0.5em} \textbf{for} \(i = 0\) \textbf{to} \(k\)
2. \hspace{1em} \(C[i] = 0\)
3. \hspace{1em} \textbf{for} \(j = 1\) \textbf{to} \(\text{length}[A]\)
Chapter 8. Lower Bounds and Sorting in Linear Time

**Count sort**

Algorithm COUNTING-SORT \((A, B, k)\) \(\{A \text{ contains } n \text{ integers; } k \text{ is the max}\}\)

1. for \(i = 0\) to \(k\)
2. \(C[i] = 0\)
3. for \(j = 1\) to \(\text{length}[A]\)
4. \(C[A[j]] = C[A[j]] + 1\)

Example: \(A: 2 5 3 0 2 3 0 3, \ k = 5, \ C: 2 0 2 3 0 1\)

**Analysis:**

\(T(n) = O(k + n)\)
Chapter 8. Lower Bounds and Sorting in Linear Time

Count sort

Algorithm **COUNTING-SORT** \((A, B, k)\) \(\{A\text{ contains }n\text{ integers; }k\text{ is the max}\}\)

1. \(\text{for } i = 0 \text{ to } k\)
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Count sort

Algorithm Counting-Sort \((A, B, k)\) \{A contains \(n\) integers; \(k\) is the max\}

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5. \hspace{1em} \{\(C[i]\) contains the number of elements whose values = \(i\)}
6. \textbf{for} \(i = 1\ \textbf{to} \ k\)
### Count sort

**Algorithm** \textsc{Counting-Sort} \((A, B, k)\) \hspace{1cm} \{A contains \(n\) integers; \(k\) is the max\}

1. \textbf{for} \(i = 0\) to \(k\)
2. \hspace{0.5cm} \(C[i] = 0\)
3. \textbf{for} \(j = 1\) to \text{length}[A]
4. \hspace{0.5cm} \(C[A[j]] = C[A[j]] + 1\)
5. \hspace{0.5cm} \{\(C[i]\) contains the number of elements whose values = \(i\}\}
6. \textbf{for} \(i = 1\) to \(k\)
7. \hspace{0.5cm} \(C[i] = C[i] + C[i - 1]\)
Count sort

Algorithm COUNTING-SORT \((A, B, k)\)  \(\{A\text{ contains } n\text{ integers}; k\text{ is the max}\}\)

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8. \(\{C[i]\text{ contains the number of elements whose values }\leq i\}\)
Count sort

Algorithm Counting-Sort \((A, B, k)\) \{\(A\) contains \(n\) integers; \(k\) is the max\}

1. \textbf{for} \(i = 0\) \textbf{to} \(k\)
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8. \{\(C[i]\) contains the number of elements whose values \(\leq i\)\}
9. \textbf{for} \(j = \text{length}[A]\) \textbf{downto} \(1\)
Count sort

Algorithm \textsc{Counting-Sort} \((A, B, k)\) \(\{A\) contains \(n\) integers; \(k\) is the max\}
1. for \(i = 0\) to \(k\)  
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8. \(\{C[i] \text{ contains the number of elements whose values } \leq i\}\)  
9. for \(j = \text{length}[A]\) \textbf{downto} 1  
10. \(B[C[A[j]]] = A[j]\)
Chapter 8. Lower Bounds and Sorting in Linear Time

Count sort

Algorithm COUNTING-SORT \((A, B, k)\) \(\{A\text{ contains }n\text{ integers; }k\text{ is the max}\}\)

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9. \(\text{for } j = \text{length}[A] \text{ downto } 1\)
10. \(B[C[A[j]]] = A[j]\)
11. \(C[A[j]] = C[A[j]] - 1\)
Count sort

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10. \(B[C[A[j]]] = A[j]\)
11. \(C[A[j]] = C[A[j]] - 1\)

Example: \(A: 2\ 5\ 3\ 0\ 2\ 3\ 0\ 3,\ \ k = 5,\)
Count sort

Algorithm Counting-Sort \((A, B, k)\) \{A contains \(n\) integers; \(k\) is the max\}
1.  \textbf{for} \(i = 0\) \textbf{to} \(k\)
2.  \hspace{1em} \(C[i] = 0\)
3.  \textbf{for} \(j = 1\) \textbf{to} length\([A]\)
4.  \hspace{1em} \(C[A[j]] = C[A[j]] + 1\)
5.  \{\(C[i]\) contains the number of elements whose values = \(i\)\}
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7.  \hspace{1em} \(C[i] = C[i] + C[i - 1]\)
8.  \{\(C[i]\) contains the number of elements whose values \(\leq\) \(i\)\}
9.  \textbf{for} \(j = \text{length}[A]\) \textbf{downto} 1
10. \hspace{1em} \(B[C[A[j]]] = A[j]\)
11. \hspace{1em} \(C[A[j]] = C[A[j]] - 1\)

Example: \(A: 2\ 5\ 3\ 0\ 2\ 3\ 0\ 3,\ \ k = 5,\ \ C: 2\ 0\ 2\ 3\ 0\ 1\)
Chapter 8. Lower Bounds and Sorting in Linear Time

Count sort

Algorithm COUNTING-SORT \((A, B, k)\) \{A contains \(n\) integers; \(k\) is the max\}

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Example: \(A: 2\ 5\ 3\ 0\ 2\ 3\ 0\ 3\), \(k = 5\), \(C: 2\ 0\ 2\ 3\ 0\ 1\)

analysis:
Count sort

Algorithm COUNTING-SORT \((A, B, k)\) \{ \(A\) contains \(n\) integers; \(k\) is the max\}

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Example: \(A: 2\ 5\ 3\ 0\ 2\ 3\ 0\ 3,\ \ k = 5,\ \ C: 2\ 0\ 2\ 3\ 0\ 1\)

analysis: \(T(n) = O(k + n)\)
Radix Sort:

Algorithm Radix-Sort\( (A,d) \)

1. for \( i = 1 \) to \( d \)
2. sort \( A \) on the \( i \)th digit

Lemma. Given \( n \) \( b \)-bit binary numbers and any positive \( r \leq b \).

Radix-Sort uses \( \Theta\left(\left\lceil \frac{b}{r}\right\rceil \left(n + 2r\right)\right) \) time.
Radix Sort:

<table>
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Chapter 8. Lower Bounds and Sorting in Linear Time

Radix Sort:

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Algorithm \textsc{Radix-Sort}(A, d)
Radix Sort:

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Algorithm **Radix-Sort**($A, d$)

1. **for** $i = 1$ **to** $d$
Chapter 8. Lower Bounds and Sorting in Linear Time

Radix Sort:

329  720  720  329
457  355  329  355
657  436  436  436
839  457  839  457
436  657  355  657
720  329  457  720
355  839  657  839

Algorithm \textsc{Radix-Sort}(A, d)
1. \textbf{for} $i = 1$ \textbf{to} $d$
2. \textbf{sort} $A$ on the $i$th digit
Radix Sort:

329    720    720    329
457    355    329    355
657    436    436    436
839    457    839    457
436    657    355    657
720    329    457    720
355    839    657    839

Algorithm \textsc{Radix-Sort}(A, d)
1. \textbf{for} $i = 1$ \textbf{to} $d$
2. \textbf{sort} $A$ on the $i$th digit

Lemma. Given $n$ $b$-bit binary numbers and any positive $r \leq b$. \textsc{Radix-Sort} uses $\Theta([b/r](n + 2^r))$ time.
Lemma. Given \( n \) \( b \)-bit binary numbers and any positive \( r \leq b \). Radix-Sort uses \( \Theta([b/r](n + 2^r)) \) time.
Lemma. Given \( n \) \( b \)-bit binary numbers and any positive \( r \leq b \). Radix-Sort uses \( \Theta([b/r] (n + 2^r)) \) time.

Proof. Each \( b \)-digit binary number can be regarded as \([b/r]\) \( r \)-digit binary numbers. These \( r \)-digit binary numbers are of integer values in the range of \( \{0, 1, \ldots, 2^r - 1\} \).
Lemma. Given $n$ $b$-bit binary numbers and any positive $r \leq b$. 
Radix-Sort uses $\Theta(\lceil b/r \rceil (n + 2^r))$ time.

Proof. Each $b$-digit binary number can be regarded as $\lceil b/r \rceil$ $r$-digit binary numbers. These $r$-digit binary numbers are of integer values in the range of $\{0, 1, \ldots, 2^r - 1\}$.

Run Radix-Sort on the original binary numbers assumed to be $\lceil b/r \rceil$ columns.
Lemma. Given $n$ $b$-bit binary numbers and any positive $r \leq b$. Radix-Sort uses $\Theta(\lceil b/r \rceil (n + 2^r))$ time.

Proof. Each $b$-digit binary number can be regarded as $\lceil b/r \rceil$ $r$-digit binary numbers. These $r$-digit binary numbers are of integer values in the range of $\{0, 1, \ldots, 2^r - 1\}$.

Run Radix-Sort on the original binary numbers assumed to be $\lceil b/r \rceil$ columns.

For every column, sorting by Counting-Sort with $2^r - 1$ being the maximum.
Lemma. Given $n$ $b$-bit binary numbers and any positive $r \leq b$. Radix-Sort uses $\Theta([b/r](n + 2^r))$ time.

Proof. Each $b$-digit binary number can be regarded as $[b/r]$ $r$-digit binary numbers. These $r$-digit binary numbers are of integer values in the range of $\{0, 1, \ldots, 2^r - 1\}$.

Run Radix-Sort on the original binary numbers assumed to be $[b/r]$ columns.

For every column, sorting by Counting-Sort with $2^r - 1$ being the maximum.

The total time is $O([b/r](n + 2^r))$, where $(n + 2^r)$ is time for Counting-Sort.
**Lemma.** Given $n$ $b$-bit binary numbers and any positive $r \leq b$. **Radix-Sort** uses $\Theta([b/r](n + 2^r))$ time.

**Proof.** Each $b$-digit binary number can be regarded as $[b/r]$ $r$-digit binary numbers. These $r$-digit binary numbers are of integer values in the range of $\{0, 1, \ldots, 2^r - 1\}$.

- Run **Radix-Sort** on the original binary numbers assumed to be $[b/r]$ columns.
- For every column, sorting by **Counting-Sort** with $2^r - 1$ being the maximum.
- The total time is $O([b/r](n + 2^r))$, where $(n + 2^r)$ is time for **Counting-Sort**.

Since all steps in the two algorithms are mandatory, the total time is also $\Omega([b/r](n + 2^r))$, thus $\Theta([b/r](n + 2^r))$.
Lemma. Given \( n \) \( b \)-bit binary numbers and any positive \( r \leq b \). \textsc{Radix-Sort} uses \( \Theta([b/r](n + 2^r)) \) time.

Proof. Each \( b \)-digit binary number can be regarded as \([b/r]\) \( r \)-digit binary numbers. These \( r \)-digit binary numbers are of integer values in the range of \( \{0, 1, \ldots, 2^r - 1\} \).

Run \textsc{Radix-Sort} on the original binary numbers assumed to be \([b/r]\) columns.

For every column, sorting by \textsc{Counting-Sort} with \( 2^r - 1 \) being the maximum.

The total time is \( O([b/r](n + 2^r)) \), where \( (n + 2^r) \) is time for \textsc{Counting-Sort}.

Since all steps in the two algorithms are mandatory, the total time is also \( \Omega([b/r](n + 2^r)) \), thus \( \Theta([b/r](n + 2^r)) \).

Once \( b \) and \( n \) are given, we can choose \( r \) to minimize the quantity \([b/r](n + 2^r)\).
Chapter 8. Lower Bounds and Sorting in Linear Time

Bucket Sort (assuming uniform distribution of inputs)
Bucket Sort (assuming uniform distribution of inputs)
Algorithm Bucket-Sort($A$)
Bucket Sort (assuming uniform distribution of inputs)

Algorithm \textsc{Bucket-Sort}(A)
1. \hspace{0.5cm} n = \text{length}[A]
Bucket Sort (assuming uniform distribution of inputs)

Algorithm Bucket-Sort\((A)\)
1. \(n = \text{length}[A]\)
2. \(\text{for } i = 1 \text{ to } n\)
Chapter 8. Lower Bounds and Sorting in Linear Time

**Bucket Sort** (assuming uniform distribution of inputs)

Algorithm **Bucket-Sort**$(A)$

1. $n = \text{length}[A]$  
2. **for** $i = 1$ **to** $n$  
3. insert $A[i]$ into list $B[[nA[i]]]$
Bucket Sort (assuming uniform distribution of inputs)

Algorithm Bucket-Sort(A)
1. \( n = \text{length}[A] \)
2. \( \text{for } i = 1 \text{ to } n \)
3. \( \text{insert } A[i] \text{ into list } B[\lfloor nA[i] \rfloor] \)
4. \( \text{for } i = 0 \text{ to } n - 1 \)
Bucket Sort (assuming uniform distribution of inputs)

Algorithm Bucket-Sort(A)
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4. \( \text{for } i = 0 \text{ to } n - 1 \)
5. \( \text{sort list } B[i] \text{ with Insertion Sort} \)
Chapter 8. Lower Bounds and Sorting in Linear Time

**Bucket Sort** (assuming uniform distribution of inputs)

Algorithm **Bucket-Sort**($A$)

1. $n = \text{length}[A]$
2. for $i = 1$ to $n$
3. insert $A[i]$ into list $B[\lfloor nA[i] \rfloor]$
4. for $i = 0$ to $n - 1$
5. sort list $B[i]$ with **Insertion Sort**
6. concatenate the lists $B[0], B[1], ..., B[n - 1]$
Bucket Sort (assuming uniform distribution of inputs)

Algorithm Bucket-Sort\( (A) \)
1. \( n = \text{length}[A] \)
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5. \( \text{sort list } B[i] \text{ with Insertion Sort} \)
6. \( \text{concatenate the lists } B[0], B[1], \ldots, B[n - 1] \)

\( A: \) .78 .17 .39 .26 .72 .94 .21 .12 .23 .68
Bucket Sort (assuming uniform distribution of inputs)

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6. \( \text{concatenate the lists } B[0], B[1], ..., B[n - 1] \)

A: 0.78 0.17 0.39 0.26 0.72 0.94 0.21 0.12 0.23 0.68

B: 0 /
Bucket Sort (assuming uniform distribution of inputs)

Algorithm Bucket-Sort(A)
1. \( n = \text{length}[A] \)
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5. \( \text{sort list } B[i] \text{ with Insertion Sort} \)
6. \( \text{concatenate the lists } B[0], B[1], \ldots, B[n - 1] \)

A: \( .78 .17 .39 .26 .72 .94 .21 .12 .23 .68 \)

B: \( 0 / \)
\( 1 \rightarrow .12 \rightarrow .17 \)
Chapter 8. Lower Bounds and Sorting in Linear Time

**Bucket Sort** (assuming uniform distribution of inputs)

Algorithm \textsc{Bucket-Sort}(A)

1. \( n = \text{length}[A] \)
2. \textbf{for} \( i = 1 \) \textbf{to} \( n \)
3. \hspace{1em} insert \( A[i] \) into list \( B[\lfloor nA[i] \rfloor] \)
4. \textbf{for} \( i = 0 \) \textbf{to} \( n - 1 \)
5. \hspace{1em} sort list \( B[i] \) with \textsc{Insertion Sort}
6. \hspace{1em} concatenate the lists \( B[0], B[1], ..., B[n - 1] \)

\begin{align*}
A: & \quad .78 \quad .17 \quad .39 \quad .26 \quad .72 \quad .94 \quad .21 \quad .12 \quad .23 \quad .68 \\
B: & \quad 0 / \\
& \hspace{1em} 1 \rightarrow .12 \rightarrow .17 \\
& \hspace{1em} 2 \rightarrow .21 \rightarrow .23 \rightarrow .26
\end{align*}
Chapter 8. Lower Bounds and Sorting in Linear Time

**Bucket Sort** (assuming uniform distribution of inputs)

Algorithm **Bucket-Sort**($A$)

1. $n = \text{length}[A]$
2. for $i = 1$ to $n$
3. insert $A[i]$ into list $B[[nA[i]]]$
4. for $i = 0$ to $n - 1$
5. sort list $B[i]$ with **Insertion Sort**
6. concatenate the lists $B[0], B[1], ..., B[n - 1]$

<table>
<thead>
<tr>
<th>A:</th>
<th>.78</th>
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<th>.94</th>
<th>.21</th>
<th>.12</th>
<th>.23</th>
<th>.68</th>
</tr>
</thead>
<tbody>
<tr>
<td>B:</td>
<td>0</td>
<td>/</td>
<td>1</td>
<td>→</td>
<td>.12</td>
<td>→</td>
<td>.17</td>
<td>2</td>
<td>→</td>
<td>.21</td>
</tr>
</tbody>
</table>
Bucket Sort (assuming uniform distribution of inputs)

Algorithm Bucket-Sort(A)
1. \( n = \text{length}[A] \)
2. for \( i = 1 \) to \( n \)
3. insert \( A[i] \) into list \( B[\lfloor nA[i] \rfloor] \)
4. for \( i = 0 \) to \( n - 1 \)
5. sort list \( B[i] \) with Insertion Sort
6. concatenate the lists \( B[0], B[1], ..., B[n - 1] \)

A: \(.78\ .17\ .39\ .26\ .72\ .94\ .21\ .12\ .23\ .68\)

B: \(0\ /
1 \rightarrow .12 \rightarrow .17
2 \rightarrow .21 \rightarrow .23 \rightarrow .26
3 \rightarrow .39
4 \)
Bucket Sort (assuming uniform distribution of inputs)

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A: .78 .17 .39 .26 .72 .94 .21 .12 .23 .68

B: 0 /
1 → .12 → .17
2 → .21 → .23 → .26
3 → .39
4 /
5 /
Chapter 8. Lower Bounds and Sorting in Linear Time

**Bucket Sort** (assuming uniform distribution of inputs)

**Algorithm** Bucket-Sort($A$)

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Chapter 8. Lower Bounds and Sorting in Linear Time

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   2 → 0.21 → 0.23 → 0.26
   3 → 0.39
   4 /
   5 /
   6 → 0.68
   7 → 0.72 → 0.78
   8 /
Chapter 8. Lower Bounds and Sorting in Linear Time

Bucket Sort (assuming uniform distribution of inputs)

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   9 → .94
Chapter 8. Lower Bounds and Sorting in Linear Time

**Bucket Sort** (assuming uniform distribution of inputs)

Algorithm **Bucket-Sort**(*A*)
1. $n = length[A]$
2. **for** $i = 1$ **to** $n$
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5. sort list $B[i]$ with **Insertion Sort**

A: .78 .17 .39 .26 .72 .94 .21 .12 .23 .68

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   1 → .12 → .17
   2 → .21 → .23 → .26
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   4 /
   5 /
   6 → .68
   7 → .72 → .78
   8 /
   9 → .94
Chapter 9. Medians and Order Statistics

Chapter 9. Medians and order statistics
Chapter 9. Medians and Order Statistics

Chapter 9. Medians and order statistics

• find the maximum: linear time
Chapter 9. Medians and Order Statistics

- find the maximum: linear time
- find the minimum: linear time
- find the median (i.e., the \( n/2 \)th smallest element)?
  - the problem has upper bound O\( (n \log_2 n) \).

Why?

Can we do better?
Chapter 9. Medians and Order Statistics

Chapter 9. Medians and order statistics

- find the maximum: linear time
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Chapter 9. Medians and order statistics

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Chapter 9. Medians and Order Statistics

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Chapter 9. Medians and order statistics

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Can we do better?
Chapter 9. Medians and Order Statistics

Selection problem

\[ \text{Input: a list } A \text{ of elements, an integer } i \]

\[ \text{Output: the } i \text{th smallest element in } A \]

There are algorithms solving it in linear time.

Two types of algorithms:

• Selection in worst case linear time
• Selection in expected linear time (but worst case \( \Theta(n^2) \))
Chapter 9. Medians and Order Statistics

Selection problem

Input: a list \( A \) of elements, an integer \( i \);
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Chapter 9. Medians and Order Statistics

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Chapter 9. Medians and Order Statistics

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Chapter 9. Medians and Order Statistics

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Chapter 9. Medians and Order Statistics

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Chapter 9. Medians and Order Statistics

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There are algorithms solving it in linear time.

Two types of algorithms:

- Selection in worst case linear time
- Selection in *expected* linear time (but worst case $\Theta(n^2)$)
Selection in worst case linear time

Input: set $S$ of $n$ elements and $i$;
Output: the $i$th smallest element in $S$;

Main idea:
• find a pivot $x$ to partition the list $S$ into two sublists $S_1$ and $S_2$,
  such that $\forall y \in S_1 \ y < x$ and $\forall z \in S_2 \ z > x$;
• both $S_1$ and $S_2$ are guaranteed only a fraction of $S$;
• the $i$th smallest element is either $x$, or in $S_1$ or in $S_2$ (but not both);
• in either of the latter two cases, the algorithm is applied recursively.
Chapter 9. Medians and Order Statistics

Selection in worst case linear time
Selection in worst case linear time

**INPUT:** set $S$ of $n$ elements and $i$;
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Chapter 9. Medians and Order Statistics

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Chapter 9. Medians and Order Statistics

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Chapter 9. Medians and Order Statistics

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Chapter 9. Medians and Order Statistics

Selection in worst case linear time

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Chapter 9. Medians and Order Statistics

Assume total time complexity $T(n)$ if time for finding pivot: $c_n$ and time for the recursive step: $T(\beta n)$, for some $0 < \beta < 1$.

Then $T(n) \leq T(\beta n) + c_n$ $\leq c_n + c\beta n + T(\beta^2 n)$ $\leq c_n + c\beta n + c\beta^2 n + \cdots + c\beta^{m-1} n + T(\beta^m n)$ (where $\beta^m n = 1$).

$\leq c_n (1 - \beta^{m+1}) + c' \leq c_1 1^{m+1} + c' = O(n)$.
Chapter 9. Medians and Order Statistics

Assume total time complexity $T(n)$

- if time for finding pivot: $cn$
Assume total time complexity $T(n)$

if time for finding pivot: $cn$
and time for the recursive step: $T(\beta n)$, for some $0 < \beta < 1$
Chapter 9. Medians and Order Statistics

Assume total time complexity $T(n)$

**if** time for finding pivot: $cn$
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Chapter 9. Medians and Order Statistics

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$$T(n) \leq cn +$$
Chapter 9. Medians and Order Statistics

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and time for the recursive step: $T(\beta n)$, for some $0 < \beta < 1$

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$$T(n) \leq cn + c\beta n + T(\beta^2 n)$$
Assume total time complexity \( T(n) \)

- if time for finding pivot: \( cn \)
- and time for the recursive step: \( T(\beta n) \), for some \( 0 < \beta < 1 \)

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Chapter 9. Medians and Order Statistics

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\end{itemize}

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\[ T(n) \leq cn + c\beta n + T(\beta^2 n) \]

\[ T(n) \leq cn + c\beta n + c\beta^2 n + \ldots + c\beta^{m-1} n + T(\beta^m n) \]
Assume total time complexity $T(n)$

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$$T(n) \leq cn + c\beta n + c\beta^2 n + \cdots + c\beta^{m-1} n + T(\beta^m n) \quad (\text{where } \beta^m n = 1)$$
Chapter 9. Medians and Order Statistics

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$$\leq cn \left( \frac{1 - \beta^m}{1 - \beta} \right) + c'$$
Assume total time complexity $T(n)$

*if* time for finding pivot: $cn$

and time for the recursive step: $T(\beta n)$, for some $0 < \beta < 1$

Then $T(n) \leq T(\beta n) + cn$

$$T(n) \leq cn + c\beta n + T(\beta^2 n)$$

$T(n) \leq cn + c\beta n + c\beta^2 n + \cdots + c\beta^{m-1} n + T(\beta^m n)$ (where $\beta^m n = 1$)

$$\leq cn\left(1 - \frac{\beta^m}{1 - \beta}\right) + c' \leq c\frac{1}{1 - \beta} n + c'$$
Chapter 9. Medians and Order Statistics

Assume total time complexity $T(n)$ if time for finding pivot: $cn$
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Then $T(n) \leq T(\beta n) + cn$

$$T(n) \leq cn + c\beta n + T(\beta^2 n)$$

$$T(n) \leq cn + c\beta n + c\beta^2 n + \cdots + c\beta^{m-1} n + T(\beta^m n) \text{ (where } \beta^m n = 1)$$

$$\leq cn \left( \frac{1 - \beta^m}{1 - \beta} \right) + c' \leq c \frac{1}{1 - \beta} n + c' = O(n)$$
Assume total time complexity $T(n)$ if the time for finding pivot:

$$cn + T(\alpha n),$$

for some $0 < \alpha < 1$, and time for the recursive step:

$$T(\beta n),$$

for some $0 < \beta < 1$. Assume $\alpha + \beta < 1$, then

$$T(n) \leq cn + T(\alpha n) + T(\beta n).$$

With the recursive tree method (you draw a picture):

$$T(n) \leq cn + c\alpha n + T(\alpha^2 n) + 2T(\alpha\beta n) + c\beta n + T(\beta^2 n).$$

$$= cn + c(\alpha + \beta) n + c\alpha^2 n + 2c\alpha\beta n + c\beta^2 n + T(\alpha^3 n) + 3T(\alpha^2\beta n) + 3T(\alpha\beta^2 n) + T(\beta^3 n).$$

$$\leq cn + c(\alpha + \beta) n + c(\alpha + \beta)^2 n + T(\alpha^3 n) + 3T(\alpha^2\beta n) + 3T(\alpha\beta^2 n) + T(\beta^3 n).$$

$$\leq cn + c(\alpha + \beta) n + c(\alpha + \beta)^2 n + c(\alpha + \beta)^3 n + \cdots + c(\alpha + \beta)^m n + c\cdot\sum_{i=0}^{m-1} c(\alpha + \beta)^i n.$$

Where $m = \max\{i, j\}$, for such $i, j$ that $\alpha^i n = 1$ and $\beta^j n = 1$.

Therefore, $T(n) \leq c1 - (\alpha + \beta)^m n \leq c1 - (\alpha + \beta)^n = O(n)$. 

Chapter 9. Medians and Order Statistics

Assume total time complexity $T(n)$

- if time for finding pivot: $cn + T(\alpha n)$, for some $0 < \alpha < 1$
Chapter 9. Medians and Order Statistics

Assume total time complexity $T(n)$

\begin{align*}
\textbf{if} \text{ time for finding pivot: } & \quad cn + T(\alpha n), \text{ for some } 0 < \alpha < 1 \\
\text{and time for the recursive step: } & \quad T(\beta n), \text{ for some } 0 < \beta < 1 \\
\textbf{assume} \alpha + \beta < 1, \\
\end{align*}
Chapter 9. Medians and Order Statistics

Assume total time complexity $T(n)$

if time for finding pivot: $cn + T(\alpha n)$, for some $0 < \alpha < 1$
and time for the recursive step: $T(\beta n)$, for some $0 < \beta < 1$

assume $\alpha + \beta < 1$, then

$T(n) \leq cn + T(\alpha n) + T(\beta n)$
Chapter 9. Medians and Order Statistics

Assume total time complexity $T(n)$

- if time for finding pivot: $cn + T(\alpha n)$, for some $0 < \alpha < 1$
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**assume** $\alpha + \beta < 1$, then

$$T(n) \leq cn + T(\alpha n) + T(\beta n)$$

With the recursive tree method (you draw a picture):
Chapter 9. Medians and Order Statistics

Assume total time complexity $T(n)$

- if time for finding pivot: $cn + T(\alpha n)$, for some $0 < \alpha < 1$
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With the recursive tree method (you draw a picture):

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Chapter 9. Medians and Order Statistics

Assume total time complexity $T(n)$

if time for finding pivot: $cn + T(\alpha n)$, for some $0 < \alpha < 1$
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Chapter 9. Medians and Order Statistics

Assume total time complexity \( T(n) \)

- if time for finding pivot: \( cn + T(\alpha n) \), for some \( 0 < \alpha < 1 \)
- and time for the recursive step: \( T(\beta n) \), for some \( 0 < \beta < 1 \)

**assume** \( \alpha + \beta < 1 \), then

\[
T(n) \leq cn + T(\alpha n) + T(\beta n)
\]

With the recursive tree method (you draw a picture):

\[
T(n) \leq cn + c\alpha n + T(\alpha^2 n) + T(\alpha\beta n) + c\beta n + T(\beta\alpha n) + T(\beta^2 n)
\]

\[
= cn + c\alpha n + c\beta n + T(\alpha^2 n) + 2T(\alpha\beta n) + T(\beta^2 n)
\]
Chapter 9. Medians and Order Statistics

Assume total time complexity $T(n)$

if time for finding pivot: $cn + T(\alpha n)$, for some $0 < \alpha < 1$
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$$= cn + c\alpha n + c\beta n + T(\alpha^2 n) + 2T(\alpha \beta n) + T(\beta^2 n)$$

$$\leq cn + c(\alpha + \beta)n + c\alpha^2 n + 2c\alpha\beta n + c\beta^2 n$$
Chapter 9. Medians and Order Statistics

Assume total time complexity $T(n)$

**if** time for finding pivot: $cn + T(\alpha n)$, for some $0 < \alpha < 1$

and time for the recursive step: $T(\beta n)$, for some $0 < \beta < 1$

**assume** $\alpha + \beta < 1$, then

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T(n) \leq cn + T(\alpha n) + T(\beta n)
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With the recursive tree method (you draw a picture):

\[
T(n) \leq cn + c\alpha n + T(\alpha^2 n) + T(\alpha\beta n) + c\beta n + T(\beta\alpha n) + T(\beta^2 n)
\]

\[
= cn + c\alpha n + c\beta n + T(\alpha^2 n) + 2T(\alpha\beta n) + T(\beta^2 n)
\]

\[
\leq cn + c(\alpha + \beta)n + c\alpha^2 n + 2c\alpha\beta n + c\beta^2 n + T(\alpha^3 n) + 3T(\alpha^2\beta n) + 3T(\alpha\beta^2 n) + T(\beta^3 n)
\]

Therefore,

\[
T(n) \leq c_1 \left(1 - (\alpha + \beta)^m\right)n + c_1'\
\leq O(n)
\]
Chapter 9. Medians and Order Statistics

Assume total time complexity $T(n)$

if time for finding pivot: $cn + T(\alpha n)$, for some $0 < \alpha < 1$
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$$\leq cn + c(\alpha + \beta)n + c\alpha^2 n + 2c\alpha\beta n + c\beta^2 n + T(\alpha^3 n) + 3T(\alpha^2 \beta n) + 3T(\alpha\beta^2 n) + T(\beta^3 n)$$

$$= cn + c(\alpha + \beta)n + c(\alpha + \beta)^2 n + T(\alpha^3 n) + 3T(\alpha^2 \beta n) + 3T(\alpha\beta^2 n) + T(\beta^3 n)$$
Chapter 9. Medians and Order Statistics

Assume total time complexity $T(n)$

**if** time for finding pivot: $cn + T(\alpha n)$, for some $0 < \alpha < 1$

and time for the recursive step: $T(\beta n)$, for some $0 < \beta < 1$

**assume** $\alpha + \beta < 1$, then

$$T(n) \leq cn + T(\alpha n) + T(\beta n)$$

With the recursive tree method (you draw a picture):

$$T(n) \leq cn + c\alpha n + T(\alpha^2 n) + T(\alpha \beta n) + c\beta n + T(\beta \alpha n) + T(\beta^2 n)$$

$$= cn + c\alpha n + c\beta n + T(\alpha^2 n) + 2T(\alpha \beta n) + T(\beta^2 n)$$

$$\leq cn + c(\alpha + \beta)n + c\alpha^2 n + 2c\alpha \beta n + c\beta^2 n + T(\alpha^3 n) + 3T(\alpha^2 \beta n) + 3T(\alpha \beta^2 n) + T(\beta^3 n)$$

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Chapter 9. Medians and Order Statistics

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$$\leq cn + c(\alpha + \beta)n + c(\alpha + \beta)^2 n + c(\alpha + \beta)^3 n + \cdots + c(\alpha + \beta)^{m-1} n + c'$$
Chapter 9. Medians and Order Statistics

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where $m = \max\{i, j\}$, for such $i, j$ that $\alpha^i n = 1$ and $\beta^j n = 1.$
and $c' = 2T(1)$, the base case.
Chapter 9. Medians and Order Statistics

Assume total time complexity $T(n)$

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\]

\[
    \leq cn+c(\alpha+\beta)n+c\alpha^2 n+2c\alpha\beta n+c\beta^2 n + T(\alpha^3 n) + 3T(\alpha^2 \beta n) + 3T(\alpha \beta^2 n) + T(\beta^3 n) 
\]

\[
    = cn + c(\alpha + \beta)n + c(\alpha + \beta)^2 n + T(\alpha^3 n) + 3T(\alpha^2 \beta n) + 3T(\alpha \beta^2 n) + T(\beta^3 n)
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and $c' = 2T(1)$, the base case.

Therefore,

\[
    T(n) \leq c \frac{1-(\alpha+\beta)^m}{1-(\alpha+\beta)} n
\]
Chapter 9. Medians and Order Statistics

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- **if** time for finding pivot: $cn + T(\alpha n)$, for some $0 < \alpha < 1$
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Therefore,

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Chapter 9. Medians and Order Statistics

Assume total time complexity \( T(n) \)

\[ \text{if time for finding pivot: } cn + T(\alpha n), \text{ for some } 0 < \alpha < 1 \]

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\textbf{assume } \alpha + \beta < 1, \text{ then }

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\[ \leq cn + c(\alpha + \beta)n + c(\alpha + \beta)^2 n + c(\alpha + \beta)^3 n + \cdots + c(\alpha + \beta)^{m-1} n + c' \]

where \( m = \max\{i, j\} \), for such \( i, j \) that \( \alpha^i n = 1 \) and \( \beta^j n = 1 \).

\[ \text{and } c' = 2T(1), \text{ the base case.} \]

Therefore,

\[ T(n) \leq c \frac{1-(\alpha+\beta)^m}{1-(\alpha+\beta)} n \leq c \frac{1}{1-(\alpha+\beta)} n = O(n). \]
Chapter 9. Medians and Order Statistics

How to find such a pivot?

- the very selection algorithm is recursively called for finding the pivot
- the size of the sublist to find the pivot is also a fraction $\alpha n$ of the original list $S$, $|S| = n$;
- the total time actually is $T(n) \leq T(\alpha n) + T(\beta n) + cn$ where $\alpha + \beta < 1$. 
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Algorithm \texttt{SELECT} \((S, i); \) \{ where \( S \) contains \( n \) distinct elements\}
Algorithm **SELECT** \((S, i)\); \{ where \(S\) contains \(n\) distinct elements\}

(1) divide \(S\) into \([n/5]\) groups of 5 elements
Algorithm \textsc{Select} \((S, i)\); \{ where \(S\) contains \(n\) distinct elements\}

(1) divide \(S\) into \(\lceil n/5 \rceil\) groups of 5 elements

(2) sort each group (of 5) and find the median of each group;
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Algorithm $\text{SELECT} (S, i)$; \{ where $S$ contains $n$ distinct elements \}
(1) divide $S$ into $\lceil n/5 \rceil$ groups of 5 elements
(2) sort each group (of 5) and find the median of each group;
    let $M$ contain all these medians; where $|M| = \lceil n/5 \rceil$
Chapter 9. Medians and Order Statistics

Algorithm \texttt{Select} \((S, i)\); \{ where \(S\) contains \(n\) distinct elements\}
(1) divide \(S\) into \(\lceil n/5 \rceil\) groups of 5 elements
(2) sort each group (of 5) and find the median of each group;
    let \(M\) contain all these medians; where \(|M| = \lfloor n/5 \rfloor\)
(3) \textbf{recursively call} \texttt{Select}(\(M, \lceil n/10 \rceil\));
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Algorithm \textsc{Select} \((S, i)\); \{ where \(S\) contains \(n\) distinct elements\}
(1) divide \(S\) into \(\lceil n/5 \rceil\) groups of 5 elements
(2) sort each group (of 5) and find the median of each group;
    let \(M\) contain all these medians; where \(|M| = \lceil n/5 \rceil\)
(3) \textbf{recursively call} \textsc{Select} \((M, \lceil n/10 \rceil)\);
    let the result be \(x\) and let the rank of \(x\) be \(k\) in \(S\)
(4) if \(i = k\) return \((x)\)
(5) else use \(x\) as the pivot to partition \(S\) resulting in \(S_1\) and \(S_2\),
    such that \(\forall y \in S_1\) \(y < x\) and \(\forall z \in S_2\) \(z > x\)
(6) if \(i < k\) recursively call \textsc{Select} \((S_1, i)\)
    else recursively call \textsc{Select} \((S_2, i - k)\)
Algorithm \textsc{Select} \((S, i); \) \{ where \( S \) contains \( n \) distinct elements\}

1. divide \( S \) into \( \lceil n/5 \rceil \) groups of 5 elements
2. sort each group (of 5) and find the median of each group;
   let \( M \) contain all these medians; where \(|M| = \lceil n/5 \rceil\)
3. \textbf{recursively call} \textsc{Select}(\(M, \lceil n/10 \rceil \))
   let the result be \( x \) and let the rank of \( x \) be \( k \) in \( S \)
4. \textbf{if} \( i = k \) \textbf{return} \((x)\)
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Algorithm $\text{Select} \ (S, i); \ \{ \text{where } S \text{ contains } n \text{ distinct elements} \} $

1. divide $S$ into $\lceil n/5 \rceil$ groups of 5 elements 
2. sort each group (of 5) and find the median of each group; 
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3. recursively call $\text{Select} \ (M, \lceil n/10 \rceil)$; 
   let the result be $x$ and let the rank of $x$ be $k$ in $S$ 
4. if $i = k$ return $(x)$ 
5. else use $x$ as the pivot to partition $S$ resulting in $S_1$ and $S_2$, 
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6. if $i < k$ recursively call $\text{SELECT}(S_1, i)$
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Algorithm \texttt{Select} \((S, i)\); \{ where \(S\) contains \(n\) distinct elements \}
(1) divide \(S\) into \([n/5]\) groups of 5 elements
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    let \(M\) contain all these medians; where \(|M| = [n/5]\)
(3) \textbf{recursively call} \texttt{Select}(M, [n/10]);
    let the result be \(x\) and let the rank of \(x\) be \(k\) in \(S\)
(4) \textbf{if} \(i = k\) \textbf{return} \((x)\)
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\[ |S_1| \geq 3 \left( \lceil \frac{n}{5} \rceil \right) \geq 3 \frac{n}{10} \Rightarrow |S_2| < n - 3 \frac{n}{10} = 7 \frac{n}{10} \]

Similarly, the number of elements \( \geq x \) is at least:

\[ |S_2| \geq 3 \left( \lceil \frac{n}{5} \rceil - 2 \right) \geq 3 \frac{n}{10} - 6 \geq 3 \frac{n}{10} \Rightarrow |S_1| < n - 3 \frac{n}{10} + 6 = 7 \frac{n}{10} + 6 \]

So a time upper bound for \( \text{Select} \) is \( T(n) \leq T_{\text{mom}} + T_{\text{sub}} + cn T(n) \leq T(\lceil \frac{n}{5} \rceil) + T(\lceil 7 \frac{n}{10} + 6 \rceil) + cn \) when \( n \geq 140 \) (why?)
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Note: the number of elements $\leq x$ is at least:

$$|S_1| \geq 3\left(\frac{\lceil n/5 \rceil}{2}\right) \geq \frac{3n}{10}$$
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Note: the number of elements \( \leq x \) is at least:

\[
|S_1| \geq 3\left(\left\lceil\frac{n}{5}\right\rceil\right) \geq 3\frac{n}{10} \implies
\]

\[
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Chapter 9. Medians and Order Statistics

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|S_1| \geq 3\left( \frac{\lceil n/5 \rceil}{2} \right) \geq 3n/10 \quad \Rightarrow \quad |S_2| < n - 3n/10 = 7n/10
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Chapter 9. Medians and Order Statistics

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Chapter 9. Medians and Order Statistics

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So a time upper bound for $\text{SELECT}$ is
Chapter 9. Medians and Order Statistics

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So a time upper bound for \texttt{SELECT} is \( T(n) \leq T_{\text{mom}} + T_{\text{sub}} + cn \)
Chapter 9. Medians and Order Statistics

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So a time upper bound for \textsc{Select} is $T(n) \leq T_{mom} + T_{sub} + cn$

$$T(n) \leq T([n/5]) + T([7n/10 + 6]) + cn$$

when $n \geq 140$ (why?)
Selection in *expected* linear time
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Selection in *expected* linear time

**Input:** a list $A$ of elements, an integer $i$;
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Selection in *expected* linear time

**Input:** a list $A$ of elements, an integer $i$;
**Output:** the $i$th smallest element in $A$;
Selection in \textit{expected} linear time

\textbf{Input:} a list $A$ of elements, an integer $i$;  
\textbf{Output:} the $i$th smallest element in $A$;

Idea of the algorithm:

- randomly identify a pivot $x$ and partition the list $A$ into two sublists $A_l$ and $A_u$; assume the \textbf{rank} of $x$ is $k$;
Selection in *expected* linear time

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- randomly identify a pivot $x$ and partition the list $A$ into two sublists $A_l$ and $A_u$; assume the **rank** of $x$ is $k$;
- if $i = k$, done, return $(x)$;
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- if $i = k$, done, return ($x$);
- else if $k > i$, recursively do for $A_l$ with $i$;
Selection in *expected* linear time

**INPUT:** a list $A$ of elements, an integer $i$;
**OUTPUT:** the $i$th smallest element in $A$;

Idea of the algorithm:

- randomly identify a pivot $x$ and partition the list $A$ into two sublists $A_l$ and $A_u$; assume the *rank* of $x$ is $k$;
- if $i = k$, done, return ($x$);
- else if $k > i$, recursively do for $A_l$ with $i$;
  - else recursively do for $A_u$ with $i - k$;
Algorithm \textsc{Randomized-Select} \((A, p, r, i)\)

If pivots always partition lists into \(n^r\) for some \(r > 1\),

time \(T(n)\) would have the recurrence

\[
T(n) \leq \max\{T(n^r), T((r-1)n^r)\} + cn
\]

assuming \(r \geq 2\),

\[
T(n) \leq cn(r-1)r + cn((r-1)r)^2 + \ldots cn((r-1)r)^m = O(n)
\]

where \(((r-1)r)^n = 1\), \(m = \log r\).
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Algorithm RANDOMIZED-SELECT \((A, p, r, i)\)
1. \textbf{if} \(p = r\)
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Algorithm RANDOMIZED-SELECT (A, p, r, i)
1. if \( p = r \)
2. return \((A[p])\)
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Algorithm RANDOMIZED-SELECT \((A, p, r, i)\)
1. \textbf{if} \(p = r\)
2. \textbf{return} \((A[p])\)
3. \(q = \text{RANDOMIZED PARTITION} (A, p, r)\)
Algorithm \textsc{Randomized-Select} \((A, p, r, i)\)

1. \textbf{if} \(p = r\)
2. \textbf{return} \((A[p])\)
3. \(q = \textsc{Randomized Partition} \((A, p, r)\)\)
4. \(k = q - p + 1\)

If pivots always partition lists into \(n^r\) with \(r > 1\),
then time \(T(n)\) would have the recurrence
\[
T(n) \leq \max\{T(n^r), T((r - 1)n^r)\} + cn
\]
assuming \(r \geq 2\),
then
\[
T(n) \leq cn(r - 1)r + cn((r - 1)r)^2 + \ldots + cn((r - 1)r)^m = O(n)
\]
Algorithm \textsc{Randomized-Select} \((A, p, r, i)\)

1. \textbf{if} \(p = r\)
2. \hspace{1em} \textbf{return} \((A[p])\)
3. \(q = \textsc{Randomized Partition} \((A, p, r)\)\)
4. \hspace{1em} \(k = q - p + 1\)
5. \textbf{if} \(i = k\)
Algorithm **RANDOMIZED-SELECT** \((A, p, r, i)\)
1. if \(p = r\)
2. return \((A[p])\)
3. \(q = \text{RANDOMIZED PARTITION} (A, p, r)\)
4. \(k = q - p + 1\)
5. if \(i = k\)
6. return \((A[q])\)
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Algorithm **Randomized-Select** \((A, p, r, i)\)

1.   \textbf{if} \(p = r\)
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3.   \(q = \text{Randomized Partition} \,(A, p, r)\)
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6.      \textbf{return} \((A[q])\)
7.   \textbf{else if} \(i < k\)
Chapter 9. Medians and Order Statistics

Algorithm `RANDOMIZED-SELECT (A, p, r, i)`
1. if $p = r$
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5. if $i = k$
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7. else if $i < k$
8. return (RANDOMIZED-SELECT ($A, p, q - 1, i$))
Algorithm `Randomized-Select (A, p, r, i)`
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8. return ($\text{Randomized-Select} (A, p, q - 1, i)$)
9. else return ($\text{Randomized-Select} (A, q + 1, r, i - k)$)
Algorithm \textsc{Randomized-Select} ($A, p, r, i$)

1. \textbf{if} $p = r$
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3. \hspace{1em} $q = \textsc{Randomized\ Partition} (A, p, r)$
4. \hspace{1em} $k = q - p + 1$
5. \hspace{1em} \textbf{if} $i = k$
6. \hspace{2em} \textbf{return} ($A[q]$)
7. \hspace{1em} \textbf{else if} $i < k$
8. \hspace{2em} \textbf{return} ($\textsc{Randomized-Select} (A, p, q - 1, i)$)
9. \hspace{1em} \textbf{else return} ($\textsc{Randomized-Select} (A, q + 1, r, i - k)$)

If pivots always partition lists into $\frac{n}{r} : \frac{r-1}{r} n$, for some $r > 1$, 
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If pivots always partition lists into \( \frac{n}{r} : \frac{r-1}{r}n \), for some \( r > 1 \),
time \( T(n) \) would have the recurrence

\[
T(n) \leq \max\{T\left(\frac{n}{r}\right), T\left(\frac{(r-1)n}{r}\right)\} + nc
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assuming \(r \geq 2\),

\[
T(n) \leq cn\left(\frac{r - 1}{r}\right)^1 + cn\left(\frac{r - 1}{r}\right)^2 + cn\left(\frac{r - 1}{r}\right)^3 + \ldots cn\left(\frac{r - 1}{r}\right)^m = O(n)
\]

where \(\left(\frac{r-1}{r}\right)^m n = 1\), \(m = \log_{\frac{r-1}{r}} n\).
Performance analysis
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Performance analysis

The worst case: running time $\Theta(n^2)$.
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Performance analysis

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Performance analysis

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Average case: $E[T(n)]$

- on sublist $A[p..r]$, assume $n = r - p + 1$;
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- the pivot is chosen with probability $\frac{1}{n}$;
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Average case: $E[T(n)]$ (cont’)

- so the expected time $E[T(n)]$ needs to include the average time of recursion on the case when sublist $A[p..q]$ possibly has lengths $k = 0, 1, 2, \ldots, n - 1$
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• thus the expected time $E[T(n)]$ is computed as

$$E[T(n)] \leq \sum_{k=1}^{n} \frac{1}{n} \cdot E[T(\max\{k-1, n-k\})] + an, \text{ for some constant } a > 0$$
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E[T(n)] \leq \sum_{k=1}^{n} \frac{1}{n} \cdot E[T(\max\{k-1, n-k\})] + an, \text{ for some constant } a > 0
\]

because \( \max\{k-1, n-k\} = k-1 \) if \( k > n/2 \) and \( \max\{k-1, n-k\} = n-k \) if \( k \leq n/2 \)

\[
E[T(n)] \leq 2/n \sum_{k=n/2}^{n-1} E[T(k)] + an
\]
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We conclude that
\[ E[T(n)] \leq \frac{2}{n} \sum_{k=1}^{n-1} k = \frac{n}{2} \]

Theorem.

\[ E[T(n)] = O(n). \]

Proof (by substitution method).

We will prove that
\[ E[T(n)] \leq cn \]

for some \( c > 0 \).

\begin{itemize}
  \item Base case: \( n = \) ?, we will decide later;
  \item Assumption: for all \( k \leq n-1 \),
    \[ E[T(k)] \leq ck \];
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    \[ \leq 2c \]
    \[ = 2c \]
    \[ n \]
    \[ \leq 3cn/4 + c/2 + an \]
    \[ \leq cn \when \]
    \[ (cn/4 - c/2 - an) \geq 0 \].
\end{itemize}
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We conclude that $E[T(n)] \leq \frac{2}{n} \sum_{k=n/2}^{n-1} E[T(k)] + an$
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\]

\[
= \frac{2c}{n} \sum_{k=1}^{n-1} k - \sum_{k=1}^{n/2-1} k + an
\]

That is when \( \frac{2c}{n} \sum_{k=1}^{n-1} k - \sum_{k=1}^{n/2-1} k \geq 0 \).

• **Base case:** \( T(n) \leq cn \), for \( n < \frac{2c}{c-4a} \),

How to prove?
We conclude that $E[T(n)] \leq 2/n\sum_{k=n/2}^{n-1} E[T(k)] + an$

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  $$= \cdots = \frac{cn}{2}$$
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Chapter 9. Medians and Order Statistics

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- **Base case:** $T(n) \leq cn$, for $n < 2c/(c - 4a)$,
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We conclude that $E[T(n)] \leq 2/n \sum_{k=n/2}^{n-1} E[T(k)] + an$

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  \[
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  \]

  \[
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Summary of Algorithm Analysis Scenarios
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Given an algorithm, carry out the following in order:
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**Summary of Algorithm Analysis Scenarios**

Given an algorithm, carry out the following in order:

- analyzing time $T(n)$ of the algorithm
- obtain an expression $T(n) =$ ...
- guess an upper (or lower) bound (e.g., $T(n) = O(\cdot)$)
- prove the correctness of the bound.

For example, given *Insertion Sort*:

- we first analyzed the algorithm and obtained $T(n) =$
  
  $c_1 n + c_2 (n - 1) + c_4 (n - 1) + c_5 n \sum_{j=2}^n t_j + c_6 n \sum_{j=2}^n (t_j - 1) + c_7 n \sum_{j=2}^n (t_j - 1) + c_8 (n - 1)$

- we guessed upper bound $T(n) = O(n^2)$, i.e., $T(n) \leq cn^2$;
- and finally proved that it was indeed the case.
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For example, given Insertion Sort:

• we first analyzed the algorithm and obtained
  
  \[
  T(n) = c_1 n + c_2 (n-1) + c_3 \sum_{j=2}^{n} t_j + c_4 (n-1) + c_5 n \sum_{j=2}^{n} (t_j-1) + c_6 n \sum_{j=2}^{n} (t_j-1) + c_7 (n-1)
  \]

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- obtain an expression \( T(n) = \ldots \)
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For example, given \textsc{Insertion Sort}:

- we first analyzed the algorithm

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Summary of Algorithm Analysis Scenarios

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For example, given INSERTION SORT:

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  and obtained

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T(n) = c_1 n + c_2 (n - 1) + c_4 (n - 1) + c_5 \sum_{j=2}^{n} t_j + c_6 \sum_{j=2}^{n} (t_j - 1) + c_7 \sum_{j=2}^{n} (t_j - 1) + c_8 (n - 1)
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- and finally proved that it was indeed the case.
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Summary of Algorithm Analysis Scenarios

For recursive algorithms

For example, given Binary Search algorithm,

• we first analyze the time $T(n)$ of the algorithm

  $T(n) \leq T(\lfloor n/2 \rfloor) + c$

• we guess upper bound $T(n) = O(\log_2 n)$, i.e.,

  $T(n) \leq c \log_2 n$;

• we prove the guessed bound.

(1) we can use the recursive tree method by unfolding the time function;

or

(2) we can use the substitution method by the principle of induction.

But we need the recurrence to apply induction.

using the recurrence:

$T(n) \leq T(\lfloor n/2 \rfloor) + c$

to prove $T(n) \leq c \log_2 n$.

see previous lecture notes
Summary of Algorithm Analysis Scenarios

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Chapter 9. Medians and Order Statistics

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  (2) we can use the substitution method by the principle of induction. 

  \textbf{But we need the recurrence to apply induction.}
Summary of Algorithm Analysis Scenarios

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