Lecture Note III
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Part IV. Advanced Design and Analysis Techniques
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- Chapter 14.9 Exhaustive search
- Chapter 15. Dynamic programming
- Chapter 16. Greedy algorithms
Chapter 14.9. Exhaustive Search

Chapter 14.9. Exhaustive Search (not in the text!)
Chapter 14.9. Exhaustive Search

To enumerate all possible solutions to the problem instance

(\textit{not in the text!})
Chapter 14.9. Exhaustive Search

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How?
Chapter 14.9. Exhaustive Search

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- systematic examining all solutions
Chapter 14.9. Exhaustive Search

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How?

- systematic examining all solutions
- without repeating solutions that have been examined
Chapter 14.9. Exhaustive Search

To enumerate all possible solutions to the problem instance

How?

• systematic examining all solutions
• without repeating solutions that have been examined
• stop when a satisfactory solution is found
First, we need to be able to count total number of “things” to be enumerated.
Chapter 14.9. Exhaustive Search

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Chapter 14.9. Exhaustive Search

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- without missing one (correctness)
- without over-counting (efficiency)
- A sophisticated counting often has recursive solution.
Chapter 14.9. Exhaustive Search

Examples of counting:

1. The total number of permutations of \(1, 2, \ldots, n\) is
   \[P(n) = n \times P(n-1)\]
   with base case \(P(1) = 1\).

2. The total number of ways to choose \(k\) from \(n\) items is
   \[^nC_k = \frac{n \times (n-1) \times \ldots \times (n-k+1)}{k!}\]
   or, alternatively,
   \[^nC_k = \frac{(n-1) \times \ldots \times (n-k+1)}{(k-1)!}\]
   with base cases: (??)
Chapter 14.9. Exhaustive Search

Examples of counting:

(1) total number of permutations of \((1, 2, \ldots, n)\) is

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Chapter 14.9. Exhaustive Search

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Chapter 14.9. Exhaustive Search

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Chapter 14.9. Exhaustive Search

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Chapter 14.9. Exhaustive Search

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Chapter 14.9. Exhaustive Search

Example: Boolean Formula Satisfiability problem (SAT)
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**INPUT:** boolean formula \( f(x_1, x_2, \ldots, x_n) \),

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Chapter 14.9. Exhaustive Search

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$f(x_1, x_2, x_3) = (x_1 \lor x_2 \lor \neg x_3) \land (\neg x_1 \lor x_3)$ is satisfiable
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g(x_1, x_2) = (x_1 \lor \neg x_2) \land (\neg x_1 \lor x_2) \land (x_1 \lor x_2) \land (\neg x_1 \lor \neg x_2) \text{ is not!}
\]
Chapter 14.9. Exhaustive Search

Use exhaustive search to solve the SAT problem.

Input: boolean formula \( f(x_1, x_2, \ldots, x_n) \)

Output: "yes" if and only if \( f(x_1, x_2, \ldots, x_n) \) is satisfiable.

How?

What will you exhaustively search on?

- Enumerate all combinations of \( T \) and \( F \) for \( x_1, x_2, \ldots, x_n \).
- Can you solve it with a recursive algorithm?
- Can you solve it with an iterative algorithm?
Chapter 14.9. Exhaustive Search

Use exhaustive search to solve the SAT problem.

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Chapter 14.9. Exhaustive Search

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Chapter 14.9. Exhaustive Search

Solve SAT problem with a recursive algorithm:

• what data will the recursion be applied to?
  boolean formula \( f(x_1, \ldots, x_n) \)

• what is the terminating (base) case?
  \( n=0 \), formula without variables

• what is the recursive case?
  \[
  f(x_1, \ldots, x_{n-1}, x_n) = f(x_1, \ldots, x_{n-1}, T) \lor f(x_1, \ldots, x_{n-1}, F)
  \]

  \[
  f(x_1, \ldots, x_{n-1}, T) = \Rightarrow g(x_1, \ldots, x_{n-1})
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Chapter 14.9. Exhaustive Search

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Chapter 14.9. Exhaustive Search

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Chapter 14.9. Exhaustive Search

Algorithm SAT Solver($f(x_1, \ldots, x_{n-1}, x_n)$)
Chapter 14.9. Exhaustive Search

Algorithm SAT Solver\( (f(x_1, \ldots, x_{n-1}, x_n)) \)

1. \textbf{if} \( n = 0 \), \textbf{return} \( f \);
Algorithm SAT Solver($f(x_1, \ldots, x_{n-1}, x_n)$)

1. if $n = 0$, return ($f$);
2. else $g(x_1, \ldots, x_{n-1}) = f(x_1, \ldots, x_{n-1}, T)$

Does this algorithm exhaustively search all assignments to the variables?

- draw a search tree based on the algorithm.
- what does the tree look like?
- what does each path mean?
- how many paths?
- time? $T(n) = 2T(n-1) + cn$, $T(0) = c$, $\Rightarrow T(n) = \Theta(2^n)$
Chapter 14.9. Exhaustive Search

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4. return \( (\text{SAT Solver}(g(x_1, \ldots, x_{n-1})) \lor \text{SAT Solver}(h(x_1, \ldots, x_{n-1}))) \)
Chapter 14.9. Exhaustive Search

Algorithm \text{SAT Solver}(f(x_1, \ldots, x_{n-1}, x_n))

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2. \textbf{else} \ g(x_1, \ldots, x_{n-1}) = f(x_1, \ldots, x_{n-1}, \text{T})
3. \quad h(x_1, \ldots, x_{n-1}) = f(x_1, \ldots, x_{n-1}, \text{F})
4. \textbf{return} \ (\text{SAT Solver}(g(x_1, \ldots, x_{n-1})) \lor \text{SAT Solver}(h(x_1, \ldots, x_{n-1})))

Does this algorithm exhaustively search all assignments to the variables?

- draw a \textit{search tree} based on the algorithm.
- what does the tree look like?
Chapter 14.9. Exhaustive Search

Algorithm \textsc{SAT Solver}(f(x_1, \ldots, x_{n-1}, x_n))

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3. \; h(x_1, \ldots, x_{n-1}) = f(x_1, \ldots, x_{n-1}, F)
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- what does each path mean?
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Chapter 14.9. Exhaustive Search

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Chapter 14.9. Exhaustive Search

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Does this algorithm exhaustively search all assignments to the variables?

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- time? $T(n) = 2T(n-1) + cn$, $T(0) = c$, $T(n) = \Theta(2^n)$
Chapter 14.9. Exhaustive Search

Algorithm \textsc{SAT Solver}(f(x_1, \ldots, x_{n-1}, x_n))

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Chapter 14.9. Exhaustive Search

Solve SAT problem with iterative algorithms

- How? what to iterate on? assignments
- what is the initial value?
  \[ x_1 = F, x_2 = F, \ldots, x_n = F, \]
or simply \((F, F, \ldots, F)\)
- what to increment
  \((\ldots, F, T, \ldots, T)\) \rightarrow \((\ldots, T, F, \ldots, F)\)
  always flip the last bit.
Chapter 14.9. Exhaustive Search

Solve SAT problem with iterative algorithms

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Chapter 14.9. Exhaustive Search

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Chapter 14.9. Exhaustive Search

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Chapter 14.9. Exhaustive Search

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• what to increment

\[(\ldots, F, T, \ldots, T) \longrightarrow (\ldots, T, F, \ldots, F)\]

always flip the last bit.
Chapter 14.9. Exhaustive Search

1. for \( \langle x_1, \ldots, x_n \rangle = \langle F, \ldots, F \rangle \) to \( \langle T, \ldots, T \rangle \)
2. \( V = \text{Evaluate}(f, x_1, \ldots, x_n) \)
3. if \( V = T \), return \( (T) \)
4. return \( (F) \)

• for loop can be implemented by encoding vectors \( \langle F, \ldots, F \rangle, \ldots, \langle T, \ldots, T \rangle \) with binary numbers then further with integers
  • a decoding process is needed to converting integers back to vectors
Algorithm SAT Solver-Enum($f(x_1, \ldots, x_{n-1}, x_n)$)
Chapter 14.9. Exhaustive Search

Algorithm SAT Solver-Enum\((f(x_1, \ldots, x_{n-1}, x_n))\)

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Chapter 14.9. Exhaustive Search

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Chapter 14.9. Exhaustive Search

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Chapter 14.9. Exhaustive Search

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Chapter 14.9. Exhaustive Search

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Iterative exhaustive search seems to be more convenient.

Another example: Travel Salesman Problem (TSP)

Related problem: Hamiltonian Cycle

Input: a graph $G = (V,E)$

Output: yes if and only if $G$ contains a Hamiltonian cycle (Hamiltonian path is a cycle going through every vertex exactly once.)

How to enumerate all cycles and validate?

• enumerate all permutations of $(1,2,...,n)$

• how to encode these permutations as integers?
Iterative exhaustive search seems to be more convenient
Chapter 14.9. Exhaustive Search

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Chapter 14.9. Exhaustive Search

Exhaustive search could be non-trivial
Chapter 14.9. Exhaustive Search

Exhaustive search could be non-trivial

Maximum Independent Set

**INPUT:** a graph $G = (V, E)$

**OUTPUT:** a subset $I \subseteq V$ such that

1. $\forall u, v \in I, (u, v) \not\in E$,
2. $|I|$ is the maximum.
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Maximum Independent Set

**Input:** a graph $G = (V, E)$

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• trivial exhaustive search: check every subset of $V$ and verify
• non-trivial: use a search tree, achieving a better time upper bound.

Taking advantage of the independent set
Chapter 14.9. Exhaustive Search

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  - taking advantage of the independent set
Chapter 14.9. Exhaustive Search

The algorithm follows a logical search tree...
Chapter 14.9. Exhaustive Search

The algorithm follows a logical search tree

- given a graph $G$, it picks an arbitrary vertex $v$ from $G$;

- exhaustively, there are two cases to consider:
  1. to include $v$ in the independent set;
  2. to exclude $v$ from the independent set;

- resulting in two subgraphs $G_1$ and $G_2$ to be recursively considered,
  1. $G_1$ is the result of $G$ after $v$ and all its neighbors are removed;
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- the algorithm terminates when the considered graph is empty.
Chapter 14.9. Exhaustive Search

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Chapter 14.9. Exhaustive Search

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Chapter 14.9. Exhaustive Search

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Chapter 14.9. Exhaustive Search

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Chapter 14.9. Exhaustive Search

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Chapter 14.9. Exhaustive Search

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Algorithm \text{MaxIndSet} (G)

1. if $G = \emptyset$ return ($\emptyset$)
2. else pick an arbitrary vertex $v$ in $G$
3. let $G_1$ be $G$ with $v$ and all its neighbors removed
4. $I_1 = \{v\} \cup \text{MaxIndSet} (G_1)$
5. let $G_2$ be $G$ with $v$ removed
6. $I_2 = \text{MaxIndSet} (G_2)$
7. if $|I_1| \geq |I_2|$ return ($I_1$)
8. else return ($I_2$)

- the algorithm is a search tree
- the time complexity:
  $T(|G|) = cn^2 + T(|G_1|) + T(|G_2|)$

$T(n) = T(n-1-m) + T(n-1) + cn^2$ where $m$ is the number of neighbors of $v$'s
Algorithm \texttt{MaxIndSet} ($G$)

1. \textbf{if} \hspace{1mm} $G = \emptyset$ \textbf{return} $(\emptyset)$
Chapter 14.9. Exhaustive Search

Algorithm \textbf{MaxIndSet} \((G)\)

1. \hspace{1em} \textbf{if} \hspace{0.5em} G = \emptyset \hspace{0.5em} \textbf{return} \hspace{0.5em} (\emptyset)
2. \hspace{1em} \textbf{else} \hspace{0.5em} pick an arbitrary vertex \(v\) in \(G\)
Algorithm $\text{MaxIndSet}(G)$

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Chapter 14.9. Exhaustive Search

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Chapter 14.9. Exhaustive Search

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Chapter 14.9. Exhaustive Search

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7. \hspace{0.5cm} \textbf{if} \(|I_1| \geq |I_2|\) \textbf{return} \((I_1)\)
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• the algorithm is a search tree
Chapter 14.9. Exhaustive Search

Algorithm $\text{MaxIndSet} \ (G)$

1. \textbf{if} $G = \emptyset$ \textbf{return} ($\emptyset$)
2. \textbf{else} pick an arbitrary vertex $v$ in $G$
3. \hspace{1em} let $G_1$ be $G$ with $v$ and all its neighbors removed
4. \hspace{1em} $I_1 = \{v\} \cup \text{MaxIndSet} \ (G_1)$
5. \hspace{1em} let $G_2$ be $G$ with $v$ removed
6. \hspace{1em} $I_2 = \text{MaxIndSet} \ (G_2)$
7. \hspace{1em} \textbf{if} $|I_1| \geq |I_2|$ \textbf{return} ($I_1$)
8. \hspace{1em} \textbf{else} \textbf{return} ($I_2$)

- the algorithm is a search tree
- the time complexity:
Chapter 14.9. Exhaustive Search

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Chapter 14.9. Exhaustive Search

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$$T(n) = T(n-1-m) + T(n-1) + cn^2$$

where $m$ is the number of neighbors of $v$'s
Chapter 14.9. Exhaustive Search

\[ T(n) = T(n - 1 - m) + T(n - 1) + cn^2 \]
Chapter 14.9. Exhaustive Search

\[ T(n) = T(n - 1 - m) + T(n - 1) + cn^2 \]

- \( m \geq 0 \),
- Can we guarantee \( m \geq 1 \) so we have \( T(n) \leq T(n - 2) + T(n - 1) + cn^2 \), \( \Rightarrow T(n) = O(2^n) \).
- Or even better, to guarantee \( m \geq 2 \)? If we can, \( T(n) \leq T(n - 3) + T(n - 1) + cn^2 \), \( \Rightarrow T(n) = O(1.5^n) \).
- Can we guarantee \( m \geq 3 \)? Possible but a little more complicated.

Use the substitution method to prove \( T(n) = O(1.5^n) \).
Chapter 14.9. Exhaustive Search

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...
Chapter 14.9. Exhaustive Search

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Chapter 14.9. Exhaustive Search

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Chapter 14.9. Exhaustive Search

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Chapter 14.9. Exhaustive Search

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  \[ T(n) \leq T(n - 2) + T(n - 1) + cn^2, \implies T(n) = O(1.6181^n) \]
Chapter 14.9. Exhaustive Search

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Chapter 14.9. Exhaustive Search

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Chapter 14.9. Exhaustive Search

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Chapter 14.9. Exhaustive Search

Let $T(n) \leq T(n - 3) + T(n - 1) + cn^2$, with $T(1) = O(1)$
Chapter 14.9. Exhaustive Search

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Chapter 14.9. Exhaustive Search

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Assume that \( T(k) \leq 1.5^k \) for all \( k < n \).
Chapter 14.9. Exhaustive Search

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Assume that $T(k) \leq 1.5^k$ for all $k < n$. Then

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Chapter 14.9. Exhaustive Search

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T(n) \leq T(n - 3) + T(n - 1) + n^2 \leq 1.5^{n-3} + 1.5^{n-1} + n^2
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when $n > n_0$ ($n_0$ to be determined)
Chapter 14.9. Exhaustive Search

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\leq 1.5^{n-3}(1 + 1.5^2 + \frac{n^2}{1.5^{n-3}})
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Chapter 14.9. Exhaustive Search

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$$\leq 1.5^{n-3}(1 + 1.5^2 + \frac{n^2}{1.5^{n-3}}) \leq 1.5^{n-3}(1 + 1.5^2 + 0.1)$$
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Chapter 14.9. Exhaustive Search

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Chapter 14.9. Exhaustive Search

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Now we decide $n_0$:

$$\frac{n^2}{1.5^{n-3}} \leq 0.1$$
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Chapter 14.9. Exhaustive Search

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$$\frac{n^2}{1.5^{n-3}} \leq 0.1 \implies n^2 \leq 0.1 \times 1.5^{n-3} \text{ holds when roughly } n \geq n_0 = 29$$
Chapter 14.9. Exhaustive Search

- Algorithms for SAT and MAXINDSET run in exponential time $O(2^n)$ or $O(\gamma^n)$ for $1 < \gamma < 2$
Chapter 14.9. Exhaustive Search

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Chapter 14.9. Exhaustive Search

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- Search tree does not have obvious overlapping subproblems.
Chapter 14.9. Exhaustive Search

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Chapter 15. Dynamic Programming

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  bottom-up approaches avoid re-computation for subproblems
Chapter 15. Dynamic Programming

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Revisit the problem of computing the $n$th element of Fibonacci sequence

$F(n) = F(n-1) + F(n-2)$,

$F(1) = F(2) = 1$. 

• it divides problems into two subproblems
• there are overlapping subproblems
• direct top-down search would incur an exponential time
• bottom-up computation is much faster

compute table $T[1..n]$ from left to right, by looking up values that have already been computed
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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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compute table \( T[1..n] \) from left to right,
by looking up values that have already been computed
Chapter 15. Dynamic Programming

Assembly-line scheduling

\[ S_{i,j} \]: the \( j \)th station on line \( i \), \( i = 1, 2, \ldots \), \( j = 1, 2, \ldots, n \).

\[ a_{i,j} \] is the assembly time at station \( S_{i,j} \).

\[ t_{i,j} \] is time for transferring lines from station \( S_{i,j} \).

\[ e_1 \] and \( e_2 \) are entering time costs.

\[ x_1 \] and \( x_2 \) are exiting time costs.

To determine the fastest way through a factory, there are \( 2^n \) possible ways.
Chapter 15. Dynamic Programming

Assembly-line scheduling

- $S_{i,j}$: the $j$th station on line $i$, $i = 1, 2$, $j = 1, 2, \cdots, n$. 
Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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To determine the fastest way through a factory

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Chapter 15. Dynamic Programming

Analyzing the problem:
Chapter 15. Dynamic Programming

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The problem is to determine the fastest way through $S_{1,n}$ or $S_{2,n}$.
Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

For general \( j, 1 \leq j \leq n \), the fastest way through \( S_{1,j} \) is faster between

\( \text{• the fastest way through } S_{1,j-1} \text{ then } S_{1,j-1} \rightarrow S_{1,j} \)

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

A dynamic programming approach:

1. the above analysis
2. formulate a recursive solution to the problem

Define $f_i(j)$ to be the minimum time through station $S_{ij}$, for $i = 1, 2$ and $j = 1, \cdots , n$.

• the problem is to compute $\min\{f_1(n) + x_1, f_2(n) + x_2\}$

• where $f_1()$ and $f_2()$ are defined recursively:

1. $f_1(j) = \min\{f_1(j-1) + a_{1j}, f_2(j-1) + t_2(j-1) + a_{1j}\}$
2. $f_2(j) = \min\{f_2(j-1) + a_{2j}, f_1(j-1) + t_1(j-1) + a_{2j}\}$

• base cases $f_1(1) = e_1 + a_{11}$ and $f_2(1) = e_2 + a_{21}$
Chapter 15. Dynamic Programming

A dynamic programming approach:

**step 1**: the above analysis
A dynamic programming approach:

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\[ f_i(j) \text{ to be the minimum time through station } S_{i,j}, \] for \( i = 1, 2 \) and \( j = 1, \ldots, n \).

- the problem is to compute
  \[ \min \{ f_1(n) + x_1, f_2(n) + x_2 \} \]
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  \[ f_1(j) = \min \{ f_1(j-1) + a_{1,j}, f_2(j-1) + t_{2,j-1} + a_{1,j} \} \]

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define $f_i(j)$ to be the minimum time through station $S_{i,j}$, for $i = 1, 2$ and $j = 1, \ldots, n$. 

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- Base cases $f_1(1) = e_1 + a_{1,1}$ and $f_2(1) = e_2 + a_{2,1}$
Chapter 15. Dynamic Programming

Data dependency:
Chapter 15. Dynamic Programming

**step 3:** compute \( f_1(j) \) and \( f_2(j) \), for all \( j = 1, 2, \ldots, n \).
Chapter 15. Dynamic Programming

**step 3:** compute $f_1(j)$ and $f_2(j)$, for all $j = 1, 2, \ldots, n$.

to fill out a $2 \times n$ table, from left to right

<table>
<thead>
<tr>
<th>$f_1(1)$</th>
<th>$f_1(2)$</th>
<th>$f_1(3)$</th>
<th>...</th>
<th>$f_1(n)$</th>
</tr>
</thead>
<tbody>
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<td>$f_2(1)$</td>
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• using the recurrences to compute; and
Chapter 15. Dynamic Programming

**step 3**: compute $f_1(j)$ and $f_2(j)$, for all $j = 1, 2, \ldots, n$.

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• using the recurrences to compute; and
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**Chapter 15. Dynamic Programming**

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Chapter 15. Dynamic Programming

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1. base cases $f_1(1) = e_1 + a_{1,1}$ and $f_2(1) = e_2 + a_{2,1}$
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Chapter 15. Dynamic Programming

**step 3**: compute $f_1(j)$ and $f_2(j)$, for all $j = 1, 2, \ldots, n$.

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(2) $f_2(j) = \min\{f_2(j - 1) + a_{2,j}, f_1(j - 1) + t_{1,j-1} + a_{2,j}\}$
Chapter 15. Dynamic Programming

**step 3:** compute \( f_1(j) \) and \( f_2(j) \), for all \( j = 1, 2, \ldots, n \). to fill out a \( 2 \times n \) table, from left to right

\[
\begin{array}{cccccc}
  f_1(1) & f_1(2) & f_1(3) & \ldots & f_1(n) \\
  f_2(1) & f_2(2) & f_2(3) & \ldots & f_2(n)
\end{array}
\]

- using the recurrences to compute; and
- looking up already-computed values, where

  (0) base cases \( f_1(1) = e_1 + a_{1,1} \) and \( f_2(1) = e_2 + a_{2,1} \)

  (1) \( f_1(j) = \min\{f_1(j - 1) + a_{1,j}, f_2(j - 1) + t_{2,j-1} + a_{1,j}\} \)

  (2) \( f_2(j) = \min\{f_2(j - 1) + a_{2,j}, f_1(j - 1) + t_{1,j-1} + a_{2,j}\} \)

- the fastest time is \( f^* = \min\{f_1(n) + x_1, f_2(n) + x_2\} \)
Chapter 15. Dynamic Programming

Example
Chapter 15. Dynamic Programming

Example

\[
\begin{array}{c|c|c|c|c|c|c|c}
\hline
j & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
f_1 & 9 & 18 & 20 & 24 & 32 & 36 \\
\hline
f_2 & 12 & 16 & 22 & 26 & 31 & 38 \\
\hline
\end{array}
\]

Min finish time \( f^* = 36 + 3 = 39 \)
Chapter 15. Dynamic Programming

Algorithm ASSEMBLYLINE$(a, t, e, x, n)$
Algorithm ASSEMBLYLINE($a, t, e, x, n$)
1. $M[1, 1] = e_1 + a_{1,1}$
Chapter 15. Dynamic Programming

Algorithm ASSEMBLYLINE\((a, t, e, x, n)\)
1. \(M[1, 1] = e_1 + a_{1, 1}\)
2. \(M[2, 1] = e_2 + a_{2, 1}\)
Algorithm \textsc{AssemblyLine}(a, t, e, x, n)
1. \hspace{1em} M[1, 1] = e_1 + a_{1,1}
2. \hspace{1em} M[2, 1] = e_2 + a_{2,1}
3. \hspace{1em} \textbf{for} \ j = 2 \ \textbf{to} \ n
Algorithm ASSEMBLYLINE\((a, t, e, x, n)\)
1. \(M[1, 1] = e_1 + a_{1,1}\)
2. \(M[2, 1] = e_2 + a_{2,1}\)
3. for \(j=2\) to \(n\)
4. if \(M[1, j - 1] + a_{1,j} \leq M[2, j - 1] + t_{2,j-1} + a_{1,j}\)
Algorithm \textsc{AssemblyLine}(a, t, e, x, n)

1. \( M[1, 1] = e_1 + a_{1,1} \)
2. \( M[2, 1] = e_2 + a_{2,1} \)
3. \textbf{for} \ j=2 \textbf{to} \ n \ \\
4. \quad \textbf{if} \ M[1, j - 1] + a_{1,j} \leq M[2, j - 1] + t_{2,j - 1} + a_{1,j} \ \\
5. \quad M[1, j] = M[1, j - 1] + a_{1,j} \ \\
6. \quad P[1,j] = 1 \ \\
7. \quad \textbf{else} \ \\
8. \quad M[1, j] = M[2, j - 1] + t_{2,j - 1} + a_{1,j} \ \\
9. \quad P[1,j] = 2 \ \\
10. \textbf{if} \ M[2, j - 1] + a_{2,j} \leq M[1, j - 1] + t_{1,j - 1} + a_{2,j} \ \\
11. \quad M[2, j] = M[2, j - 1] + a_{2,j} \ \\
12. \quad P[2,j] = 2 \ \\
13. \quad \textbf{else} \ \\
14. \quad M[2, j] = M[1, j - 1] + t_{1,j - 1} + a_{2,j} \ \\
15. \quad P[2,j] = 1 \ \\
16. \textbf{return} \ \{\min\{f_1(n) + x_1, f_2(n) + x_2\}\}
Algorithm ASSEMBLYLINE\((a, t, e, x, n)\)
1. \(M[1, 1] = e_1 + a_{1,1}\)
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5. \(M[1, j] = M[1, j - 1] + a_{1,j}\)
6. \(P[1, j] = 1\)
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Algorithm\ ASSEMBLY\ LINE(a, t, e, x, n)
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2. \( M[2, 1] = e_2 + a_{2,1} \)
3. \textbf{for} \ j=2 \textbf{ to } n
4. \hspace{1em} \textbf{if} \ M[1, j - 1] + a_{1,j} \leq M[2, j - 1] + t_{2,j-1} + a_{1,j} \n5. \hspace{1em} M[1, j] = M[1, j - 1] + a_{1,j} \n6. \hspace{1em} P[1, j] = 1 \n7. \hspace{1em} \textbf{else} \ M[1, j] = M[2, j - 1] + t_{2,j-1} + a_{1,j} \n8. \hspace{1em} P[1, j] = 2
Algorithm ASSEMBLYLINE\((a, t, e, x, n)\)
1. \(M[1, 1] = e_1 + a_{1,1}\)
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5. \hspace{2em} \(M[1, j] = M[1, j - 1] + a_{1,j}\)
6. \hspace{2em} \(P[1, j] = 1\)
7. \hspace{1em} \textbf{else} \(M[1, j] = M[2, j - 1] + t_{2, j - 1} + a_{1,j}\)
8. \hspace{2em} \(P[1, j] = 2\)
9. \hspace{1em} \textbf{if} \(M[2, j - 1] + a_{2,j} \leq M[1, j - 1] + t_{1, j - 1} + a_{2,j}\)
Algorithm `ASSEMBLYLINE(a, t, e, x, n)`

1. \( M[1, 1] = e_1 + a_{1,1} \)
2. \( M[2, 1] = e_2 + a_{2,1} \)
3. for \( j = 2 \) to \( n \)
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12. \hspace{1em} \textbf{else}$ M[2, j] = M[1, j - 1] + t_{1,j-1} + a_{2,j}$
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step 4: get the fastest path, not just the faster time.
**Chapter 15. Dynamic Programming**

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Do we know the fastest path from the fastest time \( f^* \)?
Chapter 15. Dynamic Programming

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Do we know the fastest path from the fastest time $f^*$?

<table>
<thead>
<tr>
<th>$f_1(1)$</th>
<th>$f_1(2)$</th>
<th>$f_1(3)$</th>
<th>...</th>
<th>$f_1(n-1)$</th>
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</tr>
</thead>
<tbody>
<tr>
<td>$f_2(1)$</td>
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Chapter 15. Dynamic Programming

**step 4**: get the fastest path, not just the faster time.

Do we know the fastest path from the fastest time \( f^* \)?

\[
\begin{array}{cccccc}
  f_1(1) & f_1(2) & f_1(3) & \ldots & f_1(n-1) & f_1(n) \\
  f_2(1) & f_2(2) & f_2(3) & \ldots & f_2(n-1) & f_2(n) \\
\end{array}
\]

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f^* = \min\{f_1(n) + x_1, f_2(n) + x_2\}
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Chapter 15. Dynamic Programming

Algorithm PRINTPATHMAIN($M, P, f^*, x$)
1.  if $(f^* - x_1) = M[1, n]$
Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

Algorithm `PRINTPATHMAIN(M, P, f*, x)`
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Algorithm `PRINTPATH(P, i, j)`
1. if \( j \geq 1 \)
Chapter 15. Dynamic Programming

Algorithm $\text{PrintPathMain}(M, P, f^*, x)$
1. if $(f^* - x_1) = M[1, n]$
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1. if $j \geq 1$
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Chapter 15. Dynamic Programming

Algorithm \texttt{PrintPathMain}(M, P, f^*, x)
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Summary on steps for dynamic programming solving the assembly line problem
Chapter 15. Dynamic Programming

Summary on steps for dynamic programming solving the assembly line problem

1. the structure of optimal solution
Chapter 15. Dynamic Programming

Summary on steps for dynamic programming solving the assembly line problem

(1) the structure of optimal solution

(2) defining optimal cost recursively
Chapter 15. Dynamic Programming

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We consider a little more about the assembly line problem.
Chapter 15. Dynamic Programming

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Assume that each station $S_{i,j}$ produces a part $p_k$ with time cost $\tau_{i,j,k}$, where $k = 1, 2, 3$. 

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Chapter 15. Dynamic Programming

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- can we still compute the fastest time? yes.
  
  so can the fastest path and the sequence of parts be computed!

- Given a sequence of parts produced, can we know the path?
  
  No, but we may predict such a path with a confidence.
Chapter 15. Dynamic Programming

Example-2. Casino dice and decoding algorithm

The house rolls die \( n \) times: e.g., 5, 3, 2, 5, 6, 6, 1,..., 1
• not honest, switch between dice: fair (F) and loaded (L)
• a small chance to switch is small, a large chance to stay on the same die

Given the sequence of numbers, e.g., 5, 3, 2, 5, 6, 6, 1,..., 1, what is the sequence of dices used most likely?
Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

We compute the most probable path of dices to roll a given sequence of numbers $S = d_1 \ldots d_n$, where $d_i \in \{1, 2, \ldots, 6\}$.
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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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As in AssemblyLine problem, we compute the most probable path to roll $d_1 \ldots d_i$:

- $p(S, i, F)$ for which the fair die $F$ is used in the last step
- $p(S, i, L)$ for which the loaded die $L$ is used in the last step
Chapter 15. Dynamic Programming

Both $p(S, i, F)$ and $p(S, i, L)$ have recursive solutions.
Chapter 15. Dynamic Programming

Both $p(S, i, F)$ and $p(S, i, L)$ have recursive solutions.

$$p(S, i, F) = \max \{p(S, i - 1, F) \times 0.95 \times \frac{1}{6}, \ p(S, i - 1, L) \times 0.1 \times \frac{1}{6}\}$$
Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Base cases: when $i = 1$

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Chapter 15. Dynamic Programming

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- design dynamic programming algorithm to compute $\max\{P(S, n, F), P(S, n, L)\}$
  given from $d_1d_2 \ldots d_n$
Chapter 15. Dynamic Programming

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- design dynamic programming algorithm to compute \( \max \{ P(S, n, F), P(S, n, L) \} \)
  given from \( d_1 d_2 \ldots d_n \)

- design a traceback process to print out the sequence of dices used.
Consider a sequence
\[ \ldots \ 1 \ 3 \ 2 \ 4 \ 6 \ 6 \ 6 \ 4 \ 1 \ 6 \ 5 \ 6 \ 6 \ 2 \ 4 \ 2 \ 1 \ 2 \ 3 \ 5 \ldots \]
Chapter 15. Dynamic Programming

Consider a sequence

\[ \ldots 1 3 2 4 6 6 6 4 1 6 5 6 6 2 4 2 1 2 3 5 \ldots \]

By computing the maximum probability to produce the sequence, we decode the dices used.

But besides the casino interest, is such algorithm meaningful in other applications? The answer is definitely YES!
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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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By computing the maximum probability to produce the sequence, we decode the dices used

\[
\ldots 1 3 2 4 6 6 6 4 1 6 5 6 6 2 4 2 1 2 3 5 \ldots
\]

\[
\ldots F F F F L L L L L L L L L F F F F F F F \ldots
\]

But besides the casino interest, is such algorithm meaningful in other applications?
Chapter 15. Dynamic Programming

Consider a sequence

... 1 3 2 4 6 6 6 6 4 1 6 5 6 6 2 4 2 1 2 3 5...

By computing the maximum probability to produce the sequence, we decode the dices used

... 1 3 2 4 6 6 6 6 4 1 6 5 6 6 2 4 2 1 2 3 5...

... F F F F L L L L L L L L L F F F F F F F...

But besides the casino interest, is such algorithm meaningful in other applications?

The answer is definitely YES!
Chapter 15. Dynamic Programming

a segment of DNA sequence, is it meaningful?
Chapter 15. Dynamic Programming

If the highlighted segment is not random, it may be a gene.
Technically, the “linear model” is unfolded of a “more condensed model”.

A hidden Markov model (HMM)
Chapter 15. Dynamic Programming
Chapter 15. Dynamic Programming
Chapter 15. Dynamic Programming
Chapter 15. Dynamic Programming
Chapter 15. Dynamic Programming

Knapsack Problem

Input:
- \( n \) items of sizes \( s_1, \ldots, s_n \) and values \( v_1, \ldots, v_n \);
- a knapsack size \( B \).

Output:
- a subset \( A \subseteq \{1, 2, \ldots, n\} \) that maximizes \( \sum_{i \in A} v_i \) with constraint \( \sum_{i \in A} s_i \leq B \).

Analysis:
For \( n \) items and knapsack size \( B \); examine the last item \( n \), then either put item \( n \) into the knapsack or discard it; That is
1. either increase value by \( v_n \), reduce available space from \( B \) to \( B - s_n \), reduce the number of items by 1,
2. or simply reduce the number of items by 1
Both situations reduce the problem size to a subproblem.
Chapter 15. Dynamic Programming

Knapsack Problem

**INPUT:** $n$ items of sizes $s_1, \ldots, s_n$ and values $v_1, \ldots, v_n$; and a knapsack size $B$
Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

In general, let \( \{1, 2, \ldots, k\} \) be the first \( k \) items, and \( X \) be the available space in the knapsack;
Chapter 15. Dynamic Programming

In general, let \{1, 2, \ldots, k\} be the first \(k\) items, and \(X\) be the available space in the knapsack;

If we define \(A(k, X)\) to be the subset of items drawn from \{1, 2, \ldots, k\} such that \(A\) maximizes the total value while fits into the space \(X\)
Chapter 15. Dynamic Programming

In general, let \( \{1, 2, \ldots, k\} \) be the first \( k \) items, and \( X \) be the available space in the knapsack;

If we define \( A(k, X) \) to be the subset of items drawn from \( \{1, 2, \ldots, k\} \) such that \( A \) maximizes the total value while fits into the space \( X \),

Then

\[
A(k, X) = \begin{cases} 
\text{either } & A(k-1, X - s_k) \cup \{k\} & \text{value gained by } v_k \\
\text{or } & A(k-1, X) & \text{value not gained}
\end{cases}
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\end{cases}
\]

Define \( V(k, X) \) to be the maximum value of items drawn from \( \{1, 2, \ldots, k\} \) which fit into the space of \( X \), then

\[
V(k, X) = \max \left\{ V(k - 1, X - s_k) + v_k \quad X \geq s_k \right\}
\]
• For every $k \leq n$ and every $X \leq B$.

$$V(k, X) = \max \left\{ \begin{array}{ll}
V(k - 1, X - s_k) + v_k & X \geq s_k \\
V(k - 1, X) & \\
\end{array} \right.$$

base case,
Chapter 15. Dynamic Programming

- For every $k \leq n$ and every $X \leq B$.

\[
V(k, X) = \max \begin{cases} 
V(k - 1, X - s_k) + v_k & X \geq s_k \\
V(k - 1, X) & \text{base case,}
\end{cases}
\]

- What does the table look like? How to fill it out?
- How to traceback for $A$, the optimal subset of items?
For every $k \leq n$ and every $X \leq B$.

$$V(k, X) = \max \left\{ \begin{array}{ll} V(k - 1, X - s_k) + v_k & X \geq s_k \\ V(k - 1, X) & \end{array} \right.$$ 

Base case,

$$V(0, X) = 0, \ V(k, 0) = 0$$
Chapter 15. Dynamic Programming

• For every $k \leq n$ and every $X \leq B$.

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Chapter 15. Dynamic Programming

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What does the table look like? How to fill it out?

How to traceback for $A$, the optimal subset of items?
Chapter 15. Dynamic Programming

Example

The optimal knapsack should contain \{1,2\} = 7

\[
V(k, X) = \max \begin{cases} 
V(k - 1, X - s_k) + v_k & X \geq s_k \\
V(k - 1, X) & \text{otherwise}
\end{cases}
\]
Chapter 15. Dynamic Programming

**Example**
The optimal knapsack should contain \{1, 2\} = 7

<table>
<thead>
<tr>
<th>(i)</th>
<th>(0)</th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
<th>(4)</th>
<th>(5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td>7</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>7</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>7</td>
</tr>
</tbody>
</table>

\[ V(k, X) = \max \begin{cases} V(k - 1, X - s_k) + v_k & X \geq s_k \\ V(k - 1, X) & \end{cases} \]

- fill out the base case row and column;
- the order of cells to be filled;
- \(V(i, X) = V(i - 1, X)\) if \(X < s_i\).
- time complexity
Chapter 15. Dynamic Programming

Hidden Markov models and Viterbi’s algorithm
Chapter 15. Dynamic Programming

Hidden Markov models and Viterbi’s algorithm
Chapter 15. Dynamic Programming

The diagram shows a network of three states: Rain, Nice, and Snow. The edges between the states have probabilities labeled on them. For example, the edge from Rain to Nice has a probability of 0.5. The diagram illustrates the transitions and probabilities between these states.
Chapter 15. Dynamic Programming
Chapter 15. Dynamic Programming

(a)

0.99

1

0.01

2

0.9

A 0.4
C 0.1
G 0.1
T 0.4

A 0.05
C 0.4
G 0.5
T 0.05

(b)

state sequence (hidden):

... 1 1 1 1 1 2 2 2 2 1 1 ...

transitions:  ? 0.99 0.99 0.99 0.99 0.01 0.9 0.9 0.9 0.1 0.99

(c)

symbol sequence (observable):

... A T C A A G G C G A T ...

emissions:  0.4 0.4 0.1 0.4 0.4 0.5 0.5 0.4 0.4 0.4
Chapter 15. Dynamic Programming

Viterbi Algorithm

- Input: an HMM $\mathcal{M}$, and data $D = d_1 d_2 \ldots d_n$
Chapter 15. Dynamic Programming

Viterbi Algorithm

- **Input:** an HMM $\mathcal{M}$, and data $D = d_1 d_2 \ldots d_n$
- **Output:** hidden state sequences $H = s_1 s_2 \ldots s_n$ that generate $D$; such that

\[ \text{Prob}(H, D | \mathcal{M}) \text{ achieves the maximum} \]
Chapter 15. Dynamic Programming

Viterbi Algorithm

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achieves the maximum

... 1 3 2 4 6 6 6 4 1 6 5 6 6 2 4 2 1 2 3 5 ...

... F F F F L L L L L L L L L F F F F F F F F ...
A hidden Markov model (HMM) consists of \((S, T, e, t)\)
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A hidden Markov model (HMM) consists of \((S, T, e, t)\)

- a set of states \(S = \{F, L\}\);
- a transition relation \(T = \{(F, F), (F, L), (L, F), (L, L)\} \subseteq S \times S\),
Chapter 15. Dynamic Programming

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- emission probability distribution $e$
  
  $e(F, d) = \frac{1}{6}$ for $d = 1, 2, \ldots, 6$;
A hidden Markov model (HMM) consists of $(S, T, e, t)$

- a set of states $S = \{F, L\}$;
- a transition relation $T = \{(F, F), (F, L), (L, F), (L, L)\} \subseteq S \times S$,
- emission probability distribution $e$
  
  $e(F, d) = \frac{1}{6}$ for $d = 1, 2, \ldots, 6$;
  $e(L, 6) = \frac{1}{2}$ and $e(L, d) = \frac{1}{10}$ for $d = 1, 2, \ldots, 5$;
Chapter 15. Dynamic Programming

A hidden Markov model (HMM) consists of $(S, T, e, t)$

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  $e(L, 6) = \frac{1}{2}$ and $e(L, d) = \frac{1}{10}$ for $d = 1, 2, \ldots, 5$;
- transition probability distribution $t$
Chapter 15. Dynamic Programming

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  \(e(L, 6) = \frac{1}{2}\) and \(e(L, d) = \frac{1}{10}\) for \(d = 1, 2, \ldots, 5\);
- transition probability distribution \(t\)
  \(t(F, F) = 0.95, t(F, L) = 0.05, t(L, F) = 0.1, t(L, L) = 0.90\);
The goal is to identify hidden states \( s_1 s_2 \ldots s_n \) that generates \( d_1 d_2 \ldots d_n \), to maximize the associated probability.
Chapter 15. Dynamic Programming

The goal is to identify hidden states $s_1 s_2 \ldots s_n$ that generates $d_1 d_2 \ldots d_n$, to maximize the associated probability

$$P(s_1 s_2 \ldots s_n, d_1 d_2 \ldots d_n)$$
Chapter 15. Dynamic Programming

The goal is to **identify** hidden states $s_1 s_2 \ldots s_n$ that generates $d_1 d_2 \ldots d_n$, to **maximize** the associated probability

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e.g., $d_1 = 3$, $d_2 = 6$, there are 4 possible ways to generate them:
The goal is to identify hidden states $s_1 s_2 \ldots s_n$ that generates $d_1 d_2 \ldots d_n$, to maximize the associated probability $P(s_1 s_2 \ldots s_n, d_1 d_2 \ldots d_n)$.

e.g., $d_1 = 3$, $d_2 = 6$, there are 4 possible ways to generate them:

- $FF$, $FL$, $LL$, $LF$,
Chapter 15. Dynamic Programming

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e.g., $d_1 = 3$, $d_2 = 6$, there are 4 possible ways to generate them:

- $FF$, $FL$, $LL$, $LF$, but with different probabilities
Chapter 15. Dynamic Programming

- $P(FF, 36)$
Chapter 15. Dynamic Programming

- $P(FF, 3 6) = e(F, 3) \times t(F, F) \times e(F, 6)$
Chapter 15. Dynamic Programming

- \( P(FF, 36) = e(F, 3) \times t(F, F) \times e(F, 6) = \frac{1}{6} \times 0.95 \times \frac{1}{6} \)
Chapter 15. Dynamic Programming

- $P(FF, 36) = e(F, 3) \times t(F, F) \times e(F, 6) = \frac{1}{6} \times 0.95 \times \frac{1}{6} = 0.02639$
Chapter 15. Dynamic Programming

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- $P(FL, 36)$
Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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- $P(FL, 3\ 6) = e(F, 3) \times t(F, L) \times e(L, 6) = \frac{1}{6} \times 0.05 \times \frac{1}{2} = 0.00167$
Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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- \( P(LL, 36) \)
Chapter 15. Dynamic Programming

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- \( P(LL, 3 6) = e(L, 3) \times t(L, L) \times e(L, 6) \)
Chapter 15. Dynamic Programming

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- \( P(LL, 36) = e(L, 3) \times t(L, L) \times e(L, 6) = \frac{1}{10} \times 0.90 \times \frac{1}{2} \)
Chapter 15. Dynamic Programming

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- \( P(LL, 36) = e(L, 3) \times t(L, L) \times e(L, 6) = \frac{1}{10} \times 0.90 \times \frac{1}{2} = 0.045 \)
Chapter 15. Dynamic Programming

\[ P(\text{FF}, 36) = e(F, 3) \times t(F, F) \times e(F, 6) = \frac{1}{6} \times 0.95 \times \frac{1}{6} = 0.02639 \]

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\[ P(\text{LF}, 36) \]
Chapter 15. Dynamic Programming

\[ P(FF, 36) = e(F, 3) \times t(F, F) \times e(F, 6) = \frac{1}{6} \times 0.95 \times \frac{1}{6} = 0.02639 \]

\[ P(FL, 36) = e(F, 3) \times t(F, L) \times e(L, 6) = \frac{1}{6} \times 0.05 \times \frac{1}{2} = 0.00417 \]

\[ P(LL, 36) = e(L, 3) \times t(L, L) \times e(L, 6) = \frac{1}{10} \times 0.90 \times \frac{1}{2} = 0.045 \]

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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but for 10 digits, how many cases to consider?
Chapter 15. Dynamic Programming

Take a dynamic programming approach:
Take a dynamic programming approach:

For prefix $d_1 \ldots d_k$, define:

- $P(F,k)$ to be the maximum probability for states $F$ to generate $d_1 \ldots d_k$
- $P(L,k)$ to be the maximum probability for states $L$ to generate $d_1 \ldots d_k$
Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

Then recursively,
Then recursively,

\[ P(F, k) = \max \left\{ P(F, k - 1) \times t(F, F) \times e(F, d_k), \right. \]
\[ \left. P(L, k - 1) \times t(L, F') \times e(F, d_k) \right\} \]
Chapter 15. Dynamic Programming

Then recursively,

\[ P(F, k) = \max \left\{ \begin{array}{l}
  P(F, k - 1) \times t(F, F) \times e(F, d_k) \\
  P(L, k - 1) \times t(L, F) \times e(F, d_k)
\end{array} \right. \]

\[ P(L, k) = \max \left\{ \begin{array}{l}
  P(F, k - 1) \times t(F, L) \times e(L, d_k) \\
  P(L, k - 1) \times t(L, L) \times e(L, d_k)
\end{array} \right. \]
Then recursively,

\[
P(F, k) = \max \left\{ \frac{P(F, k - 1)}{6}, \frac{P(L, k - 1)}{6} \right\} \times t(F, F) \times e(F, d_k)
\]

\[
P(L, k) = \max \left\{ \frac{P(F, k - 1)}{6}, \frac{P(L, k - 1)}{6} \right\} \times t(L, F) \times e(F, d_k)
\]

\[
P(F, 1) = \frac{1}{6}
\]
Chapter 15. Dynamic Programming

Then recursively,

\[ P(F, k) = \max \left\{ P(F, k-1) \times t(F, F) \times e(F, d_k), P(L, k-1) \times t(L, F) \times e(F, d_k) \right\} \]

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\[ P(F, 1) = \frac{1}{6} \quad P(L, 1) = \begin{cases} \frac{1}{10} & 1 \leq d_1 \leq 5 \\ \frac{1}{2} & d_1 = 6 \end{cases} \]
Chapter 15. Dynamic Programming

The outlined method is Viterbi Algorithm.
Chapter 15. Dynamic Programming

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- **Start** and **Finish** states can be added into the model, but they do not emit digits (called **silent states**);
Chapter 15. Dynamic Programming

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- The state in phase $k - 1$ determines the state in phase $k$;
The outlined method is Viterbi Algorithm.

- **Start** and **Finish** states can be added into the model, but they do not emit digits (called **silent states**);
- The state in phase $k - 1$ determines the state in phase $k$;
- Given a state in current phase $k$, the previous state in phase $k - 1$ can only be one of those having transition to the current state.
Chapter 15. Dynamic Programming

A hidden Markov model (HMM) consists of \((S, T, e, t)\), where

- \(S\) is a set of states: \(s_0, s_1, \ldots, s_m\); where \(s_0\) is the begin state;
- transitions \(T \subseteq S \times S\);
- each state \(s_i\) is associated with a (emission) probability distribution \(e(i, a)\) for \(a \in \Sigma\), such that \(\sum_{a \in \Sigma} e(i, a) = 1\) for every \(i, 1 \leq i < m\);
- every \((s_i, s_j)\) \(\in T\) is associated with a probability distribution \(t(i, j)\), such that \(\sum_{1 \leq j \leq m} t(i, j) = 1\) for every \(i, 0 \leq i < m\).
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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

HMM decoding problem:

Input: HMM $M$, sequence of symbols $x \in \Sigma^*$

Output: $y^*$, a sequence of states such that $y^* = \arg\max_y \{\text{Prob}(y, x | M)\}$ where $y$ begins from $s_0$. 

Diagram: State transitions with transition probabilities $t(i, k) = 0.6$, $t(i, j) = 0.4$, and emission probabilities $e(i, a) = 0.2$, $e(i, b) = 0.8$. States $\Sigma = \{a, b\}$. 

Diagram:

- States: $s_0, s_1, s_2, s_3, s_4, \ldots, s_m$
- Edges: $s_0 \rightarrow s_1$, $s_1 \rightarrow s_2$, $s_2 \rightarrow s_3$, $s_3 \rightarrow s_4$, $s_4 \rightarrow s_5$, $s_5 \rightarrow s_6$, $s_6 \rightarrow s_7$, $s_7 \rightarrow s_8$, $s_8 \rightarrow s_9$, $s_9 \rightarrow s_10$, $s_{10} \rightarrow s_{11}$
- Transition probabilities: $t(i, k) = 0.6$, $t(i, j) = 0.4$
- Emission probabilities: $e(i, a) = 0.2$, $e(i, b) = 0.8$
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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

The idea of Viterbi algorithm:
Chapter 15. Dynamic Programming

The idea of Viterbi algorithm:

For any prefix $x_1 \ldots x_k$, $k \geq 0$, 

$$\max_{s_0 \ldots s_j} \{ \text{Prob}(y, x_1 \ldots x_k | M) \}$$

the maximum probability of a path $s_0 \ldots s_j$ to generate $x_1 \ldots x_k$;
Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

\[ f(j,k) = \max_{y=s_0 \ldots s_j} \{ \text{Prob}(y,x_1 \ldots x_k) \} \]

Recursively,
\[ f(j,k) = \max_i \{ f(i,k-1) \times t(i,j) \times e(j,x_k) \} \]

Base case:
\[ f(0,0) = 1, f(i,0) = 0 \text{ for all } i \geq 1. \]
Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

Define \( f(j, k) = \max_{y=s_0...s_j} \{ \text{Prob}(y, x_1...x_k) \} \)

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Base case: \( f(0, 0) = 1 \)
Chapter 15. Dynamic Programming

Define \( f(j, k) = \max_{y=s_0...s_j} \{ \text{Prob}(y, x_1 ... x_k) \} \)

Recursively, \( f(j, k) = \max_i \{ f(i, k - 1) \times t(i, j) \times e(j, x_k) \} \)

Base case: \( f(0, 0) = 1, \quad f(i, 0) \)
Chapter 15. Dynamic Programming

Define \( f(j, k) = \max_{y=s_0 \ldots s_j} \{ \text{Prob}(y, x_1 \ldots x_k) \} \)

Recursively, \( f(j, k) = \max_i \{ f(i, k-1) \times t(i, j) \times e(j, x_k) \} \)

Base case: \( f(0, 0) = 1, \quad f(i, 0) = 0 \) for all \( i \geq 1 \).
Algorithm VITERBI\((x, n, t, e, m)\)

0. \(T(0, 0) = 1; P(0, 0) = 0;\)
Algorithm VITERBI($x, n, t, e, m$)

0. $T(0, 0) = 1; P(0, 0) = 0$

1. for $j = 1$ to $m$
   2. $T(j, 0) = 0$
Algorithm \textsc{Viterbi}(x, n, t, e, m)

0. \quad T(0, 0) = 1; \quad P(0, 0) = 0;
1. \quad \textbf{for} \; j = 1 \; \textbf{to} \; m
2. \quad \quad T(j, 0) = 0;
3. \quad \textbf{for} \; k = 1 \; \textbf{to} \; n
4. \quad \quad \quad \textbf{for} \; i = 0 \; \textbf{to} \; m
5. \quad \quad \quad \quad \text{premax} = 0 \; \text{initialize the prefix probability}
6. \quad \quad \quad \quad \textbf{for} \; i = 0 \; \textbf{to} \; m \; \textbf{try every 'predecessor state' } s_i
7. \quad \quad \quad \quad \quad \quad \text{if} \; \text{premax} < T(i, k - 1) \times t(i, j) \times e(j, x_k)
8. \quad \quad \quad \quad \quad \quad \quad \text{premax} = T(i, k - 1) \times t(i, j) \times e(j, x_k) \; \text{update probability}
9. \quad \quad \quad \quad \quad \quad P(j, k) = i \; \text{memorize the predecessor}
10. \quad \quad \quad \quad T(j, k) = \text{premax}
11. \quad \quad \quad \quad \textbf{laststate} = 1, \; \text{max} = T(1, n)
12. \quad \quad \textbf{for} \; j = 2 \; \textbf{to} \; m \; \textbf{identify maximum probability}
13. \quad \quad \quad \textbf{if} \; \text{max} < T(j, n)
14. \quad \quad \quad \quad \textbf{laststate} = j
15. \quad \quad \quad \quad \text{max} = T(j, n)
16. \quad \quad \quad \quad \textbf{return} (T, P, \text{max}, \textbf{laststate})

Time complexity: $O(nm^2)$
Chapter 15. Dynamic Programming

Algorithm VITERBI($x, n, t, e, m$)

0. $T(0, 0) = 1; \ P(0, 0) = 0$;
1. for $j = 1$ to $m$
2. \hspace{1cm} $T(j, 0) = 0$;
3. for $k = 1$ to $n$ \hspace{1cm} for every prefix upto $k$th symbol
4. for $j = 1$ to $m$
5. \hspace{1cm} $\text{premax} = 0$
6. \hspace{1cm} for $i = 0$ to $m$
7. \hspace{2cm} try every 'predecessor state' $s_i$
8. \hspace{1cm} if $\text{premax} < T(i, k - 1) \times t(i, j) \times e(j, x_k)$
9. \hspace{2cm} $\text{premax} = T(i, k - 1) \times t(i, j) \times e(j, x_k)$
10. \hspace{1cm} $P(j, k) = i$
11. \hspace{1cm} memorize the predecessor
12. $T(j, k) = \text{premax}$
13. laststate = 1; max = $T(1, n)$
14. for $j = 2$ to $m$
15. \hspace{1cm} identify maximum probability
16. \hspace{1cm} if $\text{max} < T(j, n)$
17. \hspace{2cm} laststate = $j$
18. \hspace{2cm} $\text{max} = T(j, n)$
19. return ($T, P, \text{max}, \text{laststate}$)

Time complexity: $O(nm^2)$
Algorithm \textsc{Viterbi}(x, n, t, e, m)

0. \hspace{1em} T(0, 0) = 1; \hspace{0.5em} P(0, 0) = 0;
1. \hspace{1em} \textbf{for} \hspace{0.5em} j = 1 \hspace{0.5em} \textbf{to} \hspace{0.5em} m
2. \hspace{1em} \hspace{1em} T(j, 0) = 0;
3. \hspace{1em} \textbf{for} \hspace{0.5em} k = 1 \hspace{0.5em} \textbf{to} \hspace{0.5em} n \hspace{1em} \textbf{for every prefix upto} \hspace{0.5em} k\text{th symbol}
4. \hspace{1em} \hspace{1em} \textbf{for} \hspace{0.5em} j = 1 \hspace{0.5em} \textbf{to} \hspace{0.5em} m

8. \hspace{1em} \text{premax} = T(i, k - 1) \times t(i, j) \times e(j, x_k)
9. \hspace{1em} \text{premax} = T(i, k - 1) \times t(i, j) \times e(j, x_k)

10. \hspace{1em} P(j, k) = \text{premax}
11. \hspace{1em} \text{laststate} = 1 ; \hspace{1em} \max = T(1, n)
12. \hspace{1em} \textbf{for} \hspace{0.5em} j = 2 \hspace{0.5em} \textbf{to} \hspace{0.5em} m \hspace{1em} \text{identify maximum probability}
13. \hspace{1em} \textbf{if} \hspace{0.5em} \max < T(j, n)
14. \hspace{1em} \hspace{1em} \text{laststate} = j
15. \hspace{1em} \hspace{1em} \max = T(j, n)
16. \hspace{1em} \textbf{return} \hspace{0.5em} (T, P, \max, \text{laststate})

\textbf{Time complexity:} \hspace{1em} O(nm^2)
Chapter 15. Dynamic Programming

Algorithm VITERBI($x, n, t, e, m$)

0. $T(0, 0) = 1; P(0, 0) = 0$
1. for $j = 1$ to $m$
2. $T(j, 0) = 0$
3. for $k = 1$ to $n$  
   for every prefix upto $k$th symbol
4. for $j = 1$ to $m$  
   for every ‘last state’ $s_j$

Time complexity: $O(nm^2)$
Chapter 15. Dynamic Programming

Algorithm \textsc{Viterbi}(x, n, t, e, m)
0. \hspace{0.5cm} T(0, 0) = 1; \hspace{0.2cm} P(0, 0) = 0;
1. \hspace{0.5cm} \textbf{for} \hspace{0.2cm} j = 1 \hspace{0.2cm} \textbf{to} \hspace{0.2cm} m
2. \hspace{0.5cm} T(j, 0) = 0;
3. \hspace{0.5cm} \textbf{for} \hspace{0.2cm} k = 1 \hspace{0.2cm} \textbf{to} \hspace{0.2cm} n \hspace{0.5cm} \text{for every prefix upto kth symbol}
4. \hspace{0.5cm} \textbf{for} \hspace{0.2cm} j = 1 \hspace{0.2cm} \textbf{to} \hspace{0.2cm} m \hspace{0.5cm} \text{for every ‘last state’ } s_j
5. \hspace{0.5cm} \text{premax} = 0
6. \hspace{0.5cm} \textbf{for} \hspace{0.2cm} i = 0 \hspace{0.2cm} \textbf{to} \hspace{0.2cm} m \hspace{0.5cm} \text{try every ‘predecessor state’ } s_i
7. \hspace{0.5cm} \textbf{if} \hspace{0.2cm} \text{premax} < T(i, k - 1) \times t(i, j) \times e(j, x_k)
8. \hspace{0.5cm} \text{premax} = T(i, k - 1) \times t(i, j) \times e(j, x_k)
9. \hspace{0.5cm} P(j, k) = i \hspace{0.5cm} \text{memorize the predecessor}
10. \hspace{0.5cm} T(j, k) = \text{premax}
11. \hspace{0.5cm} \text{laststate} = 1 \hspace{0.5cm} \text{max} = T(1, n)
12. \hspace{0.5cm} \textbf{for} \hspace{0.2cm} j = 2 \hspace{0.2cm} \textbf{to} \hspace{0.2cm} m \hspace{0.5cm} \text{identify maximum probability}
13. \hspace{0.5cm} \textbf{if} \hspace{0.2cm} \text{max} < T(j, n)
14. \hspace{0.5cm} \text{laststate} = j
15. \hspace{0.5cm} \text{max} = T(j, n)
16. \hspace{0.5cm} \textbf{return} \hspace{0.2cm} (T, P, \text{max, laststate})

Time complexity: \(O(nm^2)\)
Chapter 15. Dynamic Programming

Algorithm VITERBI$(x, n, t, e, m)$

0. $T(0, 0) = 1; \ P(0, 0) = 0$
1. for $j = 1$ to $m$
2. $T(j, 0) = 0$
3. for $k = 1$ to $n$ for every prefix upto $k$th symbol
4. for $j = 1$ to $m$ for every ‘last state’ $s_j$
5. $premax = 0$ initialize the prefix probability
6. for $i = 0$ to $m$ try every ‘predecessor state’ $s_i$
7. if $premax < T(i, k-1) \times t(i, j) \times e(j, x_k)$
8. $premax = T(i, k-1) \times t(i, j) \times e(j, x_k)$ update probability
9. $P(j, k) = i$ memorize the predecessor
10. $T(j, k) = premax$
11. laststate = 1
12. max = $T(1, n)$
13. for $j = 2$ to $m$ identify maximum probability
14. if $max < T(j, n)$
15. laststate = $j$
16. max = $T(j, n)$
17. return ($T$, $P$, $max$, laststate)
Chapter 15. Dynamic Programming

Algorithm \textsc{Viterbi}(x, n, t, e, m)

0. \( T(0, 0) = 1; \ P(0, 0) = 0; \)
1. \( \text{for } j = 1 \text{ to } m \)
2. \( T(j, 0) = 0; \)
3. \( \text{for } k = 1 \text{ to } n \) \quad \text{for every prefix upto } k\text{th symbol}
4. \( \text{for } j = 1 \text{ to } m \) \quad \text{for every ‘last state’ } s_j
5. \( \text{premax} = 0 \) \quad \text{initialize the prefix probability}
6. \( \text{for } i = 0 \text{ to } m \) \quad \text{try every ‘predecessor state’ } s_i

7. \( \text{if } \text{premax} < T(i, k-1) \times t(i, j) \times e(j, x_k) \)
8. \( \text{premax} = T(i, k-1) \times t(i, j) \times e(j, x_k) \) \quad \text{update probability}
9. \( P(j,k) = i \) \quad \text{memorize the predecessor}
10. \( T(j,k) = \text{premax} \)
11. \( \text{laststate} = 1; \text{max} = T(1,n) \)
12. \( \text{for } j = 2 \text{ to } m \) \quad \text{identify maximum probability}
13. \( \text{if } \text{max} < T(j,n) \)
14. \( \text{laststate} = j; \text{max} = T(j,n) \)
15. \( \text{return } (T, P, \text{max}, \text{laststate}) \)

Time complexity: \( O(nm^2) \)
Chapter 15. Dynamic Programming

Algorithm VITERBI($x, n, t, e, m$)

0. $T(0, 0) = 1; P(0, 0) = 0$
1. for $j = 1$ to $m$
2. $T(j, 0) = 0$
3. for $k = 1$ to $n$ for every prefix upto $k$th symbol
4. for $j = 1$ to $m$ for every ‘last state’ $s_j$
5. premax = 0 initialize the prefix probability
6. for $i = 0$ to $m$ try every ‘predecessor state’ $s_i$
7. if premax < $T(i, k - 1) \times t(i, j) \times e(j, x_k)$
Chapter 15. Dynamic Programming

Algorithm VITERBI\((x, n, t, e, m)\)

0. \(T(0, 0) = 1; P(0, 0) = 0;\)
1. for \(j = 1\) to \(m\)
2. \(T(j, 0) = 0;\)
3. for \(k = 1\) to \(n\) for every prefix upto \(k\)th symbol
4. for \(j = 1\) to \(m\) for every ‘last state’ \(s_j\)
5. \(\text{premax} = 0\) initialize the prefix probability
6. for \(i = 0\) to \(m\) try every ‘predecessor state’ \(s_i\)
7. if \(\text{premax} < T(i, k - 1) \times t(i, j) \times e(j, x_k)\)
8. \(\text{premax} = T(i, k - 1) \times t(i, j) \times e(j, x_k)\)

9. \(P(j, k) = i\) memorize the predecessor
10. \(T(j, k) = \text{premax}\)
11. \(\text{laststate} = 1; \text{max} = T(1, n)\)
12. for \(j = 2\) to \(m\) identify maximum probability
13. if \(\text{max} < T(j, n)\)
14. \(\text{laststate} = j; \text{max} = T(j, n)\)
15. return \((T, P, \text{max}, \text{laststate})\)

Time complexity: \(O(nm^2)\)
Chapter 15. Dynamic Programming

Algorithm VITERBI(x, n, t, e, m)

0. $T(0, 0) = 1; P(0, 0) = 0$
1. for $j = 1$ to $m$
2. $T(j, 0) = 0$
3. for $k = 1$ to $n$ for every prefix upto $k$th symbol
4. for $j = 1$ to $m$ for every ‘last state’ $s_j$
5. premax = 0 initialize the prefix probability
6. for $i = 0$ to $m$ try every ‘predecessor state’ $s_i$
7. if $\text{premax} < T(i, k - 1) \times t(i, j) \times e(j, x_k)$
8. $\text{premax} = T(i, k - 1) \times t(i, j) \times e(j, x_k)$ update probability
9. $P(j, k) = i$ memorize the predecessor
10. $T(j, k) = \text{premax}$
11. laststate = 1 max = $T(1, n)$
12. for $j = 2$ to $m$ identify maximum probability
13. if $\text{max} < T(j, n)$
14. laststate = $j$
15. $\text{max} = T(j, n)$
16. return $(T, P, \text{max}, \text{laststate})$

Time complexity: $O(n m^2)$
Algorithm VITERBI($x, n, t, e, m$)

0. $T(0, 0) = 1; P(0, 0) = 0$
1. for $j = 1$ to $m$
2. $T(j, 0) = 0$
3. for $k = 1$ to $n$ for every prefix upto $k$th symbol
4. for $j = 1$ to $m$ for every ‘last state’ $s_j$
5. $premax = 0$ initialize the prefix probability
6. for $i = 0$ to $m$ try every ‘predecessor state’ $s_i$
7. if $premax < T(i, k - 1) \times t(i, j) \times e(j, x_k)$
8. $premax = T(i, k - 1) \times t(i, j) \times e(j, x_k)$ update probability
9. $P(j, k) = i$
10. $laststate = 1; max = T(1, n)$
11. for $j = 2$ to $m$ identify maximum probability
12. if $max < T(j, n)$
13. $laststate = j$
14. $max = T(j, n)$
15. return $(T, P, max, laststate)$

Time complexity: $O(nm^2)$
Chapter 15. Dynamic Programming

Algorithm \textsc{Viterbi}(x, n, t, e, m)

0. \quad T(0, 0) = 1; \quad P(0, 0) = 0;
1. \quad \textbf{for} \; j = 1 \; \textbf{to} \; m
2. \quad T(j, 0) = 0;
3. \quad \textbf{for} \; k = 1 \; \textbf{to} \; n \quad \text{for every prefix upto } k\text{th symbol}
4. \quad \textbf{for} \; j = 1 \; \textbf{to} \; m \quad \text{for every 'last state' } s_j
5. \quad \text{premax} = 0 \quad \text{initialize the prefix probability}
6. \quad \textbf{for} \; i = 0 \; \textbf{to} \; m \quad \text{try every 'predecessor state' } s_i
7. \quad \textbf{if} \; \text{premax} < T(i, k - 1) \times t(i, j) \times e(j, x_k)
8. \quad \text{premax} = T(i, k - 1) \times t(i, j) \times e(j, x_k) \quad \text{update probability}
9. \quad P(j, k) = i \quad \text{memorize the predecessor}

\text{Time complexity: } O(nm^2)
Algorithm VITERBI\((x, n, t, e, m)\)

0. \(T(0, 0) = 1; \ P(0, 0) = 0;\)
1. for \(j = 1\) to \(m\)
2. \(T(j, 0) = 0;\)
3. for \(k = 1\) to \(n\) for every prefix upto \(k^\text{th}\) symbol
4. for \(j = 1\) to \(m\) for every ‘last state’ \(s_j\)
5. \(\text{premax} = 0\) initialize the prefix probability
6. for \(i = 0\) to \(m\) try every ‘predecessor state’ \(s_i\)
7. if \(\text{premax} < T(i, k - 1) \times t(i, j) \times e(j, x_k)\)
8. \(\text{premax} = T(i, k - 1) \times t(i, j) \times e(j, x_k)\) update probability
9. \(P(j, k) = i\) memorize the predecessor
10. \(T(j, k) = \text{premax}\)

Time complexity: \(O(nm^2)\)
Algorithm \textsc{Viterbi}(x, n, t, e, m)

0. \( T(0, 0) = 1; P(0, 0) = 0; \)
1. \textbf{for} \( j = 1 \) to \( m \)
2. \( \quad T(j, 0) = 0; \)
3. \textbf{for} \( k = 1 \) to \( n \) \hspace{1cm} \text{for every prefix upto \( k \)th symbol}
4. \textbf{for} \( j = 1 \) to \( m \) \hspace{1cm} \text{for every ‘last state’} \( s_j \)
5. \( \quad \text{premax} = 0 \) \hspace{1cm} \text{initialize the prefix probability}
6. \textbf{for} \( i = 0 \) to \( m \) \hspace{1cm} \text{try every ‘predecessor state’} \( s_i \)
7. \hspace{1cm} \textbf{if} \ \text{premax} \ < \ T(i, k - 1) \times t(i, j) \times e(j, x_k) \\
8. \hspace{1cm} \quad \text{premax} = T(i, k - 1) \times t(i, j) \times e(j, x_k) \hspace{1cm} \text{update probability}
9. \hspace{1cm} \quad P(j, k) = i \hspace{1cm} \text{memorize the predecessor}
10. \hspace{1cm} T(j, k) = \text{premax}
11. \hspace{1cm} \text{laststate} = 1; \ \text{max} = T(1, n)

\text{Time complexity:} \ O(nm^2)
Algorithm VITERBI($x, n, t, e, m$)

0. $T(0, 0) = 1; P(0, 0) = 0$
1. for $j = 1$ to $m$
2. $T(j, 0) = 0$
3. for $k = 1$ to $n$ for every prefix upto $k$th symbol
4. for $j = 1$ to $m$ for every ‘last state’ $s_j$
5. $\text{premax} = 0$ initialize the prefix probability
6. for $i = 0$ to $m$ try every ‘predecessor state’ $s_i$
7. if $\text{premax} < T(i, k - 1) \times t(i, j) \times e(j, x_k)$
8. $\text{premax} = T(i, k - 1) \times t(i, j) \times e(j, x_k)$ update probability
9. $P(j, k) = i$ memorize the predecessor
10. $T(j, k) = \text{premax}$
11. $\text{laststate} = 1; \text{max} = T(1, n)$
12. for $j = 2$ to $m$ identify maximum probability
Chapter 15. Dynamic Programming

Algorithm VITERBI($x, n, t, e, m$)

0. $T(0, 0) = 1; P(0, 0) = 0;$
1. for $j = 1$ to $m$
2. $T(j, 0) = 0;$
3. for $k = 1$ to $n$ for every prefix upto $k$th symbol
4. for $j = 1$ to $m$ for every ‘last state’ $s_j$
5. prem$\max = 0$ initialize the prefix probability
6. for $i = 0$ to $m$ try every ‘predecessor state’ $s_i$
7. if prem$\max < T(i, k - 1) \times t(i, j) \times e(j, x_k)$
8. prem$\max = T(i, k - 1) \times t(i, j) \times e(j, x_k)$ update probability
9. $P(j, k) = i$ memorize the predecessor
10. $T(j, k) =$ prem$\max$
11. last$\max = 1; max = T(1, n)$
12. for $j = 2$ to $m$ identify maximum probability
13. if $max < T(j, n)$
Chapter 15. Dynamic Programming

Algorithm VITERBI\((x, n, t, e, m)\)

0. \(T(0, 0) = 1; P(0, 0) = 0;\)
1. for \(j = 1\) to \(m\)  
2. \(T(j, 0) = 0;\)
3. for \(k = 1\) to \(n\) for every prefix upto \(k\)th symbol  
4. for \(j = 1\) to \(m\) for every ‘last state’ \(s_j\)  
5. \(\text{premax} = 0\) initialize the prefix probability  
6. for \(i = 0\) to \(m\) try every ‘predecessor state’ \(s_i\)  
7. if \(\text{premax} < T(i, k - 1) \times t(i, j) \times e(j, x_k)\)  
8. \(\text{premax} = T(i, k - 1) \times t(i, j) \times e(j, x_k)\) update probability  
9. \(P(j, k) = i\) memorize the predecessor  
10. \(T(j, k) = \text{premax}\)  
11. \(\text{laststate} = 1; \text{max} = T(1, n)\)  
12. for \(j = 2\) to \(m\) identify maximum probability  
13. if \(\text{max} < T(j, n)\)  
14. \(\text{laststate} = j;\)
Chapter 15. Dynamic Programming

Algorithm VITERBI($x$, $n$, $t$, $e$, $m$)

0. $T(0, 0) = 1$; $P(0, 0) = 0$
1. for $j = 1$ to $m$
2. \hspace{1cm} $T(j, 0) = 0$
3. for $k = 1$ to $n$
\hspace{1cm} for every prefix upto $k$th symbol
4. \hspace{1cm} for $j = 1$ to $m$
\hspace{2cm} for every ‘last state’ $s_j$
5. \hspace{2cm} premax = 0
\hspace{2cm} initialize the prefix probability
6. \hspace{2cm} for $i = 0$ to $m$
\hspace{3cm} try every ‘predecessor state’ $s_i$
7. \hspace{3cm} if premax < $T(i, k - 1) \times t(i, j) \times e(j, x_k)$
8. \hspace{3cm} premax = $T(i, k - 1) \times t(i, j) \times e(j, x_k)$
\hspace{3cm} update probability
9. \hspace{3cm} $P(j, k) = i$
\hspace{3cm} memorize the predecessor
10. \hspace{1cm} $T(j, k) = $ premax
11. laststate = 1; max = $T(1, n)$
12. for $j = 2$ to $m$
\hspace{1cm} identify maximum probability
13. \hspace{1cm} if max < $T(j, n)$
14. \hspace{2cm} laststate = $j$
15. \hspace{2cm} max = $T(j, n)$

Time complexity: $O(nm^2)$
Algorithm VITERBI\((x, n, t, e, m)\)

0. \(T(0, 0) = 1; P(0, 0) = 0;\)
1. for \(j = 1\) to \(m\)
2. \(T(j, 0) = 0;\)
3. for \(k = 1\) to \(n\) \(\text{for every prefix up to kth symbol}\)
4. for \(j = 1\) to \(m\) \(\text{for every 'last state' } s_j\)
5. \(\text{premax} = 0\) \(\text{initialize the prefix probability}\)
6. for \(i = 0\) to \(m\) \(\text{try every 'predecessor state' } s_i\)
7. if \(\text{premax} < T(i, k - 1) \times t(i, j) \times e(j, x_k)\)
8. \(\text{premax} = T(i, k - 1) \times t(i, j) \times e(j, x_k)\) \(\text{update probability}\)
9. \(P(j, k) = i\) \(\text{memorize the predecessor}\)
10. \(T(j, k) =\text{premax}\)
11. \(\text{laststate} = 1; \text{max} = T(1, n)\)
12. for \(j = 2\) to \(m\) \(\text{identify maximum probability}\)
13. if \(\text{max} < T(j, n)\)
14. \(\text{laststate} = j;\)
15. \(\text{max} = T(j, n)\)
16. return \((T, P, \text{max}, \text{laststate})\)
Algorithm \textsc{Viterbi}(x, n, t, e, m)

0. \( T(0, 0) = 1; \ P(0, 0) = 0; \)
1. \( \text{for } j = 1 \text{ to } m \)
2. \( \ T(j, 0) = 0; \)
3. \( \text{for } k = 1 \text{ to } n \) for every prefix upto \( k \)th symbol
4. \( \text{for } j = 1 \text{ to } m \) for every ‘last state’ \( s_j \)
5. \( \ premax = 0 \) initialize the prefix probability
6. \( \text{for } i = 0 \text{ to } m \) try every ‘predecessor state’ \( s_i \)
7. \( \text{if } premax < T(i, k - 1) \times t(i, j) \times e(j, x_k) \)
8. \( \ premax = T(i, k - 1) \times t(i, j) \times e(j, x_k) \) update probability
9. \( P(j, k) = i \) memorize the predecessor
10. \( T(j, k) = premax \)
11. \( \text{laststate} = 1; \ max = T(1, n) \)
12. \( \text{for } j = 2 \text{ to } m \) identify maximum probability
13. \( \text{if } max < T(j, n) \)
14. \( \text{laststate} = j; \)
15. \( max = T(j, n) \)
16. \( \text{return } (T, P, max, \text{laststate}) \)

Time complexity:

\( O(nm^2) \)
Chapter 15. Dynamic Programming

Algorithm VITERBI\((x, n, t, e, m)\)

0. \(T(0, 0) = 1; \ P(0, 0) = 0;\)
1. for \(j = 1\) to \(m\)
2. \(T(j, 0) = 0;\)
3. for \(k = 1\) to \(n\) for every prefix upto \(k\)th symbol
4. for \(j = 1\) to \(m\) for every ‘last state’ \(s_j\)
5. \(\text{premax} = 0\) initialize the prefix probability
6. for \(i = 0\) to \(m\) try every ‘predecessor state’ \(s_i\)
7. if \(\text{premax} < T(i, k - 1) \times t(i, j) \times e(j, x_k)\)
8. \(\text{premax} = T(i, k - 1) \times t(i, j) \times e(j, x_k)\) update probability
9. \(P(j, k) = i\) memorize the predecessor
10. \(T(j, k) = \text{premax}\)
11. \(\text{laststate} = 1; \ \text{max} = T(1, n)\)
12. for \(j = 2\) to \(m\) identify maximum probability
13. if \(\text{max} < T(j, n)\)
14. \(\text{laststate} = j;\)
15. \(\text{max} = T(j, n)\)
16. return \((T, P, \text{max}, \text{laststate})\)

Time complexity: \(O(nm^2)\)
Use the computed table $P$ and $laststate$ to traceback the hidden states.
Use the computed table $P$ and $laststate$ to traceback the hidden states.

Algorithm $\text{TRACEHIDDENSTATES}(P, j, k, x)$

1. $\textbf{if } j > 0$
Chapter 15. Dynamic Programming

Use the computed table $P$ and laststate to traceback the hidden states

Algorithm TraceHiddenStates($P, j, k, x$)

1. if $j > 0$
2. TraceHiddenStates($P, P[j, k], k - 1, x$)
Use the computed table $P$ and $laststate$ to traceback the hidden states

Algorithm $\text{TraceHiddenStates}(P, j, k, x)$

1. if $j > 0$
2. $\text{TraceHiddenStates}(P, P[j, k], k - 1, x)$
3. print ('State' $j$ 'emits symbol' $x_k$)
Chapter 15. Dynamic Programming

Use the computed table $P$ and $laststate$ to traceback the hidden states

Algorithm $\text{TraceHiddenStates}(P, j, k, x)$

1. \textbf{if} $j > 0$
2. \hspace{1em} $\text{TraceHiddenStates}(P, P[j \!, k \!], k - 1, x)$
3. \hspace{1em} \textbf{print} ('State' $j$ 'emits symbol' $x_k$)

First, call $\text{TraceHiddenStates}(P, laststate, n, x)$
Chapter 15. Dynamic Programming

There are other DP algorithms associated with HMMs.
Chapter 15. Dynamic Programming

There are other DP algorithms associated with HMMs.

- Viterbi algorithm belongs to the type of forward algorithms because it computes for prefixes of the given symbol sequence.

How to find objective function?

Recalled for forward, we defined:

$$f(j, k) = \max_{y = s_0 \ldots s_j} \{ \text{Prob}(y, x_1 \ldots x_k) \}$$

$f(j, k)$ is the maximum probability to generate prefix $x_1 \ldots x_k$ beginning from state $s_0$ and ending with state $s_j$.

For backward, we may define:

$$b(l, k) = \max_{y = s_l \ldots s_\omega} \{ \text{Prob}(y, x_k \ldots x_n) \}$$

$b(l, k)$ is the maximum probability to generate suffix $x_k \ldots x_n$ beginning from state $s_l$ and ending with state $s_\omega$. 

Chapter 15. Dynamic Programming

There are other DP algorithms associated with HMMs.

- **Viterbi** algorithm belongs to the type of forward algorithms because it computes for prefixes of the given symbol sequence.

A DP algorithm can be designed so it computes for suffixes of the given symbols sequence, a backward algorithm.
Chapter 15. Dynamic Programming

There are other DP algorithms associated with HMMs.

- **Viterbi** algorithm belongs to the type of **forward** algorithms because it computes for prefixes of the given symbol sequence.

  A DP algorithm can be designed so it computes for suffixes of the given symbols sequence, a **backward** algorithm.

**Algorithm IBRETIV**
There are other DP algorithms associated with HMMs.

- **Viterbi** algorithm belongs to the type of *forward* algorithms because it computes for prefixes of the given symbol sequence.

A DP algorithm can be designed so it computes for suffixes of the given symbols sequence, a *backward* algorithm.

**Algorithm IβRETIV**

How to find objective function?
Recalled for forward, we defined

\[
    f(j, k) = \max_{y=s_0 \ldots s_j} \{\text{Prob}(y, x_1 \ldots x_k)\}
\]

\(f(j, k)\) is the maximum probability to generate prefix \(x_1 \ldots x_k\) beginning from state \(s_0\) and ending with state \(s_j\);
There are other DP algorithms associated with HMMs.

- **Viterbi** algorithm belongs to the type of forward algorithms because it computes for prefixes of the given symbol sequence.

A DP algorithm can be designed so it computes for suffixes of the given symbol sequence, a backward algorithm.

**Algorithm IbreTiv**

How to find objective function?

Recalled for forward, we defined

\[
  f(j, k) = \max_{y=s_0 \ldots s_j} \{ \text{Prob}(y, x_1 \ldots x_k) \}
\]

\(f(j, k)\) is the maximum probability to generate prefix \(x_1 \ldots x_k\) beginning from state \(s_0\) and ending with state \(s_j\);

For backward, we may define

\[
  b(l, k) = \max_{y=s_l \ldots s_\omega} \{ \text{Prob}(y, x_k \ldots x_n) \}
\]

\(b(l, k)\) is the maximum probability to generate suffix \(x_k \ldots x_n\) beginning from state \(s_l\) and ending with state \(s_\omega\);
Chapter 15. Dynamic Programming

- We need to amend the HMM model with a silent state $s_\omega$;

\[ b(l, k) = \max_{y=s_l \ldots s_\omega} \{ Prob(y, x_k \ldots x_n) \} \]
Chapter 15. Dynamic Programming

- We need to amend the HMM model with a silent state $s_\omega$;

$$b(l, k) = \max_{y=s_1...s_\omega} \{ \text{Prob}(y, x_k...x_n) \}$$

- Recurrence (draw a diagram)

$$b(l, k) =$$
Chapter 15. Dynamic Programming

- We need to amend the HMM model with a silent state $s_\omega$;

$$b(l, k) = \max_{y=s_l \ldots s_\omega} \{ \text{Prob}(y, x_k \ldots x_n) \}$$

- Recurrence (draw a diagram)

$$b(l, k) = \max_i$$
We need to amend the HMM model with a silent state $s_\omega$;

\[
b(l, k) = \max_{y=s_1\ldots s_\omega} \{\text{Prob}(y, x_k \ldots x_n)\}
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Recurrence (draw a diagram)

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b(l, k) = \max_i \{b(i, k + 1)\}
\]
Chapter 15. Dynamic Programming

• We need to amend the HMM model with a silent state $s_\omega$;

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• Recurrence (draw a diagram)

$$b(l, k) = \max_i \{b(i, k + 1) \times t(l, i)\}$$
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the larger the $k$ value, the shortest the suffix is
Chapter 15. Dynamic Programming

- We need to amend the HMM model with a silent state \( s_\omega \);

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b(l, k) = \max_{y=s_1...s_\omega} \{\text{Prob}(y, x_k ... x_n)\}
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- Base cases:
Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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$k = n + 1$ is the case when the suffix is empty.
Chapter 15. Dynamic Programming

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**What is the relationship between $\max_j \{f(j, n)\}$ and $\max_l \{b(l, 1)\}$?**
Chapter 15. Dynamic Programming

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What is the relationship between $\max_j f(j, n)$ and $\max_l b(l, 1)$? (draw diagram to show)
Chapter 15. Dynamic Programming

Computing the total probability (instead of the maximum prob)
Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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$$p(j, k) \text{ to be the total probability for } M \text{ to generate prefix } x_1 \ldots x_k \text{ ending at state } s_j.$$
Chapter 15. Dynamic Programming

Define $p(j, k)$ to be the total probability for $M$ to generate prefix $x_1 \ldots x_k$ ending at state $s_j$. 

$$p(j, k) = \sum_i p(i, k-1) \times t(i, j) \times e(j, x_k)$$

with base cases:

- $p(0, 0) = 1$
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---

![Diagram](image-url)
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Then

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Chapter 15. Dynamic Programming

Implementation of the total probability algorithm

- almost the same as for Viterbi algorithm.
- applications, for example
  - screen a stream of signals (symbols) for occasional but desired pattern
  - use HMM to model the desired pattern
  - scanning window on the long stream; computes the total probability
  - report segments with 'significant' probability values

but what does it mean by 'significant probability values'?
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Chapter 15. Dynamic Programming

Necessary elements for problems solvable with DP
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(1) it is an optimization problem
Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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(1) it is an optimization problem
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   solutions are recursively definable
   e.g., a fastest path consists of other fastest subpaths
Chapter 15. Dynamic Programming

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e.g., solution: a path of stations,
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solutions are recursively definable

e.g., a fastest path consists of other fastest subpaths

(3) overlapping subproblems

e.g., two or more paths share a subpath.
Chapter 15. Dynamic Programming

Longest Common Subsequence (LCS)
Chapter 15. Dynamic Programming

Longest Common Subsequence (LCS)

\[ x = \text{ACCGGTCGAGTGCG} \]
\[ y = \text{GTCGTTCGGATGCCC} \]

To see how much the two sequences are related, we may examine how much the two are in common. e.g., a common subsequence for the two sequences

\[ \text{ACCGGTCGAGTGCG} \]
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\[ \text{CGTCGATGC} \]

\[ \text{ACCGGTC GAGTGCG} \]

\[ \text{|| || || ||} \]

\[ \text{CG TC GA TGC} \]

\[ \text{|| || || ||} \]

\[ \text{GTCGTTCGGGA TGCCC} \]
Chapter 15. Dynamic Programming

Longest Common Subsequence (LCS)

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

Longest Common Subsequence (LCS)

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\begin{align*}
  x &= \text{ACCGGTCGAGTGCG} \\
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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

Let $X = x_1 x_2 \cdots x_m$ be a sequence, $x_i \in \Sigma$. 

$X = \text{ACCGGTCGAGTGCG}$

$Z = \text{CG TCGA TGC}$

the corresponding indexes in $X$ are:

$3 < 4 < 6 < 7 < 8 < 9 < 11 < 12 < 13$
Chapter 15. Dynamic Programming

Let \( X = x_1 x_2 \cdots x_m \) be a sequence, \( x_i \in \Sigma \).

\( Z = z_1 z_2 \cdots z_k \) is a subsequence of \( X \).
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Chapter 15. Dynamic Programming

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\[ z_j = x_{i_j} \quad \text{for} \quad j = 1, \cdots , k. \]
Chapter 15. Dynamic Programming

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\[
\begin{array}{cccccccc}
34 & 6789 & 11 & 12 & 13 \\
X = & AC & CG & GT & CG & A & GT & GC \\
    & || & || & || & || & || & || & || \\
Z = & CG & TCGA & TGC \\
    & 12 & 3456 & 789 \\
\end{array}
\]
Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

$Z$ is a *common subsequence* of $X$ and $Y$
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  || || || ||
CG  TC  GA  TGC ← common sequence
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Chapter 15. Dynamic Programming

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|| || || ||
CG  TC  GA  TGC  ← common sequence
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**Longest Common Subsequence (LCS) problem:**

**Input:** $X = x_1x_2 \cdots x_m$, $Y = y_1y_2 \cdots y_n$.

**Output:** $Z$, a common subsequence of $X$ and $Y$ such that $\text{length}(Z)$ is the maximum.
Step 1: Analyze the LCS problem to see if it has the desired features:
Chapter 15. Dynamic Programming

**Step 1**: Analyze the LCS problem to see if it has the desired features:

- optimal substructure
- overlapping subproblems
**Step 1**: Analyze the LCS problem to see if it has the desired features:

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![LCS Example](image_url)
Chapter 15. Dynamic Programming

Step 1: Analyze the LCS problem to see if it has the desired features:

- optimal substructure
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We examine prefixes $x_1x_2\ldots x_i, i \leq m$
and $y_1y_2\ldots y_j, j \leq n$
Step 1: Analyze the LCS problem to see if it has the desired features:

- optimal substructure
- overlapping subproblems

We examine prefixes $x_1 x_2 \ldots x_i, i \leq m$
and $y_1 y_2 \ldots y_j, j \leq n$

- how is the LCS of $x_1 x_2 \ldots x_i$ and $y_1 y_2 \ldots y_j$ related to the LCS of shorter prefixes of $X$ and $Y$?
Chapter 15. Dynamic Programming

\[ x_1 \ldots x_{i-1} x_i \]
\[ y_1 \ldots y_{j-1} y_j \]
Chapter 15. Dynamic Programming

\[ x_1 \ldots x_{i-1} x_i \]
\[ y_1 \ldots y_{j-1} y_j \]

\[ x_1 \ldots x_{i-1} x_i \]
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if \( x_i = y_j \), include \( x_i \) in the LCS, reducing to LCS for \( x_1 \ldots x_{i-1} \) and \( y_1 \ldots y_{j-1} \).
Chapter 15. Dynamic Programming

\[ x_1 \cdots x_{i-1} x_i \]
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if \( x_i \neq y_j \), should we abandon either?
Chapter 15. Dynamic Programming

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If \( x_i = y_j \), include \( x_i \) in the LCS, reducing to LCS for \( x_1 \ldots x_{i-1} \) and \( y_1 \ldots y_{j-1} \)

If \( x_i \neq y_j \), should we abandon either? two options

(1) keep \( x_i \) but abandon \( y_j \)
    resulting in prefixes: \( x_1x_2 \ldots x_i \) and \( y_1y_2 \ldots y_{j-1} \), or
Chapter 15. Dynamic Programming

\[ x_1 \ldots x_{i-1} x_i \]
\[ y_1 \ldots y_{j-1} y_j \]

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if \( x_i \neq y_j \), should we abandon either? two options

(1) keep \( x_i \) but abandon \( y_j \)
resulting in prefixes: \( x_1 x_2 \ldots x_i \) and \( y_1 y_2 \ldots y_{j-1} \), or

(2) keep \( y_j \) but abandon \( x_i \)
resulting in prefixes: \( x_1 x_2 \ldots x_{i-1} \) and \( y_1 y_2 \ldots y_j \)
Let $LCS(i, j)$ be the longest common sequence for $x_1 x_2 \ldots x_i$ and $y_1 y_2 \ldots y_j$, then
Let $LCS(i, j)$ be the longest common sequence for $x_1x_2\ldots x_i$ and $y_1y_2\ldots y_j$, then

- if $x_i = y_j$, then $LCS(i, j) = \text{concat}(LCS(i - 1, j - 1), x_i)$

We conclude: $LCS$ problem has the optimal substructure property; $LCS$ problem has overlapping subproblems.
Let $LCS(i, j)$ be the longest common sequence for $x_1 x_2 \ldots x_i$ and $y_1 y_2 \ldots y_j$, then

- if $x_i = y_j$, then $LCS(i, j) = \text{concat}(LCS(i - 1, j - 1), x_i)$

- if $x_i \neq y_j$, then

  $$LCS(i, j) = LCS(i, j - 1) \text{ OR } LCS(i, j) = LCS(i - 1, j)$$

  (depending on which is of a larger length.)
Chapter 15. Dynamic Programming

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(depending on which is of a larger length.)

We conclude:

LCS problem has the optimal substructure property;
LCS problem has overlapping subproblems.
Chapter 15. Dynamic Programming

**Step 2:** define objective function and formulate recurrence

Define $l(i,j)$ to be the length of the LCS for prefixes $x_1\ldots x_i$ and $y_1\ldots y_j$.

$$l(i,j) = \begin{cases} l(i-1,j-1) + 1 & \text{if } x_i = y_j; \\ \max\{l(i,j-1), l(i-1,j), l(i-1,j-1)\} & \text{if } x_i \neq y_j. \end{cases}$$

**Base cases:**

- $l(i,0) = 0$, for $0 \leq i \leq m$, either prefix is empty → LCS is empty
- $l(0,j) = 0$, for $0 \leq j \leq n$
Chapter 15. Dynamic Programming

Step 2: define objective function and formulate recurrence

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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**base cases:**
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l(i, 0) = 0, \quad \text{for } 0 \leq i \leq m, \quad \text{either prefix is empty} \\
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\]
Chapter 15. Dynamic Programming

**Step 2:** define objective function and formulate recurrence

Define $l(i, j)$ to be the **length** of the LCS for prefixes $x_1 \ldots x_i$ and $y_1 \ldots y_j$. Then

$$l(i, j) = \begin{cases} 
l(i - 1, j - 1) + 1 & \text{if } x_i = y_j; \\
\max \left\{ l(i, j - 1); l(i - 1, j) \right\} & \text{if } x_i \neq y_j
\end{cases}$$

the value of $l(i, j)$ depends on values of $l(i - 1, j - 1)$, $l(i, j - 1)$, and $l(i - 1, j)$. 

---

*base cases:* $l(i, 0) = 0$, for $0 \leq i \leq m$, either prefix is empty $\rightarrow$ LCS is empty; $l(0, j) = 0$, for $0 \leq j \leq n$. 
Step 2: define objective function and formulate recurrence

Define \( l(i, j) \) to be the length of the LCS for prefixes \( x_1 \ldots x_i \) and \( y_1 \ldots y_j \). then

\[
l(i, j) = \begin{cases} 
  l(i - 1, j - 1) + 1 & \text{if } x_i = y_j; \\
  \max \left\{ l(i, j - 1); l(i - 1, j) \right\} & \text{if } x_i \neq y_j 
\end{cases}
\]

the value of \( l(i, j) \) depends on values of \( l(i - 1, j - 1), l(i, j - 1), \) and \( l(i - 1, j) \).

base cases: \( l(i, 0) = 0 \), for \( 0 \leq i \leq m \), either prefix is empty \( \rightarrow \) LCS is empty
Step 2: define objective function and formulate recurrence

Define \( l(i, j) \) to be the length of the LCS for prefixes \( x_1 \ldots x_i \) and \( y_1 \ldots y_j \). then

\[
l(i, j) = \begin{cases} 
  l(i-1, j-1) + 1 & \text{if } x_i = y_j; \\
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\end{cases}
\]

the value of \( l(i, j) \) depends on values of \( l(i-1, j-1), l(i, j-1), \) and \( l(i-1, j) \).

base cases: \( l(i, 0) = 0, \) for \( 0 \leq i \leq m, \) either prefix is empty \( \rightarrow \) LCS is empty

\( l(0, j) = 0, \) for \( 0 \leq j \leq n \)
Chapter 15. Dynamic Programming

**Step 3:** bottom-up table building for function $l(i, j)$.

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Chapter 15. Dynamic Programming

**Step 3:** bottom-up table building for function $l(i, j)$.

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Step 3: bottom-up table building for function $l(i, j)$.
Chapter 15. Dynamic Programming

**Step 3:** bottom-up table building for function $l(i,j)$.

LCS result between prefixes GCCCTAGC and GCG:

GCCCTAGC

G   C   G
Algorithm LCS\((x, y)\)
1. \( m = \text{length}(x), n = \text{length}(y); \)
Algorithm LCS \((x, y)\)

1. \(m = \text{length}(x)\), \(n = \text{length}(y)\); 
2. \textbf{for} \(i = 0 \text{ to } m\)
Algorithm LCS($x, y$)
1. \( m = \text{length}(x), \ n = \text{length}(y); \)
2. \textbf{for} i = 0 \textbf{to} m
3. \( T[i, 0] = 0 \)

\begin{align*}
\text{for } j = 1 \text{ to } n \\
\text{if } x_i = y_j \\
\quad T[i, j] = T[i-1, j-1] + 1 \\
\quad P[i, j] = \text{'}\textbf{↑}\text{'} \\
\text{else if } T[i, j-1] > T[i-1, j] \\
\quad T[i, j] = T[i, j-1] \\
\quad P[i, j] = \text{'}\textbf{←}\text{'}
\end{align*}

return (\( T, P \))

Time complexity: \( O(mn) \)
Algorithm LCS(x, y)
1. \( m = \text{length}(x) \), \( n = \text{length}(y) \);
2. for \( i = 0 \) to \( m \)
3. \( T[i, 0] = 0 \)
4. for \( j = 0 \) to \( n \)
Algorithm LCS($x, y$)
1. $m = \text{length}(x)$, $n = \text{length}(y)$;
2. \textbf{for} $i = 0$ \textbf{to} $m$
3. \hspace{1em} $T[i, 0] = 0$
4. \textbf{for} $j = 0$ \textbf{to} $n$
5. \hspace{1em} $T[0, j] = 0$
Chapter 15. Dynamic Programming

Algorithm LCS \((x, y)\)
1. \(m = \text{length}(x), n = \text{length}(y)\);
2. \(\text{for } i = 0 \text{ to } m\)
3. \(T[i, 0] = 0\)
4. \(\text{for } j = 0 \text{ to } n\)
5. \(T[0, j] = 0\)
6. \(\text{for } i = 1 \text{ to } m\)
7. \(\text{if } x_i = y_j\)
8. \(T[i, j] = T[i-1, j-1] + 1\); \(P[i, j] = '↖'\)
9. \(\text{else if } T[i, j-1] > T[i-1, j]\)
10. \(T[i, j] = T[i, j-1]\); \(P[i, j] = '←'\)
11. \(\text{else}\)
12. \(T[i, j] = T[i-1, j]\); \(P[i, j] = '↑'\)
13. \(\text{return } (T, P)\)

Time complexity: \(O(mn)\)
Algorithm LCS(\(x, y\))
1. \(m = \text{length}(x), n = \text{length}(y)\);
2. \text{for} \(i = 0\) \text{to} \(m\)
3. \hspace{1em} \(T[i, 0] = 0\)
4. \text{for} \(j = 0\) \text{to} \(n\)
5. \hspace{1em} \(T[0, j] = 0\)
6. \text{for} \(i = 1\) \text{to} \(m\)
7. \hspace{1em} \text{for} \(j = 1\) \text{to} \(n\)
Algorithm LCS(x, y)
1. \( m = \text{length}(x) \), \( n = \text{length}(y) \);
2. \textbf{for} \ i = 0 \textbf{ to } m \\
3. \hspace{1em} T[i, 0] = 0 \\
4. \textbf{for} \ j = 0 \textbf{ to } n \\
5. \hspace{1em} T[0, j] = 0 \\
6. \textbf{for} \ i = 1 \textbf{ to } m \\
7. \hspace{1em} \textbf{for} \ j = 1 \textbf{ to } n \\
8. \hspace{2em} \textbf{if} \ x_i = y_j
Chapter 15. Dynamic Programming

Algorithm LCS(x, y)
1. \( m = \text{length}(x) \), \( n = \text{length}(y) \);
2. \( \text{for } i = 0 \text{ to } m \)
3. \( T[i, 0] = 0 \)
4. \( \text{for } j = 0 \text{ to } n \)
5. \( T[0, j] = 0 \)
6. \( \text{for } i = 1 \text{ to } m \)
7. \( \text{for } j = 1 \text{ to } n \)
8. \( \text{if } x_i = y_j \)
9. \( T[i, j] = T[i - 1, j - 1] + 1; \) \( P[i, j] = \left\uparrow \right\downarrow \)
Algorithm LCS($x, y$)
1. $m = \text{length}(x)$, $n = \text{length}(y)$;
2. for $i = 0$ to $m$
3.    $T[i, 0] = 0$
4. for $j = 0$ to $n$
5.    $T[0, j] = 0$
6. for $i = 1$ to $m$
7.    for $j = 1$ to $n$
8.       if $x_i = y_j$
9.          $T[i, j] = T[i - 1, j - 1] + 1$; $P[i, j] = \uparrow$
10. else if $T[i, j - 1] > T[i - 1, j]$
11. return $(T, P)$

Time complexity: $O(mn)$
Algorithm LCS \((x, y)\)

1. \(m = \text{length}(x), n = \text{length}(y);\)
2. \textbf{for} \(i = 0 \text{ to } m\)
3. \hspace{1em} \(T[i, 0] = 0\)
4. \textbf{for} \(j = 0 \text{ to } n\)
5. \hspace{1em} \(T[0, j] = 0\)
6. \textbf{for} \(i = 1 \text{ to } m\)
7. \hspace{1em} \textbf{for} \(j = 1 \text{ to } n\)
8. \hspace{2em} \textbf{if} \(x_i = y_j\)
9. \hspace{3em} \(T[i, j] = T[i - 1, j - 1] + 1; P[i, j] = \text{'\(\wedge\)'\}\)
10. \hspace{2em} \textbf{else if} \(T[i, j - 1] > T[i - 1, j]\)
11. \hspace{3em} \(T[i, j] = T[i - 1, j]; P[i, j] = \text{'\(\leftarrow\)'\}\)
12. \textbf{return} \((T, P)\)

Time complexity: \(O(mn)\)
Algorithm LCS\((x, y)\)

1. \(m = \text{length}(x), n = \text{length}(y)\);
2. \(\text{for } i = 0 \text{ to } m\)
3. \(T[i, 0] = 0\)
4. \(\text{for } j = 0 \text{ to } n\)
5. \(T[0, j] = 0\)
6. \(\text{for } i = 1 \text{ to } m\)
7. \(\text{for } j = 1 \text{ to } n\)
8. \(\text{if } x_i = y_j\)
9. \(T[i, j] = T[i - 1, j - 1] + 1; P[i, j] = '↖'\)
10. \(\text{else if } T[i, j - 1] > T[i - 1, j]\)
11. \(T[i, j] = T[i, j - 1]; P[i, j] = '←'\)
12. \(\text{else } T[i, j] = T[i - 1, j]; P[i, j] = '↑'\)

Time complexity: \(O(mn)\)
Algorithm LCS($x, y$)
1. $m = \text{length}(x)$, $n = \text{length}(y)$;
2. for $i = 0$ to $m$
3. \hspace{1em} $T[i, 0] = 0$
4. for $j = 0$ to $n$
5. \hspace{1em} $T[0, j] = 0$
6. for $i = 1$ to $m$
7. \hspace{1em} for $j = 1$ to $n$
8. \hspace{2em} if $x_i = y_j$
9. \hspace{3em} $T[i, j] = T[i - 1, j - 1] + 1$; $P[i, j] = ^\searrow$
10. \hspace{2em} else if $T[i, j - 1] > T[i - 1, j]$
11. \hspace{3em} $T[i, j] = T[i, j - 1]$; $P[i, j] = ^\leftarrow$
12. \hspace{2em} else $T[i, j] = T[i - 1, j]$; $P[i, j] = ^\uparrow$
13. return ($T$, $P$)

Time complexity: $O(mn)$
Algorithm LCS($x, y$)
1. $m = \text{length}(x), n = \text{length}(y)$;
2. for $i = 0$ to $m$
3. \hspace{.5cm} $T[i, 0] = 0$
4. for $j = 0$ to $n$
5. \hspace{.5cm} $T[0, j] = 0$
6. for $i = 1$ to $m$
7. \hspace{.5cm} for $j = 1$ to $n$
8. \hspace{1.5cm} if $x_i = y_j$
9. \hspace{2cm} $T[i, j] = T[i - 1, j - 1] + 1; \ P[i, j] = \text{'$\searrow$'}$
10. \hspace{1.5cm} else if $T[i, j - 1] > T[i - 1, j]$
11. \hspace{2cm} $T[i, j] = T[i, j - 1]; \ P[i, j] = \text{'$\leftarrow$'}$
12. \hspace{1.5cm} else $T[i, j] = T[i - 1, j]; \ P[i, j] = \text{'$\uparrow$'}$
13. return $(T, P)$

Time complexity: $O(mn)$
Chapter 15. Dynamic Programming

Traceback the longest common subsequence
either recursive or iterative algorithm
Chapter 15. Dynamic Programming

Traceback the longest common subsequence either recursive or iterative algorithm

Algorithm PrintLCS($P, x, i, j$)
Traceback the longest common subsequence

either recursive or iterative algorithm

Algorithm `PRINTLCS(P, x, i, j)`
1. `if (i > 0) \land (j > 0)`
Chapter 15. Dynamic Programming

Traceback the longest common subsequence

either recursive or iterative algorithm

Algorithm PRINTLCS($P, x, i, j$)
1. if ($i > 0$) ∧ ($j > 0$)
2. if $P[i,j] = '↖'
3. PRINTLCS($P,x,i−1,j−1$)
4. PRINT($x_i$)
5. else if $P[i,j] = '←'
6. PRINTLCS($P,x,i,j−1$)
7. else PRINTLCS($P,x,i−1,j$)
8. else return ()
Traceback the longest common subsequence

either recursive or iterative algorithm

Algorithm PRINTLCS(P, x, i, j)
1. if (i > 0) ∧ (j > 0)
2. if P[i, j] = '↖'
3. PRINTLCS(P, x, i − 1, j − 1)
Algorithm PRINTLCS$(P, x, i, j)$

1. \textbf{if} $(i > 0) \land (j > 0)$
2. \hspace{1em} \textbf{if} $P[i, j] = '\\backslash'$
3. \hspace{2em} PRINTLCS$(P, x, i - 1, j - 1)$
4. \hspace{1em} \textbf{Print} $(x_i)$

Traceback the longest common subsequence

either recursive or iterative algorithm
Traceback the longest common subsequence

either recursive or iterative algorithm

Algorithm \textsc{PrintLCS}(P, x, i, j)
\begin{enumerate}
\item \textbf{if} \((i > 0) \land (j > 0)\)
\item \textbf{if} \(P[i, j] = '\backslash'\)
\item \textbf{Print} (\(x_i\))
\item \textbf{else if} \(P[i, j] = '←'\)
\end{enumerate}
Chapter 15. Dynamic Programming

Traceback the longest common subsequence

either recursive or iterative algorithm

Algorithm PRINTLCS($P, x, i, j$)
1. if ($i > 0$) $\land$ ($j > 0$)
2. if $P[i, j]$ = '↖'
3. PRINTLCS($P, x, i - 1, j - 1$)
4. Print ($x_i$)
5. else if $P[i, j]$ = '←'
6. PRINTLCS($P, x, i, j - 1$)
Chapter 15. Dynamic Programming

Traceback the longest common subsequence

either recursive or iterative algorithm

Algorithm $\text{PrintLCS}(P, x, i, j)$
1. \textbf{if} $(i > 0) \land (j > 0)$
2. \textbf{if} $P[i, j] = \text{'↖'}$
3. \text{PrintLCS}(P, x, i - 1, j - 1)$
4. \text{Print} $(x_i)$
5. \textbf{else if} $P[i, j] = \text{'←'}$
6. \text{PrintLCS}(P, x, i, j - 1)$
7. \textbf{else} $\text{PrintLCS}(P, x, i - 1, j)$
Traceback the longest common subsequence
either recursive or iterative algorithm

Algorithm \textsc{PrintLCS}(P, x, i, j)
1. \textbf{if} \ (i > 0) \land (j > 0)
2. \ \ \textbf{if} \ P[i, j] = '↖'
3. \ \ \ \textsc{PrintLCS}(P, x, i - 1, j - 1)
4. \ \ \ \textbf{Print} \ (x_i)
5. \ \ \ \textbf{else if} \ P[i, j] = '←'
6. \ \ \ \ \textsc{PrintLCS}(P, x, i, j - 1)
7. \ \ \ \textbf{else} \ \textsc{PrintLCS}(P, x, i - 1, j)
8. \ \ \ \textbf{else return} \ ()
Chapter 15. Dynamic Programming

Pairwise Alignment
Chapter 15. Dynamic Programming

Pairwise Alignment

ACCGGGTCGAGTGCG
CGTCGATGC ← common sequence
GTCGTTCGGA TGCCC

blue letters are conserved; red letters are mutated; purple letters are inserted or deleted;
Chapter 15. Dynamic Programming

Pairwise Alignment

ACCGGTTCGAGTGCG
    CGTCGATGC ← common sequence
GTCGTTCGGATGCC

biologically more meaningful view:

ACCGGTC GAGTGC
    || || || |||
    CG TC GA TGC ← common sequence
    || || || |||
GTCGTTCGGA TGCCC
Chapter 15. Dynamic Programming

Pairwise Alignment

ACCGGTCGAGTGCG

CGTCGATGC ← common sequence

GTCGTTCGGATGCCC

biologically more meaningful view:

ACCGGTC  GAGTGCG

|| || || || || || ||

CG  TC  GA  TGC ← common sequence

|| || || || || || ||

GTCGTTCGGA  TGCCC

blue letters are conserved;
red letters are mutated;
purple letters are inserted or deleted;
Chapter 15. Dynamic Programming

Alignment between two sequences:

ACCGGTC

GAGTGCG

|| || || |||

CG TC GA TGC ← common sequence

|| || || |||

GTCGTTCGGA

TGCCC

• two sequences padded with 's called gaps;
• two sequences are of the same length, aligned;
• sum of column scores is used to identify the best alignment

For example, a score scheme

+5 for conservation columns, (reward)
-2 for substitution columns, (penalty)
-6 for delete/insert columns, (penalty)

the above alignment has the total score

\((−2) + (−2) + 5 + 5 + (−2) + 5 + 5 + (−6) + 5 + 5 + (−6) + 5 + 5 + 5 + (−2) + (−6))\)
Chapter 15. Dynamic Programming

Alignment between two sequences:

```
ACCGGTC_GAGTGCG_
   || || || ||
  CG TC GA TGC ← common sequence
   || || || ||
GTCGTTCGGA_TGCC
```

• two sequences padded with 's called gaps;
• two sequences are of the same length, aligned;
• sum of column scores is used to identify the best alignment

For example, a score scheme
+5 for conservation columns, (reward)
-2 for substitution columns, (penalty)
-6 for delete/insert columns, (penalty)

the above alignment has the total score

\[
(-2) + (-2) + 5 + 5 + (-2) + 5 + 5 + (-6) + 5 + 5 + (-6) + 5 + 5 + 5 + (-2) + (-6)
\]

LCS is a special case

0 + 0 + 1 + 1 + 0 + 1 + 1 + 0 + 1 + 1 + 0 + 1 + 1 + 1 + 0 + 0
Chapter 15. Dynamic Programming

Alignment between two sequences:

```
ACCGGTC_GAGTGCG_
   || || || |||
  CG TC GA TGC  ← common sequence
   || || || |||
GTCGTTCGGA_TGCC
```

- two sequences padded with '_'s called **gaps**;
Chapter 15. Dynamic Programming

Alignment between two sequences:

```
ACCGGTC_GAGTGCG_

CG TC GA TGC ← common sequence

GTCGTTCGGA_TGCC
```

- two sequences padded with ‘_’s called **gaps**;
- two sequences are of the same length, aligned;

For example, a score scheme
+5 for conservation columns, (reward)
-2 for substitution columns, (penalty)
-6 for delete/insert columns, (penalty)
The above alignment has the total score
\[
-2 + -2 + 5 + 5 + -2 + 5 + 5 + -6 + 5 + 5 + -6 + 5 + 5 + -2 + -6
\]
Alignment between two sequences:

\[
\begin{align*}
\text{ACCGGTC}_&\_GAGTGCG_\_ \\
\text{||} &\text{||} &\text{||} &\text{||} \\
\text{CG} &\text{ TC} &\text{ GA} &\text{TGC} \quad \leftarrow \text{common sequence} \\
\text{||} &\text{||} &\text{||} &\text{||} \\
\text{GTCGTTCGGA}_&\_TGCCC \\
\end{align*}
\]

- two sequences padded with ' \_ 's called \textit{gaps};
- two sequences are of the same length, aligned;
- sum of column scores is used to identify the best alignment
Alignment between two sequences:

ACCGGTC_GAGTGCG_

\[\begin{array}{cccc}
\text{CG} & \text{TC} & \text{GA} & \text{TGC} \\
\end{array}\]  \rightarrow \text{common sequence}

GTCGGTCGGA_TGCC

- two sequences padded with ‘ ’s called gaps;
- two sequences are of the same length, aligned;
- sum of column scores is used to identify the best alignment.
Chapter 15. Dynamic Programming

Alignment between two sequences:

ACCGGTC_GAGTGCG_
|| || || |||
CG TC GA TGC ← common sequence
|| || || |||
GTCGTTCGGA_TGCCC

- two sequences padded with ’_’s called gaps;
- two sequences are of the same length, aligned;
- sum of column scores is used to identify the best alignment

For example, a score scheme
  +5 for conservation columns, (reward)
  -2 for substitution columns, (penalty)
  -6 for delete/insert columns, (penalty)
Chapter 15. Dynamic Programming

Alignment between two sequences:

\[
\begin{align*}
\text{ACCGGTC}_- & \text{GAGTGCG}_- \\
\text{CG} & \text{TC} \quad \text{GA} \quad \text{TGC} \quad \text{← common sequence} \\
\text{GTCGTTCGGA}_- & \text{TGCC} \\
\end{align*}
\]

- Two sequences padded with ‘_’s called gaps;
- Two sequences are of the same length, aligned;
- Sum of column scores is used to identify the best alignment

For example, a score scheme

\[+5\] for conservation columns, (reward)
\[-2\] for substitution columns, (penalty)
\[-6\] for delete/insert columns, (penalty)

The above alignment has the total score

\[
(-2) + (-2) + 5 + 5 + (-2) + 5 + 5 + (-6) + 5 + 5 + (-6) + 5 + 5 + 5 + (-2) + (-6)
\]
Alignment between two sequences:

\[
\begin{align*}
&\text{ACCGGTC}_-\text{GAGTGC}_- \\
&\quad | | | | | | | \\
&\quad \text{CG TC GA TGC} \quad \text{← common sequence} \\
&\quad | | | | | | | \\
&\text{GTCGTTCGGA}_-\text{TGCC}_- \\
\end{align*}
\]

- two sequences padded with ‘’s called \textit{gaps};
- two sequences are of the same length, aligned;
- sum of column scores is used to identify the best alignment

For example, a score scheme

\begin{itemize}
\item +5 for conservation columns, (reward)
\item -2 for substitution columns, (penalty)
\item -6 for delete/insert columns, (penalty)
\end{itemize}

the above alignment has the total score

\[
(-2) + (-2) + 5 + 5 + (-2) + 5 + 5 + (-6) + 5 + 5 + (-6) + 5 + 5 + 5 + (-2) + (-6)
\]

LCS is a special case
Alignment between two sequences:

```
ACCGGTC_GAGTGCG_
     || || || |||
   CG TC GA TGC ← common sequence
     || || || |||
GTCGTTCGGA_TGCC
```

- two sequences padded with ‘_’s called **gaps**;
- two sequences are of the same length, aligned;
- sum of column scores is used to identify the best alignment

For example, a score scheme
  - +5 for conservation columns, (reward)
  - -2 for substitution columns, (penalty)
  - -6 for delete/insert columns, (penalty)

the above alignment has the total score

```
(−2) + (−2) + 5 + 5 + (−2) + 5 + 5 + (−6) + 5 + 5 + (−6) + 5 + 5 + 5 + (−2) + (−6)
```

LCS is a special case

```
0 + 0 + 1+1+ 0 + 1+1+ 0 + 1+1+ 0 + 1+1+1+ 0 + 0
```
Chapter 15. Dynamic Programming

Significance of sequence alignments:

**Sequence Homology Reveals Functions**

- **Homology reveals evolution of structure/function**

<table>
<thead>
<tr>
<th>Sequence</th>
<th>Alignment</th>
<th>Consensus</th>
</tr>
</thead>
<tbody>
<tr>
<td>FOS_RAT</td>
<td>MMFSGFNADYEAASSSRCSSASPAGDSLSYHSPADSFSSSGPSPVNTQDFCDLSSVSSAF 60</td>
<td></td>
</tr>
<tr>
<td>FOS_MOUSE</td>
<td>MMFSGFNADYEAASSSRCSSASPAGDSLSYHSPADSFSSSGPSPVNTQDFCDLSSVSSAF 60</td>
<td></td>
</tr>
<tr>
<td>FOS_CHICK</td>
<td>MMYQGFAGEYEAASSSRCSSASPAGDSLTYPAPADSFSSSGPSPVNSDQFCTDLSSVSSAF 60</td>
<td></td>
</tr>
<tr>
<td>FOSB_MOUSE</td>
<td>MFQAFPGYDS-GSRCSS-SPSAESQ--YLSVDSFGSPPTAAASQE-CAGLGMPSGF 54</td>
<td></td>
</tr>
<tr>
<td>FOSB_HUMAN</td>
<td>MFQAFPGYDS-GSRCSS-SPSAESQ--YLSVDSFGSPPTAAASQE-CAGLGMPSGF 54</td>
<td></td>
</tr>
<tr>
<td><strong>Consensus</strong></td>
<td><em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<em>:</em>:<e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<td></td>
</tr>
</tbody>
</table>
Chapter 15. Dynamic Programming

Pairwise alignment problem:

**Input:** sequences $x = x_1 x_2 \ldots x_m$, $y = y_1 y_2 \ldots y_n$, and scoring scheme $\text{score}$

**Output:** $x'$ and $y'$, which are $x$ and $y$ padded with '-'s, respectively such that total score $\sum_{i=1}^{l} \text{score}(x'_i, y'_i)$ is the maximum,
where $l = |x'| = |y'|$.

How to solve it?
Chapter 15. Dynamic Programming

Pairwise alignment problem:

**INPUT:** sequences \( x = x_1 x_2 \ldots x_m \), \( y = y_1 y_2 \ldots y_n \), and scoring scheme \( \text{score} \)

**OUTPUT:** \( x' \) and \( y' \), which are \( x \) and \( y \) padded with ‘_’s, respectively such that total score \( \sum_{i=1}^{l} \text{score}(x'_i, y'_i) \) is the maximum, where \( l = |x'| = |y'| \).

How to solve it?

Similar to LCS problem, we consider prefixes \( x_1 \ldots x_i \) of \( x \), \( y_1 \ldots y_j \) of \( y \).
Chapter 15. Dynamic Programming

Pairwise alignment problem:

**INPUT**: sequences \( x = x_1x_2 \ldots x_m \), \( y = y_1y_2 \ldots y_n \), and scoring scheme \( \text{score} \)

**OUTPUT**: \( x' \) and \( y' \), which are \( x \) and \( y \) padded with 's, respectively such that total score \( \sum_{i=1}^{l} \text{score}(x'_i, y'_i) \) is the maximum, where \( l = |x'| = |y'| \).

How to solve it?

Similar to LCS problem, we consider prefixes \( x_1 \ldots x_i \) of \( x \), \( y_1 \ldots y_j \) of \( y \).

(1) \[
\begin{array}{c}
\ldots x_{i-1} x_i \\
y_1 \ldots y_{j-1} y_j
\end{array}
\]
(2) \[
\begin{array}{c}
\ldots x_{i-1} x_i \\
y_1 \ldots y_{j-1} y_j
\end{array}
\]
(3) \[
\begin{array}{c}
\ldots x_{i-1} x_i \\
y_1 \ldots y_{j-1} y_j
\end{array}
\]
Chapter 15. Dynamic Programming

Pairwise alignment problem:

**Input**: sequences \( x = x_1 x_2 \ldots x_m \), \( y = y_1 y_2 \ldots y_n \), and scoring scheme \( \text{score} \)

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How to solve it?

Similar to LCS problem, we consider prefixes \( x_1 \ldots x_i \) of \( x \), \( y_1 \ldots y_j \) of \( y \).

\[
\begin{align*}
\text{(1)} \quad & x_1 \ldots x_{i-1} x_i \\
& y_1 \ldots y_{j-1} y_j \\
\text{(2)} \quad & x_1 \ldots x_{i-1} x_i \\
& y_1 \ldots y_j-1 y_j \\
\text{(3)} \quad & x_1 \ldots x_{i-1} x_i \\
& y_1 \ldots y_j-1 y_j
\end{align*}
\]

case (1) contributes to a match/mismatch score
Chapter 15. Dynamic Programming

Pairwise alignment problem:

**INPUT:** sequences \( x = x_1 x_2 \ldots x_m \), \( y = y_1 y_2 \ldots y_n \), and scoring scheme \( \text{score} \)

**OUTPUT:** \( x' \) and \( y' \), which are \( x \) and \( y \) padded with \( \_ \)'s, respectively, such that total score \( \sum_{i=1}^{l} \text{score}(x'_i, y'_i) \) is the maximum, where \( l = |x'| = |y'|. \)

How to solve it?

Similar to LCS problem, we consider prefixes \( x_1 \ldots x_i \) of \( x \), \( y_1 \ldots y_j \) of \( y \).

\[ x_1 \ldots x_i \]
\[ y_1 \ldots y_{j-1} y_j \]

\[ x_1 \ldots x_{i-1} x_i \]
\[ y_1 \ldots y_{j-1} y_j \]

\[ x_1 \ldots x_{i-1} x_i \]
\[ y_1 \ldots y_{j-1} y_j \]

(1) \hspace{2cm} (2) \hspace{2cm} (3)

case (1) contributes to a match/mismatch score

cases (2) and (3) contribute to an insertion/deletion score.
Scoring function for pairwise alignment

Example (1)
+5 for conservation columns, (reward)
-2 for substitution columns, (penalty)
-6 for delete/insert columns, (penalty)

Example (2)
Why sum of column scores?
product of independent column probabilities
Chapter 15. Dynamic Programming

Scoring function for pairwise alignment

Example (1)
- +5 for conservation columns, (reward)
- -2 for substitution columns, (penalty)
- -6 for delete/insert columns, (penalty)
Chapter 15. Dynamic Programming

Scoring function for pairwise alignment

Example (1)
- +5 for conservation columns, (reward)
- -2 for substitution columns, (penalty)
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Example (2)

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>C</th>
<th>G</th>
<th>T</th>
<th>-</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>5</td>
<td>-1</td>
<td>-2</td>
<td>-1</td>
<td>-4</td>
</tr>
<tr>
<td>C</td>
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<td>5</td>
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<td>-4</td>
</tr>
<tr>
<td>-</td>
<td>-4</td>
<td>-4</td>
<td>-4</td>
<td>-4</td>
<td>*</td>
</tr>
</tbody>
</table>
Scoring function for pairwise alignment

Example (1)
+5 for conservation columns, (reward)
-2 for substitution columns, (penalty)
-6 for delete/insert columns, (penalty)

Example (2)

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Why sum of column scores?
Scoring function for pairwise alignment

Example (1)

+5 for conservation columns, (reward)
-2 for substitution columns, (penalty)
-6 for delete/insert columns, (penalty)

Example (2)

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<td>-1</td>
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<td>-1</td>
<td>-4</td>
</tr>
<tr>
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Why sum of column scores?

product of independent column probabilities
Chapter 15. Dynamic Programming

As for the LCS problem, we can define

\[ A(i, j) \] is the maximum score alignment between \( x_1 \ldots x_i \)
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\[
A(i, j) = \begin{cases} 
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\[ S(i, j) = \max \begin{cases} 
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S(i - 1, j) + \text{score}(x_i) & i \geq 1;
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Chapter 15. Dynamic Programming

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scoring function \( \text{score} \)

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**exercise:** Table filling and traceback
Chapter 15. Dynamic Programming

Affine gap penalty: a challenge for DP

Consider: new gap penalty score:

\[ \text{op} \]: penalty to open a gap;

\[ \text{ext} \]: penalty to extend an already opened gap

\[ \text{penalty} = -\text{op} - \text{ext} (l - 1) \]

for a gap covering \( l \) positions, where \( \text{op} > \text{ext} \).

How to modify recurrences:

\[ S(i, j) = \max \left\{ \begin{array}{l}
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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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S(i, j) = \max \left\{ S(i - 1, j - 1) + score(x_i, y_j) \right\}
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Chapter 15. Dynamic Programming

Issue: we do not know how long a gap is.

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\[ S(i, j) = \max \begin{cases} 
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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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But the complexity increased to: \( O(nm \times \max\{n, m\}) \)

Why does the complexity increase? a lot of recomputations, but where?
Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Why does the complexity increase?
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Chapter 15. Dynamic Programming

So we want to do recursion "one step at a time":

- Substitution
- Insertion
- Deletion
Chapter 15. Dynamic Programming

So we want to do recursion "one step at a time":

but at the same time, differentiate different "states"
where the two tails are

Substitute
Chapter 15. Dynamic Programming

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**Substitute**

**Insertion**
Chapter 15. Dynamic Programming

So we want to do recursion "one step at a time": 

**but at the same time, differentiate different "states"**
where the two tails are

**Substitute**

- **S(i, j)**
  - $x_1 \ldots x_{i-1} x_i$
  - $y_1 \ldots y_{j-1} y_j$

- **S(i-1, j-1)**
  - $x_1 \ldots x_{i-1} x_i$
  - $y_1 \ldots y_{j-1} y_j$

**Insertion**

- **I(i, j)**
  - $x_1 \ldots x_i$
  - $y_1 \ldots y_{j-1} y_j$

- **I(i-1, j-1)**
  - $x_1 \ldots x_i$
  - $y_1 \ldots y_{j-1} y_j$

**Deletion**

- **D(i, j)**
  - $x_1 \ldots x_{i-1} x_i$
  - $y_1 \ldots y_j$

- **D(i-1, j)**
  - $x_1 \ldots x_{i-1} x_i$
  - $y_1 \ldots y_j$
Chapter 15. Dynamic Programming

Define $S(i, j)$ to be the maximum score of alignment between $x_1 \ldots x_i$ and $y_1 \ldots y_j$ with $x_i$ being aligned to $y_j$.

Define $I(i, j)$ to be the maximum score of alignment between $x_1 \ldots x_i$ and $y_1 \ldots y_j$ with $y_j$ being insertion.
Chapter 15. Dynamic Programming

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$$I(i, j) = \max \begin{cases} 
I(i, j-1) - \text{ext} \\
S(i, j-1) - \text{op}
\end{cases}$$
Chapter 15. Dynamic Programming

Define $D(i,j)$ to be the maximum score of alignment between $x_1 \ldots x_i$ and $y_1 \ldots y_j$ with $x_i$ being deletion.

$D(i,j) = \max\{D(i-1,j) - \text{ext} S(i-1,j) - \text{op}\}$

• Note that there is no deletion right after insertion, neither insertion right after deletion. Why?

• What is the desired value? $\max\{S(m,n), I(m,n), D(m,n)\}$

• How to fill the table(s) and to traceback the optimal alignment?
Define $D(i,j)$ to be the maximum score of alignment between $x_1 \ldots x_i$ and $y_1 \ldots y_j$ with $x_i$ being deletion.
Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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\begin{align*}
\text{ACAC} & \text{ GT A} \quad \text{ACAC GT A} \\
\text{ACCT} & \text{ _ _ A} \quad \text{ACCT} \text{ _ A}
\end{align*}
Chapter 15. Dynamic Programming

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Define $D(i, j)$ to be the maximum score of alignment between $x_1 \ldots x_i$ and $y_1 \ldots y_j$ with $x_i$ being deletion.

$$D(i, j) = \max \left\{ D(i-1, j) - \text{ext}, S(i-1, j) - \text{op} \right\}$$

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Chapter 15. Dynamic Programming

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D(i, j) = \max \left\{ \begin{array}{l}
D(i-1, j) - ext \\
S(i-1, j) - op
\end{array} \right. 
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S(m, n) \\
I(m, n) \\
D(m, n)
\end{array} \right. \]

- How to fill the table(s) and to traceback the optimal alignment?
Chapter 15. Dynamic Programming

DP applied to summation problems

Unmatched Socks Problem

There are $n$ pairs of socks, each a different color;

- each day, remove arbitrarily two individual socks;
- compute the probability that, on none of the $k$ days, a pair of color-matched socks is removed.

When $n = 5$, $k = 1$

$$P_1 = \frac{\text{number of color-not-matched pairs}}{\text{number of total pairs}} = \frac{(10 \times 8)}{2^{10}} = \frac{40}{45} = \frac{8}{9}$$

When $n = 5$, $k = 2$

$$P_2 = P_1 \times \left( P_u + P_m + P_{u,m} \right) = \frac{8}{9} \times \left( \frac{1}{2^{10}} + \frac{6 \times 4}{2^{10}} + \frac{2 \times 6}{2^{10}} \right) = \frac{8}{9} \times \frac{25}{28}$$
Chapter 15. Dynamic Programming

**DP applied to summation problems**

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\[
P_3 = P_2 \times \text{??}
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Chapter 15. Dynamic Programming

DP applied to summation problems

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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DP applied to summation problems

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Chapter 15. Dynamic Programming

**DP applied to summation problems**

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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$$2m_k + u_k = 2n - 2k, \text{ for every } k \leq n$$
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We define \( P_k(m_k, u_k) \) to be the total probability that not a pair of matched socks is selected on any day of the first \( k \) days, resulting in \( m_k \) and \( u_k \). Then
Chapter 15. Dynamic Programming

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$$P_k(m_k, u_k) = \sum \left\{ P_{k-1}(m_k + 2, u_k - 2) \times \frac{(m_k + 2)(2m_k + 2)}{2m_k + 2 + u_k} \right.$$ 

$$P_{k-1}(m_k + 1, u_k) \times \frac{2(m_k + 1)u_k}{2m_k + 2 + u_k} \left. \right.$$ 

$$P_{k-1}(m_k, u_k + 2) \times \frac{(u_k + 2)^2}{2m_k + 2 + u_k} \right.$$
Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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(Most of these problems are solvable with "backward DP" as well).

Matrix Chain Multiplication problem

Input: matrices $A_1, \ldots, A_n$, where $A_i$ has dimension $p_{i-1} \times p_i$;

Output: a parenthesization by which the product $A_1 \times A_2 \times \cdots \times A_n$ uses the minimum number of scalar multiplications.
Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

scalable multiplications

\[ \text{5x5} \times \text{5x4} \times \text{4x8} \times \text{8x2} = \text{5x2} \]
Chapter 15. Dynamic Programming

scalable multiplications

= $5 \times 4 \times 8$
Chapter 15. Dynamic Programming

Many possible parenthesizations
Chapter 15. Dynamic Programming

Many possible parenthesizations

1. \((A ((B (C D)))\))
   \[5(5)2 + 5(4)2 + 4(8)2 = 154\]
2. \((A ((B (C D)))\))
   \[5(5)2 + 5(4)8 + 5(8)2 = 290\]
3. \(((A B) (C D))\)
   \[5(5)4 + 4(8)2 + 5(4)2 = 204\]
4. \(((A (B C)) D)\)
   \[5(4)8 + 5(5)8 + 5(8)2 = 440\]
5. \(((A B) C) D)\)
   \[5(5)4 + 5(4)8 + 5(8)2 = 340\]
Chapter 15. Dynamic Programming

Many possible parenthesizations

Note: The number of all possible parenthesizations is

\[ P(n) = \sum_{k=1}^{n-1} P(k)P(n-k), \text{ Catalan number} \]
Chapter 15. Dynamic Programming

**Step 1**: identify *optimal substructure*:
Optimal solution in terms of optimal solutions to subproblems:

1. The best parenthesization of $A_i A_{i+1} \cdots A_j$ must be $(A_i \cdots A_k)(A_{k+1} \cdots A_j)$ for some $k, i \leq k < j$.
2. The optimal cost of $A_i A_{i+1} \cdots A_j$ must be the smallest among optimal costs for $(A_i \cdots A_k)(A_{k+1} \cdots A_j)$ for $k = i, i+1, \ldots, j-1$.
3. For each $k$, the optimal cost for the above is the optimal cost for $(A_i \cdots A_k) +$ the optimal cost for $(A_{k+1} \cdots A_j)$ + the number of scalar multiplications to multiply the two terms.
Step 1: identify *optimal substructure*:

Optimal solution in terms of optimal solutions to subproblems:

(1) the best parenthesization of $A_i A_{i+1} \cdots A_j$ must be

$$(A_i \cdots A_k)(A_{k+1} \cdots A_j)$$

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Step 1: identify optimal substructure:
Optimal solution in terms of optimal solutions to subproblems:

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for some $k, i \leq k < j$.

(2) The optimal cost of $A_i A_{i+1} \cdots A_j$ must be the smallest among optimal costs for

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Step 1: identify optimal substructure:
Optimal solution in terms of optimal solutions to subproblems:

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   the optimal cost for $(A_{k+1} \cdots A_j) +$
   the number of scalar multiplications to multiply the two terms.
Chapter 15. Dynamic Programming

**Step 2**: objective function and recurrence

Define $f(i,j)$ to be the minimum number of scalar multiplications needed for $A_iA_{i+1}\cdots A_j$.

Then $f(i,j) = \min_{i \leq k < j} \{ f(i,k) + f(k+1,j) + p_{i-1}p_kp_{j-1} \}$

where base case is $f(i,j) = 0$ when $i = j$, for single matrix $A_i$.
Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

**Step 3.** Fill out the DP table for function $f$. 
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Chapter 15. Dynamic Programming

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- Table size $n \times n$;
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- smaller instances are of smaller $j-1$ values;
- diagonal cells have the same $j-1$ values;
Chapter 15. Dynamic Programming

<table>
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<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Side lengths: 1, 4, 5, 3, 6, 7, 1

\[ A_{1}^{(1 \times 4)} \times A_{2}^{(4 \times 5)} \times A_{3}^{(5 \times 3)} \times A_{4}^{(3 \times 6)} \times A_{5}^{(6 \times 7)} \times A_{6}^{(7 \times 1)} \]
Algorithm $\text{MatrixChainOrder}(p)$
1. $n = \text{length}[p] - 1;$
Algorithm \textsc{MatrixChainOrder}(p)
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1.  \( n = \text{length}[p] - 1; \)
2.  \textbf{for} \( i = 1 \) \textbf{to} \( n \)
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Chapter 15. Dynamic Programming

Algorithm \texttt{MatrixChainOrder}(p)
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3. \hspace{3em} \texttt{M}[i, i] = 0;
4. \hspace{1em} \texttt{for} \ l = 2 \ \texttt{to} \ n
Chapter 15. Dynamic Programming

Algorithm \textsc{MatrixChainOrder}(p)
1. \hspace{1em} \( n = \text{length}[p] - 1; \)
2. \hspace{1em} \textbf{for} \hspace{0.5em} i = 1 \hspace{0.5em} \textbf{to} \hspace{0.5em} n
3. \hspace{1em} \( M[i, i] = 0; \)
4. \hspace{1em} \textbf{for} \hspace{0.5em} l = 2 \hspace{0.5em} \textbf{to} \hspace{0.5em} n
5. \hspace{1em} \textbf{for} \hspace{0.5em} i = 1 \hspace{0.5em} \textbf{to} \hspace{0.5em} n - l + 1
Algorithm **MatrixChainOrder**(\(p\))

1. \(n = \text{length}[p] - 1;\)
2. \(\textbf{for } i = 1 \textbf{ to } n\)
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5. \(\textbf{for } i = 1 \textbf{ to } n - l + 1\)
6. \(j = i + l - 1;\)
Chapter 15. Dynamic Programming

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4. for $l = 2$ to $n$
5. \hspace{1em} for $i = 1$ to $n - l + 1$
6. \hspace{2em} $j = i + l - 1$;
7. \hspace{2em} $M[i, j] = \infty$;
Algorithm MatrixChainOrder($p$)
1. $n = length[p] - 1$;
2. for $i = 1$ to $n$
3. $M[i, i] = 0$;
4. for $l = 2$ to $n$
5. for $i = 1$ to $n - l + 1$
6. $j = i + l - 1$;
7. $M[i, j] = \infty$;
8. for $k = i$ to $j - 1$
10. if $q < M[i, j]$
11. $M[i, j] = q$
12. $S[i, j] = k$;
13. return $(M, S)$

Time complexity: $O(n^2 \times n)$
Algorithm **MatrixChainOrder**($p$)

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2. **for** $i = 1$ **to** $n$
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7. $M[i, j] = \infty$
8. **for** $k = i$ **to** $j - 1$
Algorithm \textsc{MatrixChainOrder}(p)

1. \hspace{1em} \textit{n = length}[p] - 1;
2. \hspace{1em} \textbf{for} i = 1 \textbf{ to } n
3. \hspace{3em} M[i, i] = 0;
4. \hspace{1em} \textbf{for} l = 2 \textbf{ to } n
5. \hspace{3em} \textbf{for} i = 1 \textbf{ to } n - l + 1
6. \hspace{5em} j = i + l - 1;
7. \hspace{3em} M[i, j] = \infty;
8. \hspace{3em} \textbf{for} k = i \textbf{ to } j - 1
9. \hspace{5em} q = M[i, k] + M[k + 1, j] + p[i - 1]p[k]p[j]
10. \hspace{5em} \textbf{if} \hspace{1em} q < M[i, j]
Algorithm \textsc{MatrixChainOrder}(p)
1. \hspace{1em} \( n = \text{length}[p] - 1; \)
2. \hspace{1em} \textbf{for} \  i = 1 \ \textbf{to} \ n \\
3. \hspace{2em} M[i, i] = 0; \\
4. \hspace{1em} \textbf{for} \  l = 2 \ \textbf{to} \ n \\
5. \hspace{2em} \textbf{for} \  i = 1 \ \textbf{to} \ n - l + 1 \\
6. \hspace{3em} j = i + l - 1; \\
7. \hspace{2em} M[i, j] = \infty; \\
8. \hspace{2em} \textbf{for} \  k = i \ \textbf{to} \ j - 1 \\
10. \hspace{2em} \textbf{if} \  q < M[i, j] \\
11. \hspace{3em} M[i, j] = q
Algorithm `MatrixChainOrder(p)`
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2. \( \textbf{for } i = 1 \textbf{ to } n \)
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6. \( j = i + l - 1; \)
7. \( M[i,j] = \infty; \)
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10. \( \textbf{if } q < M[i,j] \)
11. \( M[i,j] = q \)
12. \( S[i,j] = k \)

Time complexity: \( \mathcal{O}(n^2 \times n) \)
Chapter 15. Dynamic Programming

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13. \textbf{return} \( (M, S) \)

Time complexity: \( O(n^2 \times n) \)
Step 4. obtain the optimization parenthesization
Step 4. obtain the optimization parenthesization

Algorithm $\text{MatrixChainParenthesization}(i, j, S)$
1. if $i < j$
2. $k = s[i, j]$
3. print $(i, j, \text{" : "}, k)$
4. $\text{MatrixChainParenthesization}(i, k, S)$
5. $\text{MatrixChainParenthesization}(k + 1, j, S)$
6. return

Usage: call $\text{MatrixChainParenthesization}(1, n, S)$

Time complexity for $\text{MatrixChainParenthesization}$: $O(n)$. 
Chapter 15. Dynamic Programming

**Step 4.** obtain the optimization parenthesization

Algorithm `MATRIXCHAINPARENTHEZISATION(i, j, S)`

1. **if** $i < j$
2. $k = s[i, j]$
3. **print** $(i, j, " : ", k)$
4. `MATRIXCHAINPARENTHEZISATION(i, k, S)`
5. `MATRIXCHAINPARENTHEZISATION(k + 1, j, S)`
6. **return**

**Usage:** call `MATRIXCHAINPARENTHEZISATION(1, n, S)`
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Usage: call $\text{MatrixChainParenthesization}(1, n, S)$

Time complexity for $\text{MatrixChainParenthesization}$:
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Usage: call \textsc{MatrixChainParenthesization}(1, n, S)

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Chapter 16. Greedy Algorithms

Chapter 16 Greedy Algorithms
Chapter 16. Greedy Algorithms

Chapter 16 Greedy Algorithms

• Dynamic programming is to consider all possible choices and select the best.
Chapter 16. Greedy Algorithms

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  a DP approach always leads to the optimal solution
Chapter 16. Greedy Algorithms

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• Dynamic programming is to \textit{consider all possible choices and select the best.}
  
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• A greedy algorithm may \textit{ignore some choices and select the one that is locally the best.}
Chapter 16. Greedy Algorithms

Chapter 16 Greedy Algorithms

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a DP approach always leads to the optimal solution

• A greedy algorithm may ignore some choices and select the one that is locally the best.
  
a greedy strategy may or may NOT lead to the optimal solution
Chapter 16. Greedy Algorithms

Activity-selection problem
Chapter 16. Greedy Algorithms

Activity-selection problem

**Input:** \( n \) activities \( S = \{a_1, \ldots, a_n\} \), each with a start time \( s_i \) and finish time \( f_i \), \( 1 \leq i \leq n \)

**Output:** subset \( A \subseteq S \) of mutually compatible activities such that \( |A| \) is the maximum.
Chapter 16. Greedy Algorithms

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\[ A = \{a_1, a_3, a_6, a_8\}, \text{ or } \]

A diagram showing the activities $a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9$ with their respective start and finish times.
Chapter 16. Greedy Algorithms

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Chapter 16. Greedy Algorithms

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Chapter 16. Greedy Algorithms

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\[
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\]
Chapter 16. Greedy Algorithms

A dynamic programming solution
Chapter 16. Greedy Algorithms

A dynamic programming solution

Step 1 analysis of problem
Chapter 16. Greedy Algorithms

A dynamic programming solution

Step 1 analysis of problem
Chapter 16. Greedy Algorithms

A dynamic programming solution

**Step 1** analysis of problem

How to solve it recursively?
Chapter 16. Greedy Algorithms

For example, selecting \( a_3 \) would result in two subsets of activities to further consider: \( \{a_1\} \) and \( \{a_6, a_7, a_8, a_9\} \); selecting \( a_5 \) would result in two subsets of activities to further consider: \( \{a_1, a_2\} \) and \( \{a_7\} \).
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selecting $a_5$ would result in two subsets of activities to further consider: $\{a_1, a_2\}$ and $\{a_7, a_8, a_9\}$;
Chapter 16. Greedy Algorithms

Choosing an activity, say $a_5$, would result in two subsets of activities to further consider:

$$\{a_1, a_2\} \text{ and } \{a_7, a_8, a_9\}$$

But if selecting again $a_8$ would result in two smaller subsets:

$$\{a_7\} \text{ and } \{\}$$

These subproblems are not "prefixes" or "suffixes", they could be "middle segments"; we approach such problems with inside-out instead of forward or backward methods.
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Chapter 16. Greedy Algorithms

Define subset $S_{i,j} = \{ a_k \in S : f_i \leq s_k < f_k \leq s_j \}$

e.g., $S_{2,9} = \{ a_5, a_6, a_7 \}$

Introducing dummy activities: $a_0$ with $s_0 = f_0$ = the earliest start time of all, and $a_{n+1}$ with $s_{n+1} = f_{n+1}$ = latest finish time of all.
Chapter 16. Greedy Algorithms

So we consider solve activity selection on subsets of tasks:

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- \( a_0 \) with \( s_0 = f_0 = \text{earliest start time of all} \),
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Chapter 16. Greedy Algorithms

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Chapter 16. Greedy Algorithms

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Chapter 16. Greedy Algorithms

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Chapter 16. Greedy Algorithms

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Chapter 16. Greedy Algorithms

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Then

\[
|A_{i,j}| = \max_{a_k \in S_{i,j}} \{|A_{i,k} \cup \{a_k\} \cup A_{k,j}|\}
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\[
|A_{i,j}| = \max_{a_k \in S_{i,j}} \{|A_{i,k} \cup \{a_k\} \cup A_{k,j}|\}
\]

e.g.,
\[
|A_{0,10}| = \max \left\{ \ldots, |A_{0,6} \cup \{a_6\} \cup A_{6,10}|, \text{ where } S_{0,6} = \{a_1, \ldots, a_4\}, S_{6,10} = \{a_8, a_9\} \right\}
\]
Chapter 16. Greedy Algorithms

Instead of solving this with DP, we wonder if all but one of the subproblems could be made empty. |

\[ A_{i,j} = \max_{a_k \in S_{i,j}} \{|A_{i,k}| \cup \{a_k\} \cup A_{k,j}| \} \]

i.e., a greedy choice of \( a_k \in S_{i,j} \) makes \( A_{i,l} \cup \{a_l\} \cup A_{l,i} \) not needed, for all \( l \neq k \)

But the chosen \( a_k \) has to guarantee to maintain \( |A_{i,j}| \) is the maximum.

called greedy-choice property (need to be proved)
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Chapter 16. Greedy Algorithms
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Activity Selection problem has the greedy-choice property.
Chapter 16. Greedy Algorithms

Activity Selection problem has the greedy-choice property.

**Theorem 16.1:** Let $S_{i,j}$ be a subproblem and $a_k \in S_{i,j}$ be the activity of the earliest finish time, then $a_k$ is in **some** optimal solution for $S_{i,j}$.
Activity Selection problem has the greedy-choice property.

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The theorem is the same as that **selecting the activity with the earliest finish time**
Chapter 16. Greedy Algorithms

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**Proof**: (Using the “swapping method”, or “exchange method”)
Chapter 16. Greedy Algorithms

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Let $A_{i,j}$ be an optimal solution for $S_{i,j}$.

(1) If $a_k \in A_{i,j}$, then the theorem is true.
Chapter 16. Greedy Algorithms

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We define $B_{i,j} = A_{i,j} - \{a_p\} \cup \{a_k\}$. Then

- because $f_p \geq f_k$, $a_k$ is compatible with other activities in $B_{i,j}$.
  So $B_{i,j}$ is a solution for $S_{i,j}$. 

Activity Selection problem has the greedy-choice property.

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(1) If $a_k \in A_{i,j}$, then the theorem is true.

(2) If $a_k \notin A_{i,j}$, assume $a_p$ is the one with earliest finish time in $A_{i,j}$.

We define $B_{i,j} = A_{i,j} - \{a_p\} \cup \{a_k\}$. Then

- because $f_p \geq f_k$, $a_k$ is compatible with other activities in $B_{i,j}$.
  So $B_{i,j}$ is a solution for $S_{i,j}$.
- $|B_{i,j}| = |A_{i,j}|$, so $B_{i,j}$ is also an optimal solution for $S_{i,j}$. 
Activity Selection problem has the greedy-choice property.

**Theorem 16.1:** Let $S_{i,j}$ be a subproblem and $a_k \in S_{i,j}$ be the activity of the earliest finish time, then $a_k$ is in some optimal solution for $S_{i,j}$.

The theorem is the same as that selecting the activity with the earliest finish time

- makes all other subproblems disappearing, and
- maintains the optimality of solution

**Proof:** (Using the “swapping method”, or “exchange method”)

Let $A_{i,j}$ be an optimal solution for $S_{i,j}$.

(1) If $a_k \in A_{i,j}$, then the theorem is true.

(2) If $a_k \notin A_{i,j}$, assume $a_p$ is the one with earliest finish time in $A_{i,j}$.

We define $B_{i,j} = A_{i,j} - \{a_p\} \cup \{a_k\}$. Then

- because $f_p \geq f_k$, $a_k$ is compatible with other activities in $B_{i,j}$.
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- $|B_{i,j}| = |A_{i,j}|$, so $B_{i,j}$ is also an optimal solution for $S_{i,j}$.

Since $a_k \in B_{i,j}$, we prove the theorem.
Chapter 16. Greedy Algorithms

Using the Theorem 16.1 and the recurrence

\[ |A_{i,j}| = \max_{a_k \in S_{i,j}} \{ |A_{i,k} \cup \{a_k\} \cup A_{k,j}| \} \]

we can derive greedy algorithms for the Activity Selection problem.
Assume that the activities are sorted according to their finish times. 
\[ f_1 \leq f_2 \leq \cdots \leq f_n. \]
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Adding hypothetical activity \( a_0 \) (with \( s_0 = f_0 = s_1 \))
Chapter 16. Greedy Algorithms

Assume that the activities are sorted according to their finish times.
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Adding hypothetical activity \( a_0 \) (with \( s_0 = f_0 = s_1 \)) and activity \( a_{n+1} \) (with \( s_{n+1} = f_{n+1} = f_n \)).
Chapter 16. Greedy Algorithms

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Algorithm \textsc{Activity-Selection}(s, f, k, n)
Assume that the activities are sorted according to their finish times. 
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**Algorithm** \textsc{Activity-Selection}(\( s, f, k, n \))

1. \( m = k + 1 \)
Chapter 16. Greedy Algorithms

Assume that the activities are sorted according to their finish times. 
\[ f_1 \leq f_2 \leq \cdots \leq f_n. \]

Adding hypothetical activity \( a_0 \) (with \( s_0 = f_0 = s_1 \)) and activity \( a_{n+1} \) (with \( s_{n+1} = f_{n+1} = f_n \)).

Algorithm Activity-Selection\((s, f, k, n)\)
1. \( m = k + 1 \)
2. while \( m \leq n \) and \( s_m < f_k \)
   
   First called with Activity-Selection\((s, f, 0, n)\)

Time complexity: \( O(n) \).
Chapter 16. Greedy Algorithms

Assume that the activities are sorted according to their finish times. 
\[ f_1 \leq f_2 \leq \cdots \leq f_n. \]

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Chapter 16. Greedy Algorithms

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1. \( m = k + 1 \)
2. while \( m \leq n \) and \( s_m < f_k \)
3. \( m = m + 1 \)
4. if \( m \leq n \)
5. \( \text{return} \ \{a_m\} \cup \text{Activity-Selection}(s, f, m, n) \)
Chapter 16. Greedy Algorithms

Assume that the activities are sorted according to their finish times.
\( f_1 \leq f_2 \leq \cdots \leq f_n \).

Adding hypothetical activity \( a_0 \) (with \( s_0 = f_0 = s_1 \)) and activity \( a_{n+1} \) (with \( s_{n+1} = f_{n+1} = f_n \)).

**Algorithm** \textsc{Activity-Selection}(\( s, f, k, n \))

1. \( m = k + 1 \)
2. \textbf{while} \( m \leq n \) and \( s_m < f_k \)
3. \hspace{1em} \( m = m + 1 \)
4. \hspace{1em} \textbf{if} \( m \leq n \)
5. \hspace{2em} \textbf{return} \( \{a_m\} \cup \textsc{Activity-Selection}(s, f, m, n) \)
6. \hspace{1em} \textbf{else return} \( (\emptyset) \)
Chapter 16. Greedy Algorithms

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First called with \textsc{Activity-Selection}(s, f_0, n)
Chapter 16. Greedy Algorithms

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6. \hspace{1em} \textbf{else return} \( (\emptyset) \)

First called with \textsc{Activity-Selection}(s, f_0, n)

Time complexity: \( O(n) \).
Chapter 16. Greedy Algorithms

Fractional Knapsack Problem:

Input:
n items of sizes $s_1, \ldots, s_n$ and values $v_1, \ldots, v_n$; and a knapsack size $B$.

Output: a subset $A \subseteq \{1, 2, \ldots, n\}$, each with a fraction $0 < f_i \leq 1$, which maximizes $\sum_{i \in A} f_i v_i$ under the constraint $\sum_{i \in A} f_i s_i \leq B$.

Define:

$K(S, X)$ be the maximum value of packing items from set $S$ into space $X$.

Then $K(S, X) = \max_{0 < f_i \leq 1, i \in S} \{ f_i v_i + K(S - \{i\}, X - f_i s_i) \}$.

- We hope to select some item $i$ that makes other subproblems disappear.
- We select item $i$ such that it has the highest density $d_i = \frac{v_i}{s_i}$.
- For convenience, assume the items are sorted according to the non-decreasing order of density.

Theorem: Fractional Knapsack problem has the greedy-choice property.
Fractional Knapsack Problem:

**INPUT:** $n$ items of sizes $s_1, \ldots, s_n$ and values $v_1, \ldots, v_n$; and a knapsack size $B$
Chapter 16. Greedy Algorithms

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Chapter 16. Greedy Algorithms

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Chapter 16. Greedy Algorithms

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Chapter 16. Greedy Algorithms

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Chapter 16. Greedy Algorithms

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Chapter 16. Greedy Algorithms

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**Theorem:** Fractional Knapsack problem has the greedy-choice property.
Chapter 16. Greedy Algorithms

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Given a subset of items $S \subseteq \{1, 2, \ldots, n\}$ and space $X$. Assume item $i$ has the highest density in $S$. Then there is an optimal solution for $S$ which contains the fraction $f_i$ of the item $i$, for some maximum $f_i$, $0 \leq f_i \leq 1$ such that $f_i s_i \leq X$. 
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(This is left as an exercise)
Solve the following problem of Coin Change problem with DP:

**Input:** \( X \) cents of money and coin denominations \( \{d_1, d_2, d_3\} \) where \( 1 \leq d_1 < d_2 < d_3 \);

**Output:** the minimum number of coins with total amount = \( X \).

1. What does your recursive solution look like?
2. What should the objective function be?
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e.g., $X = 12, d_1 = 1, d_2 = 3, d_3 = 5$,
then an optimal solution is: $5, 5, 1, 1$
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Review for Midterm I

- Big-$O$, Big-$\Omega$, lower bound, upper bound
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Chapter 16. Greedy Algorithms

Huffman Code
Chapter 16. Greedy Algorithms

Huffman Code

compressing data using binary bits
Chapter 16. Greedy Algorithms

Huffman Code

compressing data using binary bits
code: a compressing scheme,
Chapter 16. Greedy Algorithms

Huffman Code

compressing data using binary bits
code: a compressing scheme, fixed length vs variable length code
**Chapter 16. Greedy Algorithms**

**Huffman Code**

compressing data using binary bits
code: a compressing scheme, fixed length vs variable length code

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<th>c</th>
<th>d</th>
<th>e</th>
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<td>111</td>
<td>1101</td>
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Chapter 16. Greedy Algorithms

Huffman Code

compressing data using binary bits

code: a compressing scheme, fixed length vs variable length code

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Chapter 16. Greedy Algorithms

**prefix code**: a prefix of a codeword cannot be another codeword.
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The Optimal Prefix Code Problem:
Chapter 16. Greedy Algorithms

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Note that

- $d_T(c)$ is the length of codeword for $c$ under the code scheme $T$. 
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Note that

- $d_T(c)$ is the length of codeword for $c$ under the code scheme $T$.
- $B(T) \times n$ is the number of bits required to code a file (of $n$ characters).
Algorithm \texttt{HUFFMAN}(C, f)
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1. \( n = |C| \)
Algorithm \textsc{Huffman}(C, f)

1. \( n = |C| \)
2. \( Q = C \)
Chapter 16. Greedy Algorithms

Algorithm $\text{HUFFMAN}(C, f)$

1. $n = |C|$
2. $Q = C$
3. for $i = 1$ to $n - 1$

4. $\text{newnode}(z)$
5. $x = \text{Extract-Min}(Q)$
6. $z.\text{leftchild} = x$
7. $y = \text{Extract-Min}(Q)$
8. $z.\text{rightchild} = y$
9. $f(z) = f(x) + f(y)$
10. $\text{Insert}(Q, z)$
11. return $\text{Extract-Min}(Q)$
Chapter 16. Greedy Algorithms

Algorithm **HUFFMAN**(\(C, f\))

1. \( n = |C| \)
2. \( Q = C \)
3. **for** \( i = 1 \) **to** \( n - 1 \)
4. **newnode**\((z)\)
Algorithm **HUFFMAN**(C, f)

1. \( n = |C| \)
2. \( Q = C \)
3. for \( i = 1 \) to \( n - 1 \)
4. \hspace{1em} **newnode**(z)
5. \hspace{1em} \( x = \text{EXTRACT-MIN}(Q) \) \hspace{1em} extract two least frequent characters
Algorithm \textsc{Huffman}(C, f)

1. \( n = |C| \)
2. \( Q = C \)
3. \textbf{for} \( i = 1 \text{ to } n - 1 \)
4. \hspace{1em} \textbf{newnode}(z)
5. \hspace{1em} \( x = \textsc{Extract-Min}(Q) \) \quad \text{extract two least frequent characters}
6. \hspace{1em} \( z.\text{left}\text{child} = x \)
Algorithm \textsc{Huffman}(C, f)

1. \( n = |C| \)
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6. \hspace{1cm} \( z\.\text{leftchild} = x \)
7. \hspace{1cm} \( y = \textsc{Extract-Min}(Q) \)
Chapter 16. Greedy Algorithms

Algorithm HUFFMAN(C, f)

1. \( n = |C| \)
2. \( Q = C \)
3. for \( i = 1 \) to \( n - 1 \)
4. \( \text{newnode}(z) \)
5. \( x = \text{EXTRACT-MIN}(Q) \quad \text{extract two least frequent characters} \)
6. \( z.\text{leftchild} = x \)
7. \( y = \text{EXTRACT-MIN}(Q) \)
8. \( z.\text{rightchild} = y \)
Chapter 16. Greedy Algorithms

Algorithm \textsc{Huffman}(C, f)

1. \( n = |C| \)
2. \( Q = C \)
3. \textbf{for} \( i = 1 \) \textbf{to} \( n - 1 \)
4. \textbf{newnode}(z)
5. \( x = \textsc{Extract-Min}(Q) \) \hspace{1cm} extract two least frequent characters
6. \( z.\text{leftchild} = x \)
7. \( y = \textsc{Extract-Min}(Q) \)
8. \( z.\text{rightchild} = y \)
9. \( f(z) = f(x) + f(y) \)
Algorithm HUFFMAN($C, f$)

1. $n = |C|$
2. $Q = C$
3. for $i = 1$ to $n - 1$
4.   newnode($z$)
5.   $x = \text{Extract-Min}(Q)$  \[\text{extract two least frequent characters}\]
6.   $z.\text{leftchild} = x$
7.   $y = \text{Extract-Min}(Q)$
8.   $z.\text{rightchild} = y$
9.   $f(z) = f(x) + f(y)$
9.   INSERT($Q, z$)
Chapter 16. Greedy Algorithms

Algorithm \textsc{Huffman}(C, f)

1. \hspace{1cm} n = |C|
2. \hspace{1cm} Q = C
3. \hspace{1cm} \textbf{for } i = 1 \textbf{ to } n - 1
4. \hspace{3cm} \textbf{newnode} (z)
5. \hspace{3cm} x = \textsc{Extract-Min}(Q) \quad \text{extract two least frequent characters}
6. \hspace{3cm} z.\text{leftchild} = x
7. \hspace{3cm} y = \textsc{Extract-Min}(Q)
8. \hspace{3cm} z.\text{rightchild} = y
9. \hspace{3cm} f(z) = f(x) + f(y)
9. \hspace{3cm} \textsc{Insert}(Q, z)
9. \hspace{1cm} \textbf{return } (\textsc{Extract-Min}(Q))
Chapter 16. Greedy Algorithms

An example of Huffman algorithm

- Symbols: A, B, C, D, E, F, G
  freq. : 2, 3, 5, 8, 13, 15, 18

- Huffman codes:
  A: 10100     B: 10101     C: 1011
  D: 100     E: 00     F: 01
  G: 11

A Huffman code Tree
Correctness of Huffman’s algorithm

Lemma 16.2
(Greedy choice property)

Let $C$ be an alphabet and $f$ be the frequency function for characters in $C$. Let $x$ and $y$ be two characters in $C$ having the lowest frequencies. Then there exists an optimal prefix code for $C$ in which the codewords for $x$ and $y$ have the same length and differ in the last bit.

Proof:
Assume $C$, $f$, $x$ and $y$ are as described in the lemma condition. Let $T$ be any optimal prefix code for $C$.

(a) If $T$ contains the situation about $x$ and $y$ stated in the lemma, done for the proof!

(b) Let $a, b \in C$ be two characters such that their codewords have the same length and differ in the last bit. Also assume that their codeword length $d_T(a) = d_T(b) \geq d_T(c)$, for every $c \in C$. We note that such $a$ and $b$ can be found from $T$ without difficulty.
Chapter 16. Greedy Algorithms

Correctness of Huffman’s algorithm

Lemma 16.2 (Greedy choice property)
Correctness of Huffman’s algorithm

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Chapter 16. Greedy Algorithms

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Chapter 16. Greedy Algorithms

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Chapter 16. Greedy Algorithms

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Chapter 16. Greedy Algorithms

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(a) if $T$ contains the situation about $x$ and $y$ stated in the lemma. Done for the proof!

(b) let $a, b \in C$ be two characters such that their codewords have the same length and differ in the last bit. Also assume that their codeword length

$$d_T(a) = d_T(b) \geq d_T(c), \text{ for every } c \in C$$

We note that such $a$ and $b$ can be found from $T$ without difficulty.
Chapter 16. Greedy Algorithms

We use the following swapping strategy to obtain another prefix code $T'$ for $C$. Without loss of generality, assume \{a, b\} ∩ \{x, y\} = ∅. The created code $T'$ is the same as $T$ except the positions $a$ and $x$ are swapped in $T'$; so are the positions of $b$ and $y$ swapped in $T'$. We have

$$B(T) - B(T') = \sum_{c \in C} f(c) d_T(c) - \sum_{c \in C} f(c) d_{T'}(c) = f(a) d_T(a) + f(b) d_T(b) + f(x) d_T(x) + f(y) d_T(y) - f(a) d_{T'}(a) - f(b) d_{T'}(b) - f(x) d_{T'}(x) - f(y) d_{T'}(y) = (f(a) - f(x)) d_T(a) + (f(x) - f(a)) d_T(x) + (f(b) - f(y)) d_T(b) + (f(y) - f(b)) d_T(y) \geq 0$$

which implies $B(T) \geq B(T')$. So $T'$ is also an optimal code for $C$. So $T'$ satisfies the lemma.
We use the following swapping strategy to obtain another prefix code \( T' \) for \( C \).

Without loss of generality, assume \( \{a, b\} \cap \{x, y\} = \phi \).
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We use the following swapping strategy to obtain another prefix code $T'$ for $C$.

Without loss of generality, assume $\{a, b\} \cap \{x, y\} = \emptyset$.

The created code $T'$ is the same as $T$ except the positions $a$ and $x$ are swapped in $T'$; so are the positions of $b$ and $y$ swapped in $T'$.

$$B(T) - B(T') = \sum_{c \in C} f(c)d_T(c) - \sum_{c \in C} f(c)d_{T'}(c)$$

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$$B(T) - B(T') = \sum_{c \in C} f(c)d_T(c) - \sum_{c \in C} f(c)d_{T'}(c)$$

$$= f(a)d_T(a) + f(b)d_T(b) + f(x)d_T(x) + f(y)d_T(y)$$

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Chapter 16. Greedy Algorithms

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\[
B(T) - B(T') = \sum_{c \in C} f(c)d_T(c) - \sum_{c \in C} f(c)d_{T'}(c)
\]

\[
= f(a)d_T(a) + f(b)d_T(b) + f(x)d_T(x) + f(y)d_T(y)
- f(a)d_{T'}(a) - f(b)d_{T'}(b) - f(x)d_{T'}(x) - f(y)d_{T'}(y)
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$$= f(a)d_T(a) + f(b)d_T(b) + f(x)d_T(x) + f(y)d_T(y)$$
$$- f(a)d_{T'}(a) - f(b)d_{T'}(b) - f(x)d_{T'}(x) - f(y)d_{T'}(y)$$
$$= (f(a) - f(x))d_T(a) + (f(x) - f(a))d_T(x) + (f(b) - f(y))d_T(b)$$
$$+ (f(y) - f(b))d_T(y)$$

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= f(a)d_T(a) + f(b)d_T(b) + f(x)d_T(x) + f(y)d_T(y) - f(a)d_{T'}(a) - f(b)d_{T'}(b) - f(x)d_{T'}(x) - f(y)d_{T'}(y)
\]

\[
= (f(a) - f(x))d_T(a) + (f(x) - f(a))d_T(x) + (f(b) - f(y))d_T(b) + (f(y) - f(b))d_T(y)
\]

\[
= (f(a) - f(x))(d_T(a) - d_T(x)) + (f(b) - f(y))(d_T(b) - d_T(y)) \geq 0
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- f(a)d_{T'}(a) - f(b)d_{T'}(b) - f(x)d_{T'}(x) - f(y)d_{T'}(y)
\]

\[
= (f(a) - f(x))d_T(a) + (f(x) - f(a))d_T(x) + (f(b) - f(y))d_T(b)
+ (f(y) - f(b))d_T(y)
\]

\[
= (f(a) - f(x))(d_T(a) - d_T(x)) + (f(b) - f(y))(d_T(b) - d_T(y)) \geq 0
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$$= f(a) d_T(a) + f(b) d_T(b) + f(x) d_T(x) + f(y) d_T(y)$$

$$- f(a) d_{T'}(a) - f(b) d_{T'}(b) - f(x) d_{T'}(x) - f(y) d_{T'}(y)$$

$$= (f(a) - f(x)) d_T(a) + (f(x) - f(a)) d_T(x) + (f(b) - f(y)) d_T(b)$$

$$+ (f(y) - f(b)) d_T(y)$$

$$= (f(a) - f(x))(d_T(a) - d_T(x)) + (f(b) - f(y))(d_T(b) - d_T(y)) \geq 0$$

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So $T'$ satisfies the lemma.
Chapter 16. Greedy Algorithms

Lemma 16.3 (Optimal substructure)
Chapter 16. Greedy Algorithms

Lemma 16.3 (Optimal substructure)

Let $C$ be an alphabet and $f$ be the frequency function for characters in $C$. Let $x$ and $y$ be two characters in $C$ having the lowest frequencies.
Lemma 16.3 (Optimal substructure)

Let $C$ be an alphabet and $f$ be the frequency function for characters in $C$. Let $x$ and $y$ be two characters in $C$ having the lowest frequencies. Let

$$C' = C - \{x, y\} \cup \{z\}$$

and the frequency function for $C'$ is the same as $f$ except that $f(z) = f(x) + f(y)$. 

Proof:
The code $T$ for $C$ constructed from $T'$, a code for $C'$, has the following objective function value

$$B(T) = \sum_{c \in C} f(c) d_T(c) = \sum_{c \in C' - \{x, y\}} f(c) d_{T'}(c) + (f(x) + f(y)) d_{T'}(z) + 1 \leq \sum_{c \in C' - \{x, y\}} f(c) d_{T'}(c) + (f(x) + f(y)) d_{T'}(z) + f(x) + f(y) = B(T') + f(x) + f(y)$$

Therefore, $T$ is an optimal prefix code tree for $C$. 


Lemma 16.3 (Optimal substructure)

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Let $T'$ be any an optimal prefix code tree for $C'$. Then the tree $T$ obtained from $T'$ by replacing the leaf node $z$ with an internal node having $x$ and $y$ as children, is an optimal prefix code tree for $C$.
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Chapter 16. Greedy Algorithms

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Proof:
The code \( T \) for \( C \) constructed from \( T' \), a code for \( C' \), has the following objective function value

\[
B(T) = \sum_{c \in C} f(c)d_T(c) = \sum_{c \in C - \{x, y\}} f(c)d_T(c) + f(x)d_T(x) + f(y)d_T(y)
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Chapter 16. Greedy Algorithms

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$$= \sum_{c \in C - \{x, y\}} f(c)d_T(c) + (f(x) + f(y))d_T(x)$$
$$= \sum_{c \in C - \{x, y\}} f(c)d_{T'}(c) + (f(x) + f(y))(d_{T'}(z) + 1)$$
Chapter 16. Greedy Algorithms

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$$ = \sum_{c \in C - \{x, y\}} f(c)d_T(c) + [f(x) + f(y)]d_T(x) $$

$$ = \sum_{c \in C - \{x, y\}} f(c)d_{T'}(c) + [f(x) + f(y)](d_{T'}(z) + 1) $$

$$ = \sum_{c \in C - \{x, y\}} f(c)d_{T'}(c) + [f(x) + f(y)]d_{T'}(z) + [f(x) + f(y)] $$
**Lemma 16.3 (Optimal substructure)**

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The code $T$ for $C$ constructed from $T'$, a code for $C'$, has the following objective function value

$$B(T) = \sum_{c \in C} f(c)d_T(c) = \sum_{c \in C - \{x, y\}} f(c)d_T(c) + f(x)d_T(x) + f(y)d_T(y)$$

$$= \sum_{c \in C - \{x, y\}} f(c)d_T(c) + [f(x) + f(y)]d_T(x)$$

$$= \sum_{c \in C - \{x, y\}} f(c)d_{T'}(c) + [f(x) + f(y)](d_{T'}(z) + 1)$$

$$= \sum_{c \in C - \{x, y\}} f(c)d_{T'}(c) + [f(x) + f(y)]d_{T'}(z) + [f(x) + f(y)]$$

$$= \sum_{c \in C - \{x, y\}} f(c)d_{T'}(c) + f(z)d_{T'}(z) + [f(x) + f(y)]$$
Lemma 16.3 (Optimal substructure)

Let $C$ be an alphabet and $f$ be the frequency function for characters in $C$. Let $x$ and $y$ be two characters in $C$ having the lowest frequencies. Let

$$C' = C - \{x, y\} \cup \{z\}$$

and the frequency function for $C'$ is the same as $f$ except that $f(z) = f(x) + f(y)$.

Let $T'$ be any an optimal prefix code tree for $C'$. Then the tree $T$ obtained from $T'$ by replacing the leaf node $z$ with an internal node having $x$ and $y$ as children, is an optimal prefix code tree for $C$.

Proof:

The code $T$ for $C$ constructed from $T'$, a code for $C'$, has the following objective function value

$$B(T) = \sum_{c \in C} f(c)d_T(c) = \sum_{c \in C-\{x,y\}} f(c)d_T(c) + f(x)d_T(x) + f(y)d_T(y)$$

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Now if $T$ is not optimal for $C$, assume optimal prefix code $S$ for $C$ such that $B(S) < B(T)$.
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Chapter 16. Greedy Algorithms

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We construct another code $S'$ for $C'$ by removing the branches for $x$ and $y$ and assign $z$ to its parent node. We have $B(S') = B(S) - f(x) - f(y)$. But

$$B(S') < B(T) - f(x) - f(y) = B(T') + f(x) + f(y) - f(x) - f(y) = B(T')$$
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Chapter 16. Greedy Algorithms

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Chapter 16. Greedy Algorithms

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So $T$ should be optimal for $C$.

**Theorem 16.4** Huffman’s algorithm produces an optimal prefix code.
Chapter 16. Greedy Algorithms

Coin change problem

Input: money of value \( n \), currency coin denominations \( D = \{ v_1, \ldots, v_m \} \)

Output: the least number of coins with denominations in \( D \) whose values sum up exactly to \( n \).

A mathematical formulation of the problem

Input: positive integer \( n \), and set of positive integers \( D = \{ v_1, \ldots, v_m \} \)

Output: a set of positive integers \( X = \{ x_1, \ldots, x_m \} \) such that

\[
\sum_{i=1}^{m} x_i v_i = n
\]


Chapter 16. Greedy Algorithms

Coin change problem

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Coin change problem

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$$\sum_{i=1}^{m} x_i v_i = n \text{ to minimize } \sum_{i=1}^{m} x_i$$
Chapter 16. Greedy Algorithms

A dynamic programming solution
A dynamic programming solution

analysis:
- for any value $k$, one coin of the $m$ denominations has to be used.
Chapter 16. Greedy Algorithms

A dynamic programming solution

**analysis:**
- for any value $k$, one coin of the $m$ denominations has to be used.
- then for the left value, one coin of the $m$ denominations has to be used
A dynamic programming solution

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  • for any value $k$, one coin of the $m$ denominations has to be used.
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  • ...

\[
c(k) = \begin{cases} 
  c(k - v_1) + 1 & \text{if } k \geq v_1 \\
  c(k - v_2) + 1 & \text{if } k \geq v_2 \\
  \vdots & \text{if } k \geq v_m \\
  1 & \text{otherwise}
\end{cases}
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A dynamic programming solution

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- for any value $k$, one coin of the $m$ denominations has to be used.
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Chapter 16. Greedy Algorithms

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Chapter 16. Greedy Algorithms

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Chapter 16. Greedy Algorithms

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Chapter 16. Greedy Algorithms

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- base case: \( c(v_1) = 1 \),
Chapter 16. Greedy Algorithms

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- base case: $c(v_1) = 1, \ldots, c(v_m) = 1.$
A greedy algorithm

Theorem: For the US coin denominations $D_{us} = \{1, 5, 10, 25\}$, the Coin Change problem has the greedy-choice property. Always use the coin of the largest possible value. But not all coin denominations have such property, e.g., $\{1, 3, 4, 10\}$, for $n = 6$. 
A greedy algorithm

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Chapter 16. Greedy Algorithms

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Chapter 16. Greedy Algorithms

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Proof: Assume $A$ is the optimal solution for $n$. 

• I. For $5 \leq n < 10$, assume $5 \not\in A$, but all coins in $A$ have to be one-cent coins. Let $B = A - \{1, 1, 1, 1, 1\} \cup \{5\}$. $|B| < |A|$, which is impossible. So $5 \in A$.

• II. For $10 \leq n < 25$, assume $10 \not\in A$, but all coins in $A$ are either five-cent or one-cent coins.
  (1) all coins in $A$ are one-cent coins, let $B = A - \{1, 1, 1, 1, 1, 1, 1, 1, 1\} \cup \{10\}$, then $|B| < |A|$, which is impossible. So $10 \in A$;
  (2) there is one five-cent coin and some one-cent coins in $A$, let $B = A - \{1, 1, 1, 1, 5\} \cup \{10\}$, $|B| < |A|$, which is impossible. So $10 \in A$;
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Chapter 16. Greedy Algorithms

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Chapter 16. Greedy Algorithms

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Chapter 16. Greedy Algorithms

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Chapter 16. Greedy Algorithms

**Proof**: Assume $A$ is the optimal solution for $n$. Consider various cases

- **I.** For $5 \leq n < 10$, assume $5 \notin A$, but all coins in $A$ have to be one-cent coins. Let $B = A - \{1, 1, 1, 1, 1\} \cup \{5\}$. $|B| < |A|$, which is impossible. So $5 \in A$.

- **II.** For $10 \leq n < 25$, assume $10 \notin A$, but all coins in $A$ are either five-cent or one-cent coins.

  (1) all coins in $A$ are one-cent coins, let $B = A - \{1, 1, 1, 1, 1, 1, 1, 1\} \cup \{10\}$, then $|B| < |A|$, which is impossible. So $10 \in A$;

  (2) there is one five-cent coin and some one-cent coins in $A$, let $B = A - \{1, 1, 1, 1, 5\} \cup \{10\}$, $|B| < |A|$, which is impossible. So $10 \in A$;
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  (2) there is one five-cent coin and some one-cent coins in $A$, let $B = A - \{1, 1, 1, 1, 1, 5\} \cup \{10\}$, $|B| < |A|$, which is impossible. So $10 \in A$;

  (3) there is at least two five-cent coins in $A$, so $B = A - \{5, 5\} \cup \{10\}$,
Chapter 16. Greedy Algorithms

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Chapter 16. Greedy Algorithms

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Chapter 16. Greedy Algorithms

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Chapter 16. Greedy Algorithms

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Chapter 16. Greedy Algorithms

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3. there is at least one ten-cent coin in $A$. Let $m = n - 10$. 
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Chapter 16. Greedy Algorithms

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Chapter 16. Greedy Algorithms

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• If \( 10 \leq m < 25 \), according to II, 10 is in the solution for \( m \). So we conclude that there are at least two ten-cent coins are in \( A \).
Chapter 16. Greedy Algorithms

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Chapter 16. Greedy Algorithms

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Chapter 16. Greedy Algorithms

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  If $10 \leq k < 25$, according to II, 10 is in the solution for $k$, then $\{10, 10, 10\} \subseteq A$, 

Chapter 16. Greedy Algorithms

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Chapter 16. Greedy Algorithms

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Review for Parts I, II, and III

Summaries for Parts I, II, and III
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Summaries for Parts I, II, and III

Part I: Fundamentals of Analysis of algorithms
Summaries for Parts I, II, and III

Part I: Fundamentals of Analysis of algorithms

- time/space complexity functions
Review for Parts I, II, and III

Summaries for Parts I, II, and III

Part I: Fundamentals of Analysis of algorithms

• time/space complexity functions
  input size
Review for Parts I, II, and III

Summaries for Parts I, II, and III

Part I: Fundamentals of Analysis of algorithms

- time/space complexity functions
  - input size
  - upper bound
Review for Parts I, II, and III

Summaries for Parts I, II, and III

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Review for Parts I, II, and III

Summaries for Parts I, II, and III

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Review for Parts I, II, and III

Summaries for Parts I, II, and III

Part I: Fundamentals of Analysis of algorithms

- time/space complexity functions
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  - big-$O$
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Part II: Sorting and order statistics
Review for Parts I, II, and III

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- sorting algorithms: heap, insertion, merge, quick
Review for Parts I, II, and III

Part II: Sorting and order statistics

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Review for Parts I, II, and III

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Review for Parts I, II, and III

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Review for Parts I, II, and III

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- linear time to find $k$th smallest element
Review for Parts I, II, and III

Part III. Exhaustive search, DP and greedy algorithms
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- exhaustive search by simple enumeration of all possible solutions
  - search tree method
  - resulting in $T(n) = T(n-1) + T(n-1-m)$, $m \geq 1$.
- dynamic programming
  - recursive solution (with optimal substructure)
  - recurrence for numerical objective function
  - iterative, bottom-up table-filling
  - traceback of solution.
  - 'forward DP', 'backward DP', and 'inside-out DP'
- greedy algorithm
  - DP solution
  - proof of greedy-choice property (via swapping method)
  - efficient algorithms
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  recursive solution (with optimal substructure)
  recurrence for numerical objective function
  iterative, bottom-up table-filling
  traceback of solution.

‘forward DP’, ‘backward DP’, and ‘inside-out DP’
Part III. Exhaustive search, DP and greedy algorithms

- exhaustive search by
  simple enumeration of all possible solutions
  search tree method resulting in \( T(n) = T(n - 1) + T(n - 1 - m) \), \( m \geq 1 \).

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  ‘forward DP’, ’backward DP’, and ’inside-out DP’

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  DP solution
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  efficient algorithms