Part IV. Advanced Design and Analysis Techniques

- Chapter 14.9 Exhaustive search
- Chapter 15. Dynamic programming
- Chapter 16. Greedy algorithms
Chapter 14.9. Exhaustive Search (not in the text!)
Chapter 14.9. Exhaustive Search

To enumerate all possible solutions to the problem instance

(not in the text!)
Chapter 14.9. Exhaustive Search

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How?
Chapter 14.9. Exhaustive Search

To enumerate all possible solutions to the problem instance

How?

- systematic examining all solutions
Chapter 14.9. Exhaustive Search

To enumerate all possible solutions to the problem instance

How?

• systematic examining all solutions

• without repeating solutions that have been examined
Chapter 14.9. Exhaustive Search

To enumerate all possible solutions to the problem instance

How?

• systematic examining all solutions
• without repeating solutions that have been examined
• stop when a satisfactory solution is found

(not in the text!)
Chapter 14.9. Exhaustive Search

First, we need to be able to count total number of “things” to be enumerated.
Chapter 14.9. Exhaustive Search

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Chapter 14.9. Exhaustive Search

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- without missing one (correctness)
- without over-counting (efficiency)
- A sophisticated counting often has recursive solution.
Examples of counting:

(1) total number of permutations of \(1, 2, \ldots, n\) is
\[P(n) = n \times P(n-1)\]
with base case \(P(1) = 1\).

\[P(n) = n \times (n-1) \times P(n-2) = n \times (n-1) \times \ldots \times 2 \times 1 = n!\]

(2) total number of ways to choose \(k\) from \(n\) items is
\[\binom{n}{k} = \frac{n!}{(n-k)!k!}\]
or, alternatively,
\[\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}\]
with base cases: ?
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Chapter 14.9. Exhaustive Search

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(n \choose k)
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Chapter 14.9. Exhaustive Search

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Chapter 14.9. Exhaustive Search

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Chapter 14.9. Exhaustive Search

Example: Boolean Formula Satisfiability problem (SAT)

Input: boolean formula $f(x_1, x_2, \ldots, x_n)$,
Output: "yes" if and only if $f(x_1, x_2, \ldots, x_n)$ is satisfiable.

$f(x_1, x_2, \ldots, x_n)$ is satisfiable if there is an assignment to boolean variables $x_i \in \{T, F\}, i = 1, 2, \ldots, n$, such that $f$ is evaluated to $T$.

E.g., $f(x_1, x_2, x_3) = (x_1 \lor x_2 \lor \neg x_3) \land (\neg x_1 \lor x_3)$ is satisfiable.

$g(x_1, x_2) = (x_1 \lor \neg x_2) \land (\neg x_1 \lor x_2) \land (x_1 \lor x_2) \land (\neg x_1 \lor \neg x_2)$ is not!
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Use exhaustive search to solve the SAT problem.
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Chapter 14.9. Exhaustive Search

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How?
Chapter 14.9. Exhaustive Search

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How? What will you exhaustively search on?
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How? What will you exhaustively search on?

- **Enumerate all combinations of T and F for** $x_1, \ldots, x_n$. 
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**How? What will you exhaustively search on?**

- **Enumerate all combinations of T and F for $x_1, \ldots, x_n$.**

- Can you solve it with a recursive algorithm?
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How? What will you exhaustively search on?

- **Enumerate all combinations of T and F for \( x_1, \ldots, x_n \).**
- Can you solve it with a recursive algorithm?
- Can you solve it with an iterative algorithm?
Chapter 14.9. Exhaustive Search

Solve SAT problem with a recursive algorithm:

- what data will the recursion be applied to? boolean formula $f(x_1, \ldots, x_n)$
- what is the terminating (base) case? $n=0$, formula without variables
- what is the recursive case?

$$f(x_1, \ldots, x_{n-1}, x_n) = f(x_1, \ldots, x_{n-1}, T) \lor f(x_1, \ldots, x_{n-1}, F)$$

$$f(x_1, \ldots, x_{n-1}, T) = \Rightarrow g(x_1, \ldots, x_{n-1})$$

$$f(x_1, \ldots, x_{n-1}, F) = \Rightarrow h(x_1, \ldots, x_{n-1})$$
Chapter 14.9. Exhaustive Search

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Chapter 14.9. Exhaustive Search

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Chapter 14.9. Exhaustive Search

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Chapter 14.9. Exhaustive Search

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  \[ f(x_1, \ldots, x_n) \]

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  \[ n=0, \text{ formula without variables} \]

- what is the recursive case?
  \[ f(x_1, \ldots, x_{n-1}, x_n) = \]
  \[ \neg f(x_1, \ldots, x_{n-1}, x_n) \]
Chapter 14.9. Exhaustive Search

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Chapter 14.9. Exhaustive Search

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Chapter 14.9. Exhaustive Search

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Chapter 14.9. Exhaustive Search

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Chapter 14.9. Exhaustive Search

Algorithm SAT Solver \( f(x_1, \ldots, x_{n-1}, x_n) \)
Chapter 14.9. Exhaustive Search

Algorithm \textsc{SAT Solver}(f(x_1, \ldots, x_{n-1}, x_n))

1. \textbf{if } n = 0, \textbf{return} (f);
Chapter 14.9. Exhaustive Search

Algorithm $\text{SAT Solver}(f(x_1, \ldots, x_{n-1}, x_n))$

1. if $n = 0$, return $(f)$;
2. else $g(x_1, \ldots, x_{n-1}) = f(x_1, \ldots, x_{n-1}, T)$
Chapter 14.9. Exhaustive Search

Algorithm SAT SOLVER\((f(x_1, \ldots, x_{n-1}, x_n))\)

1. if \( n = 0 \), return \( f \);
2. else \( g(x_1, \ldots, x_{n-1}) = f(x_1, \ldots, x_{n-1}, \top) \)
3. \( h(x_1, \ldots, x_{n-1}) = f(x_1, \ldots, x_{n-1}, \bot) \)
Algorithm \textsc{SAT Solver}(f(x_1, \ldots, x_{n-1}, x_n))

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3. \hspace{1em} $h(x_1, \ldots, x_{n-1}) = f(x_1, \ldots, x_{n-1}, \text{F})$
4. \textbf{return} (\textsc{SAT Solver}(g(x_1, \ldots, x_{n-1})) \lor \textsc{SAT Solver}(h(x_1, \ldots, x_{n-1})))
Chapter 14.9. Exhaustive Search

Algorithm $\text{SAT Solver}(f(x_1, \ldots, x_{n-1}, x_n))$

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4. return $(\text{SAT Solver}(g(x_1, \ldots, x_{n-1})) \lor$
   \hspace{1cm} SAT Solver$(h(x_1, \ldots, x_{n-1})))$

Does this algorithm exhaustively search all assignments to the variables?
Chapter 14.9. Exhaustive Search

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- draw a search tree based on the algorithm.
Chapter 14.9. Exhaustive Search

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4. return \( (\text{SAT Solver}(g(x_1, \ldots, x_{n-1})) \lor \text{SAT Solver}(h(x_1, \ldots, x_{n-1}))) \)

Does this algorithm exhaustively search all assignments to the variables?

- draw a search tree based on the algorithm.
- what does the tree look like?
Chapter 14.9. Exhaustive Search

Algorithm SAT Solver($f(x_1, \ldots, x_{n-1}, x_n)$)

1. if $n = 0$, return ($f$);
2. else $g(x_1, \ldots, x_{n-1}) = f(x_1, \ldots, x_{n-1}, T)$
3. $h(x_1, \ldots, x_{n-1}) = f(x_1, \ldots, x_{n-1}, F)$
4. return (SAT Solver($g(x_1, \ldots, x_{n-1})$) $\lor$ SAT Solver($h(x_1, \ldots, x_{n-1})$))

Does this algorithm exhaustively search all assignments to the variables?

- draw a search tree based on the algorithm.
- what does the tree look like?
- what does each path mean?
Chapter 14.9. Exhaustive Search

Algorithm $\text{SAT Solver}(f(x_1, \ldots, x_{n-1}, x_n))$

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Does this algorithm exhaustively search all assignments to the variables?

- draw a search tree based on the algorithm.
- what does the tree look like?
- what does each path mean? how many paths?

$T(n) = 2T(n-1) + cn$, $T(0) = c$ $\Rightarrow T(n) = \Theta(2^n)$
Chapter 14.9. Exhaustive Search

Algorithm SAT Solver($f(x_1, \ldots, x_{n-1}, x_n)$)

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Does this algorithm exhaustively search all assignments to the variables?

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Chapter 14.9. Exhaustive Search

Solve SAT problem with iterative algorithms

• How? what to iterate on?
• Assignments
  \[ x_1 = F, x_2 = F, \ldots, x_n = F, \]
  or simply \((F, F, \ldots, F)\)
• What to increment?
  \((\ldots, F, T, \ldots, T)\) \rightarrow \((\ldots, T, F, \ldots, F)\)
  always flip the last bit.
Chapter 14.9. Exhaustive Search

Solve SAT problem with iterative algorithms

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Solve SAT problem with iterative algorithms

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Chapter 14.9. Exhaustive Search

Solve SAT problem with iterative algorithms

- How? what to iterate on?
  - assignments
- what is the initial value?
Chapter 14.9. Exhaustive Search

Solve SAT problem with iterative algorithms

- How? what to iterate on?
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  \[ x_1 = \text{F}, \, x_2 = \text{F}, \ldots, \, x_n = \text{F}, \]
Chapter 14.9. Exhaustive Search

Solve SAT problem with iterative algorithms

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Chapter 14.9. Exhaustive Search

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  \[(\ldots,F,T,\ldots,T)\]
Chapter 14.9. Exhaustive Search

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Chapter 14.9. Exhaustive Search

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  \[ (\ldots, F, T, \ldots, T) \rightarrow (\ldots, T, F, \ldots, F) \]
  always flip the last bit.
Algorithm SAT Solver-Enum

1. for \( \langle x_1, \ldots, x_n \rangle = \langle F, \ldots, F \rangle \) to \( \langle T, \ldots, T \rangle \)
2. \( V = \text{Evaluate}(f, x_1, \ldots, x_n) \)
3. if \( V = T \), return \( (T) \)
4. return \( (F) \)

• for loop can be implemented by encoding vectors \( \langle F, \ldots, F \rangle, \ldots, \langle T, \ldots, T \rangle \) with binary numbers then further with integers
• a decoding process is needed to converting integers back to vectors
Chapter 14.9. Exhaustive Search

Algorithm SAT Solver-Enum(\(f(x_1, \ldots, x_{n-1}, x_n)\))
Algorithm SAT SOLVER-ENUM\(f(x_1, \ldots, x_{n-1}, x_n)\)

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Chapter 14.9. Exhaustive Search

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Chapter 14.9. Exhaustive Search

Algorithm $SAT\ Solver-Enum(f(x_1, \ldots, x_{n-1}, x_n))$

1. \textbf{for} $\langle x_1, \ldots, x_n \rangle = \langle F, \ldots, F \rangle$ \textbf{to} $\langle T, \ldots, T \rangle$
2. \hspace{1em} $V = Evaluate(f, x_1, \ldots, x_n)$
3. \hspace{1em} \textbf{if} $V = T$, \textbf{return} (T)
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Chapter 14.9. Exhaustive Search

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Chapter 14.9. Exhaustive Search

Iterative exhaustive search seems to be more convenient.

Another example: Travel Salesman Problem (TSP)

Related problem: Hamiltonian Cycle

Input: a graph $G = (V,E)$

Output: yes if and only if $G$ contains a Hamiltonian cycle (Hamiltonian path is a cycle going through every vertex exactly once.)

How to enumerate all cycles and validate?

• enumerate all permutations of $(1,2,...,n)$

• how to encode these permutations as integers?
Chapter 14.9. Exhaustive Search

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Chapter 14.9. Exhaustive Search

Exhaustive search could be non-trivial

Maximum Independent Set

Input: a graph $G = (V,E)$

Output: a subset $I \subseteq V$ such that

1. $\forall u,v \in I, (u,v) \not\in E$,
2. $|I|$ is the maximum.

- trivial exhaustive search: check every subset of $V$ and verify
- non-trivial: use a search tree, achieving a better time upper bound, taking advantage of the independent set
Chapter 14.9. Exhaustive Search

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Chapter 14.9. Exhaustive Search

The algorithm follows a logical search tree
Chapter 14.9. Exhaustive Search

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- given a graph $G$, it picks an arbitrary vertex $v$ from $G$;
Chapter 14.9. Exhaustive Search

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- given a graph $G$, it picks an arbitrary vertex $v$ from $G$;
- exhaustively, there are two cases to consider:
  - $G_1$ is the result of $G$ after $v$ and all its neighbors are removed;
  - $G_2$ is the result of $G$ after $v$ is removed.

The algorithm terminates when the considered graph is empty.
Chapter 14.9. Exhaustive Search

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- given a graph $G$, it picks an arbitrary vertex $v$ from $G$;
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Chapter 14.9. Exhaustive Search

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Chapter 14.9. Exhaustive Search

Algorithm $\text{MaxIndSet} (G)$
Chapter 14.9. Exhaustive Search

Algorithm \texttt{MaxIndSet} \( (G) \)

1. \textbf{if} \( G = \emptyset \) \textbf{return} \( (\emptyset) \)
Algorithm \textsc{MaxIndSet} \((G)\)

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3. \textbf{let} \(G_1\) be \(G\) with \(v\) and all its neighbors removed

\(\star\) the algorithm is a search tree

\(\star\) the time complexity:

\[ T(|G|) = cn^2 + T(|G_1|) + T(|G_2|) \]

\[ T(n) = T(n-1-m) + T(n-1) + cn^2 \]

where \(m\) is the number of neighbors of \(v\)’s
Algorithm $\text{MaxIndSet} (G)$

1. if $G = \emptyset$ return $(\emptyset)$
2. else pick an arbitrary vertex $v$ in $G$
3. let $G_1$ be $G$ with $v$ and all its neighbors removed
4. $I_1 = \{v\} \cup \text{MaxIndSet} (G_1)$
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Chapter 14.9. Exhaustive Search

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5. let \(G_2\) be \(G\) with \(v\) removed
6. \(I_2 = \text{MaxIndSet} (G_2)\)
7. if \(|I_1| \geq |I_2|\) return \((I_1)\)
Chapter 14.9. Exhaustive Search

Algorithm \text{MaxIndSet} (G)

1. \textbf{if} \ G = \emptyset \textbf{ return} (\emptyset)
2. \textbf{else} pick an arbitrary vertex \( v \) in \( G \)
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5. \hspace{1em} let \( G_2 \) be \( G \) with \( v \) removed
6. \hspace{1em} \( I_2 = \text{MaxIndSet} (G_2) \)
7. \hspace{1em} \textbf{if} \ |I_1| \geq |I_2| \textbf{ return} (I_1)
8. \hspace{1em} \textbf{else return} (I_2)
Chapter 14.9. Exhaustive Search

Algorithm $\text{MaxIndSet} \ (G)$

1. if $G = \emptyset$ return $(\emptyset)$
2. else pick an arbitrary vertex $v$ in $G$
3. let $G_1$ be $G$ with $v$ and all its neighbors removed
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5. let $G_2$ be $G$ with $v$ removed
6. $I_2 = \text{MaxIndSet} \ (G_2)$
7. if $|I_1| \geq |I_2|$ return $(I_1)$
8. else return $(I_2)$

- the algorithm is a search tree
Chapter 14.9. Exhaustive Search

Algorithm \texttt{MaxIndSet} \((G)\)

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- the algorithm is a search tree
- the time complexity:
Chapter 14.9. Exhaustive Search

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3. \quad \text{let} \ G_1 \ \text{be} \ G \ \text{with} \ v \ \text{and all its neighbors removed}
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5. \quad \text{let} \ G_2 \ \text{be} \ G \ \text{with} \ v \ \text{removed}
6. \quad I_2 = \text{MaxIndSet} \ (G_2)
7. \quad \textbf{if} \ |I_1| \geq |I_2| \ \textbf{return} \ (I_1)
8. \quad \textbf{else} \ \textbf{return} \ (I_2)

\begin{itemize}
\item the algorithm is a search tree
\item the time complexity: \ T(|G|) = cn^2 + T(|G_1|) + T(|G_2|)
\end{itemize}
Chapter 14.9. Exhaustive Search

Algorithm MaxIndSet \((G)\)

1. \textbf{if} \(G = \emptyset\) \textbf{return} \((\emptyset)\)
2. \textbf{else} pick an arbitrary vertex \(v\) in \(G\)
3. let \(G_1\) be \(G\) with \(v\) and all its neighbors removed
4. \(I_1 = \{v\} \cup \text{MaxIndSet} (G_1)\)
5. let \(G_2\) be \(G\) with \(v\) removed
6. \(I_2 = \text{MaxIndSet} (G_2)\)
7. \textbf{if} \(|I_1| \geq |I_2|\) \textbf{return} \((I_1)\)
8. \textbf{else return} \((I_2)\)

- the algorithm is a search tree
- the time complexity: \(T(|G|) = cn^2 + T(|G_1|) + T(|G_2|)\)

\[
T(n) = T(n - 1 - m) + T(n - 1) + cn^2
\]

where \(m\) is the number of neighbors of \(v\)'s
Chapter 14.9. Exhaustive Search

\[ T(n) = T(n - 1 - m) + T(n - 1) + cn^2 \]
Chapter 14.9. Exhaustive Search

\[ T(n) = T(n - 1 - m) + T(n - 1) + cn^2 \]

- \( m \geq 0 \),
Chapter 14.9. Exhaustive Search

\[ T(n) = T(n - 1 - m) + T(n - 1) + cn^2 \]

- \( m \geq 0, \ T(n) \leq T(n - 1) + T(n - 1) + cn^2 \),
Chapter 14.9. Exhaustive Search

\[ T(n) = T(n - 1 - m) + T(n - 1) + cn^2 \]

- \( m \geq 0, T(n) \leq T(n - 1) + T(n - 1) + cn^2, \implies T(n) = O(2^n) \)
Chapter 14.9. Exhaustive Search

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- Can we guarantee \( m \geq 1 \)
Chapter 14.9. Exhaustive Search

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- \( m \geq 0, \ T(n) \leq T(n - 1) + T(n - 1) + cn^2, \implies T(n) = O(2^n) \)
- Can we guarantee \( m \geq 1 \) so we have
  \[ T(n) \leq T(n - 2) + T(n - 1) + cn^2, \]

Or even better, to guarantee \( m \geq 2 \) if we can, \( T(n) = O(1.5n) \).

Use the substitution method to prove \( T(n) = O(1.5n) \).
Chapter 14.9. Exhaustive Search

\[ T(n) = T(n - 1 - m) + T(n - 1) + cn^2 \]

- \( m \geq 0, \ T(n) \leq T(n - 1) + T(n - 1) + cn^2 \), \( \implies \ T(n) = O(2^n) \)
- Can we guarantee \( m \geq 1 \) so we have
  \[ T(n) \leq T(n - 2) + T(n - 1) + cn^2, \implies \ T(n) = O(1.6181^n) \]
Chapter 14.9. Exhaustive Search

\[ T(n) = T(n - 1 - m) + T(n - 1) + cn^2 \]

- \( m \geq 0, T(n) \leq T(n - 1) + T(n - 1) + cn^2, \Rightarrow T(n) = O(2^n) \)

- Can we guarantee \( m \geq 1 \) so we have
  \[ T(n) \leq T(n - 2) + T(n - 1) + cn^2, \Rightarrow T(n) = O(1.6181^n) \]

- Or even better, to guarantee \( m \geq 2? \)
Chapter 14.9. Exhaustive Search

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Chapter 14.9. Exhaustive Search

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- Or even better, to guarantee \( m \geq 2 \)? if we can, \( T(n) \leq T(n - 3) + T(n - 1) + cn^2, \implies T(n) = O(1.5^n) \)
Chapter 14.9. Exhaustive Search

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Chapter 14.9. Exhaustive Search

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- Can we guarantee \( m \geq 3 \)?
Chapter 14.9. Exhaustive Search

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- Can we guarantee \( m \geq 3 \)? possible but a little more complicated.
Chapter 14.9. Exhaustive Search

Let $T(n) \leq T(n - 3) + T(n - 1) + cn^2$, with $T(1) = O(1)$
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**Proof (use the substitution method)**
Chapter 14.9. Exhaustive Search

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**Proof** (use the substitution method)

Assume that $T(k) \leq 1.5^k$ for all $k < n$. 

Chapter 14.9. Exhaustive Search

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Assume that \( T(k) \leq 1.5^k \) for all \( k < n \). Then

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Assume that $T(k) \leq 1.5^k$ for all $k < n$. Then

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Chapter 14.9. Exhaustive Search

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when \( n > n_0 \) (\( n_0 \) to be determined)
Chapter 14.9. Exhaustive Search

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Chapter 14.9. Exhaustive Search

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Chapter 14.9. Exhaustive Search

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Chapter 14.9. Exhaustive Search

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Chapter 14.9. Exhaustive Search

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Chapter 14.9. Exhaustive Search

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Now we decide $n_0$:

$$\frac{n^2}{1.5^{n-3}} \leq 0.1$$
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Chapter 14.9. Exhaustive Search

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Now we decide \( n_0 \):

\[
\frac{n^2}{1.5^{n-3}} \leq 0.1 \implies n^2 \leq 0.1 \times 1.5^{n-3} \text{ holds when roughly } n \geq n_0 = 29
\]
Chapter 14.9. Exhaustive Search

- Algorithms for SAT and \texttt{MAXINDSET} run in exponential time $O(2^n)$ or $O(\gamma^n)$ for $1 < \gamma < 2$
Chapter 14.9. Exhaustive Search

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Chapter 14.9. Exhaustive Search

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Chapter 14.9. Exhaustive Search

- Algorithms for SAT and \textsc{MaxIndSet} run in exponential time $O(2^n)$ or $O(\gamma^n)$ for $1 < \gamma < 2$
- search tree (solution search space) is large, \textit{inherently} large
- search tree does not have obvious overlapping subproblems, which otherwise would incur dynamic programming approaches.
Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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there are overlapping subproblems
bottom-up approaches avoid re-computation for subproblems
Chapter 15. Dynamic Programming

An example (dishonest casino)
Chapter 15. Dynamic Programming

An example (*dishonest casino*)

At the casino, there are two types of dice to use:
Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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- **fair die**: \(\text{Prob}(d = F, v = i) = \frac{1}{6}\) for \(i = 1, 2, \ldots, 6\);

- **loaded die**: \(\text{Prob}(d = L, v = 6) = \frac{1}{2}\),
  \(\text{Prob}(d = L, v = i) = \frac{1}{10}\), for \(i = 1, 2, \ldots, 5\).
Chapter 15. Dynamic Programming

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  $$\text{Prob}(d = L, v = i) = \frac{1}{10}, \text{ for } i = 1, 2, \ldots, 5.$$  

- **the house secretly switching dice**:  
  $\text{Prob}(F \Rightarrow L) = 5\%$;  
  $\text{Prob}(L \Rightarrow F) = 10\%$;
Chapter 15. Dynamic Programming

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At the casino, there are two types of dice to use:

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  \hspace{1cm} $\text{Prob}(d = L, v = i) = \frac{1}{10}$, for $i = 1, 2, \ldots, 5$.
- the house **secretly switching dice:**  
  $\text{Prob}(F \Rightarrow L) = 5\%$;  
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Chapter 15. Dynamic Programming

- If three consecutive 6’s are rolled, what is the chance the house used the loaded die?
- Given a long sequence of numbers (1 to 6) rolled, what dice were used and when they were switched?

```
1 3 2 4 6 6 6 4 1 6 5 6 6 2 4 2 1 2 3 5
```

```
F F F F L L L L L L L L L F F F F F F F
```
Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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\[ \ldots 1 \ 3 \ 2 \ 4 \ 6 \ 6 \ 6 \ 4 \ 1 \ 6 \ 5 \ 6 \ 6 \ 2 \ 4 \ 2 \ 1 \ 2 \ 3 \ 5 \ldots \]
Chapter 15. Dynamic Programming

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... 1 3 2 4 6 6 6 4 1 6 5 6 6 2 4 2 1 2 3 5...
... F F F F L L L L L L L L L F F F F F F F ...
Chapter 15. Dynamic Programming

a segment of DNA sequence, is it meaningful?
Chapter 15. Dynamic Programming

if the highlighted segment is not random, it may be a gene.
Chapter 15. Dynamic Programming

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The decoding algorithm for HMM can be accomplished with dynamic programming.
Chapter 15. Dynamic Programming

Will discuss the following examples:

• assembly-line scheduling
• casino dice problem and decoding algorithm
• probabilistic finite automata, Viterbi’s decoding algorithm
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Chapter 15. Dynamic Programming

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  compute table $T[1..n]$ from left to right,  
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Chapter 15. Dynamic Programming

Assembly-line scheduling

• $S_{i,j}$: the $j$th station on line $i$, $i = 1, 2, \ldots, n$.
• $a_{i,j}$ is the assembly time at station $S_{i,j}$
• $t_{i,j}$ is the time for transferring lines from station $S_{i,j}$
• $e_1$ and $e_2$ are entering time costs
• $x_1$ and $x_2$ are exiting time costs

To determine the fastest way through a factory, there are $2^n$ possible ways.
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For general \( j, 1 \leq j \leq n \), the fastest way through \( S_{1,j} \) is faster between

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A dynamic programming approach:

\[ f_i(j) \] to be the minimum time through station \( S_{i,j} \), for \( i = 1, 2 \) and \( j = 1, \ldots, n \).

- the problem is to compute

\[ \min \{ f_1(n) + x_1, f_2(n) + x_2 \} \]

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\[ f_1(1) = e_1 + a_{1,1} \]

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A dynamic programming approach:

step 1: the above analysis
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- where \( f_1() \) and \( f_2() \) are defined recursively:

\[
\begin{align*}
  f_1(j) &= \min\{ f_1(j-1) + a_1, j, f_2(j-1) + t_2, j-1 + a_1, j \} \\
  f_2(j) &= \min\{ f_2(j-1) + a_2, j, f_1(j-1) + t_1, j-1 + a_2, j \}
\end{align*}
\]

- base cases \( f_1(1) = e_1 + a_1, 1 \) and \( f_2(1) = e_2 + a_2, 1 \)
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- base cases \( f_1(1) = e_1 + a_{1,1} \) and \( f_2(1) = e_2 + a_{2,1} \)
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Data dependency:
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**step 3**: compute $f_1(j)$ and $f_2(j)$, for all $j = 1, 2, \ldots, n$. 

• the fastest time is $f^* = \min \{f_1(n) + x_1, f_2(n) + x_2\}$. 
step 3: compute $f_1(j)$ and $f_2(j)$, for all $j = 1, 2, \ldots, n$. 

to fill out a $2 \times n$ table, from left to right

<table>
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<tr>
<th>$f_1(1)$</th>
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• using the recurrences to compute; and
Chapter 15. Dynamic Programming

**step 3**: compute $f_1(j)$ and $f_2(j)$, for all $j = 1, 2, \ldots, n$.

to fill out a $2 \times n$ table, from left to right

\[
\begin{array}{cccccc}
  f_1(1) & f_1(2) & f_1(3) & \ldots & f_1(n) \\
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- the fastest time is $f^* = \min\{f_1(n) + x_1, f_2(n) + x_2\}$
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Example
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Example

Min finish time \( f^* = 36 + 3 = 39 \)

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f_1 )</td>
<td>9</td>
<td>18</td>
<td>20</td>
<td>24</td>
<td>32</td>
<td>36</td>
</tr>
<tr>
<td>( f_2 )</td>
<td>12</td>
<td>16</td>
<td>22</td>
<td>26</td>
<td>31</td>
<td>38</td>
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Algorithm ASSEMBLYLINE\((a, t, e, x, n)\)
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1. \(M[1, 1] = e_1 + a_{1,1}\)
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1. \( M[1, 1] = e_1 + a_{1, 1} \)
2. \( M[2, 1] = e_2 + a_{2, 1} \)
Algorithm \textsc{AssemblyLine}(a, t, e, x, n)

1. \( M[1, 1] = e_1 + a_{1,1} \)
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3. \textbf{for} \( j=2 \) \textbf{to} \( n \)
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3. \( \text{for } j=2 \text{ to } n \)
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5. \hspace{1em} \(M[1, j] = M[1, j - 1] + a_{1,j}\)

\(\text{return } (\min\{f_1(n) + x_1, f_2(n) + x_2\})\)
Algorithm ASSEMBLYLINE$(a, t, e, x, n)$
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2. $M[2, 1] = e_2 + a_{2,1}$
3. for $j=2$ to $n$
4. \hspace{1em} if $M[1, j - 1] + a_{1,j} \leq M[2, j - 1] + t_{2,j-1} + a_{1,j}$
5. \hspace{3em} $M[1, j] = M[1, j - 1] + a_{1,j}$
6. \hspace{3em} $P[1, j] = 1$
Algorithm `ASSEMBLYLINE(a, t, e, x, n)`

1. \( M[1, 1] = e_1 + a_{1,1} \)
2. \( M[2, 1] = e_2 + a_{2,1} \)
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4. \( \text{if } M[1, j - 1] + a_{1,j} \leq M[2, j - 1] + t_{2,j-1} + a_{1,j} \)
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4. \( \text{if } M[1, j - 1] + a_{1,j} \leq M[2, j - 1] + t_{2,j-1} + a_{1,j} \)
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7. \( \text{else } M[1, j] = M[2, j - 1] + t_{2,j-1} + a_{1,j} \)
8. \( P[1, j] = 2 \)
9. \( \text{return } \min\{f_1(n) + x_1, f_2(n) + x_2\} \)
Chapter 15. Dynamic Programming

Algorithm `ASSEMBLYLINE(a, t, e, x, n)`
1. $M[1, 1] = e_1 + a_{1,1}$
2. $M[2, 1] = e_2 + a_{2,1}$
3. for $j=2$ to $n$
4.    if $M[1, j - 1] + a_{1,j} \leq M[2, j - 1] + t_{2,j-1} + a_{1,j}$
5.        $M[1, j] = M[1, j - 1] + a_{1,j}$
6.        $P[1, j] = 1$
7.    else $M[1, j] = M[2, j - 1] + t_{2,j-1} + a_{1,j}$
8.        $P[1, j] = 2$
9.    if $M[2, j - 1] + a_{2,j} \leq M[1, j - 1] + t_{1,j-1} + a_{2,j}$
10.       $P[2, j] = 1$
11. else $M[2, j] = M[2, j - 1] + t_{2,j-1} + a_{2,j}$
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Algorithm \textsc{AssemblyLine}(a, t, e, x, n)
1. $M[1, 1] = e_1 + a_{1,1}$
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3. \textbf{for} \ j=2 \ \textbf{to} \ n
4. \quad \textbf{if} \ M[1, j - 1] + a_{1,j} \leq M[2, j - 1] + t_{2,j-1} + a_{1,j}
5. \quad \quad M[1, j] = M[1, j - 1] + a_{1,j}
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11. \quad \textbf{else} \ M[2, j] = M[2, j - 1] + t_{1,j-1} + a_{2,j}
12. \quad \quad P[2, j] = 1
13. \quad \textbf{end if}
14. \textbf{end for}
15. \textbf{return} (\min \{ f_1(n) + x_1, f_2(n) + x_2 \})
Algorithm \textsc{AssemblyLine}(a, t, e, x, n)
1. \hspace{1em} M[1, 1] = e_1 + a_{1,1}
2. \hspace{1em} M[2, 1] = e_2 + a_{2,1}
3. \hspace{1em} \textbf{for} j=2 \hspace{1em} \textbf{to} \hspace{1em} n
4. \hspace{4em} \textbf{if} M[1, j - 1] + a_{1,j} \leq M[2, j - 1] + t_{2,j-1} + a_{1,j}
5. \hspace{7em} M[1, j] = M[1, j - 1] + a_{1,j}
6. \hspace{7em} P[1, j] = 1
7. \hspace{4em} \textbf{else} M[1, j] = M[2, j - 1] + t_{2,j-1} + a_{1,j}
8. \hspace{7em} P[1, j] = 2
9. \hspace{4em} \textbf{if} M[2, j - 1] + a_{2,j} \leq M[1, j - 1] + t_{1,j-1} + a_{2,j}
10. \hspace{7em} M[2, j] = M[2, j - 1] + a_{2,j}
11. \hspace{7em} P[2, j] = 2
12. \hspace{1em} \textbf{return} \{ \min \{ f_1(n) + x_1, f_2(n) + x_2 \} \}
Algorithm \textsc{AssemblyLine}(a, t, e, x, n)

1. $M[1, 1] = e_1 + a_{1,1}$
2. $M[2, 1] = e_2 + a_{2,1}$
3. \textbf{for } \textit{j}=2 \textbf{ to } n
4. \hspace{1em} \textbf{if } M[1, j-1] + a_{1,j} \leq M[2, j-1] + t_{2,j-1} + a_{1,j}$
5. \hspace{2em} $M[1, j] = M[1, j-1] + a_{1,j}$
6. \hspace{2em} $P[1, j] = 1$
7. \hspace{1em} \textbf{else } $M[1, j] = M[2, j-1] + t_{2,j-1} + a_{1,j}$
8. \hspace{2em} $P[1, j] = 2$
9. \hspace{1em} \textbf{if } M[2, j-1] + a_{2,j} \leq M[1, j-1] + t_{1,j-1} + a_{2,j}$
10. \hspace{2em} $M[2, j] = M[2, j-1] + a_{2,j}$
11. \hspace{2em} $P[2, j] = 2$
12. \hspace{1em} \textbf{else } $M[2, j] = M[1, j-1] + t_{1,j-1} + a_{2,j}$

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Chapter 15. Dynamic Programming

Algorithm ASSEMBLYLINE\((a, t, e, x, n)\)
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2. \(M[2, 1] = e_2 + a_{2,1}\)
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**step 4:** get the fastest path, not just the faster time.
**Chapter 15. Dynamic Programming**

**step 4:** get the fastest path, not just the faster time.

Do we know the fastest path from the fastest time $f^*$?
Chapter 15. Dynamic Programming

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Do we know the fastest path from the fastest time \( f^* \)?

\[
\begin{array}{cccccccc}
  f_1(1) & f_1(2) & f_1(3) & \cdots & f_1(n-1) & f_1(n) \\
  f_2(1) & f_2(2) & f_2(3) & \cdots & f_2(n-1) & f_2(n) \\
\end{array}
\]
Chapter 15. Dynamic Programming

**step 4**: get the fastest path, not just the faster time.

Do we know the fastest path from the fastest time \( f^* \)?

<table>
<thead>
<tr>
<th></th>
<th>( f_1(1) )</th>
<th>( f_1(2) )</th>
<th>( f_1(3) )</th>
<th>( \ldots )</th>
<th>( f_1(n-1) )</th>
<th>( f_1(n) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f_2(1) )</td>
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\[
 f^* = \min\{f_1(n) + x_1, f_2(n) + x_2\} \\
 f_1(j) = \min\{f_1(j-1) + a_{1,j}, f_2(j-1) + t_{2,j-1} + a_{1,j}\} \\
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Algorithm PRINTPATHMAIN($M, P, f^*, x$)
1. if $(f^* - x_1) = M[1, n]$
Algorithm PRINTPATHMAIN\((M, P, f^*, x)\)
1. \textbf{if} \((f^* - x_1) = M[1, n]\)
2. \quad \textbf{i} = 1
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Algorithm PRINTPATHMAIN($M, P, f^*, x$)
1. $\textbf{if } (f^* - x_1) = M[1, n]$
2. $\quad i = 1$
3. $\textbf{else } i = 2$
Algorithm \texttt{PrintPathMain}(M, P, f^*, x)
1. \textbf{if} \ (f^* - x_1) = M[1, n]
2. \textbf{else} \ i = 2
3. \textbf{else} \ i = 2
4. \textbf{PrintPath} (P, i, n)
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Algorithm \textsc{PrintPathMain}(M, P, f^*, x)
1. \textbf{if} \ (f^* - x_1) = M[1, n]
2. \hspace{1em} i = 1
3. \textbf{else} \ i = 2
4. \textbf{PrintPath} \ (P, i, n)

Algorithm \textsc{PrintPath} \ (P, i, j)
1. \textbf{if} \ j \geq 1
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Algorithm PrintPathMain($M, P, f^*, x$)

1. if $(f^* - x_1) = M[1, n]$
2. $i = 1$
3. else $i = 2$
4. PrintPath ($P, i, n$)

Algorithm PrintPath ($P, i, j$)

1. if $j \geq 1$
2. PrintPath ($P, P[i, j], j - 1$)
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Algorithm \texttt{PrintPathMain}(M, P, f^*, x)
1. \textbf{if} \ (f^* - x_1) = M[1, n]
2. \quad i = 1
3. \textbf{else} \ i = 2
4. \texttt{PrintPath} \ (P, i, n)

Algorithm \texttt{PrintPath} \ (P, i, j)
1. \textbf{if} \ j \geq 1
2. \quad \texttt{PrintPath} \ (P, P[i, j], j - 1)
3. \quad \texttt{print} \ (i, j)
Summary on steps for dynamic programming solving the assembly line problem
Chapter 15. Dynamic Programming

Summary on steps for dynamic programming solving the assembly line problem

(1) the structure of optimal solution
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Summary on steps for dynamic programming solving the assembly line problem

(1) the structure of optimal solution

(2) defining optimal cost recursively
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Summary on steps for dynamic programming solving the assembly line problem

(1) the structure of optimal solution
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(3) computing optimal cost (bottom-up approach)
Chapter 15. Dynamic Programming

Summary on steps for dynamic programming solving the assembly line problem

(1) the structure of optimal solution
(2) defining optimal cost recursively
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(4) constructing optimal solution (traceback)
Chapter 15. Dynamic Programming

We consider a little more about the assembly line problem.
We consider a little more about the assembly line problem.

Assume that each station $S_{i,j}$ produces a part $p_k$ with time cost $\tau_{i,j,k}$, where $k = 1, 2, 3$. 
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- can we still compute the fastest time?
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  so can the fastest path and the sequence of parts be computed!
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Chapter 15. Dynamic Programming

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- can we still compute the fastest time? \text{yes.}
  
  so can the fastest path and the sequence of parts be computed!

- \textbf{Given a sequence of parts produced, can we know the path?}
  
  No, but we may predict such a path with a confidence.
Chapter 15. Dynamic Programming

Example-2. Casino dice and decoding algorithm

The house rolls die \( n \) times: e.g., 5, 3, 2, 5, 6, 6, 1, ... , 1

• not honest, switch between dice: fair (F) and loaded (L)
• a small chance to switch is small, a large chance to stay on the same die

Given the sequence of numbers, e.g., 5, 3, 2, 5, 6, 6, 1, ... , 1, what is the sequence of dices used most likely?
Example-2. Casino dice and decoding algorithm

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Given the sequence of numbers, e.g., 5, 3, 2, 5, 6, 6, 1, \ldots, 1, what is the sequence of dices used most likely?
We compute the most probable path of dices to roll a given sequence of numbers $S = d_1 \ldots d_n$, where $d_i \in \{1, 2, \ldots, 6\}$.
Chapter 15. Dynamic Programming

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![Diagram](image)

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As in AssemblyLine problem, we compute the most probable path to roll $d_1 \ldots d_i$:
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As in AssemblyLine problem, we compute the most probable path to roll $d_1 \ldots d_i$:

- $p(S, i, F)$ for which the fair die $F$ is used in the last step
- $p(S, i, L)$ for which the loaded die $L$ is used in the last step
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Both \( p(S, i, F) \) and \( p(S, i, L) \) have recursive solutions.
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$$p(S, i, F) = \max\{p(S, i - 1, F) \times 0.95 \times \frac{1}{6}, \ p(S, i - 1, L) \times 0.1 \times \frac{1}{6}\}$$
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Base cases: when $i =$
Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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• design dynamic programming algorithm to compute $\max \{ P(S, n, F), P(S, n, L) \}$
  given from $d_1 d_2 \ldots d_n$
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- design dynamic programming algorithm to compute \( \max\{P(S, n, F), P(S, n, L)\} \) given from \( d_1d_2\ldots d_n \)

- design a traceback process to print out the sequence of dices used.
Consider a sequence

\[ \ldots 1 \ 3 \ 2 \ 4 \ 6 \ 6 \ 6 \ 4 \ 1 \ 6 \ 5 \ 6 \ 6 \ 2 \ 4 \ 2 \ 1 \ 2 \ 3 \ 5 \ldots \]
Chapter 15. Dynamic Programming

Consider a sequence

\[ \ldots 1 \ 3 \ 2 \ 4 \ 6 \ 6 \ 6 \ 4 \ 1 \ 6 \ 5 \ 6 \ 6 \ 2 \ 4 \ 2 \ 1 \ 2 \ 3 \ 5 \ldots \]
Consider a sequence

... 1 3 2 4 6 6 6 4 1 6 5 6 6 2 4 2 1 2 3 5 ...

By computing the maximum probability to produce the sequence, we decode the dices used

... 1 3 2 4 6 6 6 4 1 6 5 6 6 2 4 2 1 2 3 5 ...
Consider a sequence

... 1 3 2 4 6 6 6 4 1 6 5 6 6 2 4 2 1 2 3 5 ...

By computing the maximum probability to produce the sequence, we decode the dices used

... 1 3 2 4 6 6 6 4 1 6 5 6 6 2 4 2 1 2 3 5 ...

... F F F F L L L L L L L L F F F F F F F ...
Consider a sequence

\[
\ldots \ 1 \ 3 \ 2 \ 4 \ 6 \ 6 \ 6 \ 4 \ 1 \ 6 \ 5 \ 6 \ 6 \ 2 \ 4 \ 2 \ 1 \ 2 \ 3 \ 5 \ldots
\]

By computing the maximum probability to produce the sequence, we decode the dices used

\[
\ldots \ F \ F \ F \ F \ L \ L \ L \ L \ L \ L \ L \ L \ F \ F \ F \ F \ F \ F \ F \ F \ldots
\]

But besides the casino interest, is such algorithm meaningful in other applications?
Consider a sequence
\[ \ldots 1 \ 3 \ 2 \ 4 \ 6 \ 6 \ 6 \ 4 \ 1 \ 6 \ 5 \ 6 \ 6 \ 2 \ 4 \ 2 \ 1 \ 2 \ 3 \ 5 \ldots \]
By computing the maximum probability to produce the sequence, we decode the dices used
\[ \ldots \ F \ F \ F \ F \ L \ L \ L \ L \ L \ L \ L \ F \ F \ F \ F \ F \ F \ F \ldots \]
But besides the casino interest, is such algorithm meaningful in other applications?
The answer is definitely YES!
Chapter 15. Dynamic Programming

a segment of DNA sequence, is it meaningful?
Chapter 15. Dynamic Programming

if the highlighted segment is not random, it may be a gene.
Chapter 15. Dynamic Programming

Technically, the “linear model” is unfolded of a “more condensed model”.

A hidden Markov model (HMM)
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Chapter 15. Dynamic Programming

Knapsack Problem

Input: $n$ items of sizes $s_1, \ldots, s_n$ and values $v_1, \ldots, v_n$; and a knapsack size $B$

Output: a subset $A \subseteq \{1, 2, \ldots, n\}$ that maximizes $\sum_{i \in A} v_i$ with constraint $\sum_{i \in A} s_i \leq B$

Analysis: For $n$ items and knapsack size $B$; examine the last item $n$, then either put item $n$ into the knapsack or discard it; That is (1) either increase value by $v_n$, reduce available space from $B$ to $B - s_n$, reduce the number of items by 1, or (2) or simply reduce the number of items by 1. Both situations reduce the problem size to a subproblem.
Chapter 15. Dynamic Programming

Knapsack Problem

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

In general, let \( \{1, 2, \ldots, k\} \) be the first \( k \) items, and \( X \) be the available space in the knapsack;
In general, let \{1, 2, \ldots, k\} be the first \(k\) items, and \(X\) be the available space in the knapsack;

If we define \(A(k, X)\) to be the subset of items drawn from \{1, 2, \ldots, k\}
such that \(A\) maximizes the total value while fits into the space \(X\)
Chapter 15. Dynamic Programming

In general, let \( \{1, 2, \ldots, k\} \) be the first \( k \) items, and \( X \) be the available space in the knapsack;

If we define \( A(k, X) \) to be the subset of items drawn from \( \{1, 2, \ldots, k\} \) such that \( A \) maximizes the total value while fits into the space \( X \)

Then

\[
A(k, X) = \begin{cases} 
\text{either } & A(k - 1, X - s_k) \cup \{k\} & \text{value gained by } v_k \\
\text{or } & A(k - 1, X) & \text{value not gained}
\end{cases}
\]
Chapter 15. Dynamic Programming

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\]

Define \( V(k, X) \) to be the maximum value of items drawn from \( \{1, 2, \ldots, k\} \) which fit into the space of \( X \), then

\[
V(k, X) = \max \left\{ \begin{array}{c} V(k-1, X - s_k) + v_k & X \geq s_k \\
V(k-1, X) & \end{array} \right. 
\]
Chapter 15. Dynamic Programming

• For every $k \leq n$ and every $X \leq B$.

$$V(k, X) = \max \begin{cases} 
V(k - 1, X - s_k) + v_k & X \geq s_k \\
V(k - 1, X) & 
\end{cases}$$

base case,
For every \( k \leq n \) and every \( X \leq B \).

\[
V(k, X) = \max \begin{cases} 
V(k - 1, X - s_k) + v_k & X \geq s_k \\
V(k - 1, X) & \end{cases}
\]

base case,

\[
V(0, X) = 0,
\]
Chapter 15. Dynamic Programming

• For every \( k \leq n \) and every \( X \leq B \).

\[
V(k, X) = \max \left\{ \begin{array}{ll}
V(k - 1, X - s_k) + v_k & X \geq s_k \\
V(k - 1, X) & \text{otherwise}
\end{array} \right.
\]

base case,

\[
V(0, X) = 0, \quad V(k, 0) = 0
\]
Chapter 15. Dynamic Programming

- For every \( k \leq n \) and every \( X \leq B \).

\[
V(k, X) = \max \left\{ \begin{array}{ll}
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V(k - 1, X) & \text{base case,}
\end{array} \right.
\]

- What does the table look like?

- How to fill it out?

- How to traceback for \( A \), the optimal subset of items?
Chapter 15. Dynamic Programming

- For every $k \leq n$ and every $X \leq B$.

$$V(k, X) = \max \begin{cases} V(k - 1, X - s_k) + v_k & X \geq s_k \\ V(k - 1, X) & \end{cases}$$

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Chapter 15. Dynamic Programming

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• What does the table look like? How to fill it out?

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Chapter 15. Dynamic Programming

Example

The optimal knapsack should contain \{1,2\} = 7

\[
V(k, X) = \max \begin{cases} 
V(k - 1, X - s_k) + v_k & X \geq s_k \\
V(k - 1, X) & \text{otherwise}
\end{cases}
\]
Chapter 15. Dynamic Programming

Example

The optimal knapsack should contain \(\{1,2\} = 7\)

<table>
<thead>
<tr>
<th>(i)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td>7</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>7</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>7</td>
</tr>
</tbody>
</table>

\(V(k, X) = \max \begin{cases} 
V(k - 1, X - s_k) + v_k & X \geq s_k \\
V(k - 1, X) & \end{cases} \)

- fill out the base case row and column;
- the order of cells to be filled;
- \(V(i, X) = V(i - 1, X)\) if \(X < s_i\);
- time complexity
Chapter 15. Dynamic Programming

Hidden Markov models and Viterbi’s algorithm
Chapter 15. Dynamic Programming

Hidden Markov models and Viterbi’s algorithm

[Diagram showing a transition between two states labeled 'Fair' and 'Loaded']

- From Fair to Fair: 0.95
- From Fair to Loaded: 0.05
- From Loaded to Fair: 0.1
- From Loaded to Loaded: 0.9

States and Transition Probabilities:
- Fair: 1: 1/6, 2: 1/6, 3: 1/6, 4: 1/6, 5: 1/6, 6: 1/6
- Loaded: 1: 1/10, 2: 1/10, 3: 1/10, 4: 1/10, 5: 1/10, 6: 1/2
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Chapter 15. Dynamic Programming
Chapter 15. Dynamic Programming

(a)

\[
\begin{align*}
0.99 & \quad 0.01 & \quad 0.9 \\
1 & \quad 2 & \\
A 0.4 & \quad C 0.1 & \quad A 0.05 \\
C 0.1 & \quad G 0.1 & \quad C 0.4 \\
G 0.1 & \quad T 0.4 & \quad G 0.5 \\
T 0.4 & & \quad T 0.05
\end{align*}
\]

(b)

\textit{state sequence (hidden):}
\[
\ldots 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 2 \quad 2 \quad 2 \quad 2 \quad 1 \quad 1 \quad \ldots
\]

transitions: ? 0.99 0.99 0.99 0.99 0.01 0.9 0.9 0.9 0.1 0.99

(c)

\textit{symbol sequence (observable):}
\[
\ldots A \quad T \quad C \quad A \quad A \quad G \quad G \quad C \quad G \quad A \quad T \quad \ldots
\]

emissions: 0.4 0.4 0.1 0.4 0.4 0.5 0.5 0.4 0.5 0.4 0.4
Chapter 15. Dynamic Programming

Viterbi Algorithm

- Input: an HMM $\mathcal{M}$, and data $D = d_1 d_2 \ldots d_n$
Viterbi Algorithm

- Input: an HMM $\mathcal{M}$, and data $D = d_1d_2 \ldots d_n$
- Output: hidden state sequences $H = s_1s_2 \ldots s_n$ that generate $D$; such that
Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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... 1 3 2 4 6 6 6 4 1 6 5 6 6 2 4 2 1 2 3 5...
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A hidden Markov model (HMM) consists of \((S, T, e, t)\)
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- a set of states \(S = \{F, L\}\);
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A hidden Markov model (HMM) consists of $(S, T, e, t)$

- a set of states $S = \{F, L\}$;
- a transition relation $T = \{(F, F), (F, L), (L, F), (L, L)\} \subseteq S \times S$,
A hidden Markov model (HMM) consists of 

$$(S, T, e, t)$$

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  \[e(F, d) = \frac{1}{6} \text{ for } d = 1, 2, \ldots, 6;\]
Chapter 15. Dynamic Programming

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  \[ e(L, 6) = \frac{1}{2} \text{ and } e(L, d) = \frac{1}{10} \text{ for } d = 1, 2, \ldots, 5; \]
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  \]
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- transition probability distribution \(t\)
  \[
t(F, F) = 0.95, t(F, L) = 0.05, t(L, F) = 0.1, t(L, L) = 0.90;
  \]
The goal is to identify hidden states $s_1 s_2 \ldots s_n$ that generates $d_1 d_2 \ldots d_n$, to maximize the associated probability
Chapter 15. Dynamic Programming

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$$P(s_1s_2\ldots s_n, d_1d_2\ldots d_n)$$
Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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- $FF$, $FL$, $LL$, $LF$,
The goal is to identify hidden states $s_1s_2 \ldots s_n$ that generates $d_1d_2 \ldots d_n$, to maximize the associated probability

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e.g., $d_1 = 3$, $d_2 = 6$, there are 4 possible ways to generate them:

- $FF$, $FL$, $LL$, $LF$, but with different probabilities
• $P(FF, 36)$
Chapter 15. Dynamic Programming

- \( P(FF, 36) = e(F, 3) \times t(F, F) \times e(F, 6) \)
Chapter 15. Dynamic Programming

- \( P(FF, 36) = e(F, 3) \times t(F, F) \times e(F, 6) = \frac{1}{6} \times 0.95 \times \frac{1}{6} \)
Chapter 15. Dynamic Programming

- \( P(FF, 36) = e(F, 3) \times t(F, F) \times e(F, 6) = \frac{1}{6} \times 0.95 \times \frac{1}{6} = 0.02639 \)
Chapter 15. Dynamic Programming

- $P(FF, 36) = e(F, 3) \times t(F, F) \times e(F, 6) = \frac{1}{6} \times 0.95 \times \frac{1}{6} = 0.02639$
- $P(FL, 36)$
Chapter 15. Dynamic Programming

- $P(FF, 36) = e(F, 3) \times t(F, F) \times e(F, 6) = \frac{1}{6} \times 0.95 \times \frac{1}{6} = 0.02639$
- $P(FL, 36) = e(F, 3) \times t(F, L) \times e(L, 6)$
Chapter 15. Dynamic Programming

• \( P(FF, 3 6) = e(F, 3) \times t(F, F) \times e(F, 6) = \frac{1}{6} \times 0.95 \times \frac{1}{6} = 0.02639 \)

• \( P(FL, 3 6) = e(F, 3) \times t(F, L) \times e(L, 6) = \frac{1}{6} \times 0.05 \times \frac{1}{2} \)
Chapter 15. Dynamic Programming

\begin{itemize}
  \item \( P(FF, 3\, 6) = e(F, 3) \times t(F, F) \times e(F, 6) = \frac{1}{6} \times 0.95 \times \frac{1}{6} = 0.02639 \)
  \item \( P(FL, 3\, 6) = e(F, 3) \times t(F, L) \times e(L, 6) = \frac{1}{6} \times 0.05 \times \frac{1}{2} = 0.00417 \)
\end{itemize}
Chapter 15. Dynamic Programming

- $P(FF, 36) = e(F, 3) \times t(F, F) \times e(F, 6) = \frac{1}{6} \times 0.95 \times \frac{1}{6} = 0.02639$
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- $P(LL, 36)$
Chapter 15. Dynamic Programming

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- $P(LL, 36) = e(L, 3) \times t(L, L) \times e(L, 6)$
Chapter 15. Dynamic Programming

- $P(FF, 3 6) = e(F, 3) \times t(F, F) \times e(F, 6) = \frac{1}{6} \times 0.95 \times \frac{1}{6} = 0.02639$
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- $P(LL, 3 6) = e(L, 3) \times t(L, L) \times e(L, 6) = \frac{1}{10} \times 0.90 \times \frac{1}{2}$

but for 10 digits, how many cases to consider?
Chapter 15. Dynamic Programming

- $P(FF, 36) = e(F, 3) \times t(F, F) \times e(F, 6) = \frac{1}{6} \times 0.95 \times \frac{1}{6} = 0.02639$
- $P(FL, 36) = e(F, 3) \times t(F, L) \times e(L, 6) = \frac{1}{6} \times 0.05 \times \frac{1}{2} = 0.00417$
- $P(LL, 36) = e(L, 3) \times t(L, L) \times e(L, 6) = \frac{1}{10} \times 0.90 \times \frac{1}{2} = 0.045$

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

\[
\begin{align*}
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\end{align*}
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Chapter 15. Dynamic Programming

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but for 10 digits, how many cases to consider?
Take a dynamic programming approach:

- For prefix $d_1 \ldots d_k$, define:
  - $P(F, k)$ to be the maximum probability for states $\ldots F$ to generate $d_1 \ldots d_k$
  - $P(L, k)$ to be the maximum probability for states $\ldots L$ to generate $d_1 \ldots d_k$
Chapter 15. Dynamic Programming

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Then recursively,
Chapter 15. Dynamic Programming

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\[
P(F, k) = \max \left\{ \begin{array}{l}
P(F, k - 1) \times t(F, F) \times e(F, d_k) \\
P(L, k - 1) \times t(L, F) \times e(F, d_k)
\end{array} \right\
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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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\end{array} \right.
\]

\[
P(F, 1) = \frac{1}{6}
\]
Then recursively,

\[
P(F, k) = \max \left\{ P(F, k - 1) \times t(F, F) \times e(F, d_k), P(L, k - 1) \times t(L, F) \times e(F, d_k) \right\}
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P(L, k) = \max \left\{ P(F, k - 1) \times t(F, L) \times e(L, d_k), P(L, k - 1) \times t(L, L) \times e(L, d_k) \right\}
\]

\[
P(F, 1) = \frac{1}{6}, \quad P(L, 1) = \begin{cases} \frac{1}{10} & 1 \leq d_1 \leq 5 \\ \frac{1}{2} & d_1 = 6 \end{cases}
\]
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The outlined method is Viterbi Algorithm.
Chapter 15. Dynamic Programming

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- **Start** and **Finish** states can be added into the model, but they do not emit digits (called **silent states**);
Chapter 15. Dynamic Programming

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- The state in phase $k - 1$ determines the state in phase $k$;
The outlined method is Viterbi Algorithm.

- **Start** and **Finish** states can be added into the model, but they do not emit digits (called silent states);
- The state in phase $k - 1$ determines the state in phase $k$;
- Given a state in current phase $k$, the previous state in phase $k - 1$ can only be one of those having transition to the current state.
Chapter 15. Dynamic Programming

A hidden Markov model (HMM) consists of \((S, T, e, t)\), where

- \(S\) is a set of states: \(s_0, s_1, \ldots, s_m\); where \(s_0\) is the begin state;
- transitions \(T \subseteq S \times S\);
- each state \(s_i\) is associated with a (emission) probability distribution \(e(i,a)\) for \(a \in \Sigma\), such that \(\sum_{a \in \Sigma} e(i,a) = 1\) for every \(i, 1 \leq i < m\);
- every \((s_i, s_j)\) is associated with a probability distribution \(t(i,j)\), such that \(\sum_{1 \leq j \leq m} t(i,j) = 1\) for every \(i, 0 \leq i < m\).
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Chapter 15. Dynamic Programming

HMM decoding problem:

Input: HMM $M$, sequence of symbols $x \in \Sigma^*$

Output: $y^*$, a sequence of states such that $y^* = \arg \max_y \{ \text{Prob}(y, x | M) \}$

where $y$ begins from $s_0$.
Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

The idea of Viterbi algorithm:
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For any prefix $x_1 \ldots x_k$, $k \geq 0$, 
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For any prefix $x_1 \ldots x_k$, $k \geq 0$, and state $s_j \in S$;
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For any prefix $x_1 \ldots x_k$, $k \geq 0$, and state $s_j \in S$;

$$\max_{y=s_0 \ldots s_j} \{ Prob(y, x_1 \ldots x_k | M) \}$$

the maximum probability of a path $s_0 \ldots s_j$ to generate $x_1 \ldots x_k$;
The idea of Viterbi algorithm:

For any prefix \( x_1 \ldots x_k, \; k \geq 0 \), and state \( s_j \in S \);

\[
\max_{y'=s_0\ldots s_j} \{ \text{Prob}(y, x_1 \ldots x_k | M) \}
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Chapter 15. Dynamic Programming

Define $f(j,k) = \max_{y=s_0 \ldots s_j} \{\text{Prob}(y, x_1 \ldots x_k)\}$

Recursively,$f(j,k) = \max_i \{f(i,k-1) \times t(i,j) \times e(j,x_k)\}$

Base case: $f(0,0) = 1$, $f(i,0) = 0$ for all $i \geq 1$. 

Diagram:
- $s_0$ generates $x_1x_2 \ldots x_{k-1}$
- $y$ generates $x_k$
- $s_j$
Chapter 15. Dynamic Programming

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Define $f(j, k) = \max_{y=s_0...s_j} \{ Prob(y, x_1...x_k) \}$

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Base case: $f(0, 0) = 1$, $f(i, 0) = 0$ for all $i \geq 1$. 

Diagram:
- $s_0$ generates $x_1x_2...x_{k-1}$
- $s_j$ generates $x_k$
- $s_i$ generates $x_{k-1}$
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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Base case: $f(0, 0) = 1,$
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Recursively, $f(j, k) = \max_i \{f(i, k - 1) \times t(i, j) \times e(j, x_k)\}$

Base case: $f(0, 0) = 1, \ f(i, 0)$
Define \( f(j, k) = \max_{y=s_0 \ldots s_j} \{ \text{Prob}(y, x_1 \ldots x_k) \} \)

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Base case: \( f(0, 0) = 1, \quad f(i, 0) = 0 \) for all \( i \geq 1 \).
Algorithm \textsc{Viterbi}(x, n, t, e, m)

0. \( T(0, 0) = 1; P(0, 0) = 0; \)
Chapter 15. Dynamic Programming

Algorithm VITERBI\((x, n, t, e, m)\)

0. \(T(0, 0) = 1; P(0, 0) = 0;\)
1. for \(j = 1\) to \(m\)
2. \(T(j, 0) = 0;\)
Chapter 15. Dynamic Programming

Algorithm VITERBI($x, n, t, e, m$)

0. $T(0, 0) = 1; P(0, 0) = 0$;
1. for $j = 1$ to $m$
2. $T(j, 0) = 0$;
3. for $k = 1$ to $n$
4. for $j = 1$ to $m$ for every 'last state' $s_j$
5. premax = 0 initialize the prefix probability
6. for $i = 0$ to $m$ try every 'predecessor state' $s_i$
7. if premax < $T(i, k-1) \times t(i, j) \times e(j, x_k)$
8. premax = $T(i, k-1) \times t(i, j) \times e(j, x_k)$ update probability
9. $P(j, k) = i$ memorize the predecessor
10. $T(j, k) = $ premax
11. laststate = 1; max = $T(1, n)$
12. for $j = 2$ to $m$ identify maximum probability
13. if max < $T(j, n)$
14. laststate = $j$;
15. max = $T(j, n)$
16. return ($T, P, max, laststate$)

Time complexity: $O(nm^2)$
Chapter 15. Dynamic Programming

Algorithm VITERBI($x, n, t, e, m$)

0. $T(0, 0) = 1; P(0, 0) = 0$
1. for $j = 1$ to $m$
2. $T(j, 0) = 0$
3. for $k = 1$ to $n$ for every prefix upto $k$th symbol

4. for $j = 1$ to $m$ for every 'last state' $s_j$
5. premax = 0 initialize the prefix probability
6. for $i = 0$ to $m$ try every 'predecessor state' $s_i$
7. if premax < $T(i, k - 1)$ \times $t(i, j)$ \times $e(j, x_k)$
8. premax = $T(i, k - 1)$ \times $t(i, j)$ \times $e(j, x_k)$ update probability
9. $P(j, k)$ = $i$ memorize the predecessor
10. $T(j, k)$ = premax
11. laststate = 1 \text{max} = T(1, n)$
12. for $j = 2$ to $m$ identify maximum probability
13. if max < $T(j, n)$
14. laststate = $j$
15. max = $T(j, n)$
16. return ($T, P, \text{max}, \text{laststate}$)

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Chapter 15. Dynamic Programming

Algorithm VITERBI($x, n, t, e, m$)

0. $T(0, 0) = 1; P(0, 0) = 0$;
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4. for $j = 1$ to $m$
Chapter 15. Dynamic Programming

Algorithm VITERBI($x, n, t, e, m$)

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Algorithm \textsc{Viterbi}(x, n, t, e, m)

0. \( T(0, 0) = 1; \ P(0, 0) = 0; \)

1. \textbf{for} \( j = 1 \) \textbf{to} \( m \)

2. \( T(j, 0) = 0; \)

3. \textbf{for} \( k = 1 \) \textbf{to} \( n \) \hspace{1cm} \text{for every prefix upto } k\text{th symbol}

4. \textbf{for} \( j = 1 \) \textbf{to} \( m \) \hspace{1cm} \text{for every ‘last state’ } s_j

5. \( \text{premax} = 0 \)
Algorithm VITERBI(x, n, t, e, m)
0. \(T(0, 0) = 1; P(0, 0) = 0\);
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2. \(T(j, 0) = 0;\)
3. \(\text{for } k = 1 \text{ to } n \quad \text{for every prefix upto } k\text{th symbol}\)
4. \(\text{for } j = 1 \text{ to } m \quad \text{for every ‘last state’ } s_j\)
5. \(\text{premax} = 0 \quad \text{initialize the prefix probability}\)
Algorithm VITERBI($x, n, t, e, m$)

0. $T(0, 0) = 1; P(0, 0) = 0$
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8. $P(j, k) = i$ memorize the predecessor
9. $T(j, k) = premax$
10. laststate = 1; max = $T(1, n)$
11. for $j = 2$ to $m$ identify maximum probability
12. if $max < T(j, n)$
13. laststate = $j$
14. max = $T(j, n)$
15. return ($T, P, max, laststate$)

Time complexity: $O(nm^2)$
Chapter 15. Dynamic Programming

Algorithm VITERBI(\(x, n, t, e, m\))

0. \(T(0, 0) = 1; P(0, 0) = 0;\)
1. for \(j = 1\) to \(m\)  
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9. \(P(j, k) = i\) memorize the predecessor
10. \(T(j, k) = \text{premax}\)
11. laststate = 1, \(\max = T(1, n)\)
12. for \(j = 2\) to \(m\) identify maximum probability
13. if \(\max < T(j, n)\)
14. laststate = \(j\)
15. \(\max = T(j, n)\)
16. return \((T, P, \max, \text{laststate})\)

Time complexity: \(O(nm^2)\)
Chapter 15. Dynamic Programming

Algorithm VITERBI($x, n, t, e, m$)

0. $T(0, 0) = 1; \ P(0, 0) = 0$;
1. for $j = 1$ to $m$
2. $T(j, 0) = 0$;
3. for $k = 1$ to $n$ for every prefix upto $k$th symbol
4. for $j = 1$ to $m$ for every ‘last state’ $s_j$
5. $premax = 0$ initialize the prefix probability
6. for $i = 0$ to $m$ try every ’predecessor state’ $s_i$
7. if $premax < T(i, k - 1) \times t(i, j) \times e(j, x_k)$
8. $premax = T(i, k - 1) \times t(i, j) \times e(j, x_k)$

9. $P(j, k)$ memorize the predecessor
10. $T(j, k) = premax$
11. laststate = 1 $\ max = T(1, n)$
12. for $j = 2$ to $m$ identify maximum probability
13. if $max < T(j, n)$
14. laststate = $j$
15. $max = T(j, n)$
16. return ($T, P, max, laststate$)

Time complexity: $O(nm^2)$
Algorithm VITERBI($x, n, t, e, m$)

0. $T(0, 0) = 1; P(0, 0) = 0$;
1. for $j = 1$ to $m$
2. $T(j, 0) = 0$;
3. for $k = 1$ to $n$ for every prefix upto $k$th symbol
4. for $j = 1$ to $m$ for every ‘last state’ $s_j$
5. $\text{premax} = 0$ initialize the prefix probability
6. for $i = 0$ to $m$ try every ’predecessor state’ $s_i$
7. if $\text{premax} < T(i, k - 1) \times t(i, j) \times e(j, x_k)$
8. $\text{premax} = T(i, k - 1) \times t(i, j) \times e(j, x_k)$ update probability

9. $P(j, k) = i$ memorize the predecessor
10. $T(j, k) = \text{premax}$
11. $\text{laststate} = 1, \text{max} = T(1, n)$
12. for $j = 2$ to $m$ identify maximum probability
13. if $\text{max} < T(j, n)$
14. $\text{laststate} = j$
15. $\text{max} = T(j, n)$
16. return $(T, P, \text{max}, \text{laststate})$

Time complexity: $O(nm^2)$
Chapter 15. Dynamic Programming

Algorithm VITERBI($x, n, t, e, m$)

0. $T(0, 0) = 1; P(0, 0) = 0$
1. for $j = 1$ to $m$
2. $T(j, 0) = 0$
3. for $k = 1$ to $n$ for every prefix upto $k$th symbol
4. for $j = 1$ to $m$ for every ‘last state’ $s_j$
5. $\text{premax} = 0$ initialize the prefix probability
6. for $i = 0$ to $m$ try every ‘predecessor state’ $s_i$
7. if $\text{premax} < T(i, k - 1) \times t(i, j) \times e(j, x_k)$
8. $\text{premax} = T(i, k - 1) \times t(i, j) \times e(j, x_k)$ update probability
9. $P(j, k) = i$
10. $\text{laststate} = 1; \text{max} = T(1, n)$
11. for $j = 2$ to $m$ identify maximum probability
12. if $\text{max} < T(j, n)$
13. $\text{laststate} = j$
14. $\text{max} = T(j, n)$
15. return $\left(T, P, \text{max}, \text{laststate}\right)$

Time complexity: $O(nm^2)$
Algorithm \textsc{Viterbi}(x, n, t, e, m)

0. \( T(0, 0) = 1; \ P(0, 0) = 0; \)
1. \textbf{for} \( j = 1 \text{ to } m \)
2. \( T(j, 0) = 0; \)
3. \textbf{for} \( k = 1 \text{ to } n \) \hspace{1cm} for every prefix upto \( k \)th symbol
4. \textbf{for} \( j = 1 \text{ to } m \) \hspace{1cm} for every ‘last state’ \( s_j \)
5. \( \text{premax} = 0 \) \hspace{1cm} initialize the prefix probability
6. \textbf{for} \( i = 0 \text{ to } m \) \hspace{1cm} try every ‘predecessor state’ \( s_i \)
7. \textbf{if} \( \text{premax} < T(i, k - 1) \times t(i, j) \times e(j, x_k) \)
8. \( \text{premax} = T(i, k - 1) \times t(i, j) \times e(j, x_k) \) \hspace{1cm} update probability
9. \( P(j, k) = i \) \hspace{1cm} memorize the predecessor

Time complexity: \( O(nm^2) \)
Chapter 15. Dynamic Programming

Algorithm \texttt{VITERBI}(x, n, t, e, m)

0. \( T(0, 0) = 1; \ P(0, 0) = 0; \)
1. \textbf{for} \( j = 1 \) \textbf{to} \( m \)
2. \( T(j, 0) = 0; \)
3. \textbf{for} \( k = 1 \) \textbf{to} \( n \) \textbf{for every prefix upto } \( k \)\textbf{th symbol}
4. \textbf{for} \( j = 1 \) \textbf{to} \( m \) \textbf{for every } 'last state' \( s_j \)
5. \( \text{premax} = 0 \) \textbf{initialize the prefix probability}
6. \textbf{for} \( i = 0 \) \textbf{to} \( m \) \textbf{try every } 'predecessor state' \( s_i \)
7. \textbf{if} \( \text{premax} < T(i, k - 1) \times t(i, j) \times e(j, x_k) \)
8. \( \text{premax} = T(i, k - 1) \times t(i, j) \times e(j, x_k) \) \textbf{update probability}
9. \( P(j, k) = i \) \textbf{memorize the predecessor}
10. \( T(j, k) = \text{premax} \)
Algorithm VITERBI($x, n, t, e, m$)

0. $T(0, 0) = 1; P(0, 0) = 0$
1. for $j = 1$ to $m$
2. $T(j, 0) = 0$
3. for $k = 1$ to $n$ for every prefix upto $k$th symbol
4. for $j = 1$ to $m$ for every ‘last state’ $s_j$
5. $\text{premax} = 0$ initialize the prefix probability
6. for $i = 0$ to $m$ try every ‘predecessor state’ $s_i$
7. \[ \text{if } \text{premax} < T(i, k - 1) \times t(i, j) \times e(j, x_k) \]
8. \[ \text{premax} = T(i, k - 1) \times t(i, j) \times e(j, x_k) \text{ update probability} \]
9. $P(j, k) = i$ memorize the predecessor
10. $T(j, k) = \text{premax}$
11. $\text{laststate} = 1; \text{max} = T(1, n)$
Algorithm VITERBI\((x, n, t, e, m)\)

0. \(T(0, 0) = 1; P(0, 0) = 0;\)
1. \(\text{for } j = 1 \text{ to } m\)
2. \(T(j, 0) = 0;\)
3. \(\text{for } k = 1 \text{ to } n\) for every prefix upto \(k\)th symbol
4. \(\text{for } j = 1 \text{ to } m\) for every ‘last state’ \(s_j\)
5. \(\text{premax} = 0\) initialize the prefix probability
6. \(\text{for } i = 0 \text{ to } m\) try every ‘predecessor state’ \(s_i\)
7. \(\text{if } \text{premax} < T(i, k - 1) \times t(i, j) \times e(j, x_k)\)
8. \(\text{premax} = T(i, k - 1) \times t(i, j) \times e(j, x_k)\) update probability
9. \(P(j, k) = i\) memorize the predecessor
10. \(T(j, k) = \text{premax}\)
11. \(\text{laststate} = 1; \text{max} = T(1, n)\)
12. \(\text{for } j = 2 \text{ to } m\) identify maximum probability

Time complexity: \(O(nm^2)\)
Algorithm VITERBI\((x, n, t, e, m)\)

0. \(T(0, 0) = 1; P(0, 0) = 0;\)
1. \(\text{for } j = 1 \text{ to } m\)
2. \(T(j, 0) = 0;\)
3. \(\text{for } k = 1 \text{ to } n \quad \text{for every prefix upto } k\text{th symbol}\)
4. \(\text{for } j = 1 \text{ to } m \quad \text{for every ‘last state’ } s_j\)
5. \(\text{premax } = 0 \quad \text{initialize the prefix probability}\)
6. \(\text{for } i = 0 \text{ to } m \quad \text{try every ‘predecessor state’ } s_i\)
7. \(\quad \text{if premax } < T(i, k - 1) \times t(i, j) \times e(j, x_k)\)
8. \(\quad \text{premax } = T(i, k - 1) \times t(i, j) \times e(j, x_k) \quad \text{update probability}\)
9. \(\quad P(j, k) = i \quad \text{memorize the predecessor}\)
10. \(T(j, k) = \text{premax}\)
11. \(\text{laststate } = 1; \text{ max } = T(1, n)\)
12. \(\text{for } j = 2 \text{ to } m \quad \text{identify maximum probability}\)
13. \(\quad \text{if max } < T(j, n)\)

Time complexity: \(O(nm^2)\)
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Algorithm VITERBI($x, n, t, e, m$)

0. \( T(0, 0) = 1; \ P(0, 0) = 0; \)
1. \( \text{for } j = 1 \text{ to } m \)
2. \( T(j, 0) = 0; \)
3. \( \text{for } k = 1 \text{ to } n \) for every prefix upto \( k \)th symbol
4. \( \text{for } j = 1 \text{ to } m \) for every ‘last state’ \( s_j \)
5. \( \text{premax} = 0 \) initialize the prefix probability
6. \( \text{for } i = 0 \text{ to } m \) try every 'predecessor state' \( s_i \)
7. \( \text{if premax} < T(i, k - 1) \times t(i, j) \times e(j, x_k) \)
8. \( \text{premax} = T(i, k - 1) \times t(i, j) \times e(j, x_k) \text{ update probability} \)
9. \( P(j, k) = i \text{ memorize the predecessor} \)
10. \( T(j, k) = \text{premax} \)
11. \( \text{laststate} = 1; \ \max = T(1, n) \)
12. \( \text{for } j = 2 \text{ to } m \) identify maximum probability
13. \( \text{if } max < T(j, n) \)
14. \( \text{laststate} = j; \)
Algorithm \textsc{Viterbi}(x, n, t, e, m)

0. \( T(0, 0) = 1; \ P(0, 0) = 0; \)
1. \textbf{for} \( j = 1 \) \textbf{to} \( m \)
2. \( T(j, 0) = 0; \)
3. \textbf{for} \( k = 1 \) \textbf{to} \( n \) \textbf{for every prefix upto kth symbol}
4. \textbf{for} \( j = 1 \) \textbf{to} \( m \) \textbf{for every ‘last state’} \( s_j \)
5. \( \text{premax} = 0 \) \textbf{initialize the prefix probability}
6. \textbf{for} \( i = 0 \) \textbf{to} \( m \) \textbf{try every ‘predecessor state’} \( s_i \)
7. \textbf{if} \( \text{premax} < T(i, k - 1) \times t(i, j) \times e(j, x_k) \)
8. \( \text{premax} = T(i, k - 1) \times t(i, j) \times e(j, x_k) \) \textbf{update probability}
9. \( P(j, k) = i \) \textbf{memorize the predecessor}
10. \( T(j, k) = \text{premax} \)
11. \( \text{laststate} = 1; \ \text{max} = T(1, n) \)
12. \textbf{for} \( j = 2 \) \textbf{to} \( m \) \textbf{identify maximum probability}
13. \textbf{if} \( \text{max} < T(j, n) \)
14. \( \text{laststate} = j; \)
15. \( \text{max} = T(j, n) \)

Time complexity: \( O(mn^2) \)
Chapter 15. Dynamic Programming

Algorithm \textsc{Viterbi}(x, n, t, e, m)

0. $T(0, 0) = 1; P(0, 0) = 0$
1. \textbf{for} $j = 1$ \textbf{to} $m$
2. \hspace{1em} $T(j, 0) = 0$
3. \textbf{for} $k = 1$ \textbf{to} $n$ \hspace{1em} for every prefix up to $k$th symbol
4. \hspace{2em} \textbf{for} $j = 1$ \textbf{to} $m$ \hspace{1em} for every 'last state' $s_j$
5. \hspace{3em} \textit{premax} = 0 \hspace{1em} initialize the prefix probability
6. \hspace{3em} \textbf{for} $i = 0$ \textbf{to} $m$ \hspace{1em} try every 'predecessor state' $s_i$
7. \hspace{4em} \textbf{if} \hspace{1em} \textit{premax} < $T(i, k - 1) \times t(i, j) \times e(j, x_k)$
8. \hspace{4em} \hspace{1em} \textit{premax} = $T(i, k - 1) \times t(i, j) \times e(j, x_k)$ \hspace{1em} update probability
9. \hspace{4em} $P(j, k) = i$ \hspace{1em} memorize the predecessor
10. $T(j, k) = \textit{premax}$
11. \hspace{1em} \textit{laststate} = 1; \hspace{1em} \textit{max} = $T(1, n)$
12. \textbf{for} $j = 2$ \textbf{to} $m$ \hspace{1em} identify maximum probability
13. \hspace{2em} \textbf{if} \hspace{1em} \textit{max} < $T(j, n)$
14. \hspace{3em} \textit{laststate} = $j$
15. \hspace{3em} \textit{max} = $T(j, n)$
16. \textbf{return} $(T, P, \textit{max}, \textit{laststate})$

Time complexity: $O(n^2 m)$
Chapter 15. Dynamic Programming

Algorithm VITERBI(x, n, t, e, m)

0. \( T(0, 0) = 1; \ P(0, 0) = 0; \)
1. \( \text{for } j = 1 \text{ to } m \)
2. \( T(j, 0) = 0; \)
3. \( \text{for } k = 1 \text{ to } n \) for every prefix upto \( k \)th symbol
4. \( \text{for } j = 1 \text{ to } m \) for every ‘last state’ \( s_j \)
5. \( \text{premax} = 0 \) initialize the prefix probability
6. \( \text{for } i = 0 \text{ to } m \) try every ‘predecessor state’ \( s_i \)
7. \( \text{if } \text{premax} < T(i, k - 1) \times t(i, j) \times e(j, x_k) \)
8. \( \text{premax} = T(i, k - 1) \times t(i, j) \times e(j, x_k) \) update probability
9. \( P(j, k) = i \) memorize the predecessor
10. \( T(j, k) = \text{premax} \)
11. \( \text{laststate} = 1; \ max = T(1, n) \)
12. \( \text{for } j = 2 \text{ to } m \) identify maximum probability
13. \( \text{if } \max < T(j, n) \)
14. \( \text{laststate} = j; \)
15. \( \max = T(j, n) \)
16. \( \text{return } (T, P, \max, \text{laststate}) \)

Time complexity:

\( O(n^2 m^2) \)
Chapter 15. Dynamic Programming

Algorithm $\text{VITERBI}(x, n, t, e, m)$

0. $T(0, 0) = 1; P(0, 0) = 0$
1. $\text{for } j = 1 \text{ to } m$
2. $T(j, 0) = 0$
3. $\text{for } k = 1 \text{ to } n$ for every prefix upto $k$th symbol
4. $\text{for } j = 1 \text{ to } m$ for every 'last state' $s_j$
5. $\text{premax} = 0$ initialize the prefix probability
6. $\text{for } i = 0 \text{ to } m$ try every 'predecessor state' $s_i$
7. $\text{if } \text{premax} < T(i, k - 1) \times t(i, j) \times e(j, x_k)$
8. $\text{premax} = T(i, k - 1) \times t(i, j) \times e(j, x_k)$ update probability
9. $P(j, k) = i$ memorize the predecessor
10. $T(j, k) = \text{premax}$
11. $\text{laststate} = 1; \text{max} = T(1, n)$
12. $\text{for } j = 2 \text{ to } m$ identify maximum probability
13. $\text{if } \text{max} < T(j, n)$
14. $\text{laststate} = j$
15. $\text{max} = T(j, n)$
16. $\text{return } (T, P, \text{max}, \text{laststate})$

Time complexity: $O(nm^2)$
Chapter 15. Dynamic Programming

Use the computed table $P$ and $\textit{laststate}$ to traceback the hidden states
Use the computed table $P$ and $laststate$ to traceback the hidden states

Algorithm $\text{TraceHiddenStates}(P, j, k, x)$

1. if $j > 0$
Use the computed table $P$ and laststate to traceback the hidden states

Algorithm $\text{TraceHiddenStates}(P, j, k, x)$

1. if $j > 0$
2. $\text{TraceHiddenStates}(P, P[j, k], k - 1, x)$
Use the computed table $P$ and $laststate$ to traceback the hidden states

Algorithm $\text{TraceHiddenStates}(P, j, k, x)$

1. if $j > 0$
2. $\text{TraceHiddenStates}(P, P[j,k], k-1, x)$
3. print ('State' $j$ 'emits symbol' $x_k$)
Use the computed table $P$ and $laststate$ to traceback the hidden states

**Algorithm** $\text{TraceHiddenStates}(P, j, k, x)$

1.   if $j > 0$
2.    $\text{TraceHiddenStates}(P, P[j, k], k - 1, x)$
3.    print ('State' $j$ 'emits symbol' $x_k$)

First, call $\text{TraceHiddenStates}(P, laststate, n, x)$
Chapter 15. Dynamic Programming

There are other DP algorithms associated with HMMs.
Chapter 15. Dynamic Programming

There are other DP algorithms associated with HMMs.

- **Viterbi** algorithm belongs to the type of forward algorithms because it computes for prefixes of the given symbol sequence.

To find the objective function:

For forward, we define:

$$f(j,k) = \max_{y=s_0...s_j} \{ \text{Prob}(y, x_1...x_k) \}$$

$f(j,k)$ is the maximum probability to generate prefix $x_1...x_k$ beginning from state $s_0$ and ending with state $s_j$;

For backward, we may define:

$$b(l,k) = \max_{y=s_l...s_\omega} \{ \text{Prob}(y, x_k...x_n) \}$$

$b(l,k)$ is the maximum probability to generate suffix $x_k...x_n$ beginning from state $s_l$ and ending with state $s_\omega$.
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There are other DP algorithms associated with HMMs.

- **Viterbi** algorithm belongs to the type of **forward** algorithms because it computes for **prefixes** of the given symbol sequence.

  A DP algorithm can be designed so it computes for **suffixes** of the given symbols sequence, a **backward** algorithm.
Chapter 15. Dynamic Programming

There are other DP algorithms associated with HMMs.

- **Viterbi** algorithm belongs to the type of forward algorithms because it computes for prefixes of the given symbol sequence.

  A DP algorithm can be designed so it computes for suffixes of the given symbols sequence, a backward algorithm.

**Algorithm IBRETIV**
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There are other DP algorithms associated with HMMs.

- **Viterbi** algorithm belongs to the type of forward algorithms because it computes for prefixes of the given symbol sequence.

A DP algorithm can be designed so it computes for suffixes of the given symbols sequence, a backward algorithm.

**Algorithm IBRETIV**

How to find objective function?

Recalled for forward, we defined

\[
f(j, k) = \max_{y=s_0\ldots s_j} \{ \text{Prob}(y, x_1\ldots x_k) \}
\]

\(f(j, k)\) is the maximum probability to generate prefix \(x_1\ldots x_k\) beginning from state \(s_0\) and ending with state \(s_j\);
Chapter 15. Dynamic Programming

There are other DP algorithms associated with HMMs.

- **Viterbi** algorithm belongs to the type of forward algorithms because it computes for prefixes of the given symbol sequence.

A DP algorithm can be designed so it computes for suffixes of the given symbols sequence, a backward algorithm.

**Algorithm IBRETIV**

How to find objective function?

Recalled for forward, we defined

\[
f(j, k) = \max_{y=s_0...s_j} \{\text{Prob}(y, x_1...x_k)\}
\]

\(f(j, k)\) is the maximum probability to generate prefix \(x_1...x_1\) beginning from state \(s_0\) and ending with state \(s_j\);

For backward, we may define

\[
b(l, k) = \max_{y=s_l...s_\omega} \{\text{Prob}(y, x_k...x_n)\}
\]

\(b(l, k)\) is the maximum probability to generate suffix \(x_k...x_n\) beginning from state \(s_l\) and ending with state \(s_\omega\);
• We need to amend the HMM model with a silent state $s_\omega$;

$$b(l, k) = \max_{y=s_l \ldots s_\omega} \{Prob(y, x_k \ldots x_n)\}$$
• We need to amend the HMM model with a silent state $s_\omega$;

\[
b(l, k) = \max_{y = s_1 \ldots s_\omega} \{ \text{Prob}(y, x_k \ldots x_n) \}
\]

• Recurrence (draw a diagram)

\[
b(l, k) =
\]
Chapter 15. Dynamic Programming

- We need to amend the HMM model with a silent state $s_\omega$;

\[
b(l, k) = \max_{y=s_1\ldots s_\omega} \{\text{Prob}(y, x_k \ldots x_n)\}
\]

- Recurrence (draw a diagram)

\[
b(l, k) = \max_i
\]
We need to amend the HMM model with a silent state \( s_\omega \);

\[
b(l, k) = \max_{y=s_1 \ldots s_\omega} \{ \text{Prob}(y, x_k \ldots x_n) \}
\]

Recurrence (draw a diagram)

\[
b(l, k) = \max_i \{ b(i, k + 1) \} \]
• We need to amend the HMM model with a silent state $s_\omega$;

\[ b(l, k) = \max_{y = s_l \ldots s_\omega} \{ Prob(y, x_k \ldots x_n) \} \]

• Recurrence (draw a diagram)

\[ b(l, k) = \max_i \{ b(i, k + 1) \times t(l, i) \} \]
Chapter 15. Dynamic Programming

- We need to amend the HMM model with a silent state $s_\omega$;

$$b(l, k) = \max_{y=s_l \ldots s_\omega} \{\text{Prob}(y, x_k \ldots x_n)\}$$

- Recurrence (draw a diagram)

$$b(l, k) = \max_i \{b(i, k + 1) \times t(l, i) \times e(l, x_k)\}$$

the larger the $k$ value, the shortest the suffix is
Chapter 15. Dynamic Programming

• We need to amend the HMM model with a silent state \(s_\omega\);

\[
b(l, k) = \max_{y=s_l\ldots s_\omega} \{\text{Prob}(y, x_k \ldots x_n)\}
\]

• Recurrence (draw a diagram)

\[
b(l, k) = \max_i \{b(i, k + 1) \times t(l, i) \times e(l, x_k)\}
\]

the larger the \(k\) value, the shortest the suffix is

• Base cases:
Chapter 15. Dynamic Programming

• We need to amend the HMM model with a silent state \( s_\omega \):

\[
b(l, k) = \max_{y=s_l \ldots s_\omega} \{ \text{Prob}(y, x_k \ldots x_n) \}
\]

• Recurrence (draw a diagram)

\[
b(l, k) = \max_i \{ b(i, k + 1) \times t(l, i) \times e(l, x_k) \}
\]

the larger the \( k \) value, the shortest the suffix is

• Base cases: \( b(s_\omega, \)
• We need to amend the HMM model with a silent state $s_\omega$;

$$b(l, k) = \max_{y = s_l \ldots s_\omega} \{\text{Prob}(y, x_k \ldots x_n)\}$$

• Recurrence (draw a diagram)

$$b(l, k) = \max_i \{b(i, k + 1) \times t(l, i) \times e(l, x_k)\}$$

the larger the $k$ value, the shortest the suffix is

• Base cases: $b(s_\omega, n + 1) =$
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- We need to amend the HMM model with a silent state $s_\omega$;

\[
b(l, k) = \max_{y=s_l \ldots s_\omega} \{ \operatorname{Prob}(y, x_k \ldots x_n) \}
\]

- Recurrence (draw a diagram)

\[
b(l, k) = \max_i \{ b(i, k + 1) \times t(l, i) \times e(l, x_k) \}
\]

the larger the $k$ value, the shortest the suffix is

- Base cases: $b(s_\omega, n + 1) = 1$,
We need to amend the HMM model with a silent state $s_\omega$:

$$b(l, k) = \max_{y=s_l \ldots s_\omega} \{\text{Prob}(y, x_k \ldots x_n)\}$$

Recurrence (draw a diagram)

$$b(l, k) = \max_i \{b(i, k + 1) \times t(l, i) \times e(l, x_k)\}$$

the larger the $k$ value, the shortest the suffix is

Base cases: $b(s_\omega, n + 1) = 1$, $b(j, n + 1) = 0$ for all $j \neq \omega$.

$k = n + 1$ is the case when the suffix is empty.
We need to amend the HMM model with a silent state \( s_\omega \);

\[
b(l, k) = \max_{y = s_1 \ldots s_\omega} \{ \text{Prob}(y, x_k \ldots x_n) \}
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What is the relationship between \( \max_j \{ f(j, n) \} \) and \( \max_l \{ b(l, 1) \} \)?
Chapter 15. Dynamic Programming

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(draw diagram to show)
Chapter 15. Dynamic Programming

Computing the total probability (instead of the maximum prob)
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$$Prob(x|\mathcal{M})$$
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Chapter 15. Dynamic Programming

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with base cases:

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Define $p(j, k)$ to be the total probability for $\mathcal{M}$ to generate prefix $x_1 \ldots x_k$ ending at state $s_j$.

Then

$$p(j, k) = \sum_{i} p(i, k - 1) \times t(i, j) \times e(j, x_k)$$

with base cases: $p(0, 0) = 1$, and $p(j, 0) = 0$ for all $j \geq 1$. 
Chapter 15. Dynamic Programming

Implementation of the total probability algorithm
Implementation of the total probability algorithm

- almost the same as for Viterbi algorithm.

What does it mean by 'significant probability values'?
Chapter 15. Dynamic Programming

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- applications, for example
  
  - screen a stream of signals (symbols) for occasional but desired pattern
Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

Necessary elements for problems solvable with DP
Chapter 15. Dynamic Programming

Necessary elements for problems solvable with DP

1. it is an optimization problem
Chapter 15. Dynamic Programming

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with an objective function to optimize by solutions

e.g., solution: a path of stations,
objective function: time
Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

Necessary elements for problems solvable with DP

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2. Solution has optimal substructure
   solutions are recursively definable
   e.g., a fastest path consists of other fastest subpaths
Chapter 15. Dynamic Programming

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   solutions are recursively definable
   e.g., a fastest path consists of other fastest subpaths

3. Overlapping subproblems
   e.g., two or more paths share a subpath.
Chapter 15. Dynamic Programming

Longest Common Subsequence (LCS)
Chapter 15. Dynamic Programming

Longest Common Subsequence (LCS)

\[ x = \text{ACCGGTCGAGTGCG} \]
\[ y = \text{GTCGTTCCGATGCCC} \]
Chapter 15. Dynamic Programming

Longest Common Subsequence (LCS)

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To see how much the two sequences are related, we may examine how much the two are in common.
Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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significance of common sequence, viewed as:
Chapter 15. Dynamic Programming

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\end{align*}
\]

significance of common sequence, viewed as:

\[
\begin{align*}
\text{ACCGGTC} & \quad \text{GAGTGCG} \\
\text{CG TC GA TGC} & \quad \text{← common sequence} \\
\text{GTCGTTCGGA TGCC}
\end{align*}
\]
Chapter 15. Dynamic Programming

Let $X = x_1 x_2 \cdots x_m$ be a sequence, $x_i \in \Sigma$. 

$X = \text{ACCGGTCGAGTGCG}$

$Z = \text{CG TCGA TGC}$

the corresponding indexes in $X$ are: 3 < 4 < 6 < 7 < 8 < 9 < 11 < 12 < 13.
Chapter 15. Dynamic Programming

Let $X = x_1x_2 \cdots x_m$ be a sequence, $x_i \in \Sigma$.

$Z = z_1z_2 \cdots z_k$ is a subsequence of $X$
Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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\[
z_j = x_{i_j} \quad \text{for} \quad j = 1, \cdots, k.
\]

\[
\begin{align*}
34 & \quad 6789 & \quad 11 & \quad 12 & \quad 13 \\
X & = \text{ACCGGTGCAGTGC} \\
& \quad || \quad ||\quad | | || \\
Z & = \quad \text{CG TCGA TGC} \\
& \quad 12 \quad 3456 \quad 789
\end{align*}
\]
Let $X = x_1 x_2 \cdots x_m$ be a sequence, $x_i \in \Sigma$.

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$3 < 4 < 6 < 7 < 8 < 9 < 11 < 12 < 13$
Chapter 15. Dynamic Programming

\( Z \) is a *common subsequence* of \( X \) and \( Y \) if \( Z \) is a subsequence of \( X \) and \( Z \) is a subsequence of \( Y \).
Chapter 15. Dynamic Programming

A common subsequence of $X$ and $Y$ is a sequence $Z$ that is a subsequence of $X$ and a subsequence of $Y$.

$\begin{array}{c}
\text{ACCGGTC GAGTGC}
\end{array}$

|| || || ||

$\begin{array}{c}
\text{CG TC GA TGC}
\end{array}$ ← common sequence

|| || || ||

$\begin{array}{c}
\text{GTCGTTCGGA TGCCC}
\end{array}$

The Longest Common Subsequence (LCS) problem:

Input: $X = x_1 x_2 \ldots x_m$ and $Y = y_1 y_2 \ldots y_n$.

Output: $Z$, a common subsequence of $X$ and $Y$ such that the length of $Z$ is the maximum.
Chapter 15. Dynamic Programming

$Z$ is a common subsequence of $X$ and $Y$ if $Z$ is a subsequence of $X$ and $Z$ is a subsequence of $Y$.

\begin{align*}
\text{ACCGGTC} & \quad \text{GAGTGCN} \\
\text{||} \quad \text{||} \quad \text{||} \quad \text{||} \quad \text{||} \\
	ext{CG} & \quad \text{TC} \quad \text{GA} \quad \text{TGC} & \quad \text{← common sequence} \\
\text{||} \quad \text{||} \quad \text{||} \quad \text{||} \quad \text{||} \\
\text{GTCGTTCCGA} & \quad \text{TGCCC}
\end{align*}

**Longest Common Subsequence (LCS) problem:**

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Chapter 15. Dynamic Programming

**Step 1:** Analyze the LCS problem to see if it has the desired features:
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- optimal substructure
- overlapping subproblems
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Chapter 15. Dynamic Programming

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We examine prefixes $x_1 x_2 \ldots x_i, i \leq m$

and $y_1 y_2 \ldots y_j, j \leq n$
Chapter 15. Dynamic Programming

Step 1: Analyze the LCS problem to see if it has the desired features:

- optimal substructure
- overlapping subproblems

We examine prefixes $x_1x_2 \ldots x_i$, $i \leq m$
and $y_1y_2 \ldots y_j$, $j \leq n$

- how is the LCS of $x_1x_2 \ldots x_i$ and $y_1y_2 \ldots y_j$ related to the LCS
of shorter prefixes of $X$ and $Y$?
Chapter 15. Dynamic Programming

\[ x_1 \ldots x_{i-1} x_i \]
\[ y_1 \ldots y_{j-1} y_j \]
Chapter 15. Dynamic Programming

\[ x_1 \ldots x_{i-1} x_i \]
\[ y_1 \ldots y_{j-1} y_j \]

If \( x_i = y_j \), include \( x_i \) in the LCS, reducing to LCS for \( x_1 \ldots x_{i-1} \) and \( y_1 \ldots y_{j-1} \).
Chapter 15. Dynamic Programming

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if \( x_i \neq y_j \), should we abandon either?
Chapter 15. Dynamic Programming

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if \(x_i \neq y_j\), should we abandon either? two options

(1) keep \(x_i\) but abandon \(y_j\)
resulting in prefixes: \(x_1 x_2 \cdots x_i\) and \(y_1 y_2 \cdots y_{j-1}\), or
Chapter 15. Dynamic Programming

\[ x_1 \ldots x_{i-1}x_i \]
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(1) keep \( x_i \) but abandon \( y_j \)
    resulting in prefixes: \( x_1x_2 \ldots x_i \) and \( y_1y_2 \ldots y_{j-1} \), or

(2) keep \( y_j \) but abandon \( x_i \)
    resulting in prefixes: \( x_1x_2 \ldots x_{i-1} \) and \( y_1y_2 \ldots y_j \)
Let $LCS(i, j)$ be the longest common sequence for $x_1x_2 \ldots x_i$ and $y_1y_2 \ldots y_j$, then

\begin{itemize}
  \item if $x_i = y_j$, then $LCS(i, j) = \text{concat}(LCS(i-1, j-1), x_i)$
  \item if $x_i \neq y_j$, then $LCS(i, j) = \text{LCS}(i, j-1) \text{ OR } LCS(i-1, j)$(depending on which is of a larger length.)
\end{itemize}

We conclude:

- LCS problem has the optimal substructure property;
- LCS problem has overlapping subproblems.
Chapter 15. Dynamic Programming

Let \( LCS(i, j) \) be the longest common sequence for \( x_1 x_2 \ldots x_i \) and \( y_1 y_2 \ldots y_j \), then

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- if \( x_i \neq y_j \), then \( LCS(i, j) = LCS(i, j - 1) \) OR \( LCS(i, j) = LCS(i - 1, j) \) (depending on which is of a larger length.)

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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We conclude:

LCS problem has the optimal substructure property;
LCS problem has overlapping subproblems.
Step 2: define objective function and formulate recurrence

Define $l(i,j)$ to be the length of the LCS for prefixes $x_1...x_i$ and $y_1...y_j$.

\[
\begin{align*}
    l(i,j) &= \begin{cases} 
        l(i-1,j-1) + 1 & \text{if } x_i = y_j; \\
        \max\{l(i,j-1), l(i-1,j), l(i-1,j-1)\} & \text{if } x_i \neq y_j.
    \end{cases}
\end{align*}
\]

Base cases:

- $l(i,0) = 0$, for $0 \leq i \leq m$, either prefix is empty → LCS is empty
- $l(0,j) = 0$, for $0 \leq j \leq n$
Chapter 15. Dynamic Programming

**Step 2**: define objective function and formulate recurrence

Define $l(i, j)$ to be the length of the LCS for prefixes $x_1 \ldots x_i$ and $y_1 \ldots y_j$. 

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Chapter 15. Dynamic Programming

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\end{cases}$$

the value of $l(i, j)$ depends on values of $l(i - 1, j - 1)$, $l(i, j - 1)$, and $l(i - 1, j)$. 

**Base cases**: $l(i, 0) = 0$, for $0 \leq i \leq m$, either prefix is empty $\rightarrow$ LCS is empty $l(0, j) = 0$, for $0 \leq j \leq n$. 

Step 2: define objective function and formulate recurrence

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\end{cases}
\]

the value of \( l(i, j) \) depends on values of \( l(i - 1, j - 1), l(i, j - 1), \) and \( l(i - 1, j) \).

Base cases: \( l(i, 0) = 0 \), for \( 0 \leq i \leq m \), either prefix is empty \( \rightarrow \) LCS is empty
Chapter 15. Dynamic Programming

Step 2: define objective function and formulate recurrence

Define $l(i, j)$ to be the length of the LCS for prefixes $x_1 \ldots x_i$ and $y_1 \ldots y_j$.  then

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\end{cases}$$

the value of $l(i, j)$ depends on values of $l(i - 1, j - 1)$, $l(i, j - 1)$, and $l(i - 1, j)$.

base cases: $l(i, 0) = 0$, for $0 \leq i \leq m$, either prefix is empty $\rightarrow$ LCS is empty

$l(0, j) = 0$, for $0 \leq j \leq n$
Chapter 15. Dynamic Programming

**Step 3:** bottom-up table building for function $l(i,j)$.

<table>
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<th>C</th>
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Step 3: bottom-up table building for function $l(i, j)$. 

![Table](image-url)
Chapter 15. Dynamic Programming

**Step 3**: bottom-up table building for function $l(i,j)$.

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Chapter 15. Dynamic Programming

**Step 3**: bottom-up table building for function \( l(i, j) \).

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LCS result between prefixes GCCCTAGC and GCG:

GCCCTAGC

G C G
Algorithm LCS($x, y$)
1. $m = \text{length}(x), n = \text{length}(y)$;
Chapter 15. Dynamic Programming

Algorithm LCS \((x, y)\)
1. \(m = \text{length}(x), n = \text{length}(y)\);
2. \(\text{for } i = 0 \text{ to } m\)
Algorithm LCS\((x, y)\)
1. \(m = \text{length}(x), \ n = \text{length}(y)\);
2. \textbf{for} \(i = 0\ \text{to} \ m\)
3. \(T[i, 0] = 0\)
Algorithm LCS($x, y$)
1. $m = \text{length}(x), n = \text{length}(y)$;
2. for $i = 0$ to $m$
3. \hspace{1em} $T[i, 0] = 0$
4. for $j = 0$ to $n$
5. \hspace{1em} if $x_i = y_j$
6. \hspace{2em} $T[i, j] = T[i-1, j-1] + 1$
7. \hspace{2em} $P[i, j] = \uparrow$
8. \hspace{1em} else if $T[i, j-1] > T[i-1, j]$
9. \hspace{2em} $T[i, j] = T[i, j-1]$
10. \hspace{2em} $P[i, j] = \leftarrow$
11. \hspace{1em} else
12. \hspace{2em} $T[i, j] = T[i-1, j]$
13. \hspace{2em} $P[i, j] = \uparrow$
14. return $(T, P)$

Time complexity: $O(mn)$
Algorithm LCS($x, y$)
1. $m = \text{length}(x)$, $n = \text{length}(y)$;
2. for $i = 0$ to $m$
3. \hspace{1em} $T[i, 0] = 0$
4. for $j = 0$ to $n$
5. \hspace{1em} $T[0, j] = 0$
Algorithm LCS($x, y$)
1. $m = \text{length}(x)$, $n = \text{length}(y)$;
2. for $i = 0$ to $m$
3. 	$T[i, 0] = 0$
4. for $j = 0$ to $n$
5. 	$T[0, j] = 0$
6. for $i = 1$ to $m$
7. for $j = 1$ to $n$
8. if $x[i] = y[j]$
9. 	$T[i, j] = T[i-1, j-1] + 1$
10. else if $T[i, j-1] > T[i-1, j]$
11. 	$T[i, j] = T[i, j-1]$
12. else
13. 	$T[i, j] = T[i-1, j]$
14. return ($T, P$)

Time complexity: $O(mn)$
Chapter 15. Dynamic Programming

Algorithm LCS $(x, y)$
1. $m = \text{length}(x), \ n = \text{length}(y)$;
2. for $i = 0$ to $m$
3.     $T[i, 0] = 0$
4. for $j = 0$ to $n$
5.     $T[0, j] = 0$
6. for $i = 1$ to $m$
7.     for $j = 1$ to $n$
8.         if $x[i] = y[j]$ then
9.             $T[i, j] = T[i-1, j-1] + 1$
10.            $P[i, j] = \text{'\rightarrow'}$
11.        else if $T[i, j-1] > T[i-1, j]$ then
12.             $T[i, j] = T[i, j-1]$
13.             $P[i, j] = \text{'\uparrow'}$
14.        else
15.             $T[i, j] = T[i-1, j]$
16.             $P[i, j] = \text{'\downarrow'}$
17.     end if
18. end for
19. end for
20. return $(T, P)$

Time complexity: $O(mn)$
Algorithm LCS(x, y)
1. \( m = \text{length}(x), \ n = \text{length}(y); \)
2. \( \text{for } i = 0 \text{ to } m \)
3. \( \quad T[i, 0] = 0 \)
4. \( \text{for } j = 0 \text{ to } n \)
5. \( \quad T[0, j] = 0 \)
6. \( \text{for } i = 1 \text{ to } m \)
7. \( \quad \text{for } j = 1 \text{ to } n \)
8. \( \quad \text{if } x_i = y_j \)
9. \( \qquad T[i,j] = T[i-1,j-1] + 1; \)
10. \( \quad \text{else if } T[i,j-1] > T[i-1,j] \)
11. \( \quad T[i,j] = T[i,j-1]; \)
12. \( \quad \text{else } T[i,j] = T[i-1,j]; \)
13. \( \text{return } (T, P) \)
Chapter 15. Dynamic Programming

Algorithm LCS($x$, $y$)
1. $m = \text{length}(x)$, $n = \text{length}(y)$;
2. for $i = 0$ to $m$
3.     $T[i, 0] = 0$
4. for $j = 0$ to $n$
5.     $T[0, j] = 0$
6. for $i = 1$ to $m$
7.     for $j = 1$ to $n$
8.         if $x_i = y_j$
9.             $T[i, j] = T[i - 1, j - 1] + 1$; $P[i, j] = \text{'\downarrow'}$
10. else if $T[i, j - 1] > T[i - 1, j]$
11.     $T[i, j] = T[i, j - 1]$
12.     $P[i, j] = \text{'\leftarrow'}$
13. else
14.     $T[i, j] = T[i - 1, j]$
15.     $P[i, j] = \text{'\uparrow'}$
16. return $(T, P)$

Time complexity: $O(mn)$
Algorithm LCS \((x, y)\)
1. \( m = \text{length}(x), \ n = \text{length}(y) \);
2. \( \text{for } i = 0 \text{ to } m \)
3. \( T[i, 0] = 0 \)
4. \( \text{for } j = 0 \text{ to } n \)
5. \( T[0, j] = 0 \)
6. \( \text{for } i = 1 \text{ to } m \)
7. \( \text{for } j = 1 \text{ to } n \)
8. \( \text{if } x_i = y_j \)
9. \( T[i, j] = T[i - 1, j - 1] + 1; \ P[i, j] = \text{‘} \backslash \text{‘} \)
10. \( \text{else if } T[i, j - 1] > T[i - 1, j] \)

Time complexity: \( O(mn) \)
Algorithm LCS\((x, y)\)
1. \(m = \text{length}(x), n = \text{length}(y)\);
2. \textbf{for } \(i = 0 \textbf{ to } m\)
3. \hspace{1em} \(T[i, 0] = 0\)
4. \textbf{for } \(j = 0 \textbf{ to } n\)
5. \hspace{1em} \(T[0, j] = 0\)
6. \textbf{for } \(i = 1 \textbf{ to } m\)
7. \hspace{2em} \textbf{for } \(j = 1 \textbf{ to } n\)
8. \hspace{3em} \textbf{if } \(x_i = y_j\)
9. \hspace{4em} \(T[i, j] = T[i - 1, j - 1] + 1; P[i, j] = \text{‘}\backslash\text{‘}\)
10. \textbf{else if } \(T[i, j - 1] > T[i - 1, j]\)
11. \hspace{3em} \(T[i, j] = T[i, j - 1]; P[i, j] = \text{‘}\leftarrow\text{‘}\)
12. \textbf{return } \((T, P)\)

Time complexity: \(O(mn)\)
Chapter 15. Dynamic Programming

Algorithm LCS($x, y$)
1. $m = \text{length}(x)$, $n = \text{length}(y)$;
2. for $i = 0$ to $m$
3.     $T[i, 0] = 0$
4. for $j = 0$ to $n$
5.     $T[0, j] = 0$
6. for $i = 1$ to $m$
7.     for $j = 1$ to $n$
8.         if $x_i = y_j$
9.             $T[i, j] = T[i - 1, j - 1] + 1$; $P[i, j] = \leftarrow$
10.        else if $T[i, j - 1] > T[i - 1, j]$
11.            $T[i, j] = T[i, j - 1]$; $P[i, j] = \leftarrow$
12.        else $T[i, j] = T[i - 1, j]$; $P[i, j] = \uparrow$
13. return $(T, P)$

Time complexity: $O(mn)$
Chapter 15. Dynamic Programming

Algorithm LCS(x, y)
1. \[ m = \text{length}(x), \ n = \text{length}(y); \]
2. \[ \text{for } i = 0 \text{ to } m \]
3. \[ T[i, 0] = 0 \]
4. \[ \text{for } j = 0 \text{ to } n \]
5. \[ T[0, j] = 0 \]
6. \[ \text{for } i = 1 \text{ to } m \]
7. \[ \text{for } j = 1 \text{ to } n \]
8. \[ \text{if } x_i = y_j \]
9. \[ T[i, j] = T[i - 1, j - 1] + 1; \ P[i, j] = \uparrow \angle \downarrow \]
10. \[ \text{else if } T[i, j - 1] > T[i - 1, j] \]
11. \[ T[i, j] = T[i, j - 1]; \ P[i, j] = \leftarrow \]
12. \[ \text{else } T[i, j] = T[i - 1, j]; \ P[i, j] = \uparrow \]
13. \[ \text{return } (T, P) \]
Algorithm LCS\((x, y)\)

1. \( m = \text{length}(x), n = \text{length}(y); \)
2. \( \text{for } i = 0 \text{ to } m \)
3. \( T[i, 0] = 0 \)
4. \( \text{for } j = 0 \text{ to } n \)
5. \( T[0, j] = 0 \)
6. \( \text{for } i = 1 \text{ to } m \)
7. \( \text{for } j = 1 \text{ to } n \)
8. \( \text{if } x_i = y_j \)
9. \( T[i, j] = T[i - 1, j - 1] + 1; P[i, j] = '↖' \)
10. \( \text{else if } T[i, j - 1] > T[i - 1, j] \)
11. \( T[i, j] = T[i, j - 1]; P[i, j] = '←' \)
12. \( \text{else } T[i, j] = T[i - 1, j]; P[i, j] = '↑' \)
13. \( \text{return } (T, P) \)

Time complexity: \(O(mn)\)
Chapter 15. Dynamic Programming

Traceback the longest common subsequence
either recursive or iterative algorithm
Chapter 15. Dynamic Programming

Traceback the longest common subsequence
either recursive or iterative algorithm

Algorithm \texttt{PRINTLCS}(P, x, i, j)
Chapter 15. Dynamic Programming

Traceback the longest common subsequence
  either recursive or iterative algorithm

Algorithm `PRINTLCS(P, x, i, j)`
1. `if (i > 0) ∧ (j > 0)`
Chapter 15. Dynamic Programming

Traceback the longest common subsequence

either recursive or iterative algorithm

Algorithm PRINTLCS(P, x, i, j)
1. if \((i > 0) \land (j > 0)\)
2. if \(P[i, j] = '^\wedge'\)
Traceback the longest common subsequence
either recursive or iterative algorithm

Algorithm \textsc{PrintLCS}(P, x, i, j)
1. if \((i > 0) \land (j > 0)\)
2. if \(P[i, j] = '\backslash'\)
3. \textsc{PrintLCS}(P, x, i - 1, j - 1)
Chapter 15. Dynamic Programming

Traceback the longest common subsequence

either recursive or iterative algorithm

Algorithm PRINTLCS(P, x, i, j)
1. if (i > 0) \& (j > 0)
2. if P[i, j] = "↖"
3. PRINTLCS(P, x, i - 1, j - 1)
4. Print (x_i)
Chapter 15. Dynamic Programming

Traceback the longest common subsequence

either recursive or iterative algorithm

Algorithm PRINTLCS($P, x, i, j$)
1. if $(i > 0) \land (j > 0)$
2. if $P[i, j] = '↖$'
3. PRINTLCS($P, x, i - 1, j - 1$)
4. Print $(x_i)$
5. else if $P[i, j] = '←'$
Chapter 15. Dynamic Programming

Traceback the longest common subsequence either recursive or iterative algorithm

Algorithm PRINTLCS\( (P, x, i, j) \)
1. \textbf{if} \( (i > 0) \land (j > 0) \)
2. \quad \textbf{if} \( P[i, j] = '↖' \)
3. \quad \textbf{Print} \((x_i)\)
4. \quad \textbf{else if} \( P[i, j] = '←' \)
5. \quad \textbf{PrintLCS} \((P, x, i, j - 1)\)
6. \quad \textbf{else} \textbf{PrintLCS} \((P, x, i - 1, j)\)
Chapter 15. Dynamic Programming

Traceback the longest common subsequence

either recursive or iterative algorithm

Algorithm \texttt{PrintLCS}(P, x, i, j)
1. \textbf{if} \ (i > 0) \land \ (j > 0)
2. \textbf{if} \ P[i, j] = '↖'
3. \texttt{PrintLCS}(P, x, i - 1, j - 1)
4. \texttt{Print} (x_i)
5. \textbf{else if} \ P[i, j] = '←'
6. \texttt{PrintLCS}(P, x, i, j - 1)
7. \textbf{else} \texttt{PrintLCS}(P, x, i - 1, j)
Traceback the longest common subsequence
either recursive or iterative algorithm

Algorithm \textsc{PrintLCS}(P, x, i, j)
1. \textbf{if} \ (i > 0) \land (j > 0)
2. \quad \textbf{if} \ P[i, j] = '↖'
3. \quad \textsc{PrintLCS}(P, x, i - 1, j - 1)
4. \quad \textbf{Print} \ (x_i)
5. \quad \textbf{else if} \ P[i, j] = '←'
6. \quad \textsc{PrintLCS}(P, x, i, j - 1)
7. \quad \textbf{else} \ \textsc{PrintLCS}(P, x, i - 1, j)
8. \quad \textbf{else return} \ ()
Chapter 15. Dynamic Programming

Pairwise Alignment
Chapter 15. Dynamic Programming

Pairwise Alignment

\[
\begin{align*}
\text{ACCGGTCGAGTGCG} & \quad \text{CGTCGATGC} \quad \text{common sequence} \\
\text{GTCGTTCGGATGCC} & \end{align*}
\]
Chapter 15. Dynamic Programming

Pairwise Alignment

ACCGGTGCGAGTGCG

CGTGCAGTCG ← common sequence

GTCGTTTCGGGATGCCC

biologically more meaningful view:

ACCGGTGTCAGTGC

|| || || ||

CGTCGATGC ← common sequence

|| || || ||

GTCGTTTCGGGATGCCC
Pairwise Alignment

ACCGGTTCGAGTGCG

CGTCGATGC ← common sequence

GTCGTTCTGGATGCCC

biologically more meaningful view:

ACCGGTTC GAGTGCG

|| || || |||

CG TC GA TGC ← common sequence

|| || || |||

GTCGTTCTGGATGCCC

blue letters are conserved;
red letters are mutated;
purple letters are inserted or deleted;
Chapter 15. Dynamic Programming

Alignment between two sequences:
Alignment between two sequences:

\[
\begin{array}{cccccccccccccccc}
ACCGGTC & \_ & GAGTGCG & \_ \\
| & | & | & | & | & | & | & | \\
CG & TC & GA & TGC & \leftarrow & \text{common sequence} \\
| & | & | & | & | & | & | & | \\
GTCGTTCGGA & TGCCC
\end{array}
\]
Chapter 15. Dynamic Programming

Alignment between two sequences:

```
ACCGGTC_GAGTGCG_
|| || || |||
CG TC GA TGC ← common sequence
|| || || |||
GTCGTTCGGA_TGCC
```

- two sequences padded with ‘_’s called gaps;
Chapter 15. Dynamic Programming

Alignment between two sequences:

```
ACCGGTC_GAGTGCG_  
|| || || ||||
CG TC GA TGC ← common sequence
|| || || ||||
GTCGTTCGGA_TGCC
```

- two sequences padded with ‘_’s called gaps;
- two sequences are of the same length, aligned;
Alignment between two sequences:

```
ACCGGTC_GAGTGCG_  
| | | | | | | | |
CG TC GA TGC ← common sequence
| | | | | | | | |
GTCGTTCGGA_TGCCC
```

- two sequences padded with ‘_’s called gaps;
- two sequences are of the same length, aligned;
- sum of column scores is used to identify the best alignment
Chapter 15. Dynamic Programming

Alignment between two sequences:

```
ACCGGTC_GAGTGCG_
   || || || |||
 CG TC GA TGC ← common sequence
   || || || |||
GTCGTTCGGA_TGCCC
```

- two sequences padded with ’_’s called gaps;
- two sequences are of the same length, aligned;
- sum of column scores is used to identify the best alignment

For example, a score scheme
+5 for conservation columns, (reward)
-2 for substitution columns, (penalty)
-6 for delete/insert columns, (penalty)
the above alignment has the total score

\[
(−2) + (−2) + 5 + 5 + (−2) + 5 + 5 + (−6) + 5 + 5 + (−6) + 5 + 5 + 5 + (−2) + (−6)
\]
Alignment between two sequences:

```
ACCGGTC_GAGTGCG_
  || || || |||
  CG TC GA TGC ← common sequence
  || || || |||
GTCGTTCGGA_TGCC
```

- two sequences padded with ‘_’s called **gaps**;
- two sequences are of the same length, aligned;
- sum of column scores is used to identify the best alignment

For example, a score scheme
- +5 for conservation columns, (reward)
- -2 for substitution columns, (penalty)
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Alignment between two sequences:

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ACCGGTC_GAGTGCG_
  || || || |||
 CG TC GA TGC ← common sequence
  || || || |||
GTCGTTCCGA_TGCC
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- two sequences padded with ‘_’s called gaps;
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the above alignment has the total score
\((-2) + (-2) + 5 + 5 + (-2) + 5 + 5 + (-6) + 5 + 5 + (-6) + 5 + 5 + 5 + (-2) + (-6)\)
Chapter 15. Dynamic Programming

Alignment between two sequences:

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\[ \begin{array}{cccccc}
| & | & | & | & |
\end{array} \]

CG TC GA TGC ← common sequence

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LCS is a special case
```
  0  +  0  +1+1+  0  +1+1+  0  +1+1+  0  +1+1+1+  0  +  0
```
Chapter 15. Dynamic Programming

Significance of sequence alignments:
Pairwise alignment problem:

**INPUT**: sequences \( x = x_1x_2 \ldots x_m \), \( y = y_1y_2 \ldots y_n \), and scoring scheme \( \text{score} \)

**OUTPUT**: \( x' \) and \( y' \), which are \( x \) and \( y \) padded with ' 's, respectively such that total score \( \sum_{i=1}^{l} \text{score}(x'_i, y'_i) \) is the maximum, where \( l = |x'| = |y'| \).

How to solve it?
Pairwise alignment problem:

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Similar to LCS problem, we consider prefixes \(x_1 \ldots x_i\) of \(x\), \(y_1 \ldots y_j\) of \(y\).
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How to solve it?

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Similar to LCS problem, we consider prefixes \( x_1 \ldots x_i \) of \( x \), \( y_1 \ldots y_j \) of \( y \).

(1) \( x_1 \ldots x_{i-1} x_i \\
    y_1 \ldots y_{j-1} y_j \)

(2) \( x_1 \ldots x_{i-1} x_i \\
    y_1 \ldots y_{j-1} y_j \)

(3) \( x_1 \ldots x_{i-1} x_i \\
    y_1 \ldots y_{j-1} y_j \)

case (1) contributes to a match/mismatch score
Chapter 15. Dynamic Programming

Pairwise alignment problem:

**Input:** sequences \( x = x_1x_2 \ldots x_m, \ y = y_1y_2 \ldots y_n, \) and scoring scheme `score`

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How to solve it?

Similar to LCS problem, we consider prefixes \( x_1 \ldots x_i \) of \( x \), \( y_1 \ldots y_j \) of \( y \).

\[
\begin{align*}
\text{case (1) contributes to a match/mismatch score} & \\
\text{cases (2) and (3) contribute to an insertion/deletion score.} &
\end{align*}
\]
Chapter 15. Dynamic Programming

Scoring function for pairwise alignment
Chapter 15. Dynamic Programming

Scoring function for pairwise alignment

Example (1)

+5 for conservation columns, (reward)
-2 for substitution columns, (penalty)
-6 for delete/insert columns, (penalty)
Chapter 15. Dynamic Programming

Scoring function for pairwise alignment

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-5 for conservation columns, (reward)
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Example (2)

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Chapter 15. Dynamic Programming

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Why sum of column scores?
Chapter 15. Dynamic Programming

Scoring function for pairwise alignment

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Why sum of column scores?

product of independent column probabilities
Chapter 15. Dynamic Programming

As for the LCS problem, we can define

\[ A(i, j) \text{ is the maximum score alignment between } x_1 \ldots x_i \]

and \( y_1 \ldots y_j \), then

\[ A(i, j) = \begin{cases} 
\text{concat}(A(i - 1, j - 1), x_i y_j) & \text{if this has the highest score;} \\
\end{cases} \]

Instead, we define objective function \( S(i, j) \) to be the maximum alignment score between \( x_1 \ldots x_i \) and \( y_1 \ldots y_j \), then

\[ S(i, j) = \max \begin{cases} 
S(i - 1, j - 1) + \text{score}(x_i, y_j) & i \geq 1, j \geq 1; \\
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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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scoring function \( \text{score} \)

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exercise: Table filling and traceback
Chapter 15. Dynamic Programming

Affine gap penalty: a challenge for DP

Consider: new gap penalty score:

\[ \text{penalty} = \text{op} - \text{ext} (l - 1) \]

where \( \text{op} > \text{ext} \).

how to modify recurrences:

\[
S(i, j) = \max \begin{cases} 
S(i - 1, j - 1) + \text{score}(x_i, y_j) \\
S(i, j - 1) + \text{score}(\text{gap}, y_j) \\
S(i - 1, j) + \text{score}(x_i, \text{gap}) 
\end{cases}
\]
Chapter 15. Dynamic Programming

Affine gap penalty: a challenge for DP

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- \( op \): penalty to open a gap;
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for a gap covering \( l \) positions, where \( op > ext \).

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$$S(i, j) = \max \begin{cases} 
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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

Issue: we do not know how long a gap is.

\[ S(i, j) = \max \begin{cases} S(i - 1, j - 1) + score(x_i, y_j) \\ S(i, j - 1) + score(-, y_j) \\ S(i - 1, j) + score(x_i, -) \end{cases} \]

Solution 1: we can try all different length \( l \)

\[ S(i, j) = \max \begin{cases} S(i - 1, j - 1) + score(x_i, y_j) \\ S(i - 1, j) + score(x_i, -) \end{cases} \]

But the complexity increased to:

\[ O(nm \times \max\{n, m\}) \]

Why does the complexity increase?

a lot of recomputations, but where?
Chapter 15. Dynamic Programming

Issue: we do not know how long a gap is.

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Chapter 15. Dynamic Programming

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Why does the complexity increase? a lot of recomputations, but where?
Chapter 15. Dynamic Programming

Issue: we do not know how long a gap is.

\[ S(i, j) = \max \begin{cases} 
S(i - 1, j - 1) + \text{score}(x_i, y_j) \\
S(i, j - 1) + \text{score}(\text{-}, y_j) \\
S(i - 1, j) + \text{score}(x_i, \text{-}) 
\end{cases} \]

Solution 1: we can try all different length \( l \)

\[ S(i, j) = \max \begin{cases} 
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But the complexity increased to: \( O(nm \times \max\{n, m\}) \)
Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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\max_{1 \leq l \leq i} S(i - l, j) - op - (l - 1) \times ext 
\end{cases}
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Chapter 15. Dynamic Programming

So we want to do recursion "one step at a time":

Substitute
Insertion
Deletion
Chapter 15. Dynamic Programming

So we want to do recursion "one step at a time":

**but at the same time, differentiate different "states"**

where the two tails are

**Substitute**

\[
\begin{array}{ll}
\text{Substitute} \\
\begin{array}{c}
\begin{array}{c}
\text{S}(i, j) \\
\text{S}(i-1, j-1)
\end{array}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{x}_1 \ldots \text{x}_{i-1} \text{x}_i \\
\text{y}_1 \ldots \text{y}_{j-1} \text{y}_j
\end{array}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{S}(i, j) \\
\text{I}(i-1, j-1)
\end{array}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
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\end{array}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{S}(i, j) \\
\text{D}(i-1, j-1)
\end{array}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{x}_1 \ldots \text{x}_{i-1} \text{x}_i \\
\text{y}_1 \ldots \text{y}_{j-1} \text{y}_j
\end{array}
\end{array}
\end{array}
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Chapter 15. Dynamic Programming

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**Substitute**

**Insertion**
Chapter 15. Dynamic Programming

So we want to do recursion "one step at a time": but at the same time, differentiate different "states" where the two tails are

**Substitute**

- \( S(i, j) \)
- \( S(i-1, j-1) \)

**Insertion**

- \( I(i, j) \)
- \( I(i-1, j-1) \)

**Deletion**

- \( D(i, j) \)
- \( D(i-1, j) \)
Define $S(i, j)$ to be the maximum score of alignment between $x_1 \ldots x_i$ and $y_1 \ldots y_j$ with $x_i$ being aligned to $y_j$.

Define $I(i, j)$ to be the maximum score of alignment between $x_1 \ldots x_i$ and $y_1 \ldots y_j$ with $y_j$ being insertion.

\[
S(i, j) = \max \left\{ S(i-1, j-1) + \text{score}(x_i, y_j), I(i-1, j-1) + \text{score}(x_i, y_j), D(i-1, j-1) + \text{score}(x_i, y_j) \right\}
\]

\[
I(i, j) = \max \left\{ I(i, j-1) - \text{ext}_{S(i, j-1)} - \text{op}, D(i, j-1) - \text{ext}_{S(i, j-1)} - \text{op} \right\}
\]
Chapter 15. Dynamic Programming

Define $S(i, j)$ to be the maximum score of alignment between $x_1 \ldots x_i$ and $y_1 \ldots y_j$ with $x_i$ being aligned to $y_j$.

$$S(i, j) = \max \left\{ \begin{array}{l} S(i-1, j-1) + \text{score}(x_i, y_j) \\ I(i-1, j-1) + \text{score}(x_i, y_j) \\ D(i-1, j-1) + \text{score}(x_i, y_j) \end{array} \right\}$$
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Define $I(i, j)$ to be the maximum score of alignment between $x_1 \ldots x_i$ and $y_1 \ldots y_j$ with $y_j$ being insertion.
Chapter 15. Dynamic Programming

Define \( S(i, j) \) to be the maximum score of alignment between \( x_1 \ldots x_i \) and \( y_1 \ldots y_j \) with \( x_i \) being aligned to \( y_j \).

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\[
I(i, j) = \max \left\{ I(i, j-1) - \text{ext}, S(i, j-1) - \text{op} \right\}
\]
Chapter 15. Dynamic Programming

Define $D(i, j)$ to be the maximum score of alignment between $x_1...x_i$ and $y_1...y_j$ with $x_i$ being deletion.

$D(i, j) = \max \{ D(i-1, j) - \text{ext}(i-1, j), \text{op} \}$

Note that there is no deletion right after insertion, neither insertion right after deletion. Why?

What is the desired value?

$\max \{ S(m, n), I(m, n), D(m, n) \}$

How to fill the table(s) and to traceback the optimal alignment?
Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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\[
\begin{align*}
\text{A C A _ G T A} & \quad \rightarrow \quad \text{A C A G T A} \\
\text{A C C T _ _ A} & \quad \rightarrow \quad \text{A C C T _ A}
\end{align*}
\]
Chapter 15. Dynamic Programming

Define $D(i, j)$ to be the maximum score of alignment between $x_1 \ldots x_i$ and $y_1 \ldots y_j$ with $x_i$ being deletion.

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Chapter 15. Dynamic Programming

Define $D(i, j)$ to be the maximum score of alignment between $x_1 \ldots x_i$ and $y_1 \ldots y_j$ with $x_i$ being deletion.

$D(i, j) = \max \left\{ \begin{array}{c}
D(i - 1, j) - \text{ext} \\ S(i - 1, j) - \text{op}
\end{array} \right.$

- Note that there is no deletion right after insertion, neither insertion right after deletion. **why?**

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Chapter 15. Dynamic Programming

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I(m, n) \\
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\end{cases}$

- How to fill the table(s) and to traceback the optimal alignment?
Chapter 15. Dynamic Programming

DP applied to summation problems

Unmatched Socks Problem

There are $n$ pairs of socks, each a different color; each day, remove arbitrarily two individual socks; compute the probability that, on none of the $k$ days, a pair of color-matched socks is removed.

When $n = 5$, $k = 1$

\[ P_1 = \frac{\text{number of color-not-matched pairs}}{\text{number of total pairs}} = \frac{10 \times 8}{2 \times 10^2} = \frac{40}{45} = \frac{8}{9} \]

When $n = 5$, $k = 2$

\[ P_2 = P_1 \times \left( P_u + P_m + P_{u,m} \right) = \frac{8}{9} \times \left( \frac{1}{2 \times 8^2} + \frac{6 \times 4}{2 \times 8^2} + 2 \times 6 \times 8^2 \right) = \frac{8}{9} \times \frac{25}{28} \]

\[ P_3 = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{64} \]
Chapter 15. Dynamic Programming

DP applied to summation problems

Unmatched Socks Problem
DP applied to summation problems

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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DP applied to summation problems

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When \( n = 5 \), \( k = 1 \)

\[
P_1 = \frac{\text{number of color-not-matched pairs}}{\text{number of total pairs}} = \frac{(10 \times 8)/2}{\binom{10}{2}}
\]
Chapter 15. Dynamic Programming

**DP applied to summation problems**

**Unmatched Socks Problem**

There are $n$ pairs of socks, each a different color;
- each day, remove arbitrarily two individual socks;
compute the probability that, on none of the $k$ days, a pair of color-matched socks is removed.

When $n = 5$, $k = 1$

$$P_1 = \frac{\text{number of color-not-matched pairs}}{\text{number of total pairs}} = \frac{(10 \times 8)/2}{(10 \choose 2)} = \frac{40}{45} =$$
Chapter 15. Dynamic Programming

DP applied to summation problems

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Chapter 15. Dynamic Programming

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DP applied to summation problems

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Chapter 15. Dynamic Programming

DP applied to summation problems

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Chapter 15. Dynamic Programming

DP applied to summation problems

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Chapter 15. Dynamic Programming

DP applied to summation problems

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\[
P_3 = P_2 \times ??
\]
Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Let $m_k$ be the number of pairs of color-matched socks after selection on the $k$th day.
Chapter 15. Dynamic Programming

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Let $m_k$ be the number of pairs of color-matched socks after selection on the $k$th day.

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Let $m_k$ be the number of pairs of color-matched socks after selection on the $k$th day.

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Note that

$$2m_k + u_k = 2n - 2k, \text{ for every } k \leq n$$
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We define $P_k(m_k, u_k)$ to be the total probability that not a pair of matched socks is selected on any day of the first $k$ days, resulting in $m_k$ and $u_k$. Then
Chapter 15. Dynamic Programming

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We define $P_k(m_k, u_k)$ to be the total probability that not a pair of matched socks is selected on any day of the first $k$ days, resulting in $m_k$ and $u_k$. Then

$$P_k(m_k, u_k) = \sum \begin{cases} 
P_{k-1}(m_k + 2, u_k - 2) \times \frac{(m_k+2)(2m_k+2)}{2m_k+2+u_k} \\
P_{k-1}(m_k + 1, u_k) \times \frac{2(m_k+1)u_k}{2m_k+2+u_k} \\
P_{k-1}(m_k, u_k + 2) \times \frac{u_k+2}{2m_k+2+u_k} 
\end{cases}$$
Chapter 15. Dynamic Programming

We have seen a number of examples with "forward DP" solutions.

Matrix Chain Multiplication problem
Input: matrices $A_1, \ldots, A_n$, where $A_i$ has dimension $p_{i-1} \times p_i$;
Output: a parenthesization by which the product $A_1 \times A_2 \times \cdots \times A_n$ uses the minimum number of scalar multiplications.
We have seen a number of examples with "forward DP" solutions. (Most of these problems are solvable with "backward DP" as well).

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Matrix Chain Multiplication problem
Chapter 15. Dynamic Programming

We have seen a number of examples with "forward DP" solutions.

(Most of these problems are solvable with "backward DP" as well).

We now consider an example with an "inside-out DP" solution.

Matrix Chain Multiplication problem

**INPUT:** matrices $A_1, \cdots, A_n$, where $A_i$ has dimension $p_{i-1} \times p_i$;

**OUTPUT:** a parenthesization by which the product $A_1 \times A_2 \times \cdots \times A_n$
uses the minimum number of scalar multiplications.
Chapter 15. Dynamic Programming

We have seen a number of examples with "forward DP" solutions.
(Most of these problems are solvable with "backward DP" as well).

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Chapter 15. Dynamic Programming

\[5 \times 4 \times 8 = 5 \times 2\]
Chapter 15. Dynamic Programming

scalable multiplications

= 5 \times 4 \times 8
Chapter 15. Dynamic Programming

Many possible parenthesizations

\[ P(n) = \sum_{k=1}^{n-1} P(k) \cdot P(n-k) \]

Catalan number
Chapter 15. Dynamic Programming

Many possible parenthesizations

1. \((A (B (C D)))\)
   \[5(5)2 + 5(4)2 + 4(8)2 = 154\]
2. \((A ((B C) D))\)
   \[5(5)2 + 5(4)8 + 5(8)2 = 290\]
3. \(((A B) (C D))\)
   \[5(5)4 + 4(8)2 + 5(4)2 = 204\]
4. \(((A (B C)) D)\)
   \[5(4)8 + 5(5)8 + 5(8)2 = 440\]
5. \(((A B) C) D)\)
   \[5(5)4 + 5(4)8 + 5(8)2 = 340\]

Note: The number of all possible parenthesizations is
\[P(n) = \sum_{k=1}^{n-1} P(k)P(n-k),\] where the Catalan number
Chapter 15. Dynamic Programming

Many possible parenthesizations

Note: The number of all possible parenthesizations is

\[ P(n) = \sum_{k=1}^{n-1} P(k)P(n-k), \text{ Catalan number} \]
Chapter 15. Dynamic Programming

**Step 1**: identify *optimal substructure*:
Optimal solution in terms of optimal solutions to subproblems:

1. The best parenthesization of $A_i A_{i+1} \cdots A_j$ must be $(A_i \cdots A_k) (A_{k+1} \cdots A_j)$ for some $k, i \leq k < j$.
2. The optimal cost of $A_i A_{i+1} \cdots A_j$ must be the smallest among optimal costs for $(A_i \cdots A_k) (A_{k+1} \cdots A_j)$ for $k = i, i+1, \cdots, j-1$.
3. For each $k$, the optimal cost for the above is the optimal cost for $(A_i \cdots A_k) +$ the optimal cost for $(A_{k+1} \cdots A_j) +$ the number of scalar multiplications to multiply the two terms.
Chapter 15. Dynamic Programming

Step 1: identify *optimal substructure*:

Optimal solution in terms of optimal solutions to subproblems:

(1) the best parenthesization of $A_i A_{i+1} \cdots A_j$ must be

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Chapter 15. Dynamic Programming

**Step 1**: identify *optimal substructure*:

Optimal solution in terms of optimal solutions to subproblems:

1. The best parenthesization of $A_iA_{i+1}\cdots A_j$ must be

   $$(A_i\cdots A_k)(A_{k+1}\cdots A_j)$$

   for some $k, i \leq k < j$.

2. The optimal cost of $A_iA_{i+1}\cdots A_j$ must be the smallest among optimal costs for

   $$(A_i\cdots A_k)(A_{k+1}\cdots A_j)$$

   $k = i, i + 1, \cdots, j - 1$. 
Step 1: identify optimal substructure:
  Optimal solution in terms of optimal solutions to subproblems:

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(A_i \cdots A_k)(A_{k+1} \cdots A_j)
\]

for some \( k, i \leq k < j \).

(2) The optimal cost of \( A_i A_{i+1} \cdots A_j \) must be the smallest among optimal costs for

\[
(\ A_i \cdots A_k)(A_{k+1} \cdots A_j)
\]

\( k = i, i + 1, \cdots, j - 1 \).

(3) For each \( k \), the optimal cost for the above =

  the optimal cost for \( (A_i \cdots A_k) \) +
  the optimal cost for \( (A_{k+1} \cdots A_j) \) +
  the number of scalar multiplications to multiply the two terms.
Step 2: objective function and recurrence

Define \( f(i,j) \) to be the minimum number of scalar multiplications needed for
\( A_i A_{i+1} \cdots A_j \).

Then
\[
    f(i,j) = \min_{i \leq k < j} \{ f(i,k) + f(k+1,j) + p_{i-1} p_k p_{j-1} \}
\]

where base case is
\( f(i,j) = 0 \) when \( i = j \), for single matrix \( A_i \).
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Step 3. Fill out the DP table for function $f$. 
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- Table size $n \times n$;
Chapter 15. Dynamic Programming

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- Table size $n \times n$;
- base cases are on the main diagonal;
- smaller instances are of smaller $j-1$ values;
- diagonal cells have the same $j-1$ values;
Chapter 15. Dynamic Programming

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<td></td>
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</tr>
</tbody>
</table>

Side lengths: 1, 4, 5, 3, 6, 7, 1

\[ A_1^{(1 \times 4)} \times A_2^{(4 \times 5)} \times A_3^{(5 \times 3)} \times A_4^{(3 \times 6)} \times A_5^{(6 \times 7)} \times A_6^{(7 \times 1)} \]
Algorithm \textsc{MatrixChainOrder}(p)

1. \hspace{1cm} n = length[p] − 1;
Chapter 15. Dynamic Programming

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Algorithm MatrixChainOrder($p$)
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4. for $l = 2$ to $n$
5. \hspace{1em} for $i = 1$ to $n - l + 1$
6. \hspace{2em} $j = i + l - 1$;
7. \hspace{2em} $M[i,j] = \infty$;
8. \hspace{2em} for $k = i$ to $j - 1$
10. if $q < M[i,j]$ then
11. \hspace{2em} $M[i,j] = q$;
12. \hspace{2em} $S[i,j] = k$;
13. return $(M,S)$

Time complexity: $O(n^2 \times n)$
Algorithm MatrixChainOrder(p)
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5. \hspace{1em} \textbf{for} \ i = 1 \ \textbf{to} \ n − l + 1
6. \hspace{1em} \hspace{1em} j = i + l − 1;
7. \hspace{1em} \hspace{1em} M[i, j] = \infty;
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3. \quad \quad M[i, i] = 0;
4. \quad \textbf{for } l = 2 \textbf{ to } n
5. \quad \quad \textbf{for } i = 1 \textbf{ to } n - l + 1
6. \quad \quad \quad j = i + l - 1;
7. \quad \quad \quad M[i, j] = \infty;
8. \quad \quad \quad \textbf{for } k = i \textbf{ to } j - 1
9. \quad \quad \quad \quad q = M[i, k] + M[k + 1, j] + p[i - 1]p[k]p[j]
10. \quad \quad \quad \quad \textbf{if } q < M[i, j]

Time complexity: $O(n^2 \times n)$
Algorithm **MatrixChainOrder**(\(p\))

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3.      \(M[i, i] = 0;\)
4.  for \(l = 2 \text{ to } n\)
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12. \[ S[i, j] = k \]
13. \textbf{return} \((M, S)\)

Time complexity: \( O(n^2 \times n) \)
Chapter 15. Dynamic Programming

Step 4. obtain the optimization parenthesization
Step 4. obtain the optimization parenthesization

Algorithm $\text{MatrixChainParenthesization}(i, j, S)$
1. if $i < j$
2. $k = s[i, j]$
3. print $(i, j, " : ", k)$
4. $\text{MatrixChainParenthesization}(i, k, S)$
5. $\text{MatrixChainParenthesization}(k + 1, j, S)$
6. return
Step 4. obtain the optimization parenthesization

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1. \textbf{if} \ i < j \\
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3. \hspace{1em} \textbf{print} (i, j, ” : ”, k) \\
4. \hspace{1em} \textsc{MatrixChainParenthesization}(i, k, S) \\
5. \hspace{1em} \textsc{MatrixChainParenthesization}(k + 1, j, S) \\
6. \textbf{return} \\

Usage: \textbf{call} \textsc{MatrixChainParenthesization}(1, n, S)
Step 4. obtain the optimization parenthesization

Algorithm MatrixChainParenthesization\((i, j, S)\)
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2. \(k = s[i, j]\)
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4. MatrixChainParenthesization\((i, k, S)\)
5. MatrixChainParenthesization\((k + 1, j, S)\)
6. \textbf{return}

Usage: \textbf{call} MatrixChainParenthesization\((1, n, S)\)

Time complexity for MatrixChainParenthesization:
Step 4. obtain the optimization parenthesization

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Usage: call MatrixChainParenthesization\((1, n, S)\)

Time complexity for MatrixChainParenthesization: \(O(n)\).
Chapter 16. Greedy Algorithms

Chapter 16 Greedy Algorithms
Chapter 16. Greedy Algorithms

**Chapter 16** Greedy Algorithms

- Dynamic programming is to *consider all possible choices and select the best.*
Chapter 16. Greedy Algorithms

**Chapter 16** Greedy Algorithms

- Dynamic programming is to *consider all possible choices and select the best.*
  - a DP approach always leads to the optimal solution
Chapter 16. Greedy Algorithms

Chapter 16 Greedy Algorithms

• Dynamic programming is to consider all possible choices and select the best.
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• A greedy algorithm may ignore some choices and select the one that is locally the best.
Chapter 16. Greedy Algorithms

Chapter 16 Greedy Algorithms

- Dynamic programming is to *consider all possible choices and select the best.*
  
a DP approach always leads to the optimal solution

- A greedy algorithm may *ignore some choices and select the one that is locally the best.*
  
a greedy strategy may or may NOT lead to the optimal solution
Chapter 16. Greedy Algorithms

Activity-selection problem
Chapter 16. Greedy Algorithms

Activity-selection problem

**Input:** $n$ activities $S = \{a_1, \ldots, a_n\}$, each with a start time $s_i$ and finish time $f_i$, $1 \leq i \leq n$

**Output:** subset $A \subseteq S$ of mutually compatible activities such that $|A|$ is the maximum.
Chapter 16. Greedy Algorithms

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$A = \{a_1, a_3, a_6, a_8\}$, or
Chapter 16. Greedy Algorithms

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\[
A = \{a_1, a_3, a_6, a_8\}, \text{ or } A = \{a_2, a_5, a_7, a_8\}, \text{ or }
\]
Chapter 16. Greedy Algorithms

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A = \{a_1, a_3, a_6, a_8\}, \text{ or } A = \{a_2, a_5, a_7, a_8\}, \text{ or } A = \{a_2, a_5, a_7, a_9\},
\]
Chapter 16. Greedy Algorithms

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\]
Chapter 16. Greedy Algorithms

A dynamic programming solution
Chapter 16. Greedy Algorithms

A dynamic programming solution

Step 1 analysis of problem
Chapter 16. Greedy Algorithms

A dynamic programming solution

**Step 1** analysis of problem
Chapter 16. Greedy Algorithms

A dynamic programming solution

**Step 1** analysis of problem

How to solve it recursively?
Chapter 16. Greedy Algorithms

For example, selecting $a_3$ would result in two subsets of activities to further consider:

- $\{a_1\}$
- $\{a_6, a_7, a_8, a_9\}$

Selecting $a_5$ would result in two subsets of activities to further consider:

- $\{a_1, a_2\}$
- $\{a_7\}$

- $\{a_8\}$
- $\{a_9\}$
For example, selecting $a_3$ would result in two subsets of activities to further consider: \( \{a_1\} \) and \( \{a_6, a_7, a_8, a_9\} \);
Chapter 16. Greedy Algorithms

For example, selecting $a_3$ would result in two subsets of activities to further consider: $\{a_1\}$ and $\{a_6, a_7, a_8, a_9\}$;

selecting $a_5$ would result in two subsets of activities to further consider: $\{a_1, a_2\}$ and $\{a_7, a_8, a_9\}$;
Chapter 16. Greedy Algorithms

selecting $a_5$ would result in two subsets of activities to further consider: 
\{a_1, a_2\} and \{a_7, a_8, a_9\};
but if selecting again $a_8$ would result in two smaller subsets: 
\{a_7\} and \{\};
these subproblems are not "prefixes" or "suffixes", they could be "middle segments";
we approach such problems with inside-out instead of forward or backward methods.
selecting $a_5$ would result in two subsets of activities to further consider: 
{$a_1, a_2$} and {$a_7, a_8, a_9$};
selecting $a_5$ would result in two subsets of activities to further consider: 
$\{a_1, a_2\}$ and $\{a_7, a_8, a_9\}$;

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selecting $a_5$ would result in two subsets of activities to further consider: 
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Chapter 16. Greedy Algorithms

So we consider solve activity selection on subsets of tasks:

Define subset $S_{i,j}$ as

$$S_{i,j} = \{ a_k \in S : f_i \leq s_k < f_k \leq s_j \}$$

e.g., $S_{2,9} = \{ a_5, a_6, a_7 \}$.

introducing dummy activities: $a_0$ with $s_0 = f_0 = \text{the earliest start time of all}$, and $a_{n+1}$ with $s_{n+1} = f_{n+1} = \text{latest finish time of all}$. 
So we consider solve activity selection on subsets of tasks:

Define subset \( S_{i,j} = \{ a_k \in S : f_i \leq s_k < f_k \leq s_j \} \):

- \( S_{2,9} = \{ a_5, a_6, a_7 \} \)
- \( S_{3,8} = \{ a_6, a_7 \} \)

Introducing dummy activities:
- \( a_0 \) with \( s_0 = f_0 = \text{the earliest start time of all} \)
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Chapter 16. Greedy Algorithms

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\[ S_{2,9} = \{a_k \in S : f_2 \leq s_k < f_k \leq s_9 \} = \{a_5, a_6, a_7\} \]
So we consider solve activity selection on subsets of tasks:

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$S_{3,8} = \{a_6, a_7\}$. 
Chapter 16. Greedy Algorithms

Define subset $S_{i,j} = \{ a_k \in S : f_i \leq s_k < f_k \leq s_j \}$

e.g.,
$S_{2,9} = \{ a_k \in S : f_2 \leq s_k < f_k \leq s_9 \} = \{ a_5, a_6, a_7 \}$
$S_{3,8} = \{ a_6, a_7 \}$.

Introducing dummy activities:
$a_0$ with $s_0 = f_0$ = the earliest start time of all, and
$a_{n+1}$ with $s_{n+1} = f_{n+1} =$ latest finish time of all.
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e.g.,

$$|A_{0,10}| = \max \left\{ \ldots \right\}$$

$$|A_{0,6} \cup \{a_6\} \cup A_{6,10}|$$

where

$S_{0,6} = \{a_1, \ldots, a_4\}$

$S_{6,10} = \{a_8, a_9\}$
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Instead of solving this with DP, we wonder if all but one of the subproblems could be made empty.

\[ A_{i,j} = \max_{a_k \in S_{i,j}} \{ |A_{i,k} \cup \{a_k\} \cup A_{k,j}| \} \]

i.e., a greedy choice of \( a_k \in S_{i,j} \) makes \( A_{i,l} \cup \{a_l\} \cup A_{l,i} \) not needed, for all \( l \neq k \)

But the chosen \( a_k \) has to guarantee to maintain \( |A_{i,j}| \) is the maximum.

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Activity Selection problem has the greedy-choice property.

Theorem 16.1: Let \( S_{i,j} \) be a subproblem and \( a_k \in S_{i,j} \) be the activity of the earliest finish time, then \( a_k \) is in some optimal solution for \( S_{i,j} \).

The theorem is the same as that selecting the activity with the earliest finish time

• makes all other subproblems disappearing, and
• maintains the optimality of solution

Proof: (Using the "swapping method", or "exchange method")

Let \( A_{i,j} \) be an optimal solution for \( S_{i,j} \).

(1) If \( a_k \in A_{i,j} \), then the theorem is true.

(2) If \( a_k \notin A_{i,j} \), assume \( a_p \) is the one with earliest finish time in \( A_{i,j} \).

We define \( B_{i,j} = A_{i,j} - \{a_p\} \cup \{a_k\} \). Then

• because \( f_p \geq f_k \), \( a_k \) is compatible with other activities in \( B_{i,j} \).

So \( B_{i,j} \) is a solution for \( S_{i,j} \).

• \( |B_{i,j}| = |A_{i,j}| \), so \( B_{i,j} \) is also an optimal solution for \( S_{i,j} \).

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Since $a_k \in B_{i,j}$, we prove the theorem.
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Using the Theorem 16.1 and the recurrence

\[ |A_{i,j}| = \max_{a_k \in S_{i,j}} \{|A_{i,k} \cup \{a_k\} \cup A_{k,j}|\} \]

we can derive greedy algorithms for the Activity Selection problem.
Assume that the activities are sorted according to their finish times.

\[ f_1 \leq f_2 \leq \cdots \leq f_n. \]
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Adding hypothetical activity \( a_0 \) (with \( s_0 = f_0 = s_1 \))
Assume that the activities are sorted according to their finish times. 
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Adding hypothetical activity \( a_0 \) (with \( s_0 = f_0 = s_1 \)) and activity \( a_{n+1} \) (with \( s_{n+1} = f_{n+1} = f_n \)).
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Algorithm \textsc{Activity-Selection}(s, f, k, n)
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**Algorithm** Activity-Selection\((s, f, k, n)\)

1. \( m = k + 1 \)
Assume that the activities are sorted according to their finish times.
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**Algorithm** \( \text{Activity-Selection}(s, f, k, n) \)

1. \( m = k + 1 \)
2. while \( m \leq n \) and \( s_m < f_k \)
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Assume that the activities are sorted according to their finish times. 
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5. \textbf{return} \( \{a_m\} \cup \text{Activity-Selection}(s, f, m, n) \)
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First called with \textsc{Activity-Selection}(\( s, f_0, n \))

Time complexity: \( O(n) \).
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Fractional Knapsack Problem:

**Input**: $n$ items of sizes $s_1, ..., s_n$ and values $v_1, ..., v_n$; and a knapsack size $B$.

**Output**: A subset $A \subseteq \{1, 2, ..., n\}$, each with a fraction $0 < f_i \leq 1$, which maximizes $\sum_{i \in A} f_i v_i$ under the constraint $\sum_{i \in A} f_i s_i \leq B$.

Define:

$$K(S, X)$$

be the maximum value of packing items from set $S$ into space $X$.

Then

$$K(S, X) = \max_{0 < f_i \leq 1, i \in S} \{ f_i v_i + K(S - \{i\}, X - f_i s_i) \}$$

- We hope to select some item $i$ that makes other subproblems disappear.
- We select item $i$ such that it has the highest density $d_i = \frac{v_i}{s_i}$.
- For convenience, assume the items are sorted according to the non-decreasing order of density.

**Theorem**: Fractional Knapsack problem has the greedy-choice property.
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**Input:** \( n \) items of sizes \( s_1, \ldots, s_n \) and values \( v_1, \ldots, v_n \); and a knapsack size \( B \)

**Output:** a subset \( A \subseteq \{1, 2, \ldots, n\} \), each with a fraction \( 0 < f_i \leq 1 \), which maximizes \( \sum_{i \in A} f_i v_i \) under the constraint \( \sum_{i \in A} f_i s_i \leq B \)

Define: \( K(S, X) \) be the maximum value of packing items from set \( S \) into space \( X \). Then

\[
K(S, X) = \max_{0 < f_i \leq 1, i \in S} \left\{ f_i v_i + K(S - \{i\}, X - f_i s_i) \right\}
\]

- We hope to select some item \( i \) that makes other subproblems disappear.
- We select item \( i \) such that it has the **highest density** \( d_i = \frac{v_i}{s_i} \).
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**Theorem:** Fractional Knapsack problem has the greedy-choice property.
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Given a subset of items $S \subseteq \{1, 2, \ldots, n\}$ and space $X$. Assume item $i$ has the highest density in $S$. Then there is an optimal solution for $S$ which contains the fraction $f_i$ of the item $i$, for some maximum $f_i$, $0 \leq f_i \leq 1$ such that $f_is_i \leq X$. 
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(This is left as an exercise)
Solve the following problem of **Coin Change** problem with **DP**:

**Input:** \( X \) cents of money and coin denominations \( \{d_1, d_2, d_3\} \) where \( 1 \leq d_1 < d_2 < d_3 \);

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Quiz # 3

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Review for Midterm I

- Big-$O$, Big-$\Omega$, lower bound, upper bound
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Huffman Code
Chapter 16. Greedy Algorithms

Huffman Code

compressing data using binary bits
Chapter 16. Greedy Algorithms

Huffman Code

compressing data using binary bits
code: a compressing scheme,
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Huffman Code

compressing data using binary bits
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### Huffman Code

compressing data using binary bits

code: a compressing scheme, fixed length vs variable length code

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<th>character</th>
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Huffman Code

compressing data using binary bits
code: a compressing scheme, fixed length vs variable length code

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- $d_T(c)$ is the length of codeword for $c$ under the code scheme $T$.
- $B(T) \times n$ is the number of bits required to code a file (of $n$ characters).
Chapter 16. Greedy Algorithms

Algorithm **HUFFMAN**$(C, f)$
Algorithm \textsc{Huffman}(C, f)

1. \( n = |C| \)
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Chapter 16. Greedy Algorithms

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4. \textbf{newnode}(z)
5. \( x = \textsc{Extract-Min}(Q) \) \quad \text{extract two least frequent characters}
Algorithm \textsc{Huffman}(C, f)

1. \hspace{1em} n = |C| \\
2. \hspace{1em} Q = C \\
3. \hspace{1em} \textbf{for} \ i = 1 \textbf{to} n - 1 \\
4. \hspace{2em} \textbf{newnode}(z) \\
5. \hspace{2em} x = \textsc{Extract-Min}(Q) \quad \text{extract two least frequent characters} \\
6. \hspace{3em} z.leftchild = x
Chapter 16. Greedy Algorithms

Algorithm HUFFMAN($C, f$)

1. $n = |C|$
2. $Q = C$
3. for $i = 1$ to $n - 1$
4. \hspace{1em} newnode($z$)
5. \hspace{1em} $x = \text{Extract-Min}(Q)$ \hspace{1em} extract two least frequent characters
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8. $z.rightchild = y$
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3. for $i = 1$ to $n - 1$
   4. $\text{newnode}(z)$
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9.  \hspace{1cm} \text{INSERT}($Q, z$)
9.  \hspace{1cm} return ($\text{EXTRACT-MIN}(Q)$)
Chapter 16. Greedy Algorithms

An example of Huffman algorithm

- Symbols: A, B, C, D, E, F, G
  freq.: 2, 3, 5, 8, 13, 15, 18

- Huffman codes:
  A: 10100  B: 10101  C: 1011
  D: 100    E: 00     F: 01
  G: 11

A Huffman code Tree
Correctness of Huffman’s algorithm

Lemma 16.2 (Greedy choice property)

Let $C$ be an alphabet and $f$ be the frequency function for characters in $C$. Let $x$ and $y$ be two characters in $C$ having the lowest frequencies. Then there exists an optimal prefix code for $C$ in which the codewords for $x$ and $y$ have the same length and differ in the last bit.

Proof:
Assume $C$, $f$, $x$ and $y$ are as described in the lemma condition. Let $T$ be any optimal prefix code for $C$.

(a) if $T$ contains the situation about $x$ and $y$ stated in the lemma. Done for the proof!

(b) let $a, b \in C$ be two characters such that their codewords have the same length and differ in the last bit. Also assume that their codeword length $d_T(a) = d_T(b) \geq d_T(c)$, for every $c \in C$. We note that such $a$ and $b$ can be found from $T$ without difficulty.
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\[ (f(a) - f(x)) d_T(a) + (f(x) - f(a)) d_T(x) + (f(b) - f(y)) d_T(b) + (f(y) - f(b)) d_T(y) \geq 0 \]

which implies $B(T) \geq B(T')$. So $T'$ is also an optimal code for $C$. 

So $T'$ satisfies the lemma.
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$$B(T) - B(T') = \sum_{c \in C} f(c) d_T(c) - \sum_{c \in C} f(c) d_{T'}(c) = f(a) d_T(a) + f(b) d_T(b) + f(x) d_T(x) + f(y) d_T(y) - f(a) d_{T'}(a) - f(b) d_{T'}(b) - f(x) d_{T'}(x) - f(y) d_{T'}(y) = (f(a) - f(x))(d_T(a) - d_T(x)) + (f(x) - f(a))(d_T(x) - d_T(y)) \geq 0$$

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Chapter 16. Greedy Algorithms

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$$= f(a)d_T(a) + f(b)d_T(b) + f(x)d_T(x) + f(y)d_T(y)$$

$$- f(a)d_{T'}(a) - f(b)d_{T'}(b) - f(x)d_{T'}(x) - f(y)d_{T'}(y)$$

$$\geq 0$$

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$$= f(a) d_T(a) + f(b) d_T(b) + f(x) d_T(x) + f(y) d_T(y)$$

$$- f(a) d_{T'}(a) - f(b) d_{T'}(b) - f(x) d_{T'}(x) - f(y) d_{T'}(y)$$

$$= (f(a) - f(x)) d_T(a) + (f(x) - f(a)) d_T(x) + (f(b) - f(y)) d_T(b)$$

$$+ (f(y) - f(b)) d_T(y)$$

which implies $B(T) \geq B(T')$. So $T'$ is also an optimal code for $C$.

So $T'$ satisfies the lemma.
We use the following swapping strategy to obtain another prefix code \( T' \) for \( C \).

Without loss of generality, assume \( \{a, b\} \cap \{x, y\} = \emptyset \).

The created code \( T' \) is the same as \( T \) except the positions \( a \) and \( x \) are swapped in \( T' \); so are the positions of \( b \) and \( y \) swapped in \( T' \).

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B(T) - B(T') = \sum_{c \in C} f(c)d_T(c) - \sum_{c \in C} f(c)d_{T'}(c) \\
= f(a)d_T(a) + f(b)d_T(b) + f(x)d_T(x) + f(y)d_T(y) \\
- f(a)d_{T'}(a) - f(b)d_{T'}(b) - f(x)d_{T'}(x) - f(y)d_{T'}(y) \\
= (f(a) - f(x))d_T(a) + (f(x) - f(a))d_T(x) + (f(b) - f(y))d_T(b) \\
+ (f(y) - f(b))d_T(y) \\
= (f(a) - f(x))(d_T(a) - d_T(x)) + (f(b) - f(y))(d_T(b) - d_T(y)) \geq 0
\]
We use the following swapping strategy to obtain another prefix code $T'$ for $C$.

Without loss of generality, assume $\{a, b\} \cap \{x, y\} = \emptyset$.

The created code $T'$ is the same as $T$ except the positions $a$ and $x$ are swapped in $T'$; so are the positions of $b$ and $y$ swapped in $T'$.

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B(T) - B(T') = \sum_{c \in C} f(c)d_T(c) - \sum_{c \in C} f(c)d_{T'}(c)
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\[
= f(a)d_T(a) + f(b)d_T(b) + f(x)d_T(x) + f(y)d_T(y) - f(a)d_{T'}(a) - f(b)d_{T'}(b) - f(x)d_{T'}(x) - f(y)d_{T'}(y)
\]

\[
= (f(a) - f(x))d_T(a) + (f(x) - f(a))d_T(x) + (f(b) - f(y))d_T(b) + (f(y) - f(b))d_T(y)
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Without loss of generality, assume $\{a, b\} \cap \{x, y\} = \emptyset$.

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$$= (f(a) - f(x))d_T(a) + (f(x) - f(a))d_T(x) + (f(b) - f(y))d_T(b)$$

$$+ (f(y) - f(b))d_T(y)$$

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So $T'$ satisfies the lemma.
Lemma 16.3 (Optimal substructure)
Chapter 16. Greedy Algorithms

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Let $C$ be an alphabet and $f$ be the frequency function for characters in $C$. Let $x$ and $y$ be two characters in $C$ having the lowest frequencies.
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Let $C$ be an alphabet and $f$ be the frequency function for characters in $C$. Let $x$ and $y$ be two characters in $C$ having the lowest frequencies. Let

$$C' = C - \{x, y\} \cup \{z\}$$

and the frequency function for $C'$ is the same as $f$ except that $f(z) = f(x) + f(y)$. 
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and the frequency function for $C'$ is the same as $f$ except that $f(z) = f(x) + f(y)$.

Let $T'$ be any optimal prefix code tree for $C'$. Then the tree $T$ obtained from $T'$ by replacing the leaf node $z$ with an internal node having $x$ and $y$ as children, is an optimal prefix code tree for $C$. 

Proof: 

The code $T$ for $C$ constructed from $T'$, a code for $C'$, has the following objective function value

$$B(T) = \sum_{c \in C} f(c) d_T(c) = \sum_{c \in C - \{x, y\}} f(c) d_{T'}(c) + (f(x) + f(y)) d_{T'}(z) + 1$$

$$= \sum_{c \in C} f(c) d_{T'}(c) + (f(x) + f(y)) (d_{T'}(z) + 1)$$

$$= \sum_{c \in C - \{x, y\}} f(c) d_{T'}(c) + (f(x) + f(y)) d_{T'}(z) + f(x) + f(y)$$

$$= \sum_{c \in C - \{x, y\}} f(c) d_{T'}(c) + f(x) + f(y) = B(T') + f(x) + f(y)$$

Therefore, $T$ is an optimal prefix code tree for $C$. 

Lemma 16.3 (Optimal substructure)

Let $C$ be an alphabet and $f$ be the frequency function for characters in $C$. Let $x$ and $y$ be two characters in $C$ having the lowest frequencies. Let

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The code $T$ for $C$ constructed from $T'$, a code for $C'$, has the following objective function value

$$B(T) = \sum_{c \in C} f(c)d_T(c) = \sum_{c \in C - \{x, y\}} f(c)d_T(c) + f(x)d_T(x) + f(y)d_T(y)$$
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Proof:
The code $T$ for $C$ constructed from $T'$, a code for $C'$, has the following objective function value

$$B(T) = \sum_{c \in C} f(c)d_T(c) = \sum_{c \in C \setminus \{x, y\}} f(c)d_T(c) + f(x)d_T(x) + f(y)d_T(y)$$
$$= \sum_{c \in C \setminus \{x, y\}} f(c)d_T(c) + \left[f(x) + f(y)\right]d_T(x)$$
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Let $C$ be an alphabet and $f$ be the frequency function for characters in $C$. Let $x$ and $y$ be two characters in $C$ having the lowest frequencies. Let

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$$= \sum_{c \in C - \{x,y\}} f(c)d_T(c) + [f(x) + f(y)]d_T(x)$$

$$= \sum_{c \in C - \{x,y\}} f(c)d_{T'}(c) + [f(x) + f(y)](d_{T'}(z) + 1)$$
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$$= \sum_{c \in C - \{x, y\}} f(c)d_T(c) + [f(x) + f(y)]d_T(x)$$
$$= \sum_{c \in C - \{x, y\}} f(c)d_{T'}(c) + [f(x) + f(y)](d_{T'}(z) + 1)$$
$$= \sum_{c \in C - \{x, y\}} f(c)d_{T'}(c) + [f(x) + f(y)]d_{T'}(z) + [f(x) + f(y)]$$
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**Lemma 16.3 (Optimal substructure)**

Let $C$ be an alphabet and $f$ be the frequency function for characters in $C$. Let $x$ and $y$ be two characters in $C$ having the lowest frequencies. Let

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**Proof:**

The code $T$ for $C$ constructed from $T'$, a code for $C'$, has the following objective function value

$$B(T) = \sum_{c \in C} f(c)d_T(c) = \sum_{c \in C - \{x, y\}} f(c)d_T(c) + f(x)d_T(x) + f(y)d_T(y)$$

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Lemma 16.3 (Optimal substructure)

Let \( C \) be an alphabet and \( f \) be the frequency function for characters in \( C \). Let \( x \) and \( y \) be two characters in \( C \) having the lowest frequencies. Let

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The code \( T \) for \( C \) constructed from \( T' \), a code for \( C' \), has the following objective function value

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\]

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= \sum_{c \in C - \{x, y\}} f(c)d_{T'}(c) + [f(x) + f(y)](d_{T'}(z) + 1)
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= \sum_{c \in C'} f(c)d_{T'}(c) + [f(x) + f(y)] = B(T') + f(x) + f(y)
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Let \( C \) be an alphabet and \( f \) be the frequency function for characters in \( C \). Let \( x \) and \( y \) be two characters in \( C \) having the lowest frequencies. Let

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= \sum_{c \in C - \{x, y\}} f(c)d_T(c) + [f(x) + f(y)]d_T(x)
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Now if $T$ is not optimal for $C$, assume optimal prefix code $S$ for $C$ such that $B(S) < B(T)$.
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Now if $T$ is not optimal for $C$, assume optimal prefix code $S$ for $C$ such that $B(S) < B(T)$.

By Lemma 16.2, we assume $S$ to be such that codewords for $x$ and $y$ are of the same depth and differ in the last bit.
Now if $T$ is not optimal for $C$, assume optimal prefix code $S$ for $C$ such that $B(S) < B(T)$.

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We construct another code $S'$ for $C'$ by removing the branches for $x$ and $y$ and assign $z$ to its parent node. We have $B(S') = B(S) - f(x) - f(y)$. But

$$B(S') < B(T) - f(x) - f(y) = B(T') + f(x) + f(y) - f(x) - f(y) = B(T')$$
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$$B(S') < B(T) - f(x) - f(y) = B(T') + f(x) + f(y) - f(x) - f(y) = B(T')$$

This implies $T'$ is not optimal code for $C'$. Contradict.
Now if $T$ is not optimal for $C$, assume optimal prefix code $S$ for $C$ such that $B(S) < B(T)$.

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So $T$ should be optimal for $C$. 

Theorem 16.4

Huffman’s algorithm produces an optimal prefix code.
Now if $T$ is not optimal for $C$, assume optimal prefix code $S$ for $C$ such that $B(S) < B(T)$.

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We construct another code $S'$ for $C'$ by removing the branches for $x$ and $y$ and assign $z$ to its parent node. We have $B(S') = B(S) - f(x) - f(y)$. But

$$B(S') < B(T) - f(x) - f(y) = B(T') + f(x) + f(y) - f(x) - f(y) = B(T')$$

This implies $T'$ is not optimal code for $C'$. Contradict.

So $T$ should be optimal for $C$.

**Theorem 16.4** Huffman’s algorithm produces an optimal prefix code.
Chapter 16. Greedy Algorithms

Coin change problem
Coin change problem

**Input:** money of value $n$, currency coin denominations $D = \{v_1, \ldots, v_m\}$
Chapter 16. Greedy Algorithms

Coin change problem

**Input:** money of value $n$, currency coin denominations $D = \{v_1, \ldots, v_m\}$

**Output:** the least number of coins with denominations in $D$ whose values sum up exactly to $n$. 
Coin change problem

**Input**: money of value \( n \), currency coin denominations \( D = \{v_1, \ldots, v_m\} \)

**Output**: the least number of coins with denominations in \( D \) whose values sum up exactly to \( n \).

A mathematical formulation of the problem

**Input**: positive integer \( n \), and set of positive integers \( D = \{v_1, \ldots, v_m\} \)
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**Input:** positive integer $n$, and set of positive integers $D = \{v_1, \ldots, v_m\}$

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**Input:** money of value $n$, currency coin denominations $D = \{v_1, \ldots, v_m\}$

**Output:** the least number of coins with denominations in $D$ whose values sum up exactly to $n$.

A mathematical formulation of the problem

**Input:** positive integer $n$, and set of positive integers $D = \{v_1, \ldots, v_m\}$

**Output:** a set of positive integers $X = \{x_1, \ldots, x_m\}$ such that

$$\sum_{i=1}^{m} x_i v_i = n$$

to minimize $\left(\sum_{i=1}^{m} x_i\right)$.
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A dynamic programming solution
Chapter 16. Greedy Algorithms

A dynamic programming solution

**analysis:**
- for any value \( k \), one coin of the \( m \) denominations has to be used.
Chapter 16. Greedy Algorithms

A dynamic programming solution

analysis:
● for any value $k$, one coin of the $m$ denominations has to be used.
● then for the left value, one coin of the $m$ denominations has to be used.
A dynamic programming solution

**analysis:**
- for any value \( k \), one coin of the \( m \) denominations has to be used.
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- \( \ldots \)
Chapter 16. Greedy Algorithms

A dynamic programming solution

analysis:
- for any value $k$, one coin of the $m$ denominations has to be used.
- then for the left value, one coin of the $m$ denominations has to be used
  - ...

define objective function:
- $c(k)$ to be the least number of coins to change value $k$. Then

\[
\begin{align*}
\text{for any value } k, \text{ one coin of the } m \text{ denominations has to be used.} \\
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\text{...} \\
\text{define objective function:} \\
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\end{align*}
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A dynamic programming solution

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- ...  

define objective function:
- $c(k)$ to be the least number of coins to change value $k$. Then

\[
c(k) = \min \left\{ \begin{array}{l}
c(k - v_1) + 1 \\
c(k - v_2) + 1 \\
\vdots \\
c(k - v_m) + 1
\end{array} \right. \\
\text{for } k \geq v_1, k \geq v_2, \ldots, k \geq v_m
\]

- base case: $c(v_1) = 1, \ldots, c(v_m) = 1$. 

Chapter 16. Greedy Algorithms

A dynamic programming solution

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$$c(k) = \min \begin{cases} 
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A dynamic programming solution

**analysis:**
- for any value $k$, one coin of the $m$ denominations has to be used.
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Chapter 16. Greedy Algorithms

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A greedy algorithm

Theorem: For the US coin denominations $D_{us} = \{1, 5, 10, 25\}$, the Coin Change problem has the greedy-choice property.

• always use the coin of the largest possible value.

but not all coin denominations have such property, e.g., $\{1, 3, 4, 10\}$, for $n = 6$. 
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Proof: Assume $A$ is the optimal solution for $n$. 

• I. For $5 \leq n < 10$, assume $5 \not\in A$, but all coins in $A$ have to be one-cent coins. Let $B = A - \{1, 1, 1, 1, 1\} \cup \{5\}$. $|B| < |A|$, which is impossible. So $5 \in A$.

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**Proof:** Assume $A$ is the optimal solution for $n$. Consider various cases

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    If $10 \leq k < 25$, according to II, 10 is in the solution for $k$, then $\{10, 10, 10\} \subseteq A$, ...
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• III. For $25 \leq n$, assume $25 \notin A$, but all coins in $A$ are either ten-cents, five-cents or one-cent.

(1) all coins in $A$ are one cent coins, \ldots So $25 \in A$;

(2) there are some five-cent and one-cent coins in $A$. $B = A - C \cup \{25\}$, where $C$ contains five-cent and one-cent coins making up to 25 cents, $|C| > 1$ So $|B| < |A|$, which is impossible. So $25 \in A$;

(3) there is at least one ten-cent coin in $A$. Let $m = n - 10$.
  • If $25 \leq m$, the solution for $m$ has to be either case III.(1) or III.(2), then $25$ is in the solution for $m$, thus $25 \in A$.
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  - big-O
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  - iterative, bottom-up table-filling
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    - 'forward DP', 'backward DP', and 'inside-out DP'

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