CSCI 4470/6470 Algorithms, Fall 2015

Lecture Note III

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Part IV. Advanced Design and Analysis Techniques

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- Chapter 14.9 Exhaustive search
- Chapter 15. Dynamic programming
- Chapter 16. Greedy algorithms

Chapter 14.9.Exhaustive Search

Chapter 14.9. Exhaustive Search (not in the text!)

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To enumerate all possible solutions to the problem instance

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How?

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- systematic examining all solutions

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- systematic examining all solutions
- without repeating solutions that have been examined

Chapter 14.9.Exhaustive Search

Chapter 14.9. Exhaustive Search (not in the text!)

To enumerate all possible solutions to the problem instance

How?

- systematic examining all solutions
- without repeating solutions that have been examined
- stop when a satisfactory solution is found

Chapter 14.9.Exhaustive Search

Example: Boolean Formula Satisfiability problem (SAT)

Chapter 14.9.Exhaustive Search

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INPUT: boolean formula $f(x_1, x_2, \ldots, x_n)$,
OUTPUT: "yes" if and only if $f(x_1, x_2, \ldots, x_n)$ is satisfiable.

Chapter 14.9.Exhaustive Search

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$f(x_1, x_2, x_3) = (x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_3)$ is satisfiable

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$f(x_1, x_2, x_3) = (x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_3)$ is satisfiable

$g(x_1, x_2) = (x_1 \vee \neg x_2) \wedge (\neg x_1 \vee x_2) \wedge (x_1 \vee x_2) \wedge (\neg x_1 \vee \neg x_2)$ is not!

Chapter 14.9.Exhaustive Search

Use exhaustive search to solve the SAT problem.

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How? What will you exhaustively search on?

- Enumerate all combinations of `T` and `F` for $x_1, \ldots, x_n$.

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- Can you solve it with a recursive algorithm?

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$$f(x_1, \ldots, x_{n-1}, x_n) =$$

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Chapter 14.9.Exhaustive Search

Algorithm $\text{SAT SOLVER}(f(x_1, \ldots, x_{n-1}, x_n))$

Chapter 14.9.Exhaustive Search

Algorithm SAT SOLVER$(f(x_1, \ldots, x_{n-1}, x_n))$

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Does this algorithm exhaustively search all assignments to the variables?

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Does this algorithm exhaustively search all assignments to the variables?

- draw a *search tree* based on the algorithm.
- what does the tree look like?
- what does each path mean? how many paths?

Chapter 14.9.Exhaustive Search

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- time?

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- time? $T(n) = 2T(n-1) + cn$, $T(0) = c$, $\Longrightarrow T(n) = \Theta(2^n)$

Chapter 14.9.Exhaustive Search

Solve SAT problem with iterative algorithms

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- How? what to iterate on?

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  $x_1 = \mathtt{F}, x_2 = \mathtt{F}, \ldots, x_n = \mathtt{F},$

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- what to increment

  $(\ldots, \mathtt{F}, \mathtt{T}, \ldots, \mathtt{T})$

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- what to increment

  $(\ldots, \mathtt{F}, \mathtt{T}, \ldots, \mathtt{T}) \longrightarrow (\ldots, \mathtt{T}, \mathtt{F}, \ldots, \mathtt{F})$

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- what is the initial value?

  $x_1 = \mathtt{F}, x_2 = \mathtt{F}, \ldots, x_n = \mathtt{F}$, or simply $(\mathtt{F}, \mathtt{F}, \ldots, \mathtt{F})$

- what to increment

  $(\ldots, \mathtt{F}, \mathtt{T}, \ldots, \mathtt{T}) \longrightarrow (\ldots, \mathtt{T}, \mathtt{F}, \ldots, \mathtt{F})$
  always flip the last bit.

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Algorithm SAT SOLVER-ENUM$(f(x_1, \ldots, x_{n-1}, x_n))$

Chapter 14.9.Exhaustive Search

Algorithm SAT SOLVER-ENUM$(f(x_1, \ldots, x_{n-1}, x_n))$

1. **for** $\langle x_1, \ldots, x_n \rangle = \langle \mathtt{F}, \ldots, \mathtt{F} \rangle$ **to** $\langle \mathtt{T}, \ldots, \mathtt{T} \rangle$

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1. **for** $\langle x_1, \ldots, x_n \rangle = \langle \mathtt{F}, \ldots, \mathtt{F} \rangle$ **to** $\langle \mathtt{T}, \ldots, \mathtt{T} \rangle$
2. $\quad V = Evaluate(f, x_1, \ldots, x_n)$

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2. $\quad V = Evaluate(f, x_1, \ldots, x_n)$
3. $\quad$ **if** $V = \mathtt{T}$, **return** $(\mathtt{T})$

Chapter 14.9.Exhaustive Search

Algorithm SAT SOLVER-ENUM$(f(x_1, \ldots, x_{n-1}, x_n))$

```
for ⟨x_1, ..., x_n⟩ = ⟨F, ..., F⟩ to ⟨T, ..., T⟩
    V = Evaluate(f, x_1, ..., x_n)
    if V = T, return (T)
return (F)
```

Chapter 14.9.Exhaustive Search

Algorithm SAT SOLVER-ENUM$(f(x_1, \ldots, x_{n-1}, x_n))$

1. **for** $\langle x_1, \ldots, x_n \rangle = \langle \mathtt{F}, \ldots, \mathtt{F} \rangle$ **to** $\langle \mathtt{T}, \ldots, \mathtt{T} \rangle$
2. $V = Evaluate(f, x_1, \ldots, x_n)$
3. **if** $V = \mathtt{T}$, **return** $(\mathtt{T})$
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- **for** loop can be implemented by encoding vectors $\langle \mathtt{F}, \ldots, \mathtt{F} \rangle$, $\ldots$, $\langle \mathtt{T}, \ldots, \mathtt{T} \rangle$ with

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- **for** loop can be implemented by encoding vectors $\langle \mathtt{F}, \ldots, \mathtt{F} \rangle$, $\ldots$, $\langle \mathtt{T}, \ldots, \mathtt{T} \rangle$ with binary numbers then further with integers
- a decoding process is needed to converting integers back to vectors

Chapter 14.9.Exhaustive Search

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(Hamiltonian path is a cycle going through every vertex exactly once.)

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- how to encode these permutations as integers ?

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Exhaustive search could be non-trivial

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  taking advantage of the independent set

Chapter 14.9.Exhaustive Search

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- the algorithm terminates when the considered graph is empty.

Chapter 14.9.Exhaustive Search

Algorithm MaxIndSet (G)

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1. **if** $G = \emptyset$ **return** $(\emptyset)$

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3. let G_1 be G with v and all its neighbors removed
4. $I_1 = \{v\} \cup$ MaxIndSet (G_1)

Chapter 14.9.Exhaustive Search

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if  G = ∅ return (∅)
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where m is the number of neighbors of v's,
how to guarantee a large m?

Chapter 14.9.Exhaustive Search

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Chapter 14.9.Exhaustive Search

$$T(n) = T(n-1-m) + T(n-1) + cn^2$$

- $m \geq 0$, $T(n) \leq T(n-1) + T(n-1) + cn^2$,

Chapter 14.9.Exhaustive Search

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Chapter 14.9.Exhaustive Search

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  - can we guarantee $m \geq 3$? possible but a little more complicated.

Chapter 14.9.Exhaustive Search

- Algorithms for SAT and $\textsc{MaxIndSet}$ run in exponential time $O(2^n)$ or $O(\gamma^n)$ for $1 < \gamma < 2$

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Revisit the problem of computing the nth element of Fibonacci sequence

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$F(n) = F(n-1) + F(n-2)$, $F(1) = F(2) = 1$.

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  compute table $T[1..n]$ from left to right,
  by looking up values that have already been computed

Chapter 15. Dynamic Programming

Assembly-line scheduling

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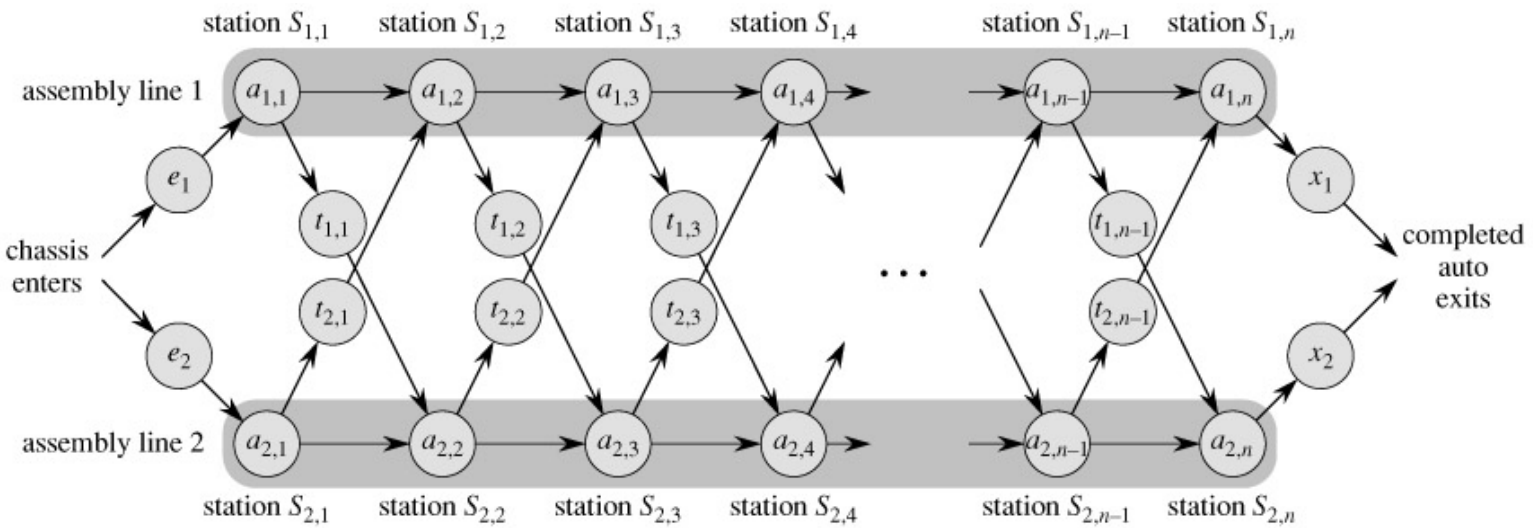

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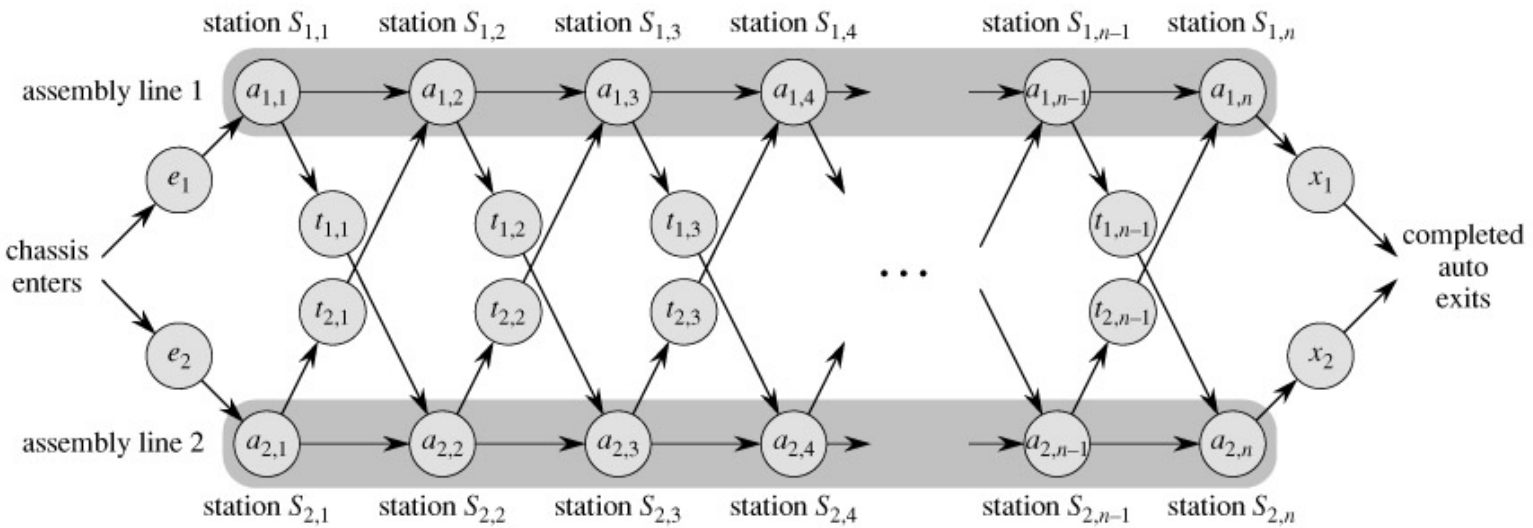
Chapter 15. Dynamic Programming

Assembly-line scheduling



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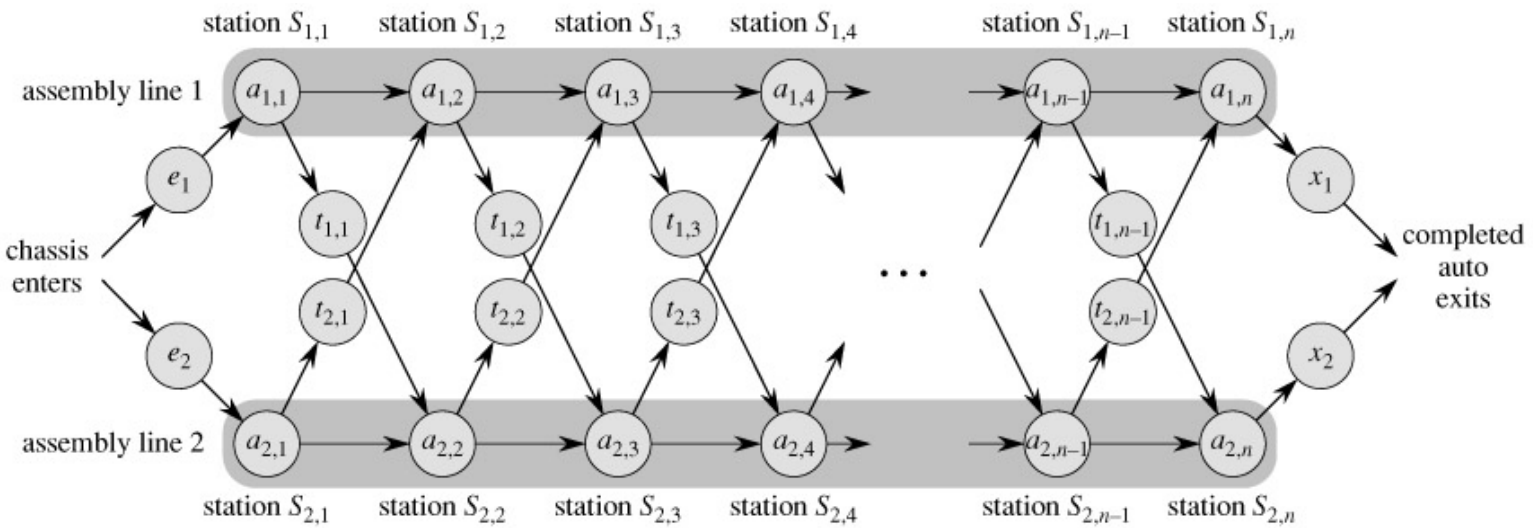
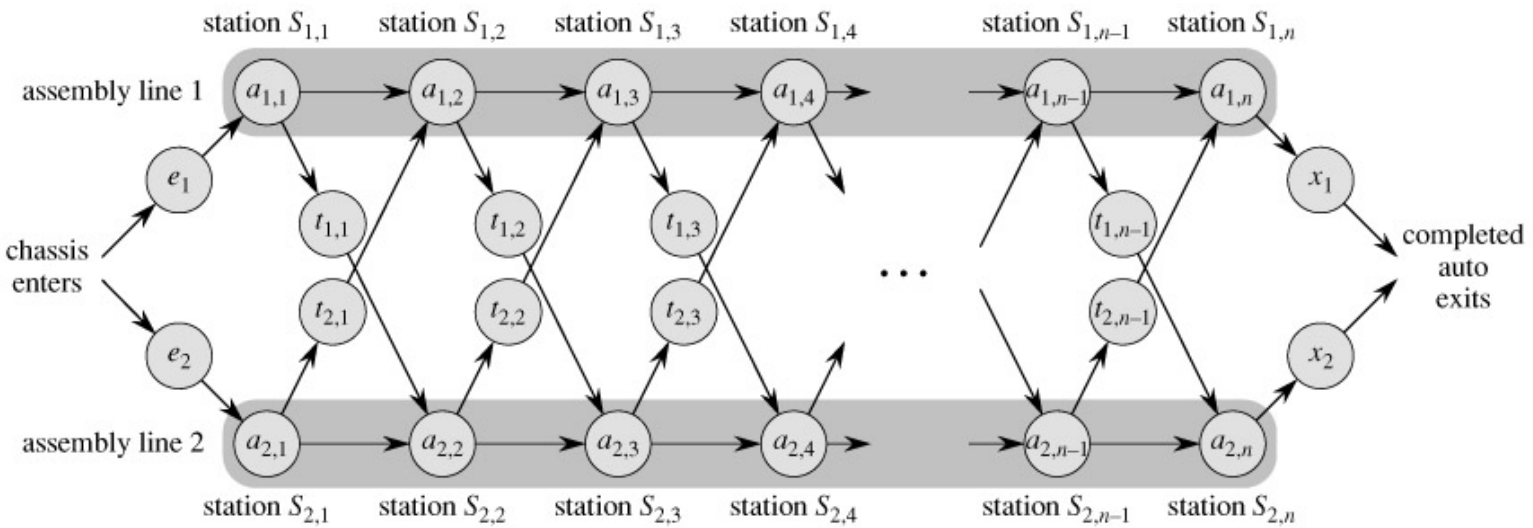


- $S_{i,j}$: the jth station on line i, $i = 1, 2$, $j = 1, 2, \cdots, n$.
- $a_{i,j}$ is the assembly time at station $S_{i,j}$
- $t_{i,j}$ is time for transferring lines from station $S_{i,j}$

Chapter 15. Dynamic Programming

Assembly-line scheduling



- $S_{i,j}$: the jth station on line i, $i = 1, 2$, $j = 1, 2, \cdots, n$.
- $a_{i,j}$ is the assembly time at station $S_{i,j}$
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Chapter 15. Dynamic Programming

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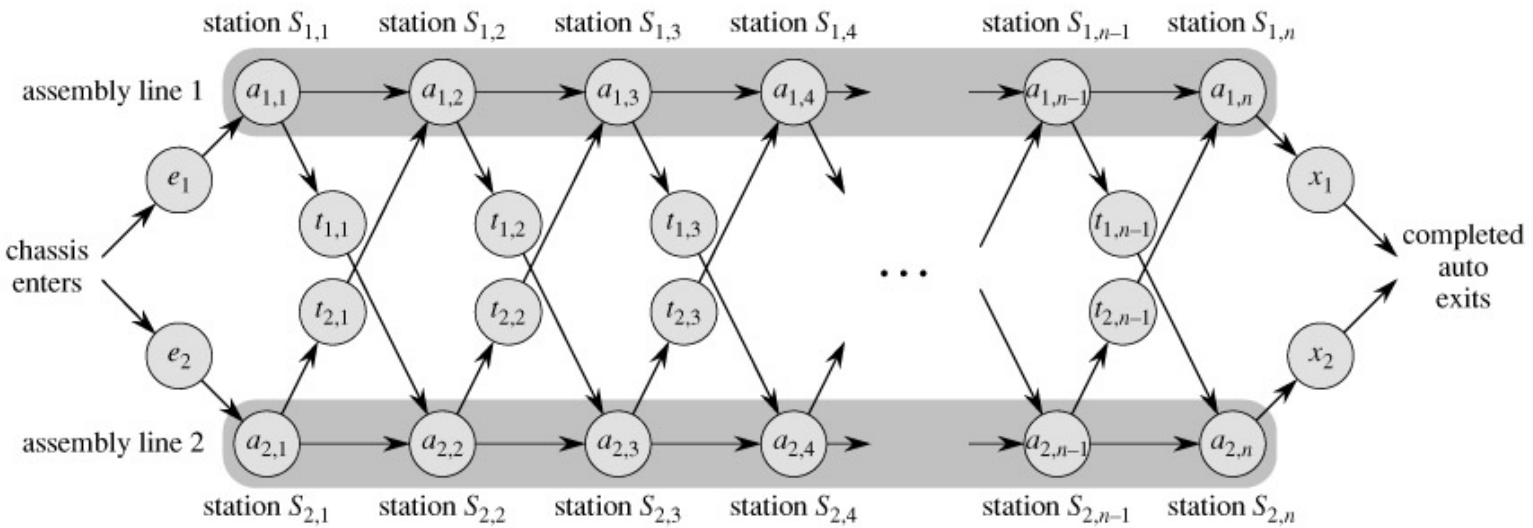

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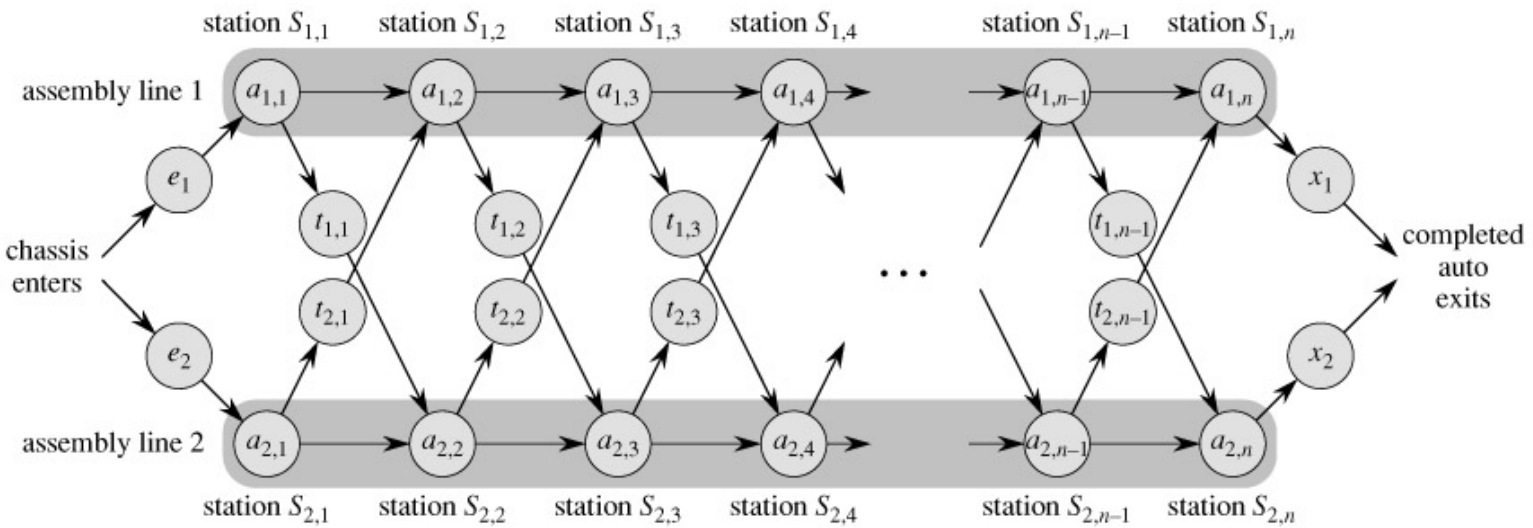

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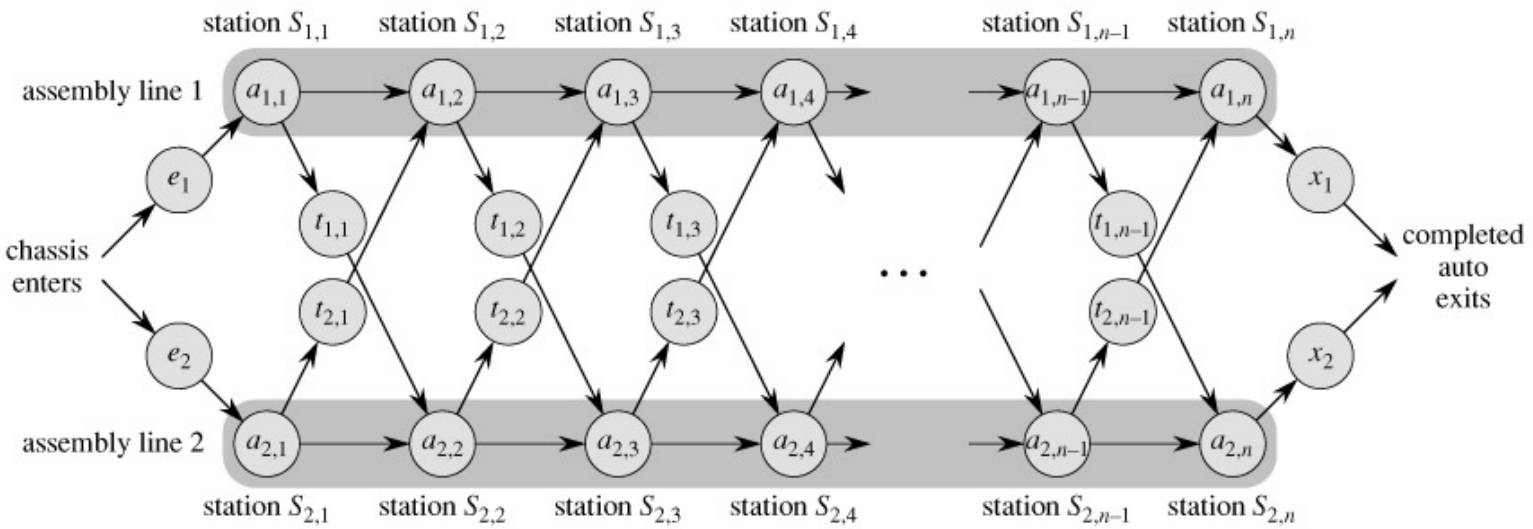
To determine the fastest way through a factory

Chapter 15. Dynamic Programming

Assembly-line scheduling



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To determine the fastest way through a factory

There are 2^n possible ways.

Chapter 15. Dynamic Programming

Analyzing the problem:

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The problem is to determine the fastest way through $S_{1,n}$ or $S_{2,n}$

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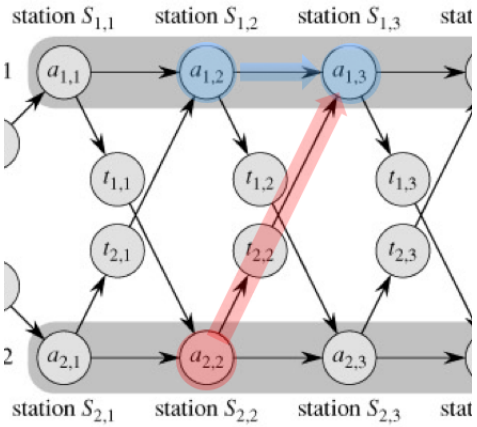

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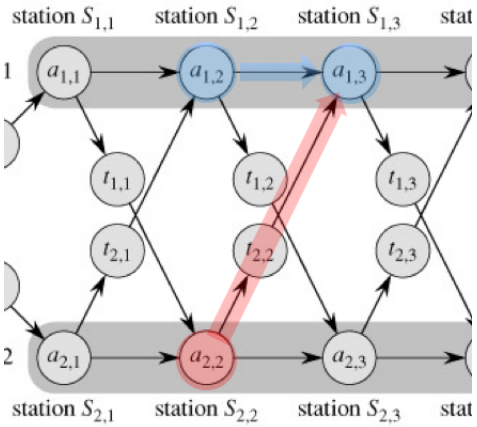

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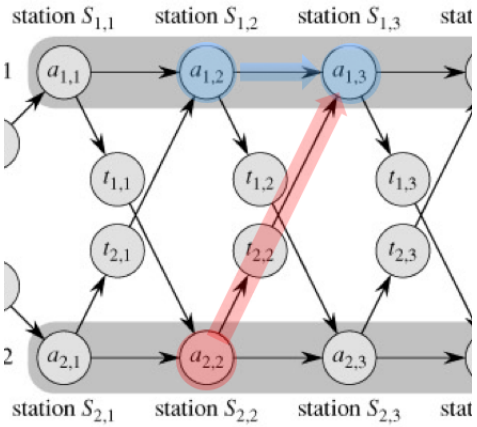

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Chapter 15. Dynamic Programming

Take a dynamic programming approach:

Chapter 15. Dynamic Programming

Take a dynamic programming approach:

step 1: the above analysis

Chapter 15. Dynamic Programming

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step 2: formulate a recursive solution to the problem

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define $f_i(j)$ to be the minimum time **through** station $S_{i,j}$,
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- base cases $f_1(1) = e_1 + a_{1,1}$ and $f_2(1) = e_2 + a_{2,1}$

step 3: compute $f_1(j)$ and $f_2(j)$, for all $j = 1, 2, \ldots, n$.

Chapter 15. Dynamic Programming

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to fill out a $2 \times n$ table, from left to right

$f_1(1)$	$f_1(2)$	$f_1(3)$	$\ldots$	$f_1(n)$
$f_2(1)$	$f_2(2)$	$f_2(3)$	$\ldots$	$f_2(n)$

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- the fastest time is $f^* = \min\{f_1(n) + x_1, f_2(n) + x_2\}$

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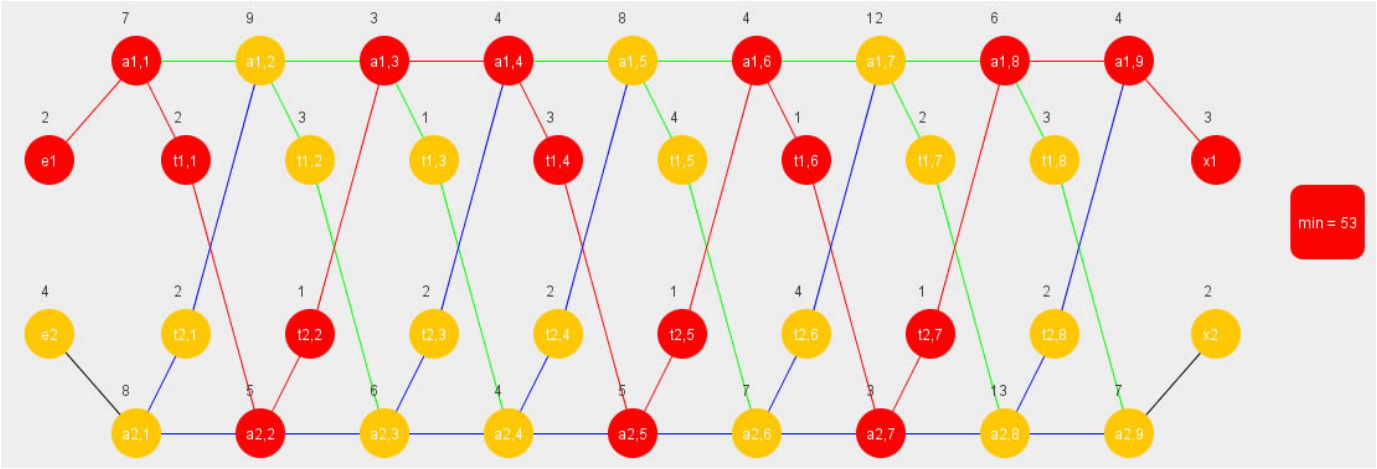
step 4: get the fastest path, not just the faster time.



Chapter 15. Dynamic Programming

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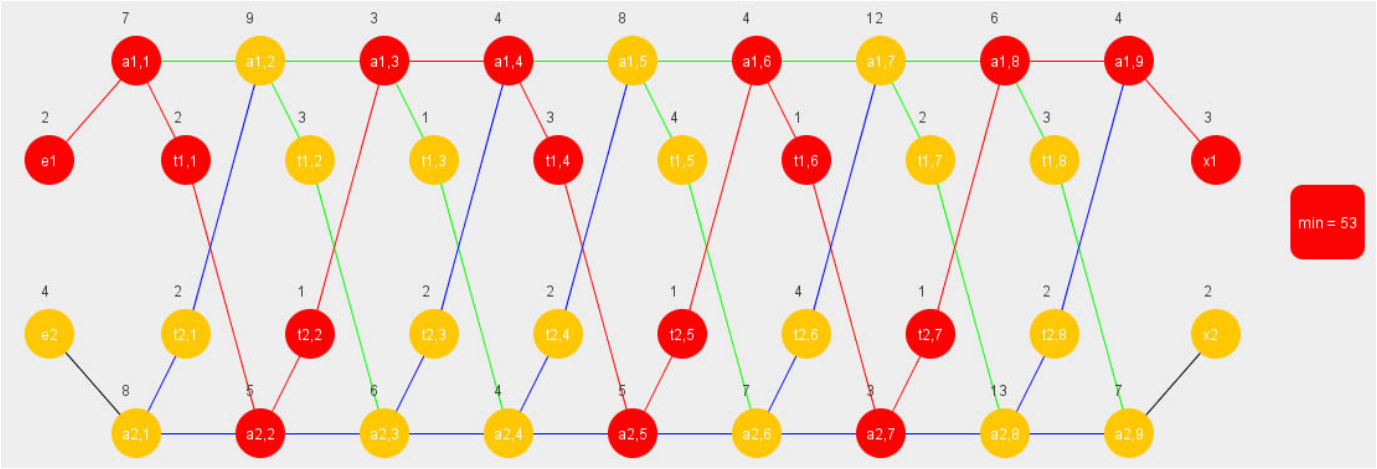

Do we know the fastest path from the fastest time f^*?

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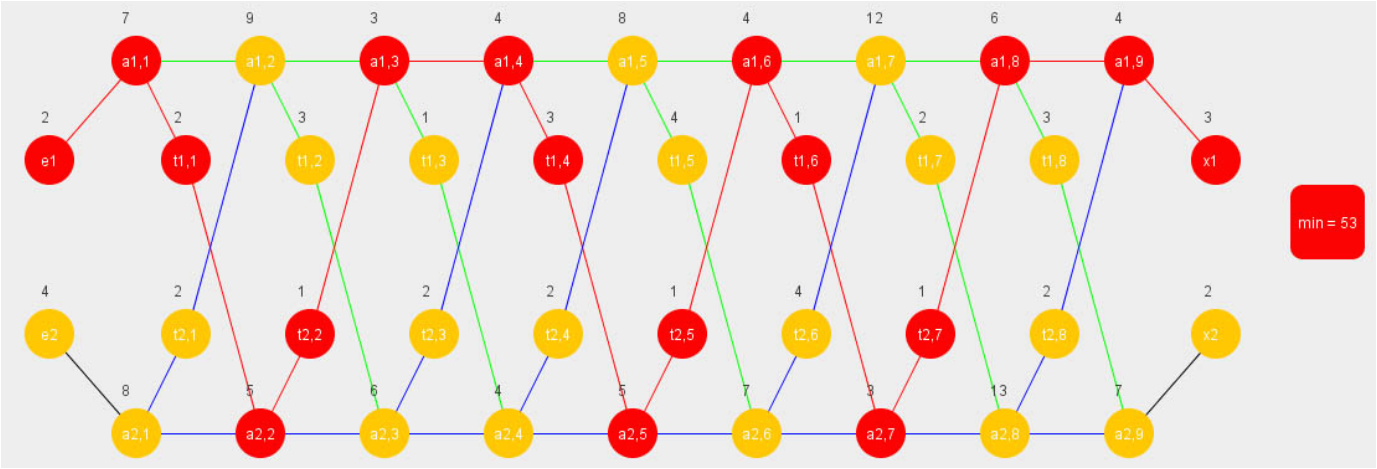

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$f_1(1)$	$f_1(2)$	$f_1(3)$	$\ldots$	$f_1(n-1)$	$f_1(n)$
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Chapter 15. Dynamic Programming

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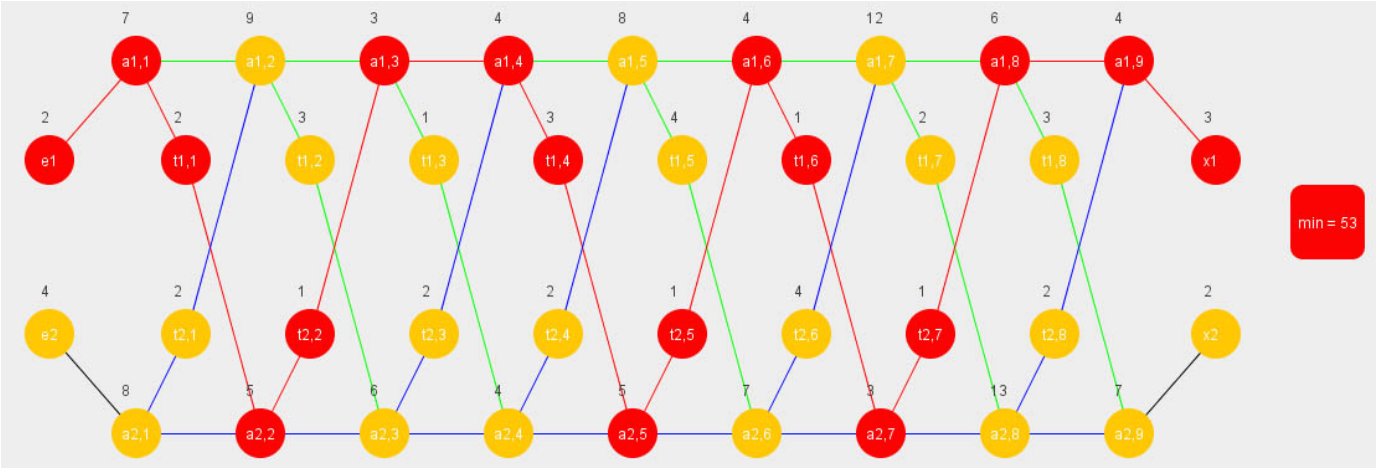

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$$
\begin{aligned}
f^* &= \min\{f_1(n) + x_1, f_2(n) + x_2\} \\
f_1(j) &= \min\{f_1(j-1) + a_{1,j}, f_2(j-1) + t_{2,j-1} + a_{1,j}\} \\
f_2(j) &= \min\{f_2(j-1) + a_{2,j}, f_1(j-1) + t_{1,j-1} + a_{2,j}\}
\end{aligned}
$$

Chapter 15. Dynamic Programming

Algorithm $\textsc{AssemblyLine}(a, t, e, x, n)$

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

Algorithm $\textsc{AssemblyLine}(a, t, e, x, n)$

```
M[1,1] = e_1 + a_{1,1}
M[2,1] = e_2 + a_{2,1}
for j=2 to n
```

Chapter 15. Dynamic Programming

Algorithm $\textsc{AssemblyLine}(a, t, e, x, n)$

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M[1,1] = e_1 + a_{1,1}
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    if M[1, j-1] + a_{1,j} ≤ M[2, j-1] + t_{2,j-1} + a_{1,j}
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            M[2,j] = M[2,j-1] + a_{2,j}
            P[2,j] = 2
         else M[2,j] = M[1,j-1] + t_{1,j-1} + a_{2,j}
                P[2,j] = 1
```

Chapter 15. Dynamic Programming

```
Algorithm AssemblyLine(a, t, e, x, n)
    M[1,1] = e_1 + a_{1,1}
    M[2,1] = e_2 + a_{2,1}
    for j=2 to n
        if M[1,j-1] + a_{1,j} ≤ M[2,j-1] + t_{2,j-1} + a_{1,j}
            M[1,j] = M[1,j-1] + a_{1,j}
            P[1,j] = 1
        else M[1,j] = M[2,j-1] + t_{2,j-1} + a_{1,j}
                P[1,j] = 2
        if M[2,j-1] + a_{2,j} ≤ M[1,j-1] + t_{1,j-1} + a_{2,j}
            M[2,j] = M[2,j-1] + a_{2,j}
            P[2,j] = 2
        else M[2,j] = M[1,j-1] + t_{1,j-1} + a_{2,j}
                P[2,j] = 1
    return (min{f_1(n) + x_1, f_2(n) + x_2})
```

Algorithm $\textsc{PrintPathMain}(M, P, f^*, x)$

1. **if** $(f^* - x_1) = M[1, n]$

Algorithm $\textsc{PrintPathMain}(M, P, f^*, x)$

```
if (f* − x_1) = M[1, n]
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Algorithm PRINTPATHMAIN(M, P, f^*, x)

1. **if** $(f^* - x_1) = M[1, n]$
2. $\quad i = 1$
3. **else** $i = 2$
4. PRINTPATH (P, i, n)

Algorithm PRINTPATH (P, i, j)

1. **if** $j \geq 1$

Algorithm PRINTPATHMAIN(M, P, f^*, x)

1. **if** $(f^* - x_1) = M[1, n]$
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Algorithm PRINTPATH (P, i, j)

1. **if** $j \geq 1$
2. $\quad$ PRINTPATH $(P, P[i, j], j - 1)$

Algorithm PRINTPATHMAIN(M, P, f^*, x)

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if (f* − x_1) = M[1, n]
    i = 1
else i = 2
PRINTPATH (P, i, n)
```

Algorithm PRINTPATH (P, i, j)

```
if j ≥ 1
    PRINTPATH (P, P[i, j], j − 1)
    print (i, j)
```

Chapter 15. Dynamic Programming

Summary on steps for dynamic programming solving the assembly line problem

Chapter 15. Dynamic Programming

Summary on steps for dynamic programming solving the assembly line problem

(1) the structure of optimal solution

Chapter 15. Dynamic Programming

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(1) the structure of optimal solution

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Chapter 15. Dynamic Programming

Summary on steps for dynamic programming solving the assembly line problem

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(4) constructing optimal solution (traceback)

Chapter 15. Dynamic Programming

We consider a little more about the assembly line problem.

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Assume that each station $S_{i,j}$ produces a part p_k with time cost $\tau_{i,j,k}$, where $k = 1, 2, 3$.

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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- Given a sequence of parts produced, can we know the path?

  No, but we may predict such a path with a confidence.

Chapter 15. Dynamic Programming

Example-2. Casino dice and decoding algorithm

Chapter 15. Dynamic Programming

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The house rolls die n times: e.g., $5, 3, 2, 5, 6, 6, 1, \ldots, 1$

Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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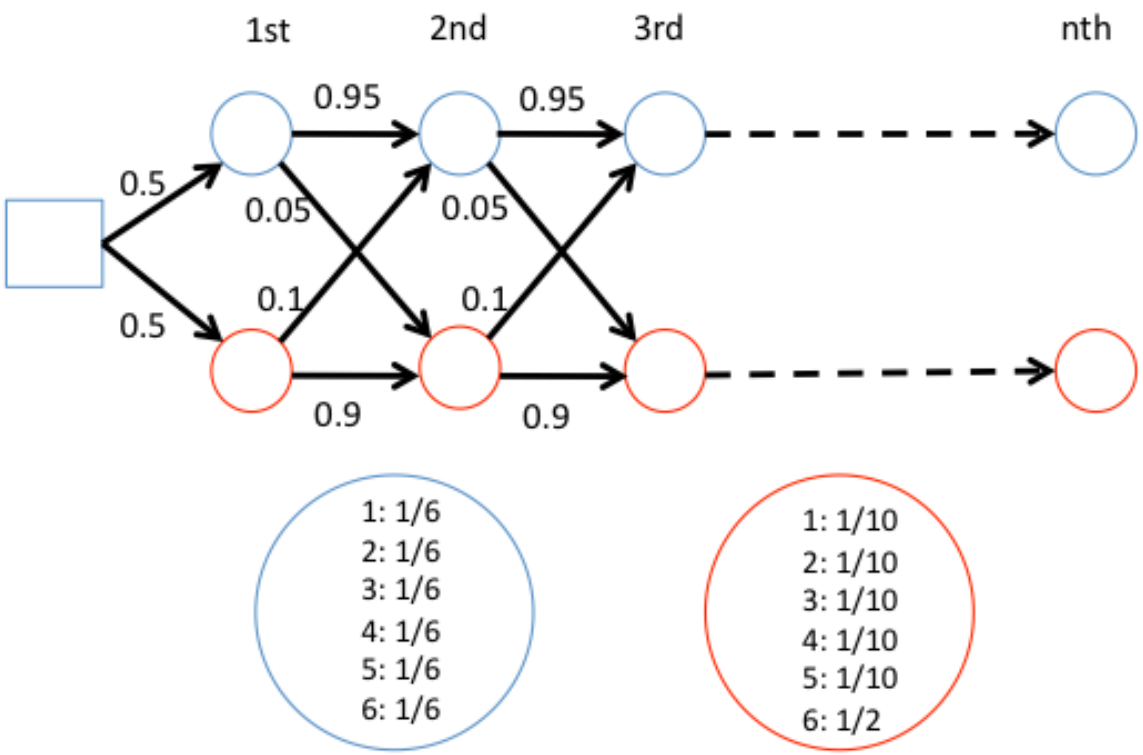
- a small chance to switch is small, a large chance to stay on the same die



Chapter 15. Dynamic Programming

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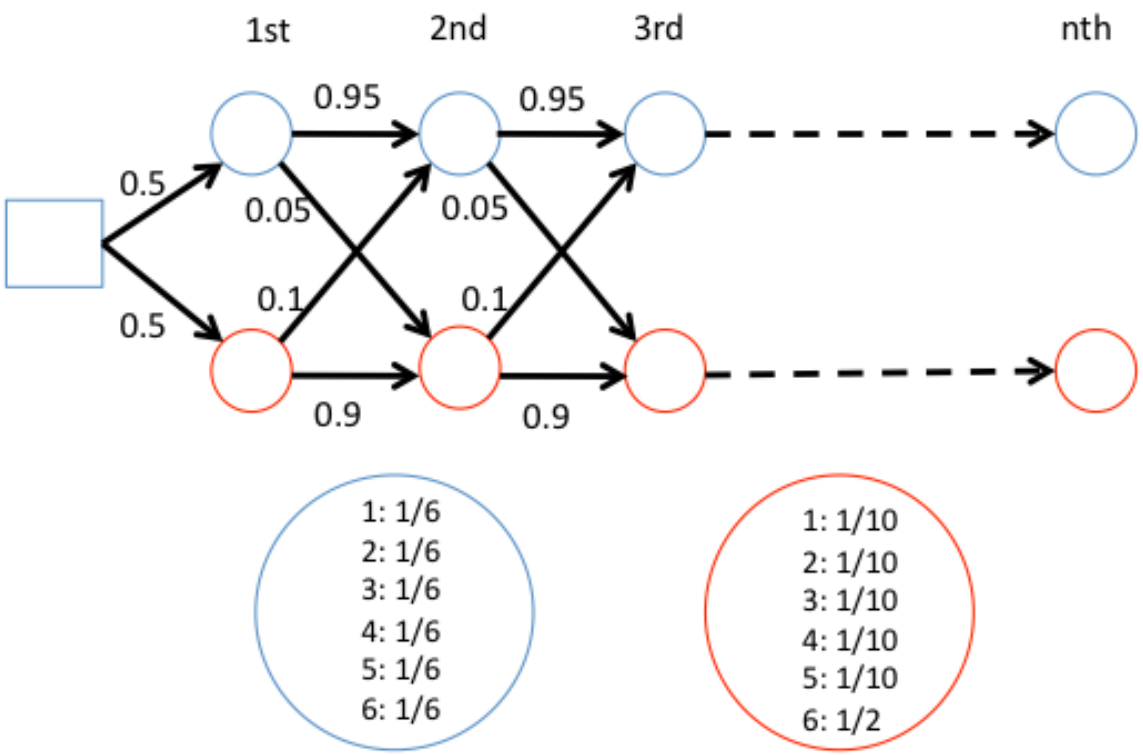

Given the sequence of numbers, e.g., $5, 3, 2, 5, 6, 6, 1, \ldots, 1$, what is the sequence of dices used

Chapter 15. Dynamic Programming

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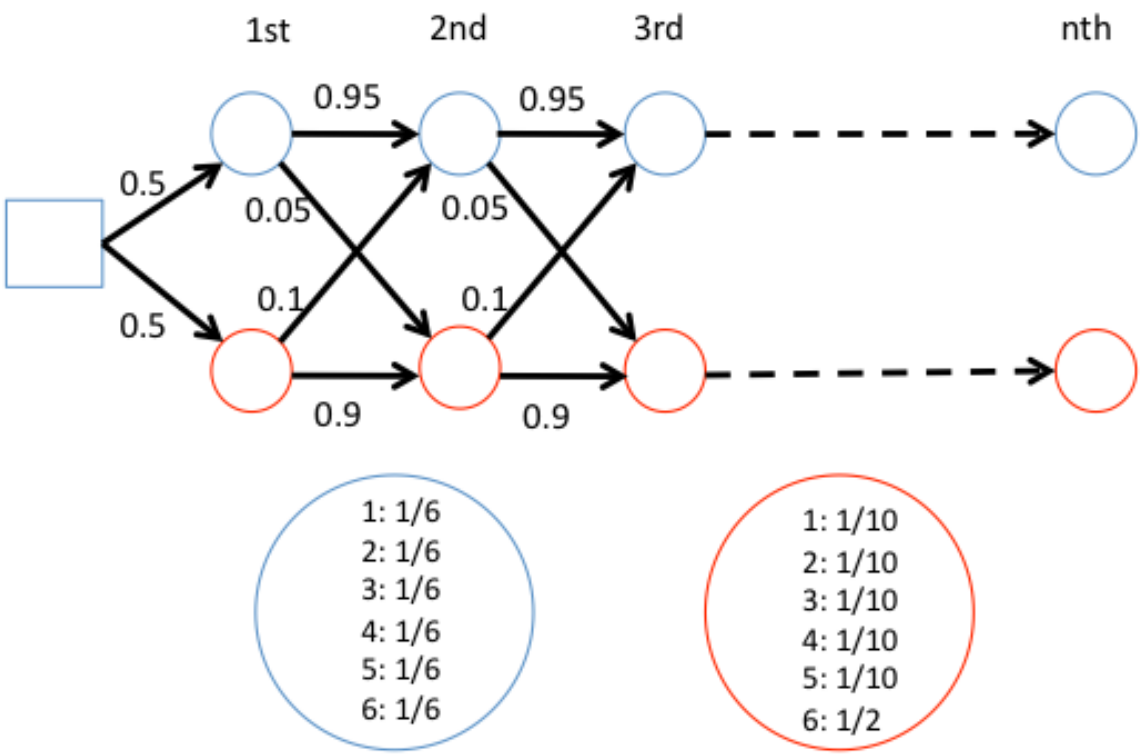
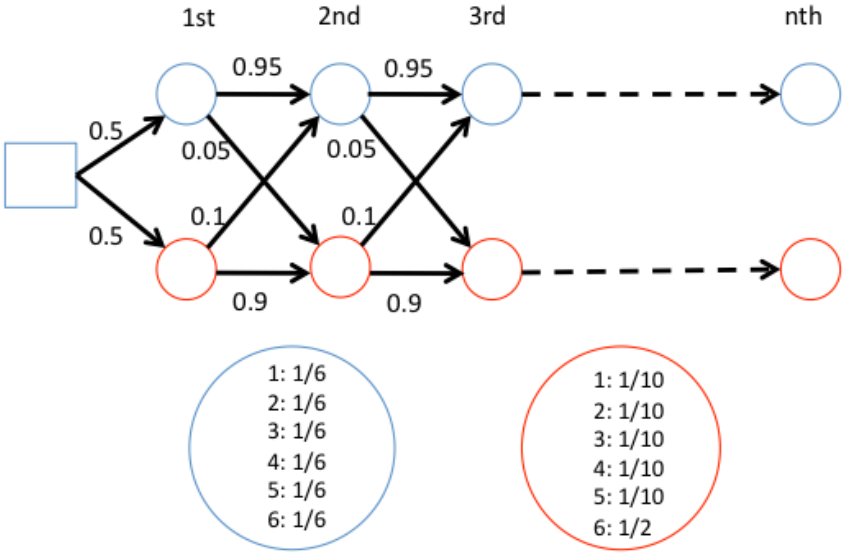


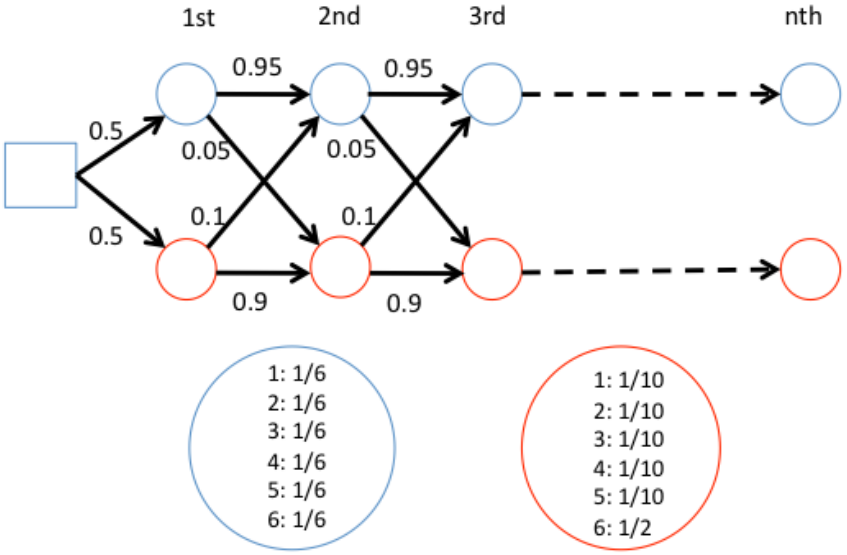
Given the sequence of numbers, e.g., $5, 3, 2, 5, 6, 6, 1, \ldots, 1$, what is the sequence of dices used **most likely**?

Chapter 15. Dynamic Programming



We compute the most probable path of dices to roll a given sequence of numbers $S = d_1 \dots d_n$, where $d_i \in \{1, 2, \dots, 6\}$

Chapter 15. Dynamic Programming



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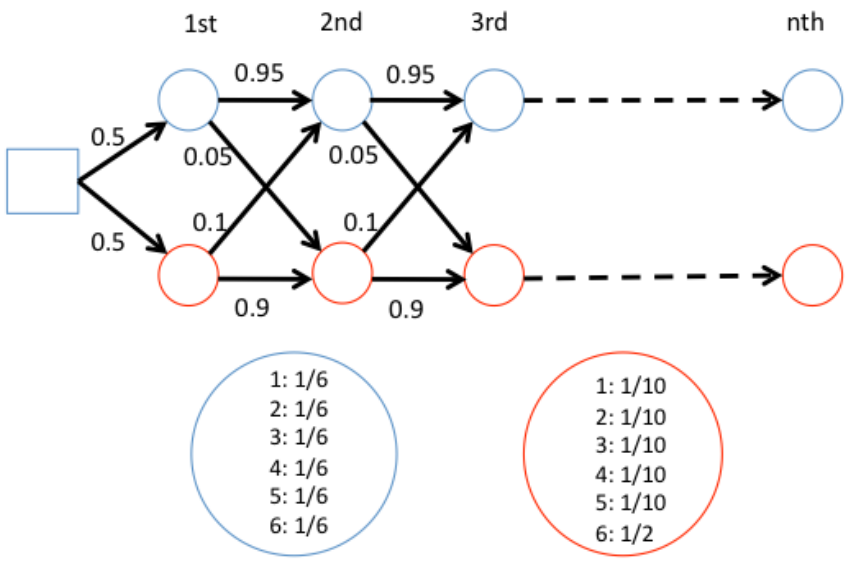
Chapter 15. Dynamic Programming



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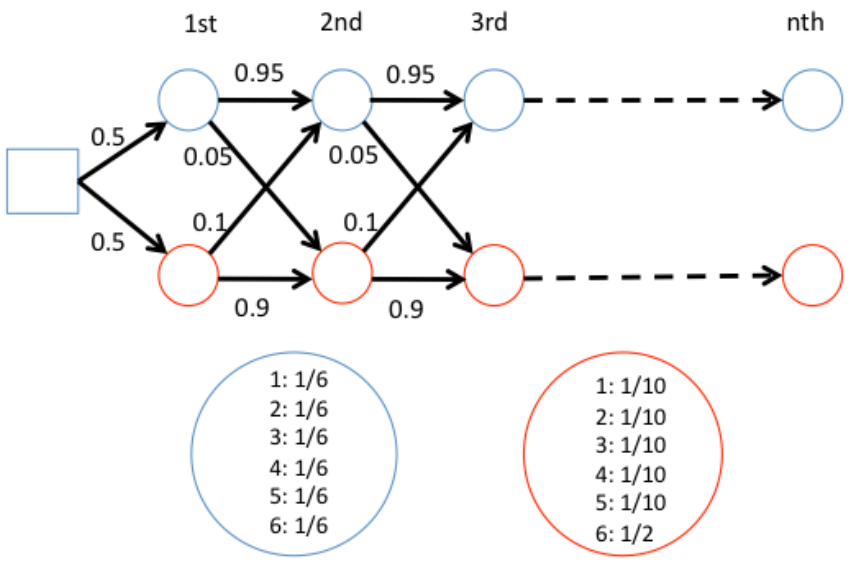
Chapter 15. Dynamic Programming



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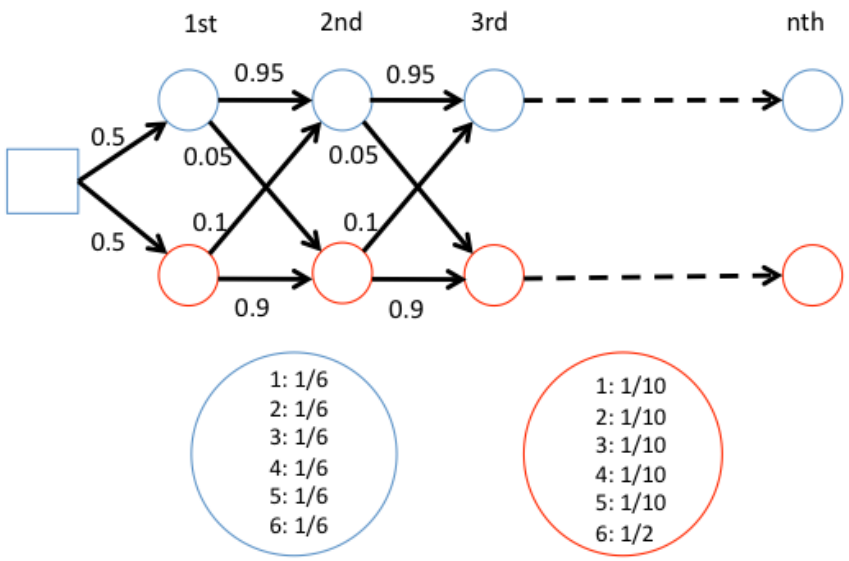
As in AssemblyLine problem, we compute the most probable path to roll $d_1 \dots d_i$:

Chapter 15. Dynamic Programming



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As in AssemblyLine problem, we compute the most probable path to roll $d_1 \dots d_i$:

- $p(S, i, F)$ for which the fair die F is used in the last step
- $p(S, i, L)$ for which the loaded die L is used in the last step

Chapter 15. Dynamic Programming

Both $p(S, i, F)$ and $p(S, i, L)$ have recursive solutions.

Chapter 15. Dynamic Programming

Both $p(S, i, F)$ and $p(S, i, L)$ have recursive solutions.

$$p(S, i, F) = \max\{p(S, i-1, F) \times 0.95 \times \frac{1}{6},\ p(S, i-1, L) \times 0.1 \times \frac{1}{6}\}$$

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Chapter 15. Dynamic Programming

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$p(S, 1, F) = 0.5 \times \frac{1}{6}$

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- design dynamic programming algorithm to compute $\max\{P(S, n, F), P(S, n, L)\}$ given from $d_1 d_2 \ldots d_n$

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- design dynamic programming algorithm to compute $\max\{P(S, n, F), P(S, n, L)\}$ given from $d_1 d_2 \ldots d_n$
- design a traceback process to print out the sequence of dices used.

Chapter 15. Dynamic Programming

Consider a sequence

... 1 3 2 4 6 6 6 4 1 6 5 6 6 2 4 2 1 2 3 5...

Chapter 15. Dynamic Programming

Consider a sequence

... 1 3 2 4 6 6 6 4 1 6 5 6 6 2 4 2 1 2 3 5...

Chapter 15. Dynamic Programming

Consider a sequence

... 1 3 2 4 6 6 6 4 1 6 5 6 6 2 4 2 1 2 3 5...

By computing the maximum probability to produce the sequence, we decode the dices used

... 1 3 2 4 6 6 6 4 1 6 5 6 6 2 4 2 1 2 3 5...

Chapter 15. Dynamic Programming

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... F F F F L L L L L L L L L F F F F F F F...

Chapter 15. Dynamic Programming

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But besides the casino interest, is such algorithm meaningful in other applications?

Chapter 15. Dynamic Programming

Consider a sequence

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But besides the casino interest, is such algorithm meaningful in other applications?

The answer is definitely YES!

Chapter 15. Dynamic Programming

```
AGGACCATAAAACTCCAGTCAGTGAAC
AAACAAGTTAATAAACTAAAACTTTCA
TGGTTCTGGCATCGATGAAGAACGCAG
GTAATGTGAATTGCAGAATTCAGTGAA
GAACGCACATTGCGCCCCTTGGTATTC
TGTTCGAGCGTCATTTCAACCCTCAAG
TGGGCTCCGTCCTCCACGGACGCGCCT
GGTGGCGTCTTGCCTCAAGCGTAGTAG
TTGGAGCGCACGGCGTCGCCCGCCGGA
TATTTCTCAAGGTTGACCTCGGATCAT
AAGGTAAGAAAGTTTTTCCTTCCGCTG
CTGGGTGCTGGGTGCTGGGTGCTGGGT
TTGCCTTATCGCTTCGGTGAGGGGCAT
TTGGCCCGCGCTAAGCCTCGTTCGGGC
CGCATCTGGTTTTTTTGCGACCGGCGT
```

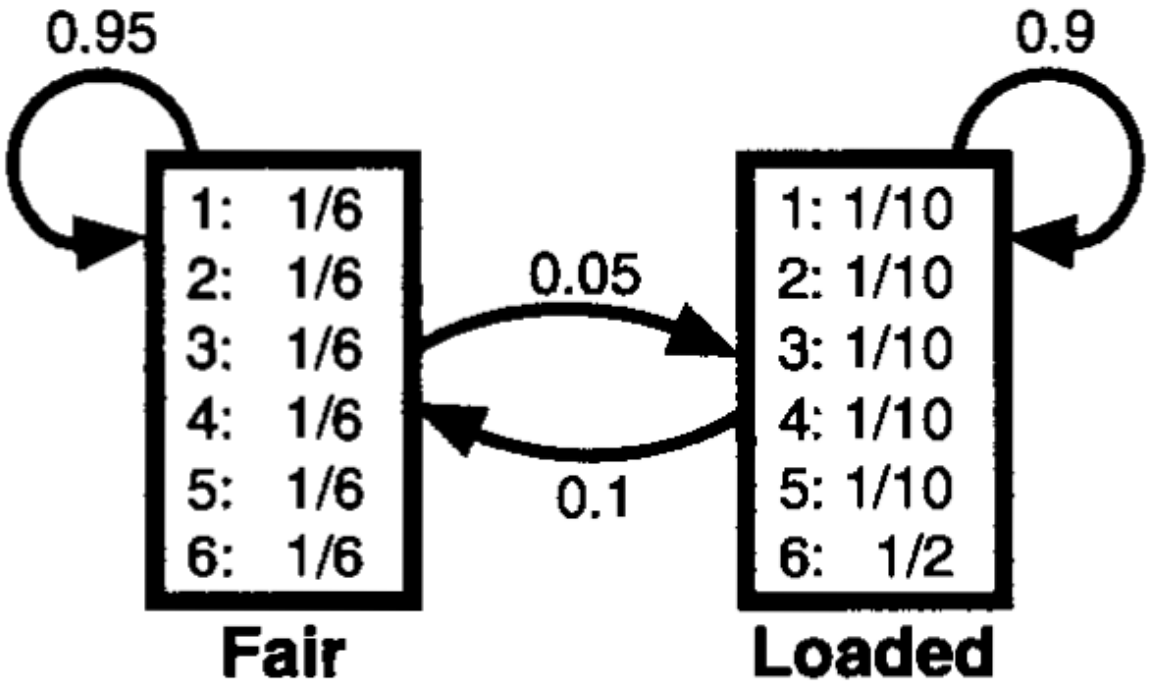
a segment of DNA sequence, is it meaningful?

Chapter 15. Dynamic Programming

if the highlighted segment is not random, it may be a gene.

Chapter 15. Dynamic Programming

Technically, the "linear model" is unfolded of a "more condensed model".



A hidden Markov model (HMM)

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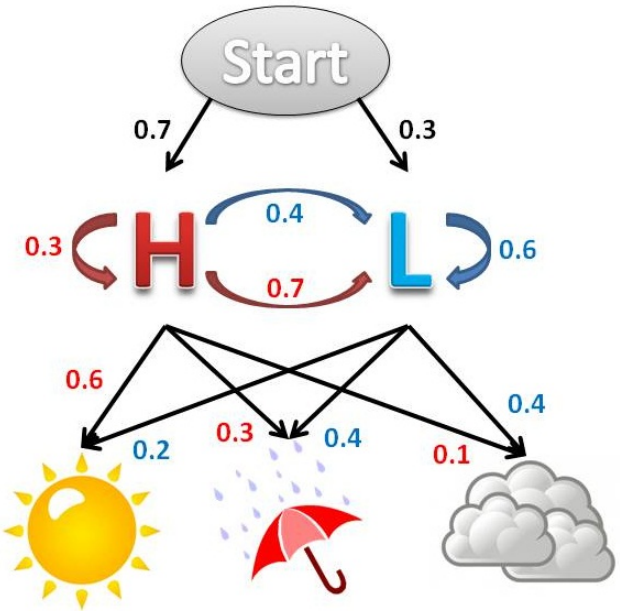

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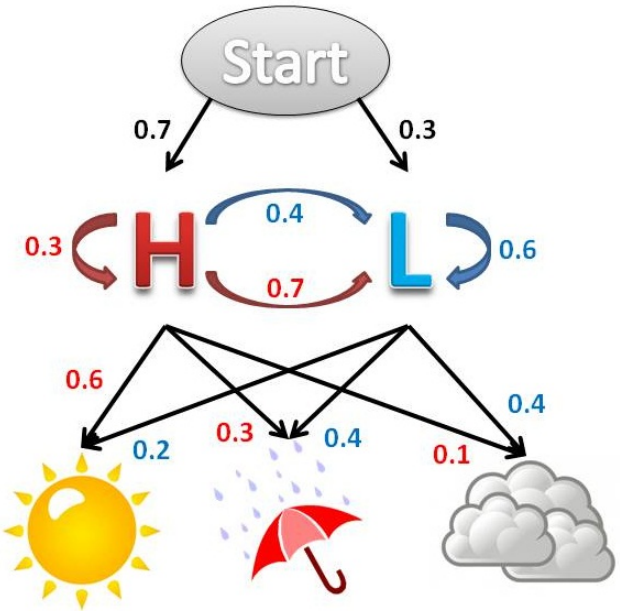

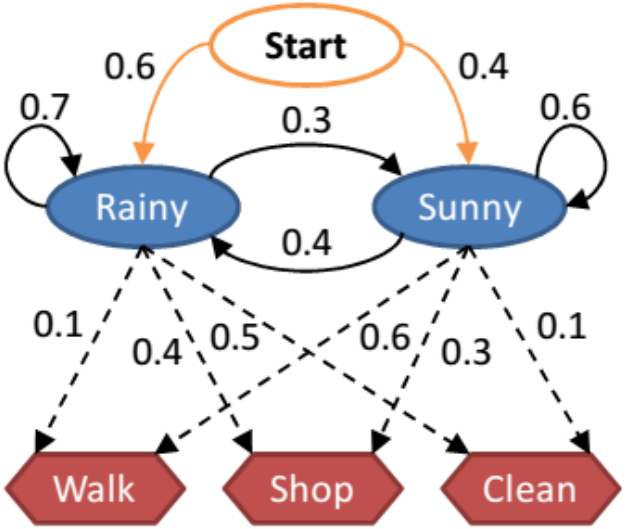

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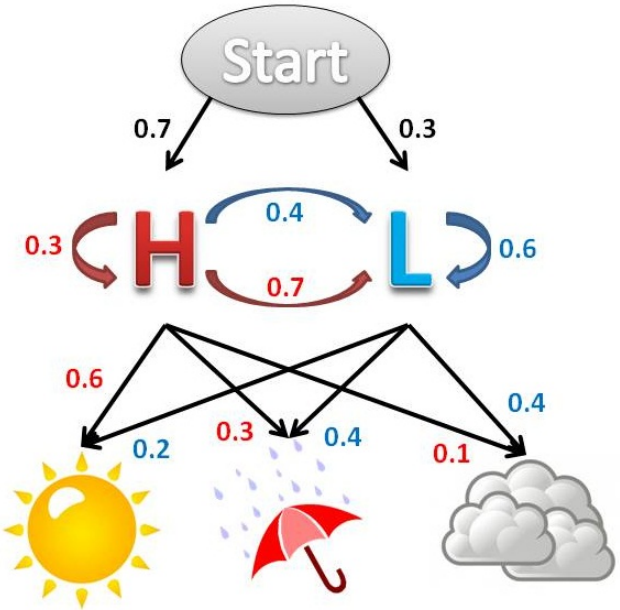
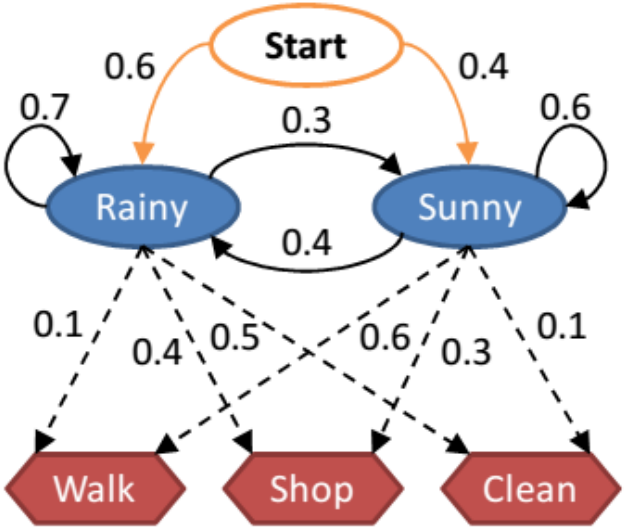
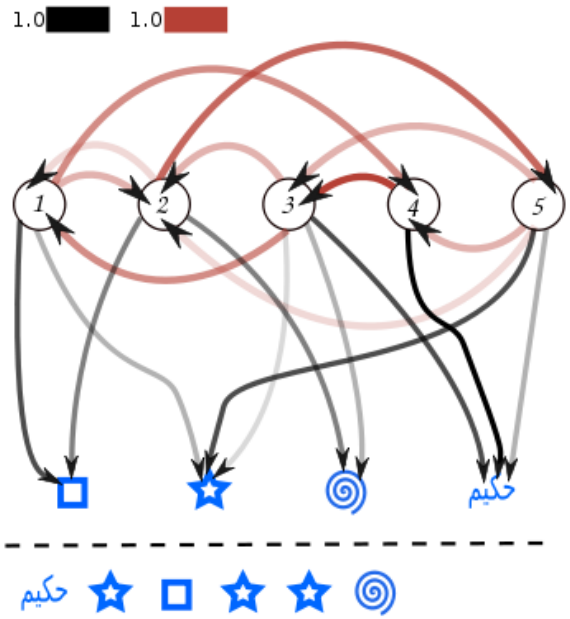
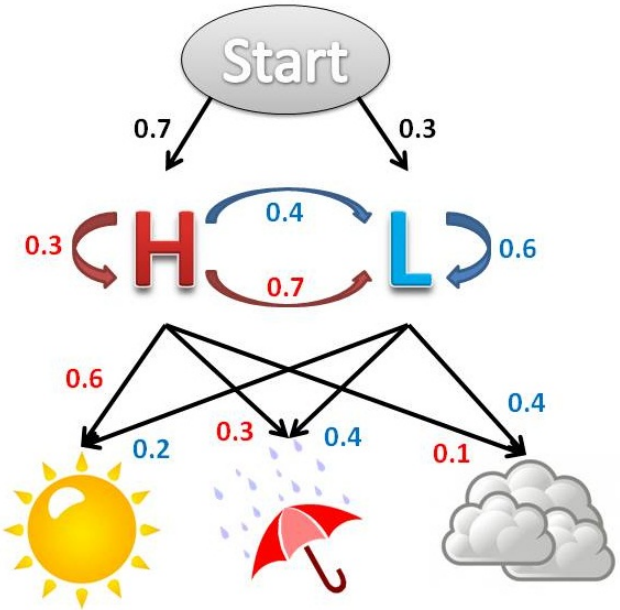
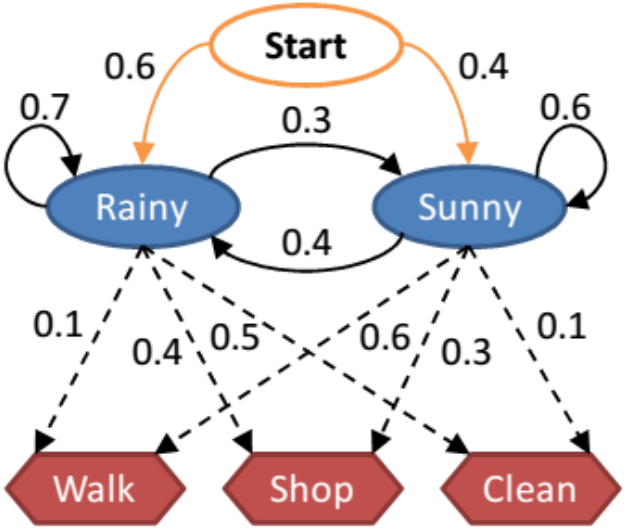


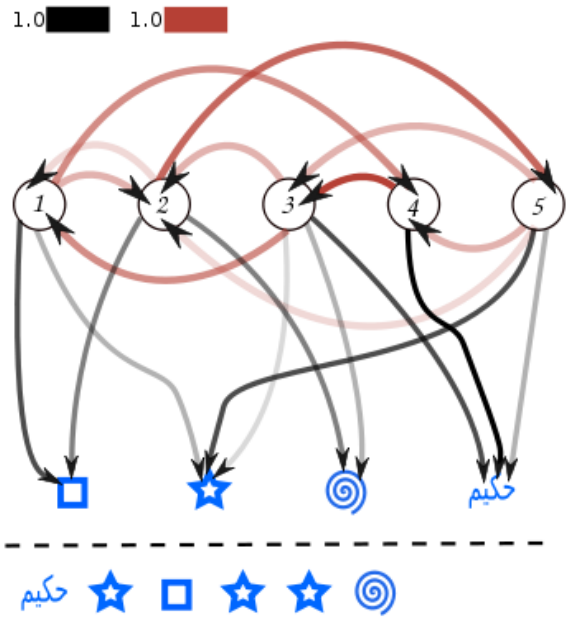


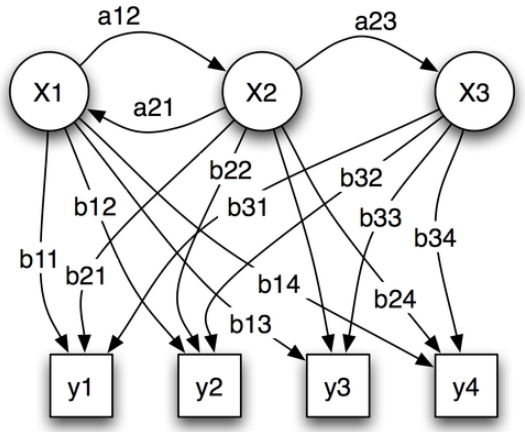


Chapter 15. Dynamic Programming



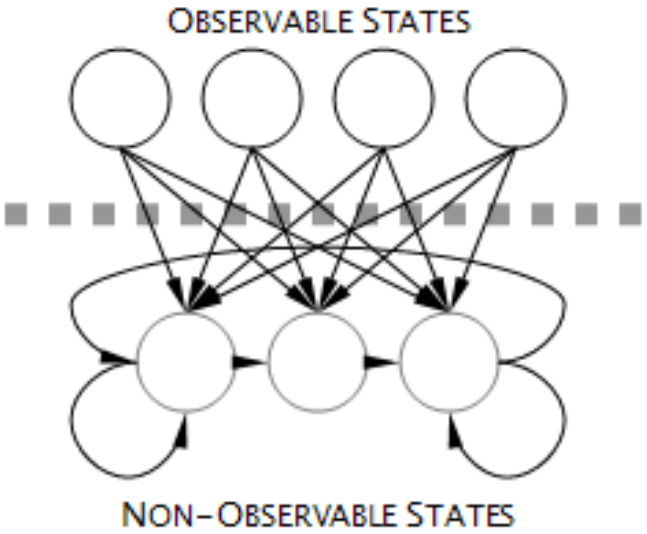






Chapter 15. Dynamic Programming

Viterbi Algorithm



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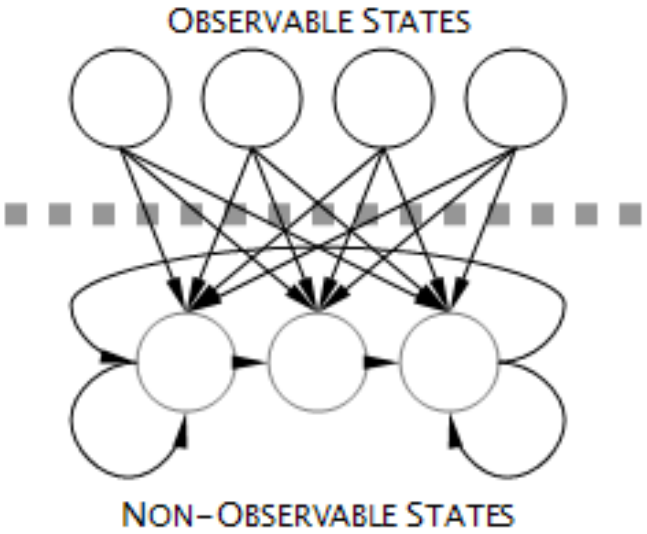

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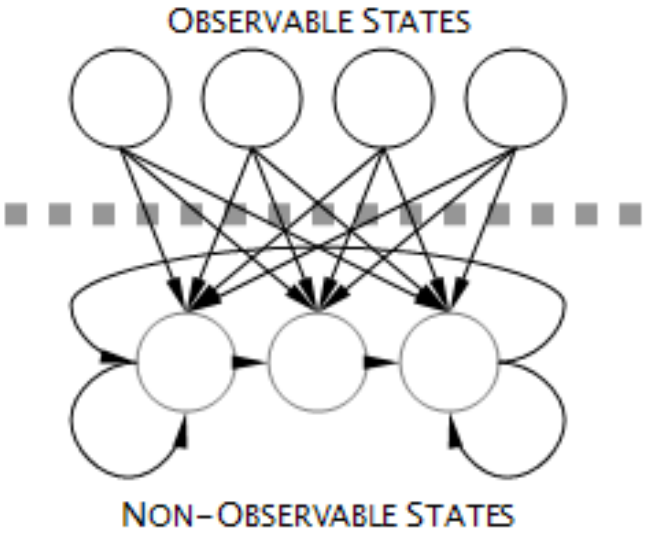

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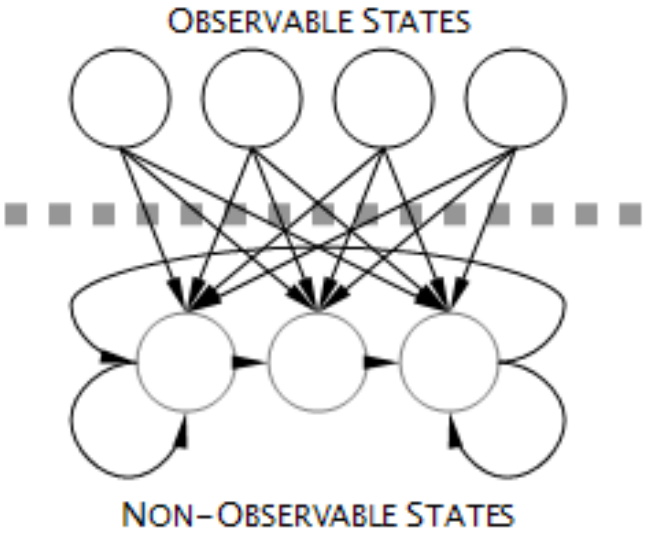

- assume an HMM $\mathcal{M}$;
- decode the hidden states H^* that have produced the given (observed) data D by computing the maximum probability

$$H^* = \arg\max_H \{Prob(H, D \mid \mathcal{M})\}$$

to produce the given data, with dynamic programming.

Chapter 15. Dynamic Programming

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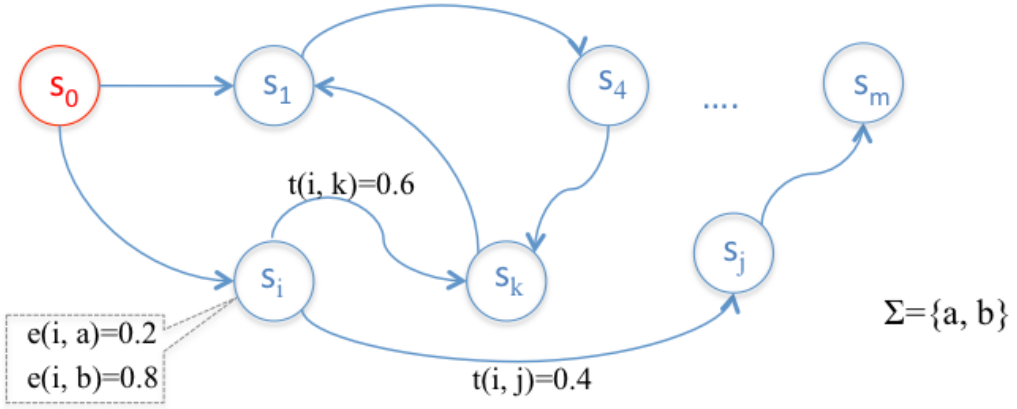
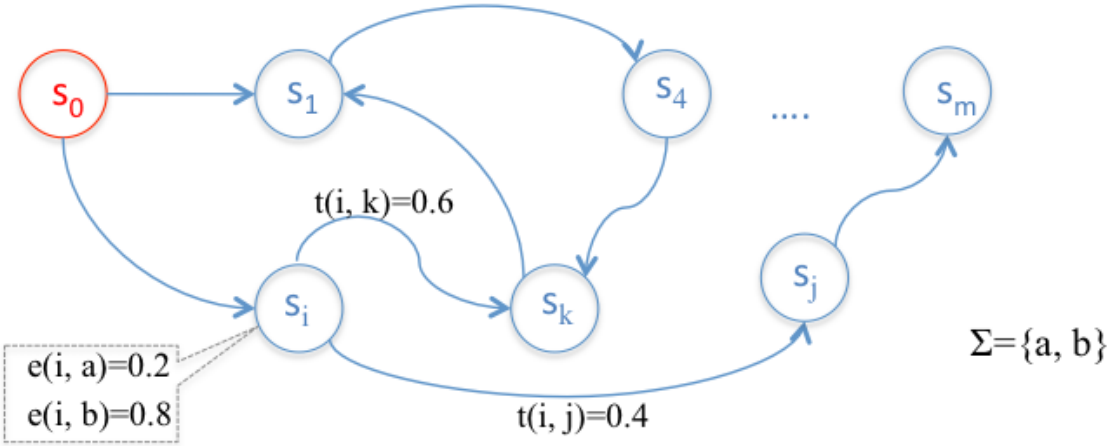


Chapter 15. Dynamic Programming



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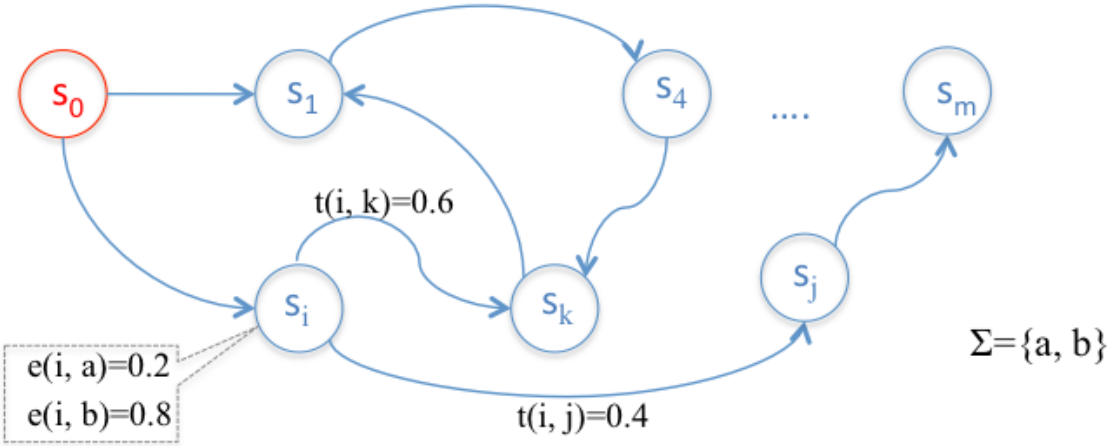

HMM decoding problem:

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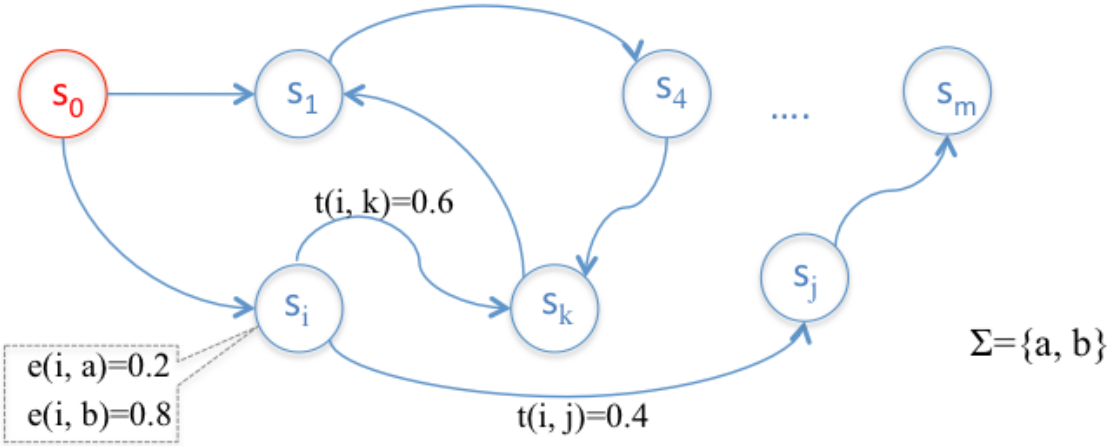

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INPUT: HMM $\mathcal{M}$, sequence of symbols $x \in \Sigma^*$
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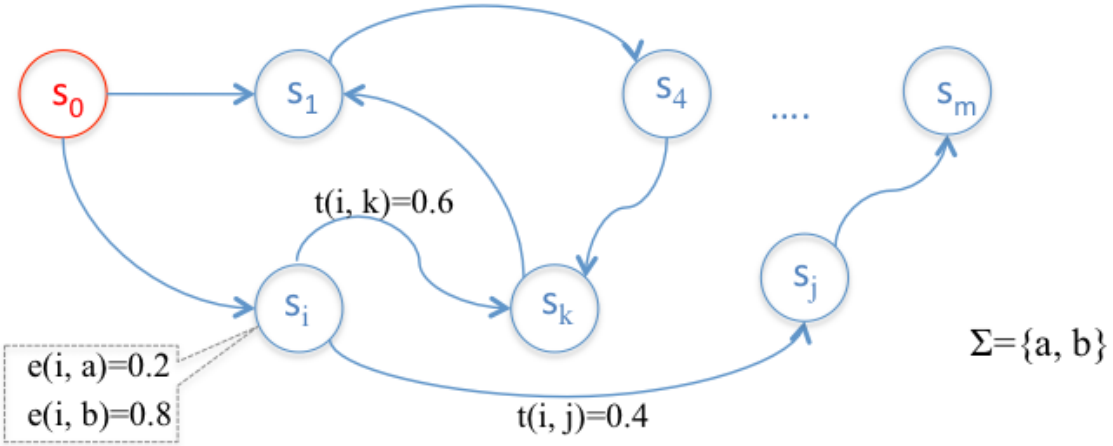

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Chapter 15. Dynamic Programming

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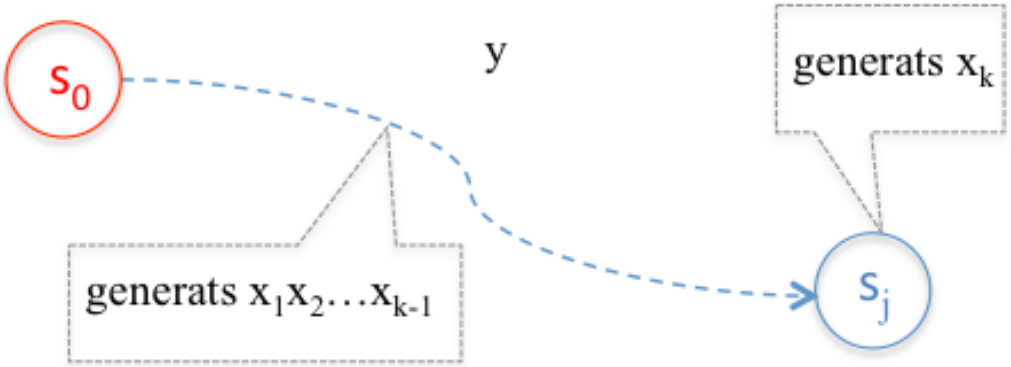
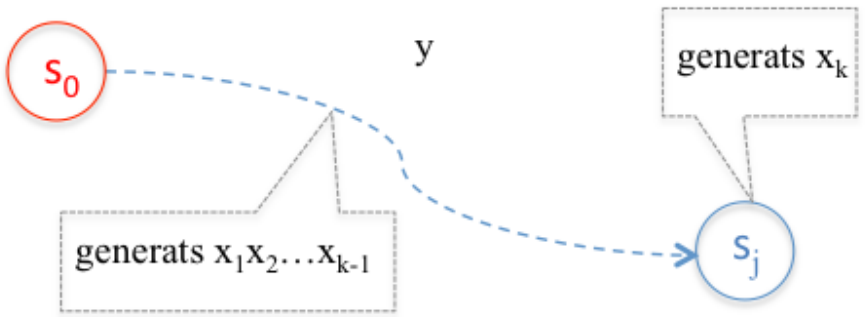
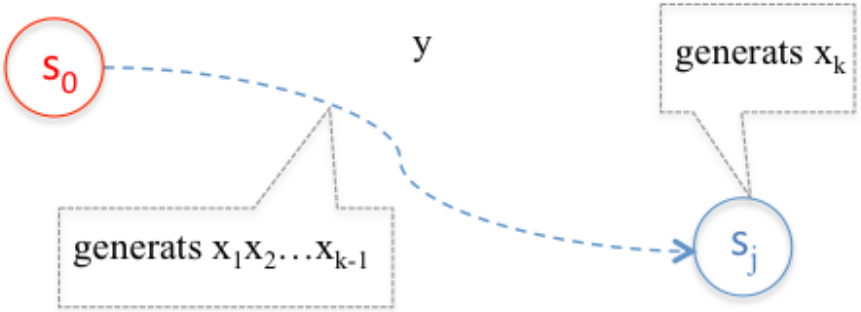


Chapter 15. Dynamic Programming

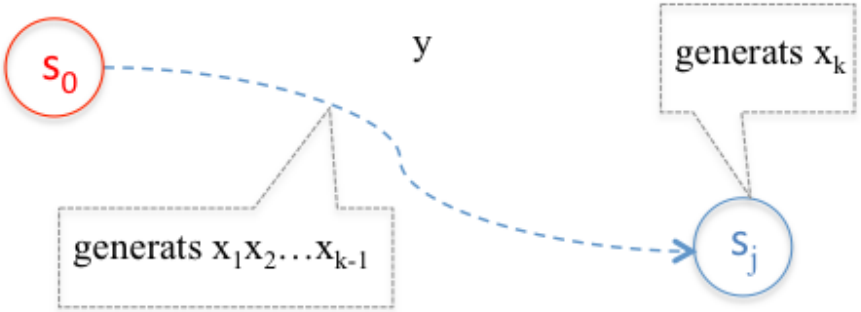


Chapter 15. Dynamic Programming



Define $f(j,k) = \max_{y=s_0\ldots s_j} \{Prob(y, x_1 \ldots x_k)\}$

Chapter 15. Dynamic Programming



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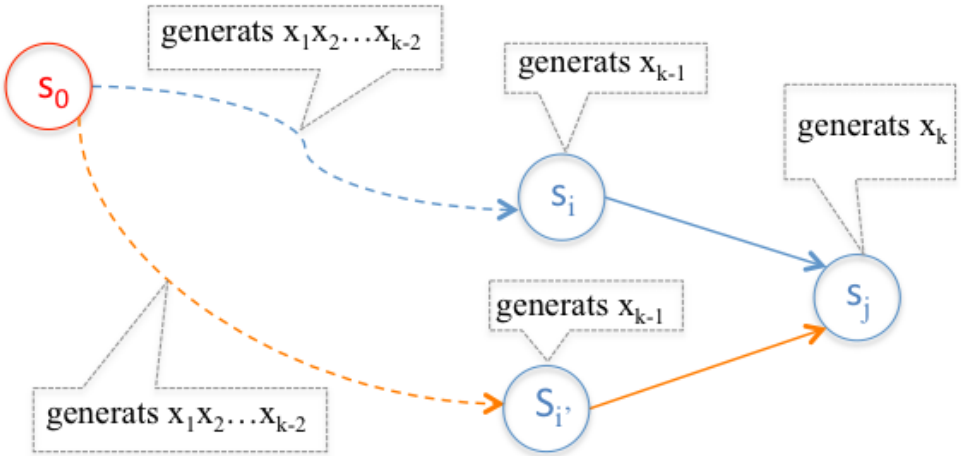

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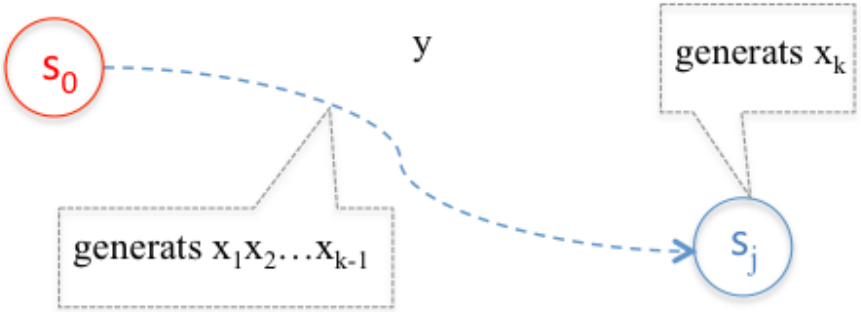

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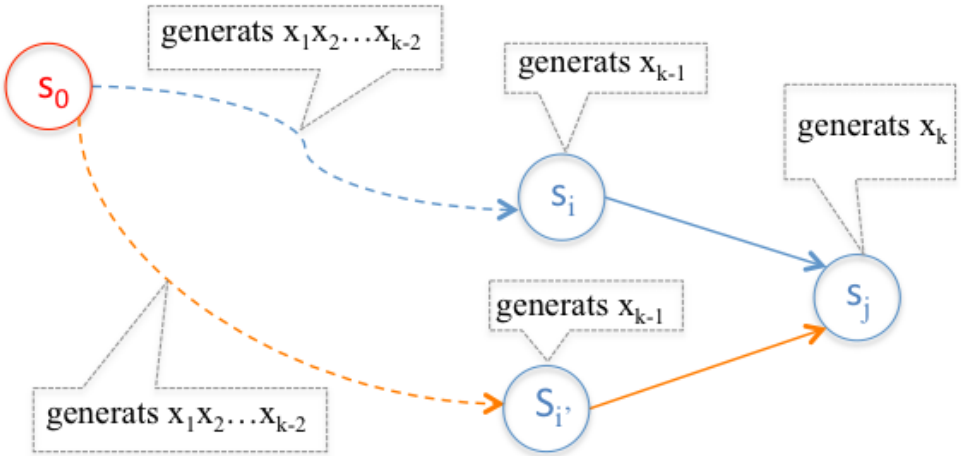

Recursively, $f(j,k) =$

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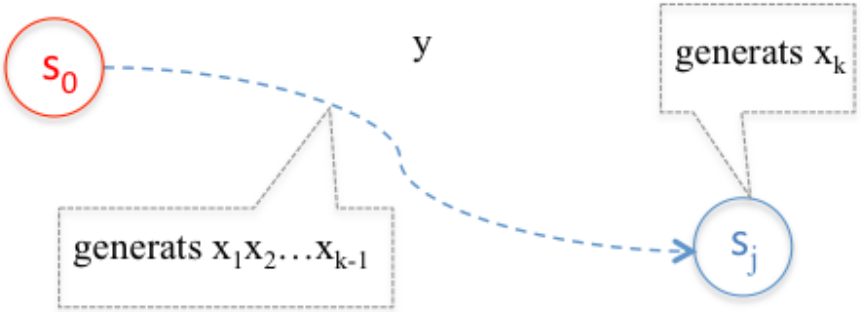

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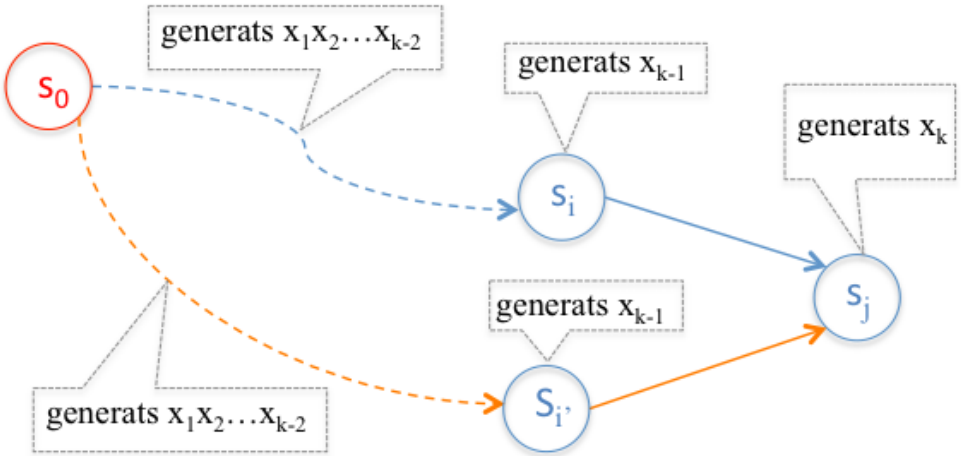

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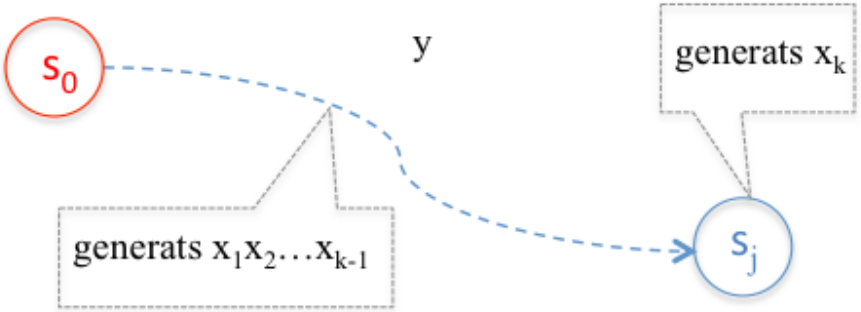

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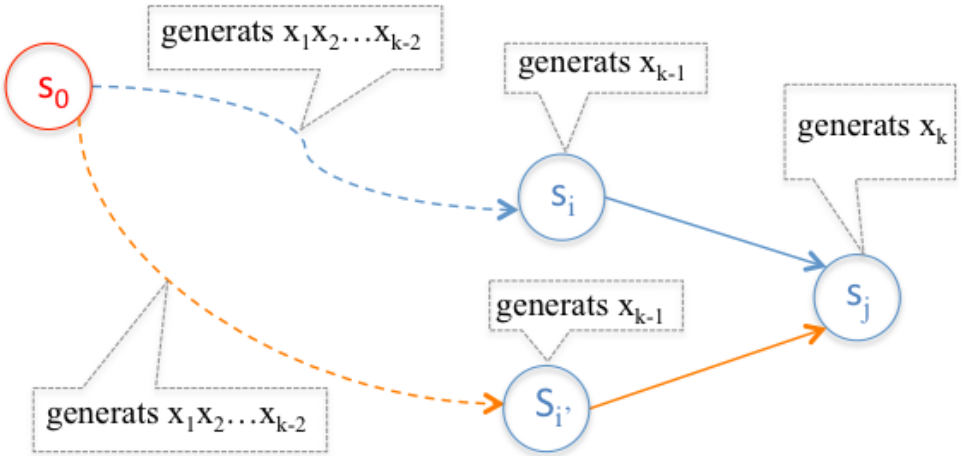

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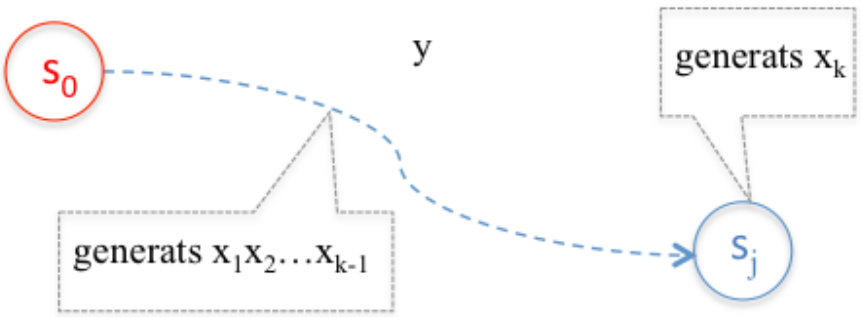

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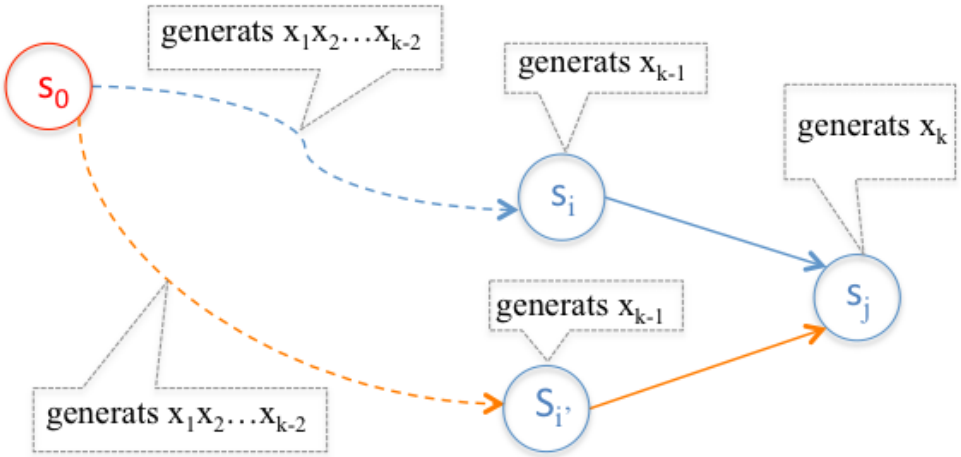

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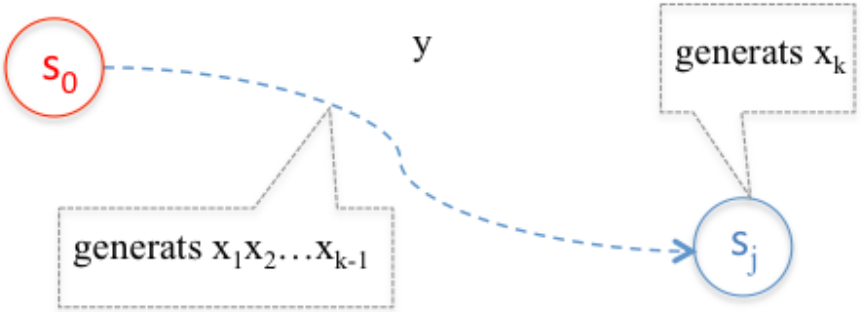

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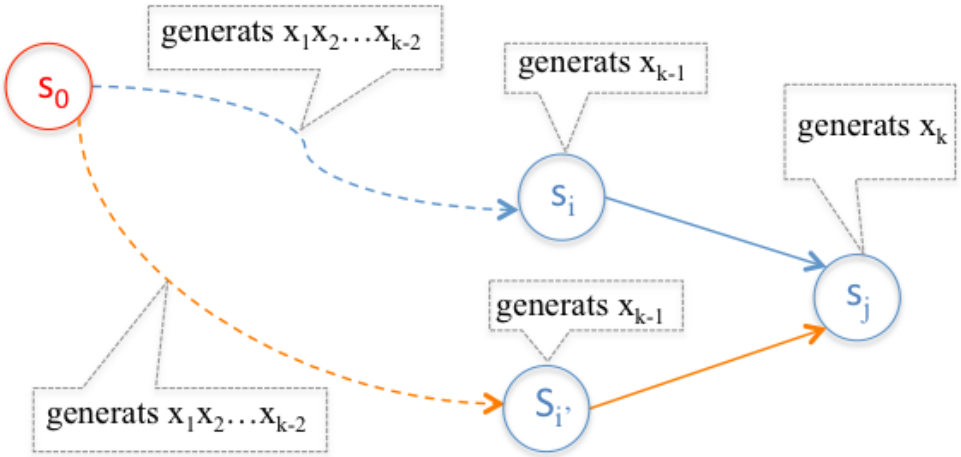

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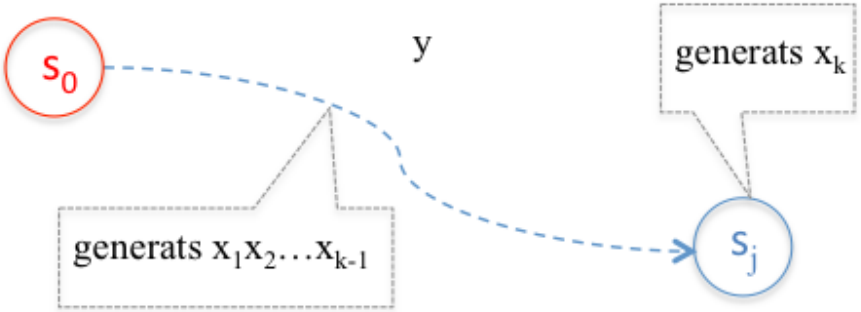

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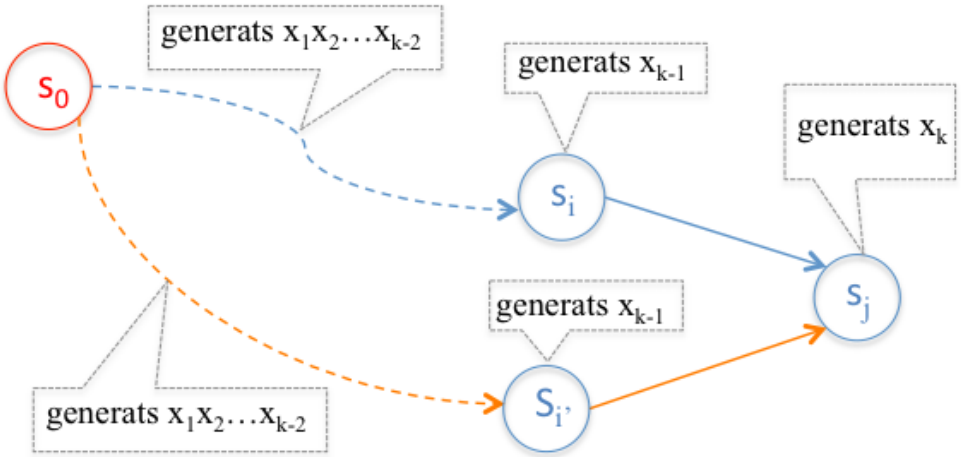

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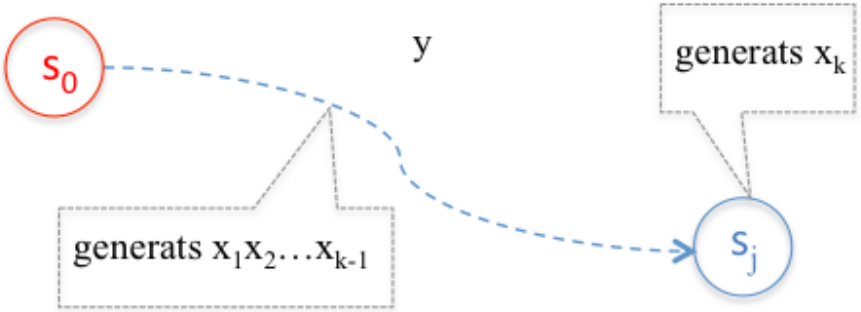

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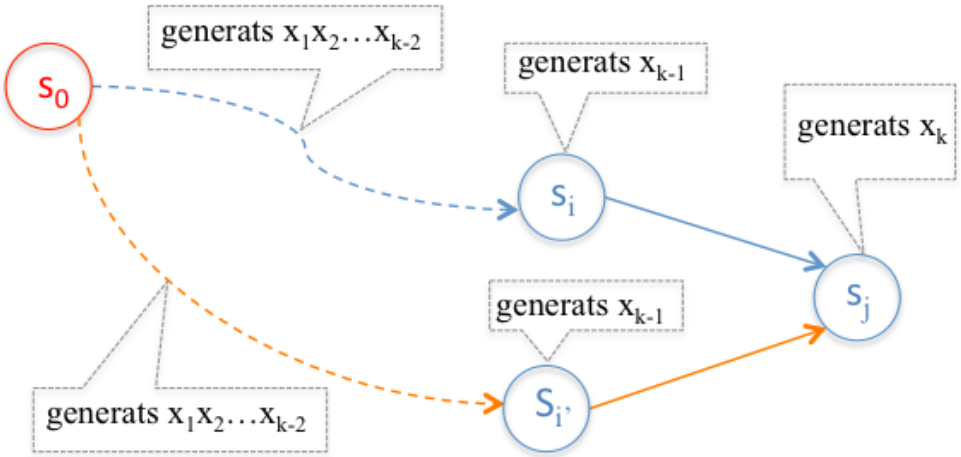

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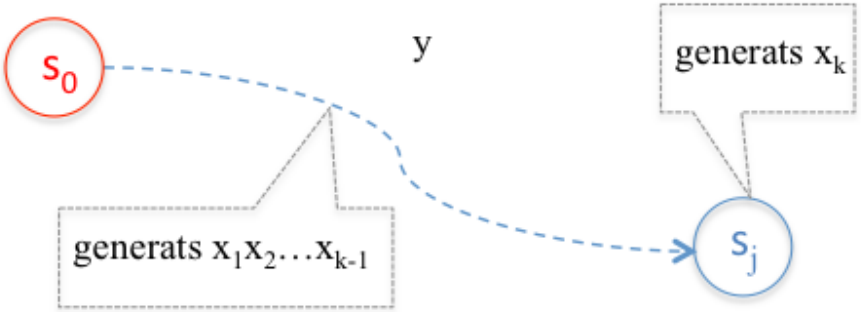

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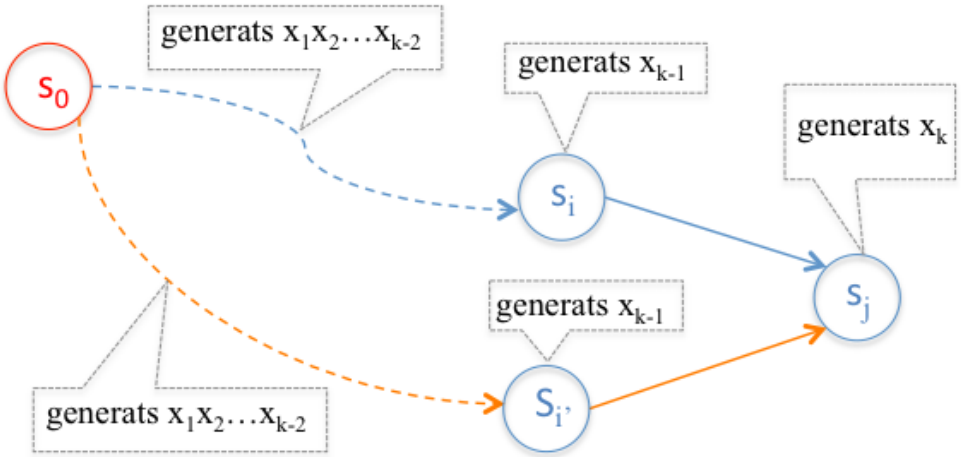

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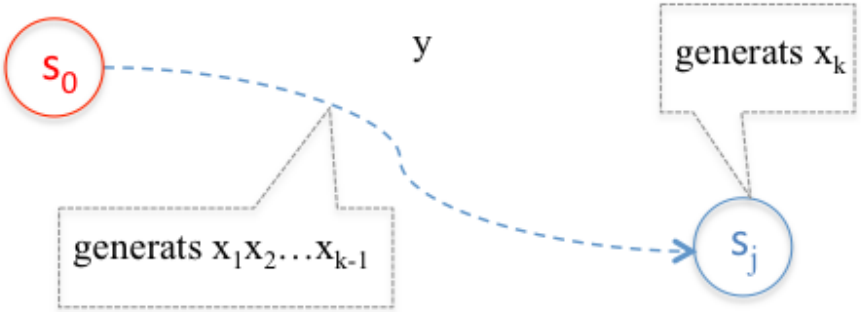

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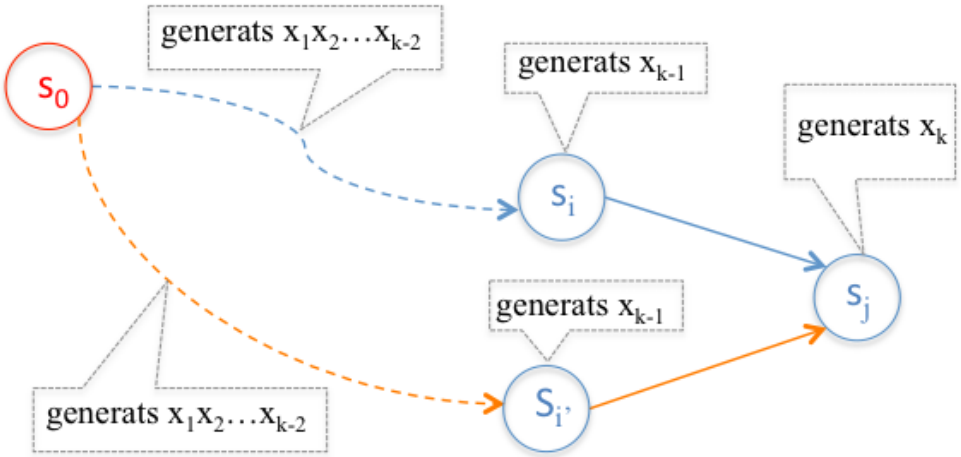

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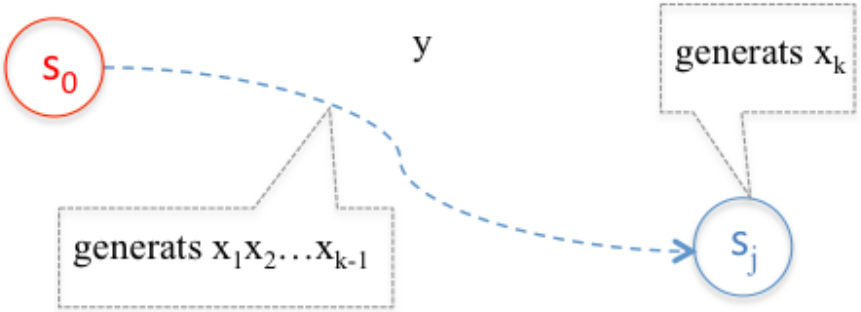

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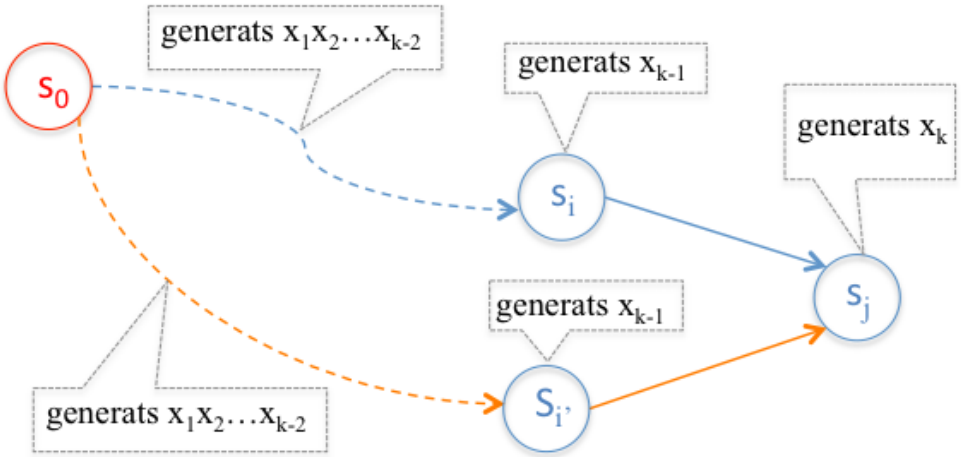

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Chapter 15. Dynamic Programming

Algorithm $\textsc{Viterbi}(x, n, t, e, m)$

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Algorithm $\textsc{Viterbi}(x, n, t, e, m)$

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T(0,0) = 1; P(0,0) = 0;
for j = 1 to m
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T(0,0) = 1; P(0,0) = 0;
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        for i = 0 to m          try every 'predecessor state' s_i
            if premax < T(i,k-1) × t(i,j) × e(j,x_k)
                premax = T(i,k-1) × t(i,j) × e(j,x_k)  update probability
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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Time complexity: $O(nm^2)$

Use the computed table P and $laststate$ to traceback the hidden states

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Algorithm $\textsc{TraceHiddenStates}(P, j, k, x)$

1. **if** $j > 0$

Chapter 15. Dynamic Programming

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1. **if** $j > 0$
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Chapter 15. Dynamic Programming

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Algorithm $\textsc{TraceHiddenStates}(P, j, k, x)$

1. **if** $j > 0$
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3. $\quad$**print** ('State' j 'emits symbol' x_k)

Chapter 15. Dynamic Programming

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Algorithm $\textsc{TraceHiddenStates}(P, j, k, x)$

```
if j > 0
    TraceHiddenStates(P, P[j, k], k − 1, x)
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First, call $\textsc{TraceHiddenStates}(P, laststate, n, x)$

Chapter 15. Dynamic Programming

There are other DP algorithms associated with HMMs.

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How to find objective function?
Recalled for forward, we defined

$$f(j,k) = \max_{y=s_0 \ldots s_j} \{Prob(y, x_1 \ldots x_k)\}$$

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For backward, we may define

$$b(l,k) = \max_{y=s_l \ldots s_\omega} \{Prob(y, x_k \ldots x_n)\}$$

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

Computing the total probability (instead of the maximum prob)

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Chapter 15. Dynamic Programming

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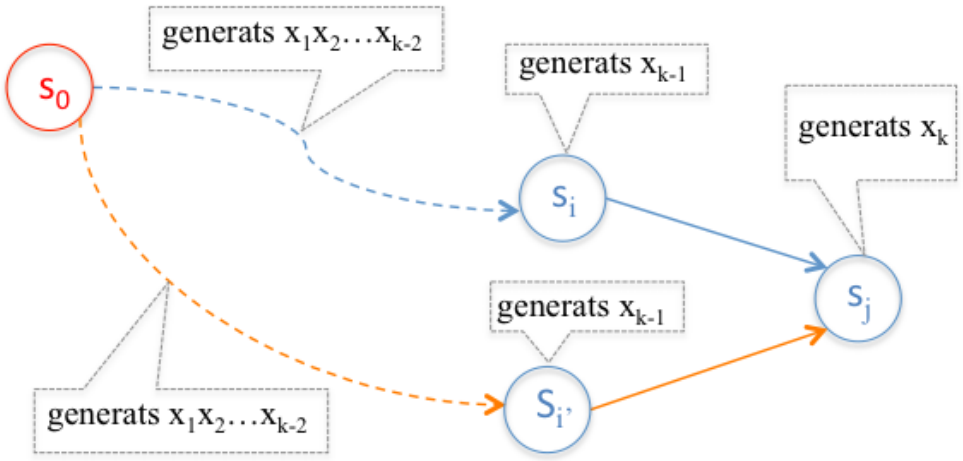

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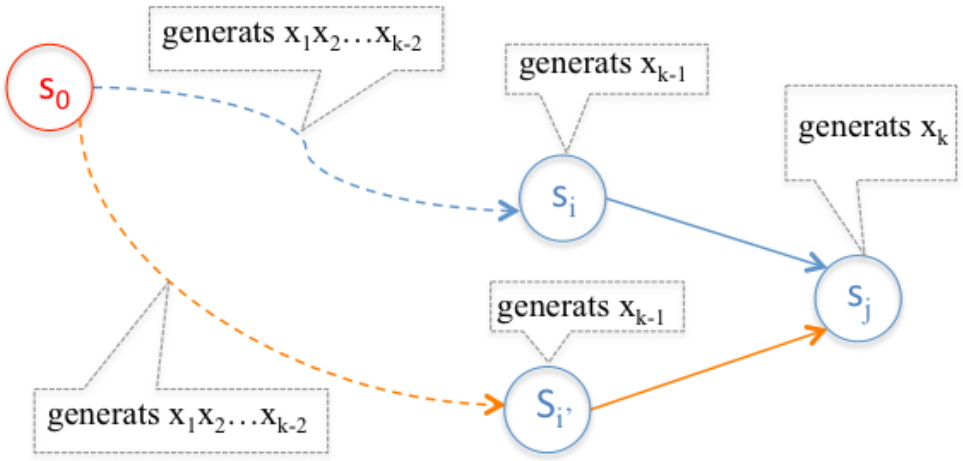

then

$$p(j,k) = \sum_i p(i,k-1) \times t(i,j) \times e(j,x_k)$$

with base cases: $p(0,0) = 1$, and $p(j,0) = 0$ for all $j \geq 1$.

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- almost the same as for Viterbi algorithm.

Chapter 15. Dynamic Programming

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  but what does it mean by 'significant probability values'?

Chapter 15. Dynamic Programming

Necessary elements for problems solvable with DP

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(3) overlapping subproblems

e.g., two or more paths share a subpath.

Chapter 15. Dynamic Programming

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```
ACCGGTCGAGTGCG
  CGTCGATGC   ⟵ common sequence
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```

Chapter 15. Dynamic Programming

Longest Common Subsequence (LCS)

$x =$ ACCGGTCGAGTGCG
$y =$ GTCGTTCGGATGCCC

To see how much the two sequences are related,
we may examine how much the two are in common.

e.g., a common subsequence for the two sequences

```
ACCGGTCGAGTGCG
   CGTCGATGC   ⟵ common sequence
GTCGTTCGGATGCCC
```

significance of common sequence

Chapter 15. Dynamic Programming

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```
ACCGGTC GAGTGCG
   || || || |||
   CG TC GA TGC   ⟵ common sequence
   || || || |||
GTCGTTCGGA TGCCC
```

Chapter 15. Dynamic Programming

Let $X = x_1 x_2 \cdots x_m$ be a sequence, $x_i \in \Sigma$.

Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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```
        34  6789 11 12 13
X =ACCGGTCGAGTGCG
        ||  ||||  |||
Z =     CG  TCGA  TGC
        12  3456  789
```

Chapter 15. Dynamic Programming

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Z =     CG  TCGA  TGC
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```

the corresponding indexes in X are:

$$3 < 4 < 6 < 7 < 8 < 9 < 11 < 12 < 13$$

Z is a *common subsequence* of X and Y
if Z is a subsequence of X and Z is a subsequence of Y.

Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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  || || || |||
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GTCGTTCGGA TGCCC
```

<u>Longest Common Subsequence</u> (LCS) problem:

INPUT: $X = x_1 x_2 \cdots x_m$, $Y = y_1 y_2 \cdots y_n$.
OUTPUT: Z, a common subsequence of X and Y such that $length(Z)$ is the maximum.

Chapter 15. Dynamic Programming

Step 1: Analyze the LCS problem to see if it has the desired features:

Chapter 15. Dynamic Programming

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- overlapping subproblems

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Chapter 15. Dynamic Programming

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We examine prefixes $x_1 x_2 \ldots x_i$, $i \leq m$
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Chapter 15. Dynamic Programming

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```

We examine prefixes $x_1 x_2 \ldots x_i$, $i \leq m$
and $y_1 y_2 \ldots y_j$, $j \leq n$

- how is the LCS of $x_1 x_2 \ldots x_i$ and $y_1 y_2 \ldots y_j$ related to the LCS of shorter prefixes of X and Y?

Chapter 15. Dynamic Programming

$$x_1 \ldots x_{i-1} x_i$$
$$y_1 \cdots y_{j-1} y_j$$

Chapter 15. Dynamic Programming

$$x_1 \ldots x_{i-1} x_i$$
$$y_1 \ldots y_{j-1} y_j$$

$$\boxed{x_1 \ldots x_{i-1}} x_i$$
$$\boxed{y_1 \ldots y_{j-1}} y_j$$

if $x_i = y_j$, include x_i in the LCS, reducing to LCS for $x_1 \ldots x_{i-1}$ and $y_1 \ldots y_{j-1}$

Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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$$\boxed{x_1 \ldots x_{i-1} x_i} \qquad \boxed{x_1 \ldots x_{i-1}} x_i$$
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if $x_i \neq y_j$, should we abandon either? two options

(1) keep x_i but abandon y_j
resulting in prefixes: $x_1 x_2 \ldots x_i$ and $y_1 y_2 \ldots y_{j-1}$, or

Chapter 15. Dynamic Programming

$$x_1 \ldots x_{i-1} x_i$$
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if $x_i \neq y_j$, should we abandon either? two options

(1) keep x_i but abandon y_j
resulting in prefixes: $x_1 x_2 \ldots x_i$ and $y_1 y_2 \ldots y_{j-1}$, or

(2) keep y_j but abandon x_i
resulting in prefixes: $x_1 x_2 \ldots x_{i-1}$ and $y_1 y_2 \ldots y_j$

Chapter 15. Dynamic Programming

Let $LCS(i, j)$ be the longest common sequence for $x_1 x_2 \ldots x_i$ and $y_1 y_2 \ldots y_j$, then

Chapter 15. Dynamic Programming

Let $LCS(i, j)$ be the longest common sequence for $x_1 x_2 \ldots x_i$ and $y_1 y_2 \ldots y_j$, then

- if $x_i = y_j$, then $LCS(i, j) = concat(\, LCS(i-1, j-1), x_i)$

Chapter 15. Dynamic Programming

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(depending on which is of a larger length.)

Chapter 15. Dynamic Programming

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We conclude:

LCS problem has the optimal substructure property;
LCS problem has overlapping subproblems.

Step 2: define objective function and formulate recurrence

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Define $l(i, j)$ to be the length of the LCS for prefixes $x_1 \ldots x_i$ and $y_1 \ldots y_j$.

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Chapter 15. Dynamic Programming

Step 2: define objective function and formulate recurrence

Define $l(i,j)$ to be the length of the LCS for prefixes $x_1 \ldots x_i$ and $y_1 \ldots y_j$. then

$$l(i,j) = \begin{cases} l(i-1,j-1)+1 & \text{if } x_i = y_j; \\ \max \begin{cases} l(i,j-1); \\ l(i-1,j); \end{cases} & \text{if } x_i \neq y_j \end{cases}$$

Chapter 15. Dynamic Programming

Step 2: define objective function and formulate recurrence

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the value of $l(i,j)$ depends on values of $l(i-1,j-1)$, $l(i,j-1)$, and $l(i-1,j)$.

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base cases: $l(i,0) = 0$, for $0 \leq i \leq m$, either prefix is empty $\rightarrow$ LCS is empty

Chapter 15. Dynamic Programming

Step 2: define objective function and formulate recurrence

Define $l(i,j)$ to be the length of the LCS for prefixes $x_1 \dots x_i$ and $y_1 \dots y_j$. then

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the value of $l(i,j)$ depends on values of $l(i-1,j-1)$, $l(i,j-1)$, and $l(i-1,j)$.

base cases: $l(i,0) = 0$, for $0 \leq i \leq m$, either prefix is empty $\rightarrow$ LCS is empty
$l(0,j) = 0$, for $0 \leq j \leq n$

Chapter 15. Dynamic Programming

Step 3: bottom-up table building for function $l(i, j)$.

		G	C	C	C	T	A	G	C	G
G										
C										
G										
C										
A										
A										
T										
G										

Chapter 15. Dynamic Programming

Step 3: bottom-up table building for function $l(i, j)$.

		G	C	C	C	T	A	G	C	G
	0	0	0	0	0	0	0	0	0	0
G	0									
C	0									
G	0									
C	0									
A	0									
A	0									
T	0									
G	0									

Chapter 15. Dynamic Programming

Step 3: bottom-up table building for function $l(i, j)$.

		G	C	C	C	T	A	G	C	G
	0	0	0	0	0	0	0	0	0	0
G	0	1	1	1	1	1	1	1	1	1
C	0	1	2	2	2	2	2	2	2	2
G	0	1	2	2	2	2	2	3	3	3
C	0	1	2	3						
A	0									
A	0									
T	0									
G	0									

Chapter 15. Dynamic Programming

Step 3: bottom-up table building for function $l(i, j)$.

		G	C	C	C	T	A	G	C	G
	0	0	0	0	0	0	0	0	0	0
G	0	1	1	1	1	1	1	1	1	1
C	0	1	2	2	2	2	2	2	2	2
G	0	1	2	2	2	2	2	3	3	3
C	0	1	2	3						
A	0									
A	0									
T	0									
G	0									

LCS result between prefixes `GCCCTAGC` and `GCG`:

```
GCCCTAGC
G  C  G
```

Chapter 15. Dynamic Programming

Algorithm LCS(x, y)

1. $m = length(x)$, $n = length(y)$;

Chapter 15. Dynamic Programming

Algorithm LCS(x, y)

1. $m = length(x)$, $n = length(y)$;
2. **for** $i = 0$ **to** m

Chapter 15. Dynamic Programming

Algorithm LCS(x, y)

```
m = length(x), n = length(y);
for i = 0 to m
    T[i, 0] = 0
```

Chapter 15. Dynamic Programming

Algorithm LCS(x, y)

```
m = length(x), n = length(y);
for i = 0 to m
    T[i, 0] = 0
for j = 0 to n
```

Chapter 15. Dynamic Programming

Algorithm LCS(x, y)

```
m = length(x), n = length(y);
for i = 0 to m
    T[i, 0] = 0
for j = 0 to n
    T[0, j] = 0
```

Chapter 15. Dynamic Programming

Algorithm LCS(x, y)

```
m = length(x), n = length(y);
for i = 0 to m
    T[i, 0] = 0
for j = 0 to n
    T[0, j] = 0
for i = 1 to m
```

Chapter 15. Dynamic Programming

```
Algorithm LCS(x, y)
    m = length(x), n = length(y);
    for i = 0 to m
        T[i, 0] = 0
    for j = 0 to n
        T[0, j] = 0
    for i = 1 to m
        for j = 1 to n
```

Chapter 15. Dynamic Programming

```
Algorithm LCS(x, y)
    m = length(x), n = length(y);
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        T[i, 0] = 0
    for j = 0 to n
        T[0, j] = 0
    for i = 1 to m
        for j = 1 to n
            if x_i = y_j
```

Chapter 15. Dynamic Programming

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Algorithm LCS(x, y)
    m = length(x), n = length(y);
    for i = 0 to m
        T[i, 0] = 0
    for j = 0 to n
        T[0, j] = 0
    for i = 1 to m
        for j = 1 to n
            if x_i = y_j
                T[i, j] = T[i − 1, j − 1] + 1; P[i, j] = ‘ ↖ ‘
```

Chapter 15. Dynamic Programming

```
Algorithm LCS(x, y)
    m = length(x), n = length(y);
    for i = 0 to m
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    for j = 0 to n
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            else if T[i, j − 1] > T[i − 1, j]
```

Chapter 15. Dynamic Programming

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            else if T[i, j − 1] > T[i − 1, j]
                        T[i, j] = T[i, j − 1]; P[i, j] = ‘ ← ‘
```

Chapter 15. Dynamic Programming

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Algorithm LCS(x, y)
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            else if T[i, j − 1] > T[i − 1, j]
                        T[i, j] = T[i, j − 1]; P[i, j] = ‘ ← ‘
                else T[i, j] = T[i − 1, j]; P[i, j] = ‘ ↑ ‘
```

Chapter 15. Dynamic Programming

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Algorithm LCS(x, y)
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        T[i, 0] = 0
    for j = 0 to n
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                T[i, j] = T[i − 1, j − 1] + 1; P[i, j] = ‘ ↖ ‘
            else if T[i, j − 1] > T[i − 1, j]
                        T[i, j] = T[i, j − 1]; P[i, j] = ‘ ← ‘
                else T[i, j] = T[i − 1, j]; P[i, j] = ‘ ↑ ‘
    return (T, P)
```

Chapter 15. Dynamic Programming

```
Algorithm LCS(x, y)
    m = length(x), n = length(y);
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    for j = 0 to n
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    return (T, P)
```

Time complexity: $O(mn)$

Chapter 15. Dynamic Programming

Traceback the longest common subsequence

either recursive or iterative algorithm

Traceback the longest common subsequence

either recursive or iterative algorithm

Algorithm $\textsc{PrintLCS}(P, x, i, j)$

Chapter 15. Dynamic Programming

Traceback the longest common subsequence

either recursive or iterative algorithm

Algorithm $\textsc{PrintLCS}(P, x, i, j)$

1. **if** $(i > 0) \wedge (j > 0)$

Traceback the longest common subsequence

either recursive or iterative algorithm

Algorithm $\textsc{PrintLCS}(P, x, i, j)$

1. **if** $(i > 0) \wedge (j > 0)$
2. **if** $P[i, j] = '\nwarrow'$

Traceback the longest common subsequence

either recursive or iterative algorithm

Algorithm $\textsc{PrintLCS}(P, x, i, j)$

```
if (i > 0) ∧ (j > 0)
    if P[i, j] = '↖'
        PrintLCS(P, x, i − 1, j − 1)
```

Traceback the longest common subsequence

either recursive or iterative algorithm

Algorithm $\textsc{PrintLCS}(P, x, i, j)$

```
if (i > 0) ∧ (j > 0)
    if P[i, j] = '↖'
        PrintLCS(P, x, i − 1, j − 1)
        Print (x_i)
```

Traceback the longest common subsequence

either recursive or iterative algorithm

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```
if (i > 0) ∧ (j > 0)
    if P[i, j] = '↖'
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```

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    else if P[i, j] = '←'
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```

Traceback the longest common subsequence

either recursive or iterative algorithm

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    else if P[i, j] = '←'
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        else PrintLCS(P, x, i − 1, j)
```

Traceback the longest common subsequence

either recursive or iterative algorithm

Algorithm $\textsc{PrintLCS}(P, x, i, j)$

```
if (i > 0) ∧ (j > 0)
    if P[i, j] = '↖'
        PrintLCS(P, x, i − 1, j − 1)
        Print (x_i)
    else if P[i, j] = '←'
            PrintLCS(P, x, i, j − 1)
        else PrintLCS(P, x, i − 1, j)
else return ()
```

Chapter 15. Dynamic Programming

Pairwise Alignment

Chapter 15. Dynamic Programming

Pairwise Alignment

```
ACCGGTCGAGTGCG
  CGTCGATGC   ⟵ common sequence
GTCGTTCGGATGCCC
```

Chapter 15. Dynamic Programming

Pairwise Alignment

```
ACCGGTCGAGTGCG
  CGTCGATGC   ⟵ common sequence
GTCGTTCGGATGCCC
```

biologically more meaningful view:

```
ACCGGTC GAGTGCG
  || || || |||
  CG TC GA TGC   ⟵ common sequence
  || || || |||
GTCGTTCGGA TGCCC
```

Pairwise Alignment

```
ACCGGTCGAGTGCG
  CGTCGATGC   ⟵ common sequence
GTCGTTCGGATGCCC
```

biologically more meaningful view:

```
ACCGGTC GAGTGCG
  || || || |||
  CG TC GA TGC   ⟵ common sequence
  || || || |||
GTCGTTCGGA TGCCC
```

blue letters are conserved;
red letters are mutated;
purple letters are inserted or deleted;

Chapter 15. Dynamic Programming

Alignment between two sequences:

Chapter 15. Dynamic Programming

Alignment between two sequences:

```
ACCGGTC_GAGTGCG_
  || || || |||
  CG TC GA TGC  ⟵ common sequence
  || || || |||
GTCGTTCGGA_TGCCC
```

Chapter 15. Dynamic Programming

Alignment between two sequences:

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```

- two sequences padded with '_'s called **gaps**;

Chapter 15. Dynamic Programming

Alignment between two sequences:

```
ACCGGTC_GAGTGCG_
  || || || |||
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  || || || |||
GTCGTTCGGA_TGCCC
```

- two sequences padded with '_'s called **gaps**;
- two sequences are of the same length, aligned;

Chapter 15. Dynamic Programming

Alignment between two sequences:

```
ACCGGTC_GAGTGCG_
  || || || |||
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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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$0 + 0 + 1 + 1 + 0 + 1 + 1 + 0 + 1 + 1 + 0 + 1 + 1 + 1 + 0 + 0$

Chapter 15. Dynamic Programming

Significance of sequence alignments:

Sequence Homology Reveals Functions

- Homology reveals evolution of structure/function

```
FOS_RAT     MMFSGFNADYEASSSRCSSASPAGDSLSYYHSPADSFSSMGSPVNTQDFCADLSVSSANF 60
FOS_MOUSE   MMFSGFNADYEASSSRCSSASPAGDSLSYYHSPADSFSSMGSPVNTQDFCADLSVSSANF 60
FOS_CHICK   MMYQGFAGEYEAPSSRCSSASPAGDSLTYYPSPADSFSSMGSPVNSQDFCTDLAVSSANF 60
FOSB_MOUSE  -MFQAFPGDYDS-GSRCSS-SPSAESQ--YLSSVDSFGSPPTAAASQE-CAGLGEMPGSF 54
FOSB_HUMAN  -MFQAFPGDYDS-GSRCSS-SPSAESQ--YLSSVDSFGSPPTAAASQE-CAGLGEMPGSF 54
Consensus    *:..* .:*:: .***** **:.:*   * *..***.*  :.. :*: *:.*.  ...*
```

- Homology reveals regulatory structure (E. Coli promoters)

```
tyr tRNA    TCTCAACGTAACACTTTACAGCGGCG··CGTCATTTGATATGATGC·GCCCCGCTTCCCGATAAGGG
rrn D1      GATCAAAAAAATACTTGTGCAAAAAA··TTGGGATCCCTATAATGCGCCTCCGTTGAGACGACAACG
rrn X1      ATGCATTTTTCCGCTTGTCTTCCTGA··GCCGACTCCCTATAATGCGCCTCCATCGACACGGCGGAT
rrn (DXE)2  CCTGAAATTCAGGGTTGACTCTGAAA··GAGGAAAGCGTAATATAC·GCCACCTCGCGACAGTGAGC
rrn E1      CTGCAATTTTTCTATTGCGGCCTGCG··GAGAACTCCCTATAATGCGCCTCCATCGACACGGCGGAT
rrn A1      TTTTAAATTTCCTCTTGTCAGGCCGG··AATAACTCCCTATAATGCGCCACCACTGACACGGAACAA
rrn A2      GCAAAAATAAATGCTTGACTCTGTAG··CGGGAAGGCGTATTATGC·ACACCCCGCGCCGCTGAGAA
λ PR        TAACACCGTGCGTGTTGACTATTTTA·CCTCTGGCGGTGATAATGG··TTGCATGTACTAAGGAGGT
λ PL        TATCTCTGGCGGTGTTGACATAAATA·CCACTGGCGGTGATACTGA··GCACATCAGCAGGACGCAC
T7 A3       GTGAAACAAAACGGTTGACAACATGA·AGTAAACACGGTACGATGT·ACCACATGAAACGACAGTGA
T7 A1       TATCAAAAAGAGTATTGACTTAAAGT·CTAACCTATAGGATACTTA·CAGCCATCGAGAGGGACACG
T7 A2       ACGAAAAACAGGTATTGACAACATGAAGTAACATGCAGTAAGATAC·AAATCGCTAGGTAACACTAG
fd VIII     GATACAAATCTCCGTTGTACTTTGTT··TCGCGCTTGGTATAATCG·CTGGGGGTCAAAGATGAGTG
                          −35                        −10                  +1
```

Chapter 15. Dynamic Programming

Pairwise alignement problem:

INPUT: sequences $x = x_1 x_2 \ldots x_m$, $y = y_1 y_2 \ldots y_n$

OUTPUT: x' and y', which are x and y padded with '_'s, respectively such that $\sum_{i=1}^{l} \mathcal{S}(x'_i, y'_i)$ is the maximum, where $l = |x'| = |y'|$ for some given scoring function $\mathcal{S}$.

How to solve it?

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case (1) contributes to a match/mismatch score

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cases (2) and (3) contribute to an insertion/deletion score.

Chapter 15. Dynamic Programming

Scoring function for pairwise alignment

Chapter 15. Dynamic Programming

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+5 for conservation columns, (reward)
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Chapter 15. Dynamic Programming

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Example (2)

	A	C	G	T	-
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C	-1	5	-3	-2	-4
G	-2	-3	5	-2	-4
T	-1	-2	-2	5	-4
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Chapter 15. Dynamic Programming

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Why sum of column scores?

product of independent column probabilities

Chapter 15. Dynamic Programming

As for the LCS problem, we can define

$A(i, j)$ is the maximum score alignment between $x_1 \ldots x_i$ and $y_1 \ldots y_j$, then

$$A(i,j) = \begin{cases} concat(A(i-1, j-1), \begin{smallmatrix} x_i \\ y_i \end{smallmatrix}) & \text{if this has the longest length;} \\ \\ \end{cases}$$

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exercise: Table filling and traceback

Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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a lot of recomputations, but where?

Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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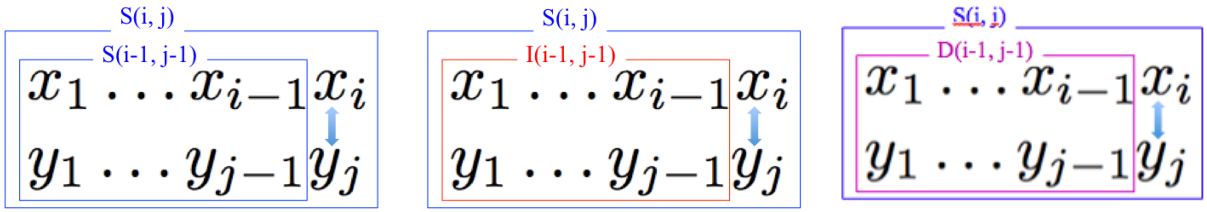
Substitute



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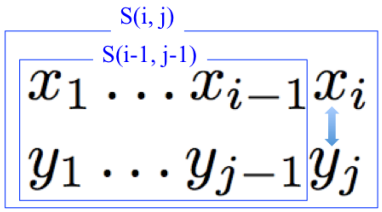
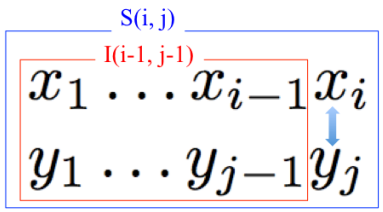
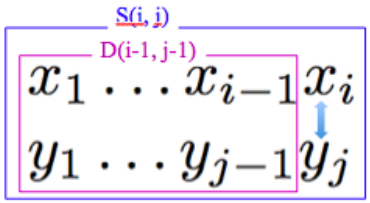
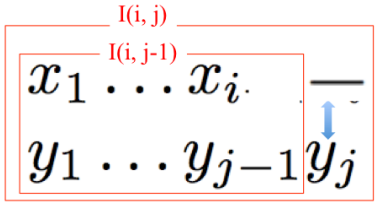
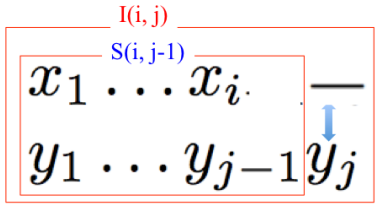






Insertion



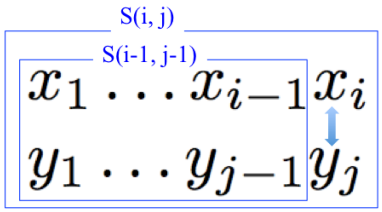


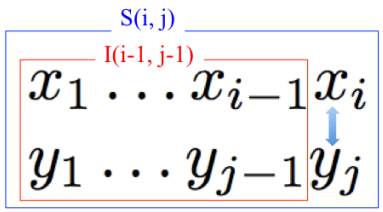
Chapter 15. Dynamic Programming

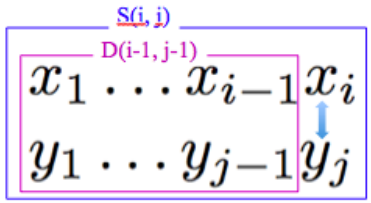
So we want to do recursion "one step at a time":

but at the same time, differentiate different "states"
where the two tails are

Substitute







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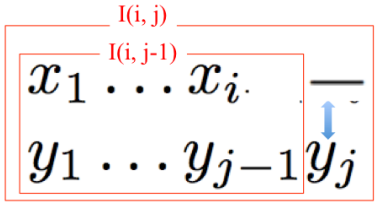
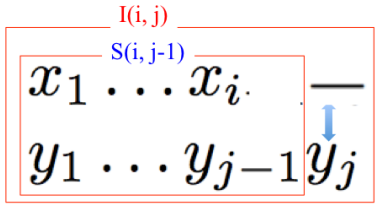
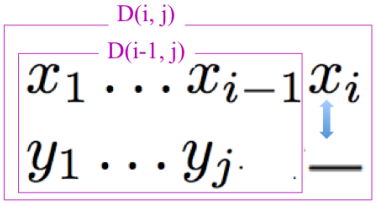
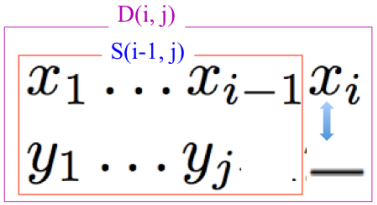




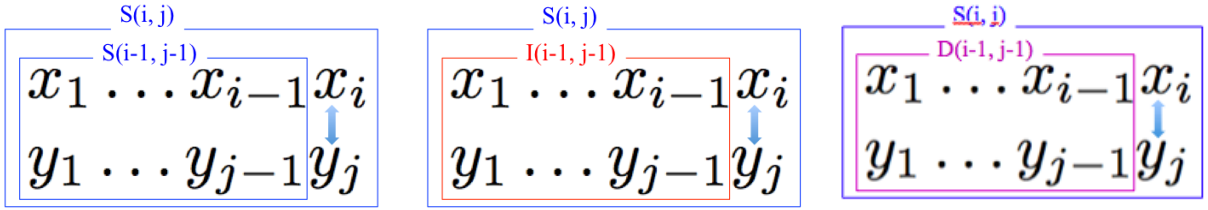
Deletion





Chapter 15. Dynamic Programming

Define $S(i, j)$ to be the maximum score of alignment between $x_1 \ldots x_i$ and $y_1 \ldots y_j$ with x_i being aligned to y_j.



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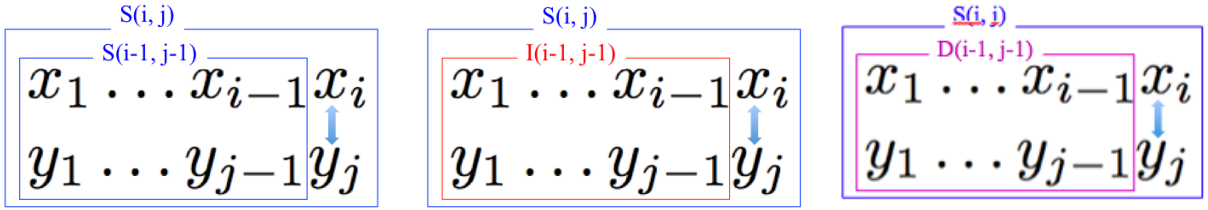

$$S(i,j) = \max \begin{cases} S(i-1,j-1) + score(x_i, y_j) \\ I(i-1,j-1) + score(x_i, y_j) \\ D(i-1,j-1) + score(x_i, y_j) \end{cases}$$

Chapter 15. Dynamic Programming

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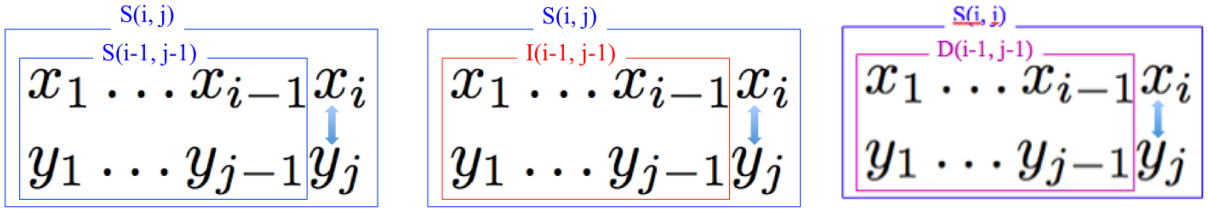

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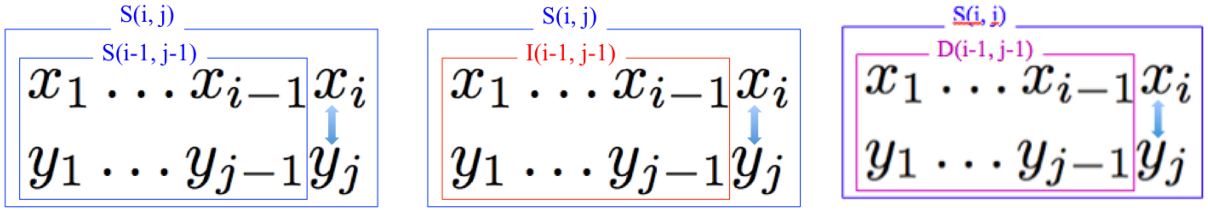

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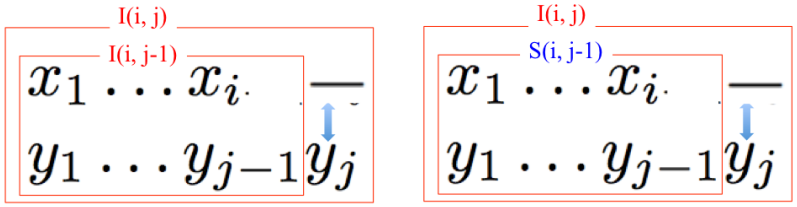

$$I(i,j) = \max \begin{cases} I(i,j-1) - ext \\ S(i,j-1) - op \end{cases}$$

Chapter 15. Dynamic Programming

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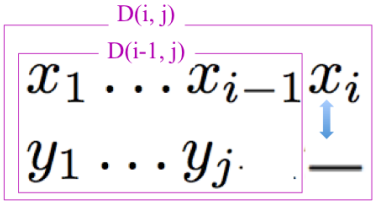

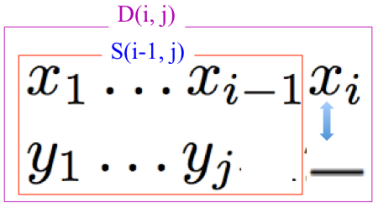

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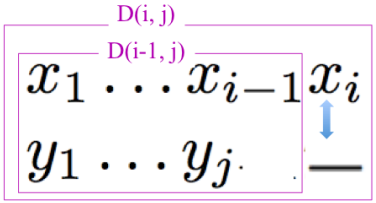

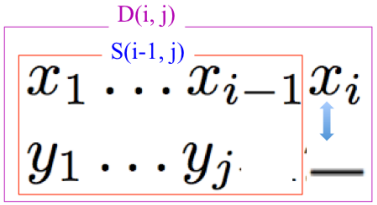

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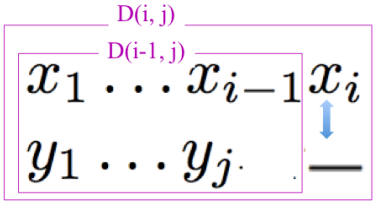

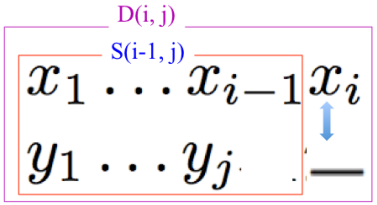

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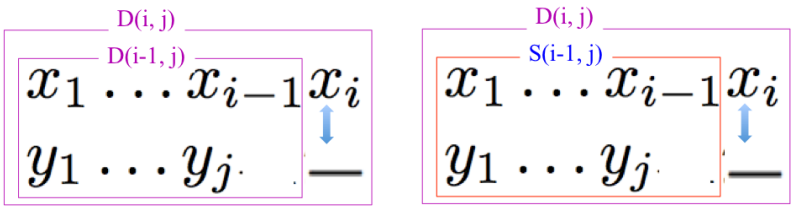

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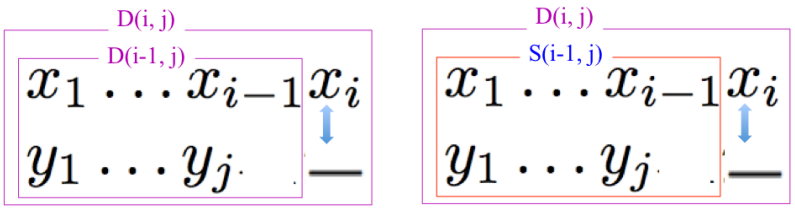

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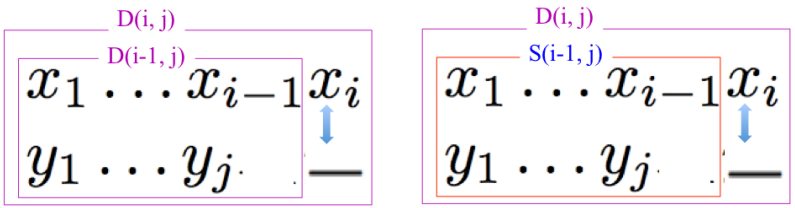

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- What is the desired value ? $\max \begin{cases} S(m,n) \\ I(m,n) \\ D(m,n) \end{cases}$
- How to fill the table(s) and to traceback the optimal alignment?

Chapter 15. Dynamic Programming

DP applied to summation problems

DP applied to summation problems

Unmatched Socks Problem

DP applied to summation problems

Unmatched Socks Problem

There are n pairs of socks, each a different color;
• each day, remove arbitrarily two individual socks;
compute the probability that, on none of the k days, a pair of color-matched socks is removed.

DP applied to summation problems

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$$P_2 = P_1 \times (P_u + P_m + P_{u,m}) =$$

DP applied to summation problems

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$$P_3 = P_2 \times ??$$

Unfortunately, the probability of individual days cannot be independently calculated,

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Note that

$$2m_k + u_k = 2n - 2k, \quad \text{for every } k \leq n$$

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$$P_k(m_k, u_k) = \sum \begin{cases} P_{k-1}(m_k + 2, u_k - 2) \times \frac{(m_k+2)(2m_k+2)}{\binom{2m_k+2+u_k}{2}} \\ P_{k-1}(m_k + 1, u_k) \times \frac{2(m_k+1)u_k}{\binom{2m_k+2+u_k}{2}} \\ P_{k-1}(m_k, u_k + 2) \times \frac{\binom{u_k+2}{2}}{\binom{2m_k+2+u_k}{2}} \end{cases}$$

Chapter 15. Dynamic Programming

Knapsack Problem

Chapter 15. Dynamic Programming

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INPUT: n items of sizes $s_1, \ldots, s_n$ and values $v_1, \ldots, v_n$;
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Chapter 15. Dynamic Programming

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INPUT: n items of sizes $s_1, \ldots, s_n$ and values $v_1, \ldots, v_n$;
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OUTPUT: a subset $A \subseteq \{1, 2, \ldots, n\}$ that maximizes $\sum_{i \in A} v_i$

Chapter 15. Dynamic Programming

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and a knapsack size B

OUTPUT: a subset $A \subseteq \{1, 2, \ldots, n\}$ that maximizes $\sum_{i \in A} v_i$
with constraint $\sum_{i \in A} s_i \leq B$

Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Chapter 15. Dynamic Programming

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Both situations reduce the problem size to a subproblem.

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Chapter 15. Dynamic Programming

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Then

$$A(k, X) = \begin{cases} \textbf{either } A(k-1, X - s_k) \cup \{k\} & \text{value gained } v_k \\ \textbf{or } A(k-1, X) & \text{value not gained} \end{cases}$$

Define $V(k, X)$ to be the maximum value of items drawn from $\{1, 2, \ldots, k\}$ which fit into the space of X, then

$$V(k, X) = \max \begin{cases} V(k-1, X - s_k) + v_k & X \geq s_k \\ V(k-1, X) \end{cases}$$

Chapter 15. Dynamic Programming

- For every $k \leq n$ and every $X \leq B$.

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Chapter 15. Dynamic Programming

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- What does the table look like? How to fill it out?
- How to traceback for A, the optimal subset of items?

Chapter 15. Dynamic Programming

Example

The optimal knapsack should contain {1,2} = 7

Items (s_i, v_i)

1: (2, 3)
2: (3, 4)
3: (4, 5)
4: (5, 6)

i\s	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
2	0	0	3	4	4	7
3	0	0	3	4	5	7
4	0	0	3	4	5	7

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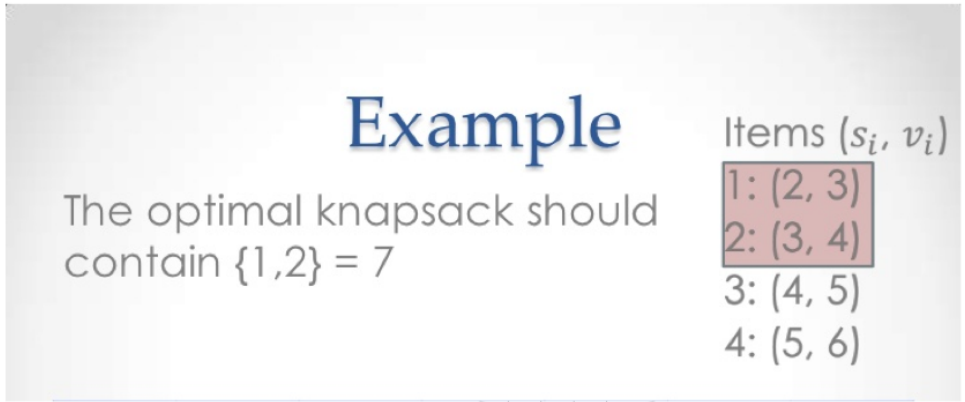
Chapter 15. Dynamic Programming



i\s	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
2	0	0	3	4	4	7
3	0	0	3	4	5	7
4	0	0	3	4	5	7

$$V(k, X) = \max \begin{cases} V(k-1, X - s_k) + v_k & X \geq s_k \\ V(k-1, X) \end{cases}$$

- fill out the base case row and column;
- the order of cells to be filled;
- $V(i, X) = V(i-1, X)$ if $X < s_i$.
- time complexity

Chapter 15. Dynamic Programming

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<u>Matrix Chain Multiplication problem</u>

INPUT: matrices $A_1, \cdots, A_n$, where A_i has dimension $p_{i-1} \times p_i$;
OUTPUT: a parenthesization by which the product $A_1 \times A_2 \times \cdots \times A_n$ uses the minimum number of scalar multiplications.

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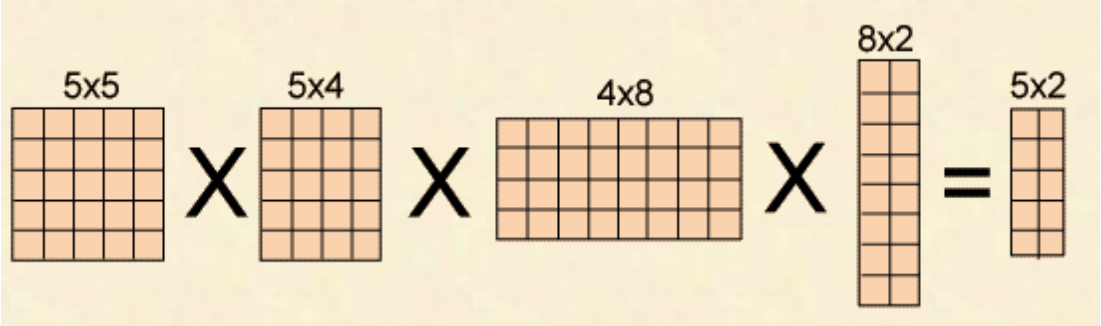
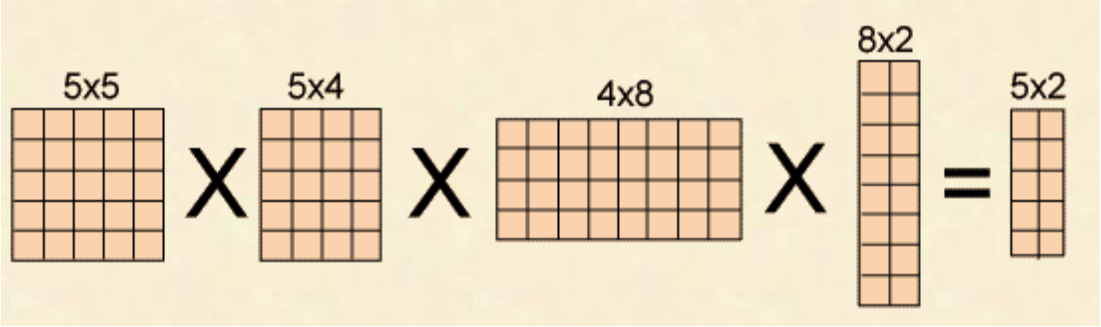


Chapter 15. Dynamic Programming



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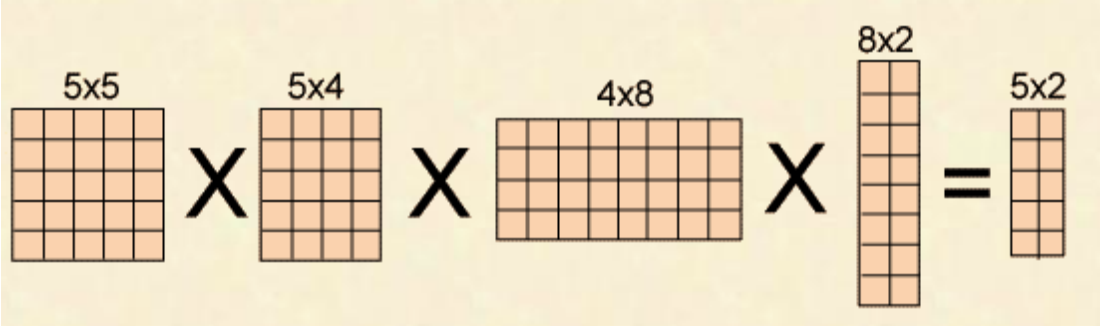
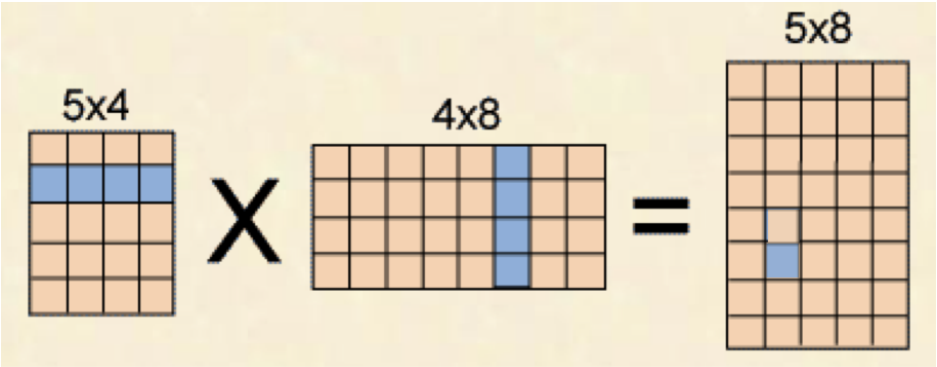


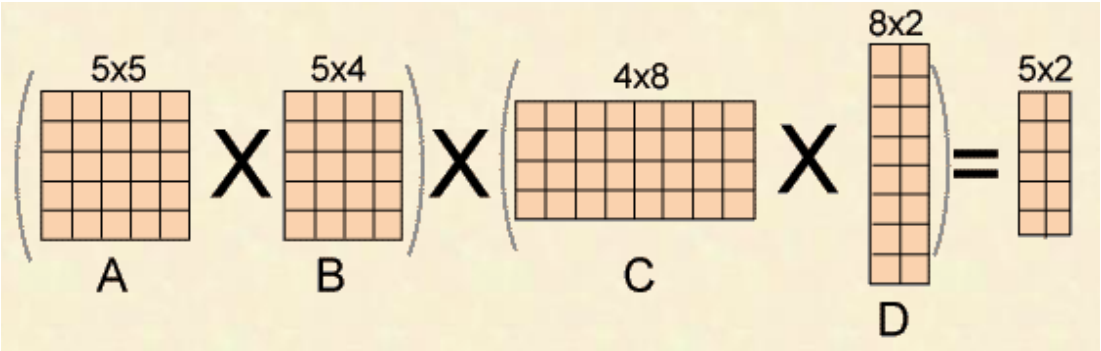
scalable multiplications



$$= 5 \times 4 \times 8$$

Chapter 15. Dynamic Programming

Many possible parenthesizations



Chapter 15. Dynamic Programming

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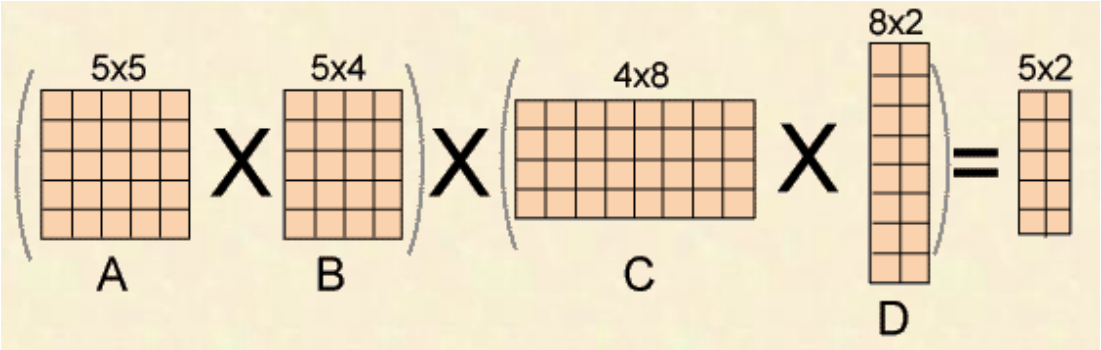

1. **(A (B (C D)))**
   5(5)2 + 5(4)2 + 4(8)2 = 154
2. **(A ((B C) D))**
   5(5)2 + 5(4)8 + 5(8)2 = 290
3. **((A B) (C D))**
   5(5)4 + 4(8)2 + 5(4)2 = 204
4. **((A (B C)) D)**
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Chapter 15. Dynamic Programming

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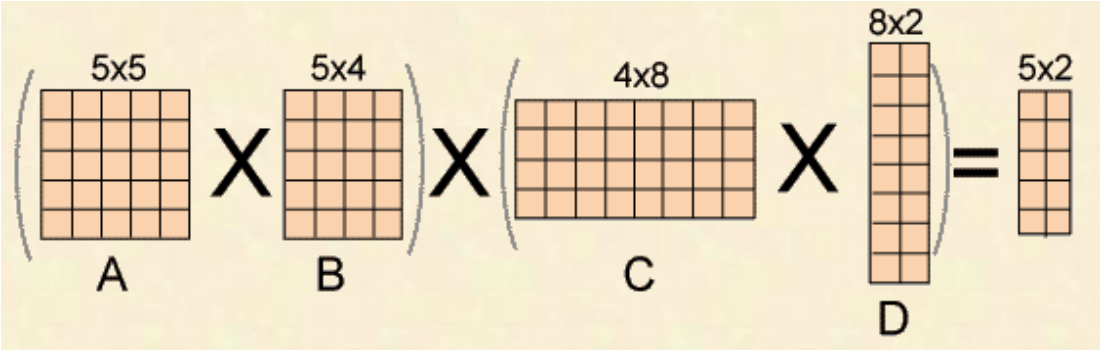

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Note: The number of all possible parenthesizations is

$$P(n) = \Sigma_{k=1}^{n-1} P(k)P(n-k), \text{ Catalan number}$$

Chapter 15. Dynamic Programming

Step 1: identify *optimal substructure*:
Optimal solution in terms of optimal solutions to subproblems:

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(1) the best parenthesization of $A_i A_{i+1} \cdots A_j$ must be

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for some $k, i \leq k < j$.

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(2) The optimal cost of $A_i A_{i+1} \cdots A_j$ must be the smallest among optimal costs for

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$k = i, i+1, \cdots, j-1$.

(3) For each k, the optimal cost for the above $=$
the optimal cost for $(A_i \cdots A_k)$ $+$
the optimal cost for $(A_{k+1} \cdots A_j)$ $+$
the number of scalar multiplications to multiply the two terms.

Step 2: objective function and recurrence

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Define $f(i, j)$ to be the minimum number of scalar multiplications needed for $A_i A_{i+1} \cdots A_j$.

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Define $f(i,j)$ to be the minimum number of scalar multiplications needed for $A_i A_{i+1} \cdots A_j$. Then

$$f(i,j) = \min_{i \le k < j} \{f(i,k) + f(k+1,j) + p_{i-1}p_k p_j\}$$

where base case is

$f(i,j) = 0$ when $i = j$, for single matrix A_i.

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Chapter 15. Dynamic Programming

	1	2	3	4	5	6
1	0 _	20 1	35 2	53 3	95 4	99 1
2		0 _	60 2	132 3	270 3	95 2
3			0 _	90 3	231 3	75 3
4				0 _	126 4	60 4
5					0 _	42 5
6						0 _

Side lengths: 1, 4, 5, 3, 6, 7, 1

$$A_1^{(1\times 4)} \times A_2^{(4\times 5)} \times A_3^{(5\times 3)} \times A_4^{(3\times 6)} \times A_5^{(6\times 7)} \times A_6^{(7\times 1)}$$

Algorithm $\textsc{MatrixChainOrder}(p)$

1. $n = length[p] - 1;$

Chapter 15. Dynamic Programming

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2. **for** $i = 1$ **to** n

Chapter 15. Dynamic Programming

Algorithm $\textsc{MatrixChainOrder}(p)$

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n = length[p] - 1;
for i = 1 to n
    M[i, i] = 0;
```

Algorithm $\textsc{MatrixChainOrder}(p)$

```
n = length[p] - 1;
for i = 1 to n
    M[i, i] = 0;
for l = 2 to n
```

Algorithm $\textsc{MatrixChainOrder}(p)$

```
n = length[p] - 1;
for i = 1 to n
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for l = 2 to n
    for i = 1 to n - l + 1
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n = length[p] - 1;
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    M[i, i] = 0;
for l = 2 to n
    for i = 1 to n - l + 1
        j = i + l - 1;
```

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        j = i + l − 1;
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        for k = i to j − 1
```

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          if q < M[i, j]
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              S[i, j] = k
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return (M, S)
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      if q < M[i, j]
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Time complexity: $O(n^2 \times n)$

Step 4. obtain the optimization parenthesization

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Algorithm $\textsc{MatrixChainParenthesization}(i, j, S)$

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if i < j
    k = s[i, j]
    print (i, j, " : ", k)
    MatrixChainParenthesization(i, k, S)
    MatrixChainParenthesization(k + 1, j, S)
return
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Time complexity for $\textsc{MatrixChainParenthesization}$:

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Time complexity for $\textsc{MatrixChainParenthesization}$: $O(n)$.

Chapter 16. Greedy Algorithms

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- A greedy algorithm may *ignore some choices and select the one that is locally the best*.

Chapter 16. Greedy Algorithms

Chapter 16 Greedy Algorithms

- Dynamic programming is to *consider all possible choices and select the best*.

  a DP approach always leads to the optimal solution

- A greedy algorithm may *ignore some choices and select the one that is locally the best*.

  a greedy strategy may or may NOT lead to the optimal solution

Chapter 16. Greedy Algorithms

Activity-selection problem

Chapter 16. Greedy Algorithms

<u>Activity-selection problem</u>

INPUT: n activities $S = \{a_1, \ldots, a_n\}$, each with a start time s_i and finish time f_i, $1 \leq i \leq n$

OUTPUT: subset $A \subseteq A$ of mutually compatible activities such that $|A|$ is the maximum.

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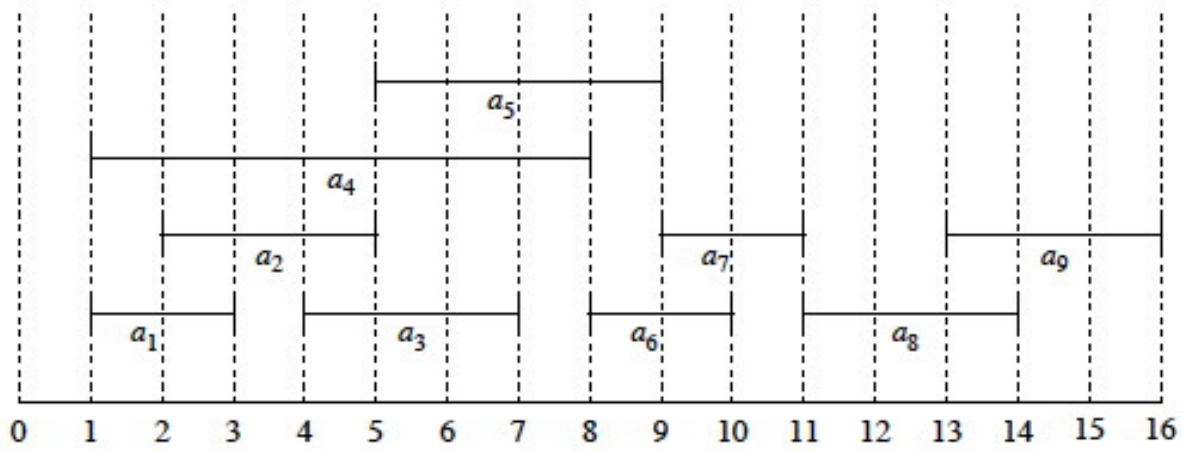

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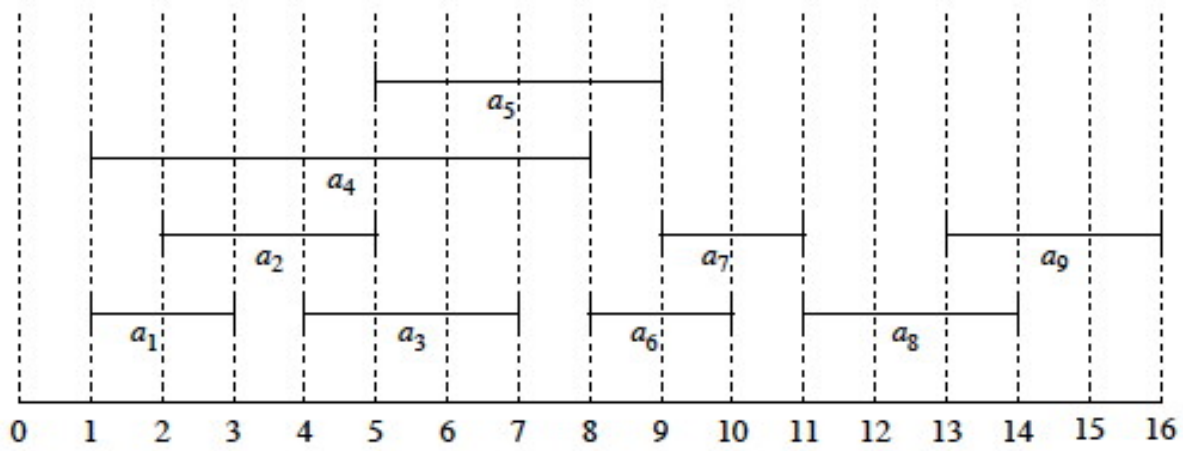

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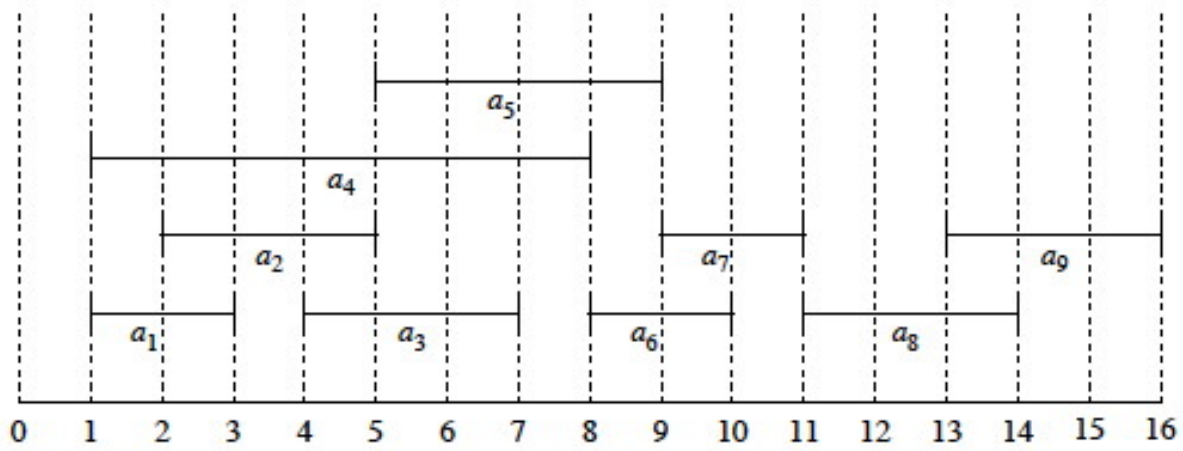

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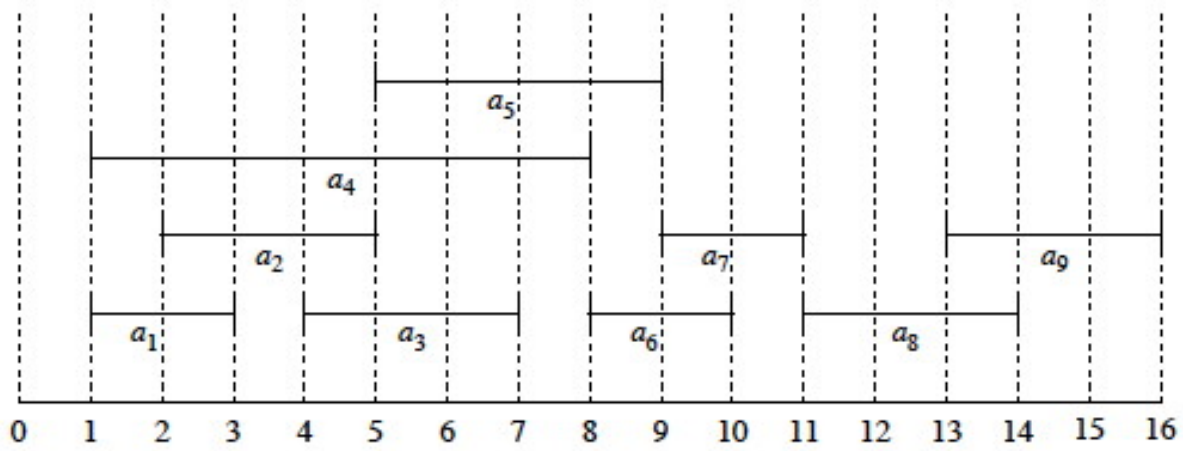

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Chapter 16. Greedy Algorithms

A dynamic programming solution

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Step 1 analysis of problem

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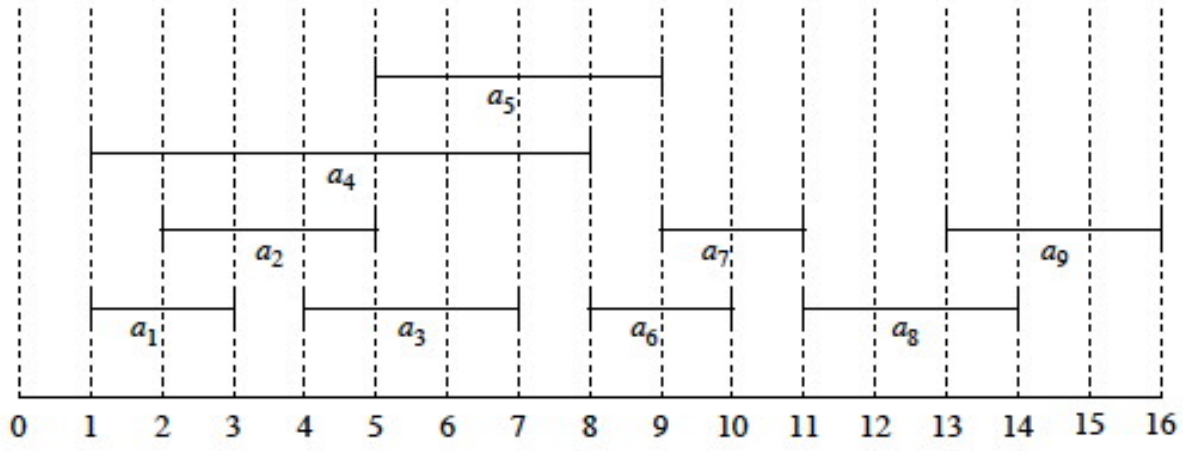
  but how to put the activities in order?
  perhaps there is no "forward DP" for this problem.

- instead, like Matrix Chain Multiplication problem,
  we consider "inside-out DP" method
  exhaustively consider to **include** a_k, for some k.

Chapter 16. Greedy Algorithms



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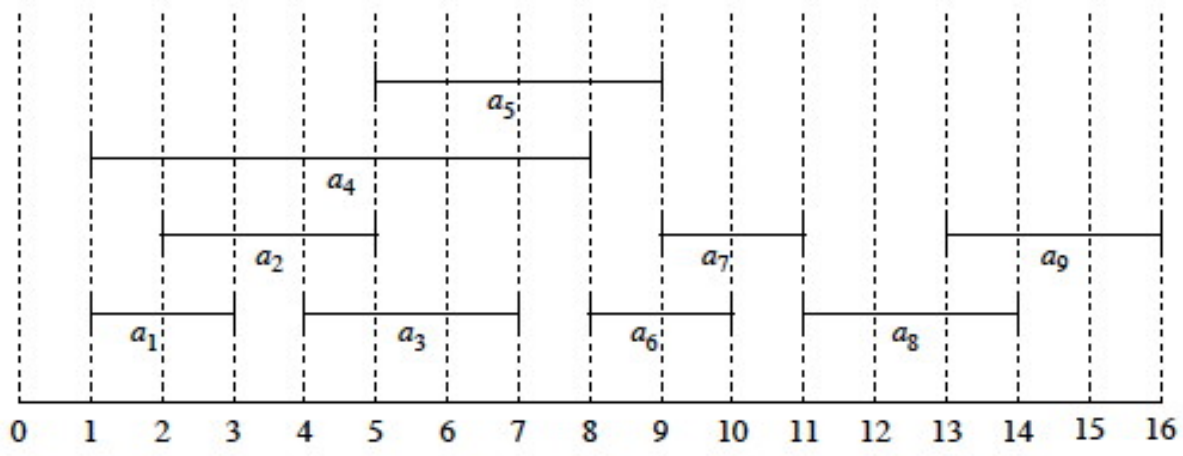
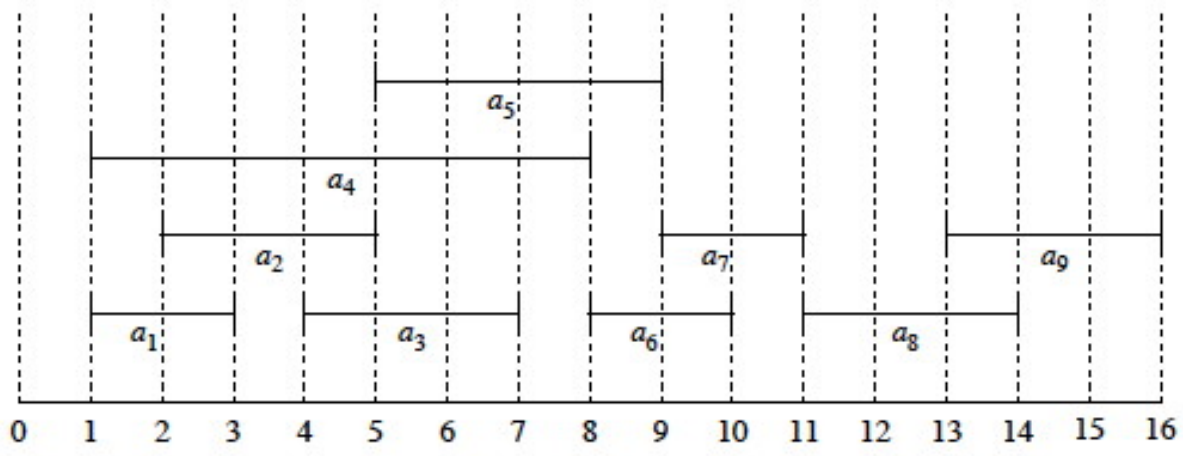


For example, selecting a_3 would result in two subsets of activities to further
consider: $\{a_1\}$ and $\{a_6, a_7, a_8, a_9\}$;

Chapter 16. Greedy Algorithms



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selecting a_5 would result in two subsets of activities to further consider: $\{a_1, a_2\}$ and $\{a_7, a_8, a_9\}$;

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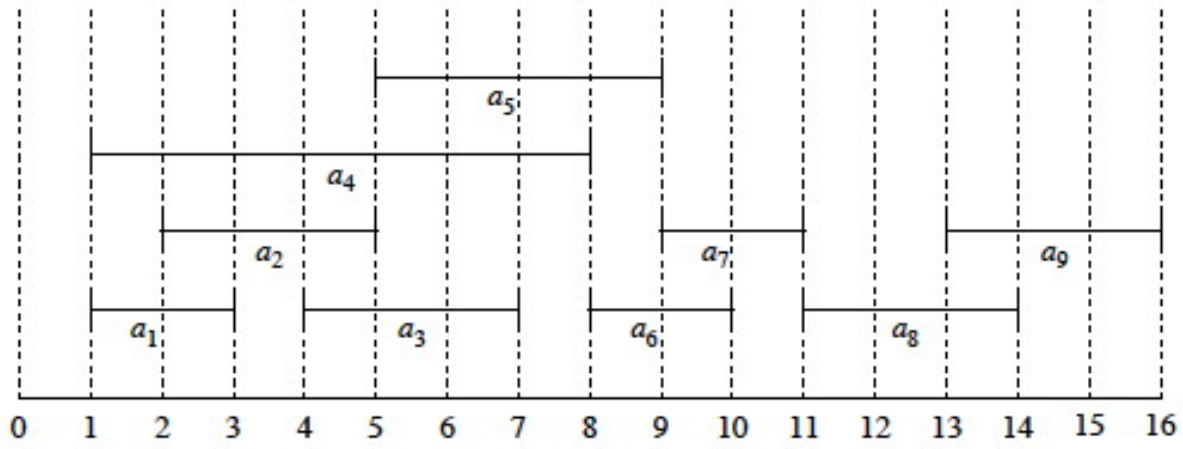

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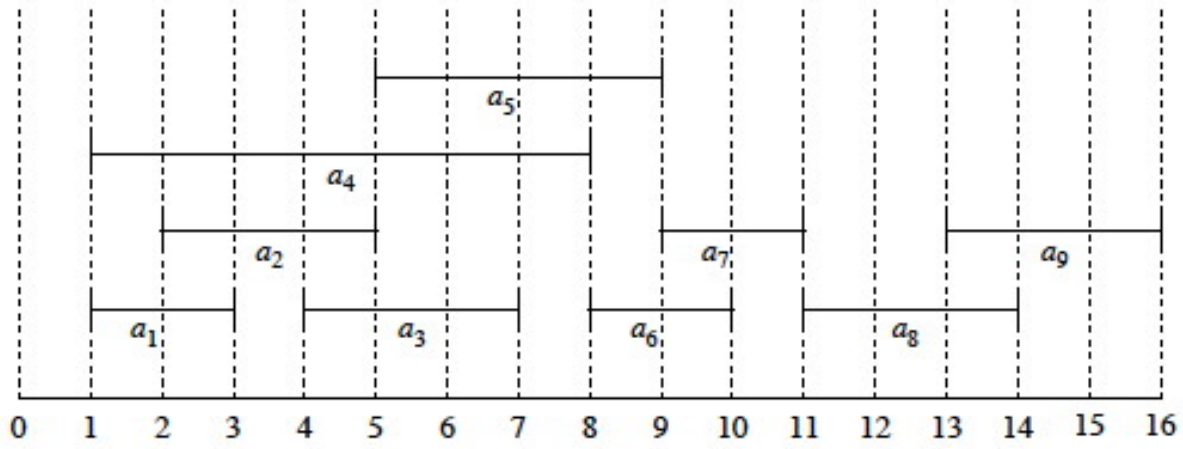

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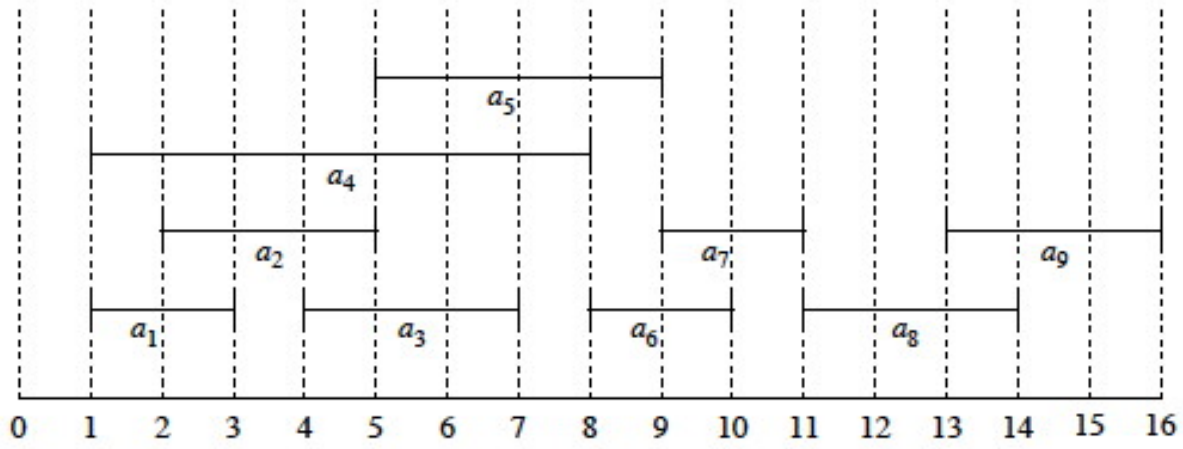

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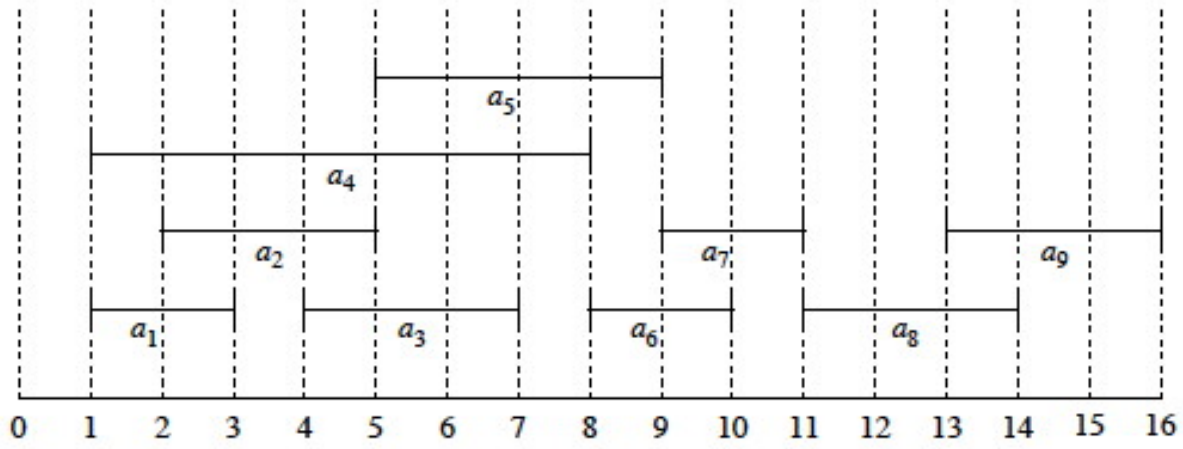
Chapter 16. Greedy Algorithms



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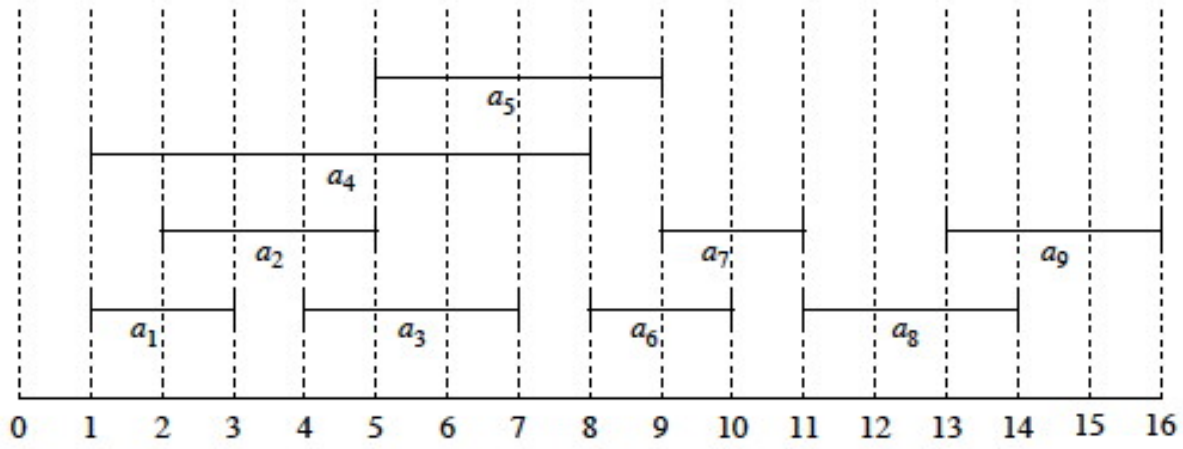

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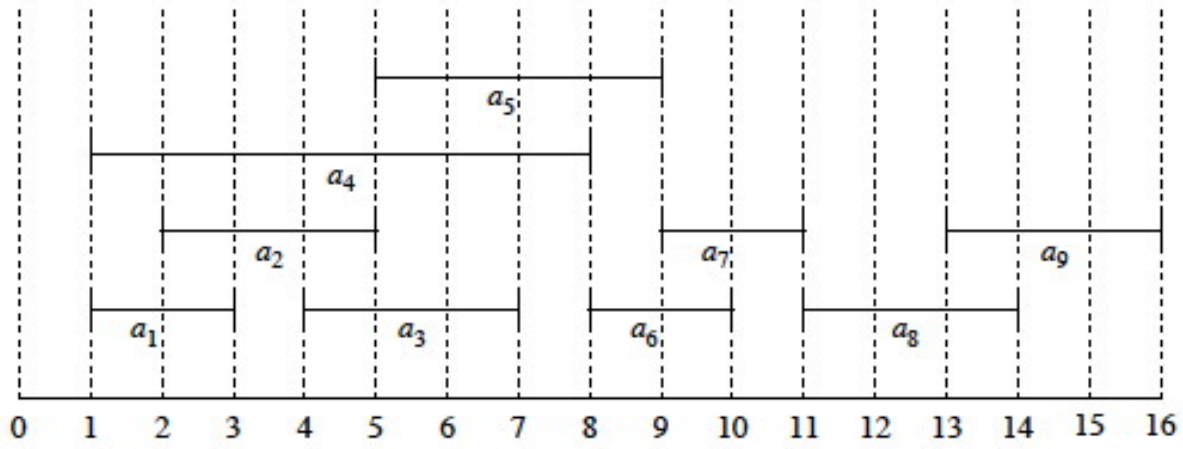

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introducing dummy activities:

a_0 with f_0 =the earliest start time of all, and

a_{n+1} with s_{n+1} = latest finish time of all.

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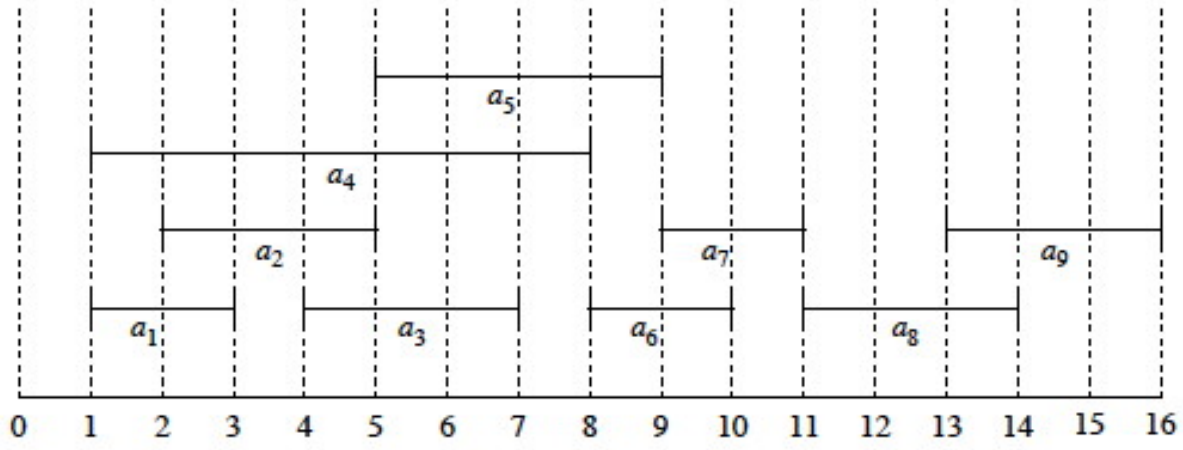

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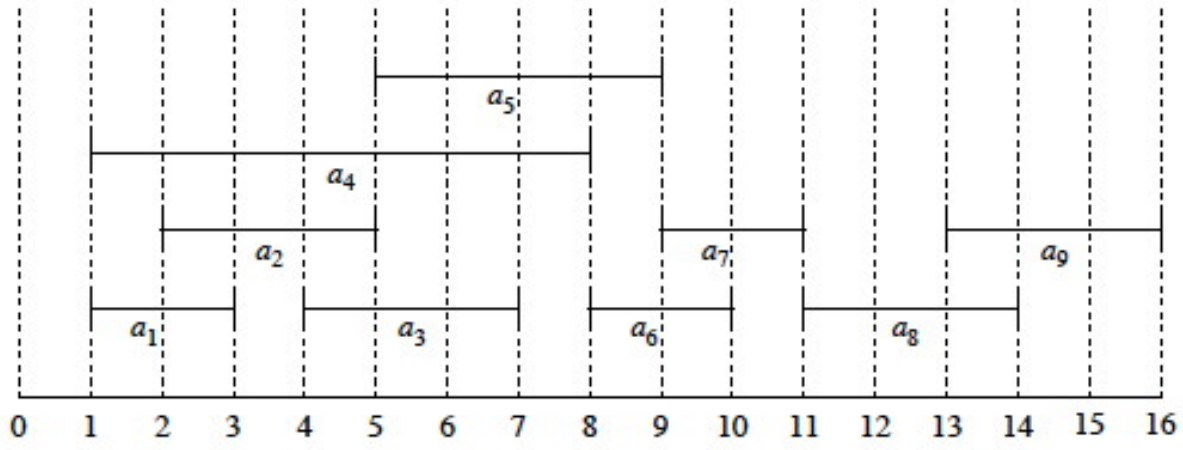

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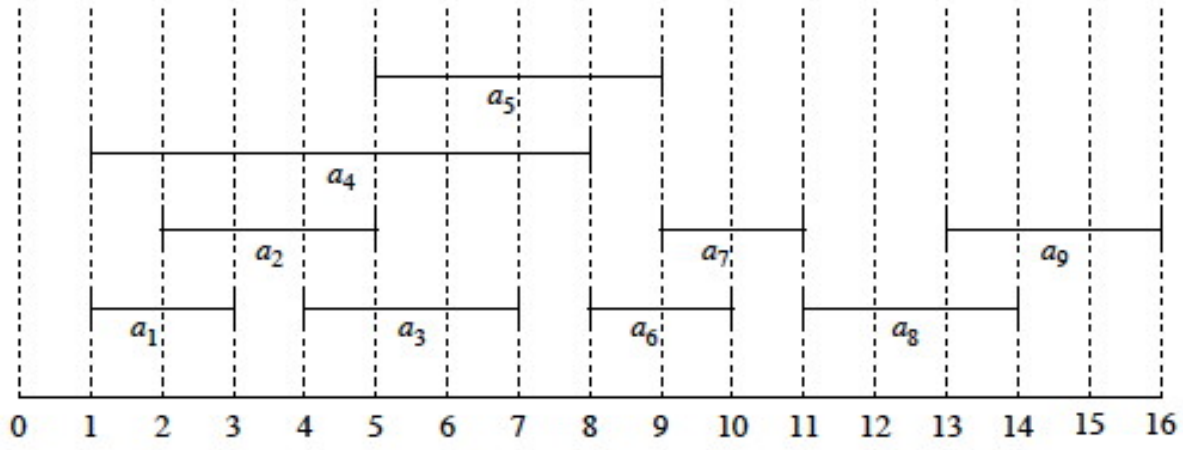

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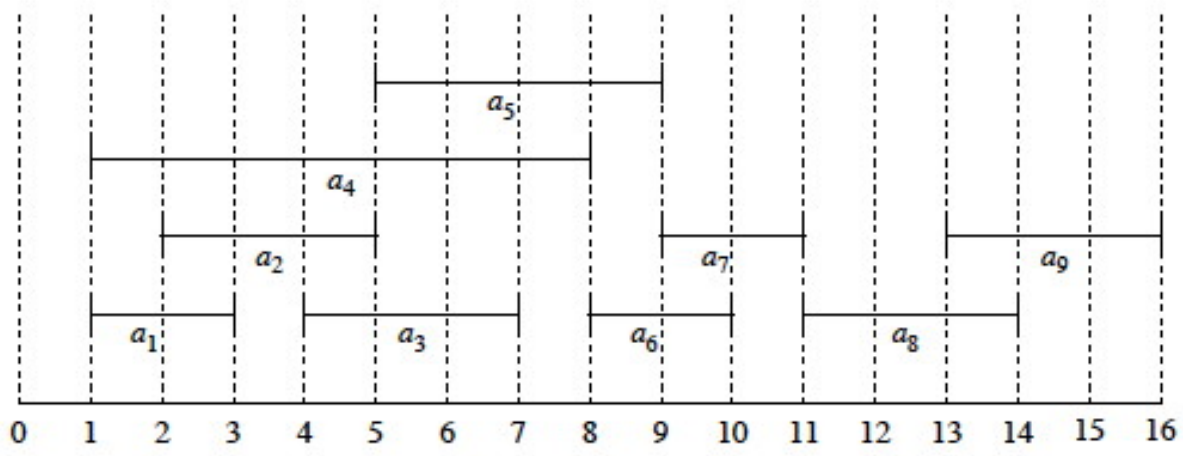

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$$|A_{0,10}| = \max \begin{cases} \dots \\ |A_{0,6} \cup \{a_6\} \cup A_{6,10}|, \text{ where } \quad S_{0,6} = \{a_1, \dots, a_4\} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad S_{6,10} = \{a_8, a_9\} \\ \dots \end{cases}$$

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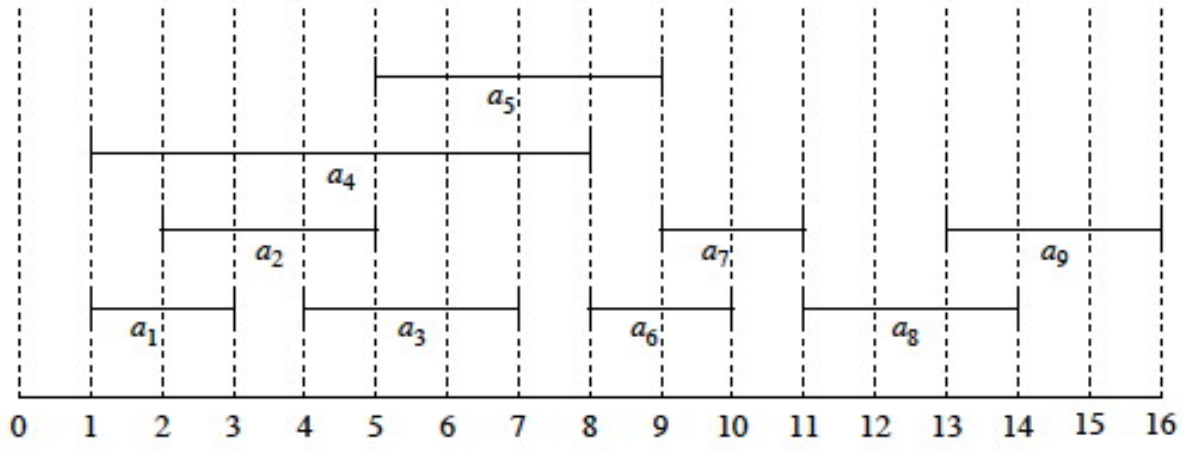

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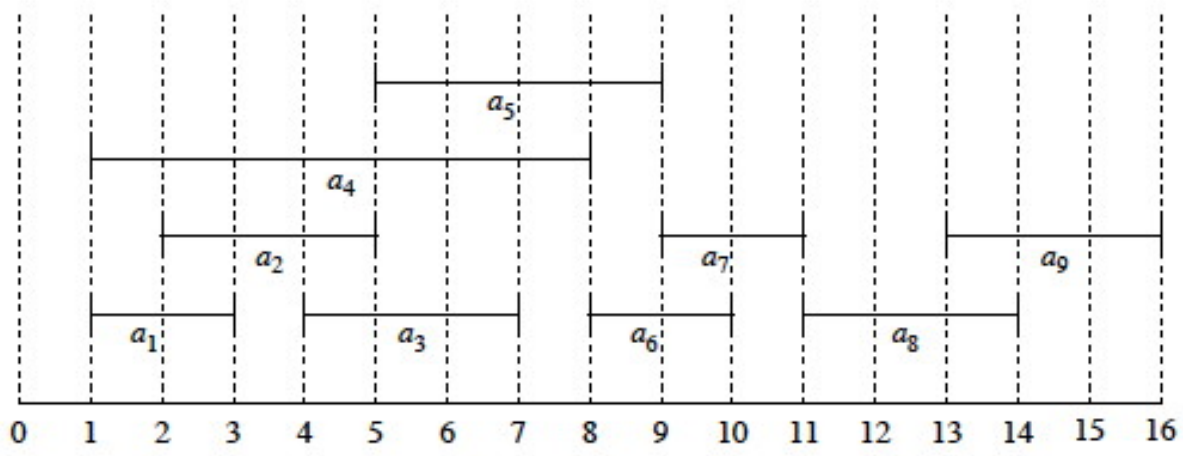

Instead of go forward with the DP, we wonder if all but one **of the subproblems** could be made empty.

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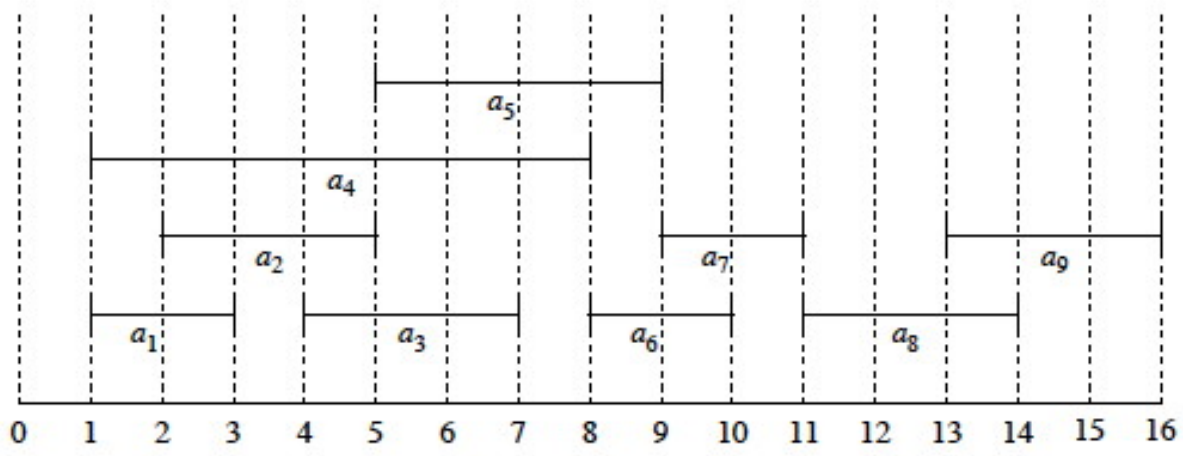

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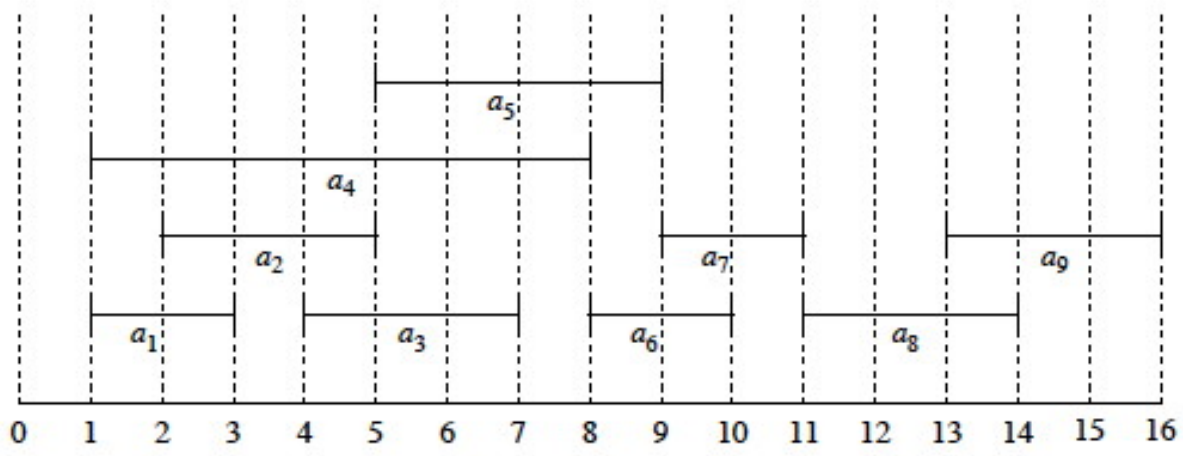

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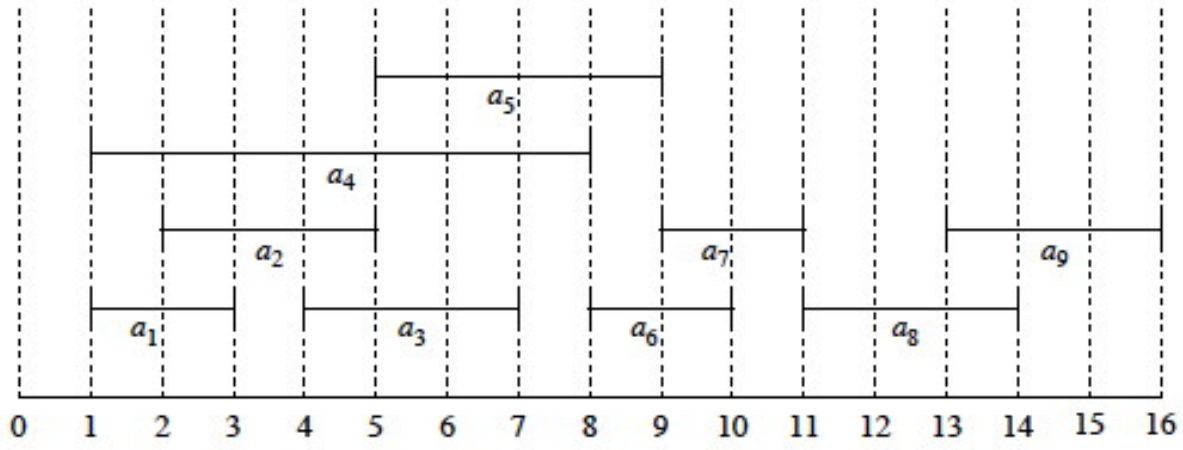

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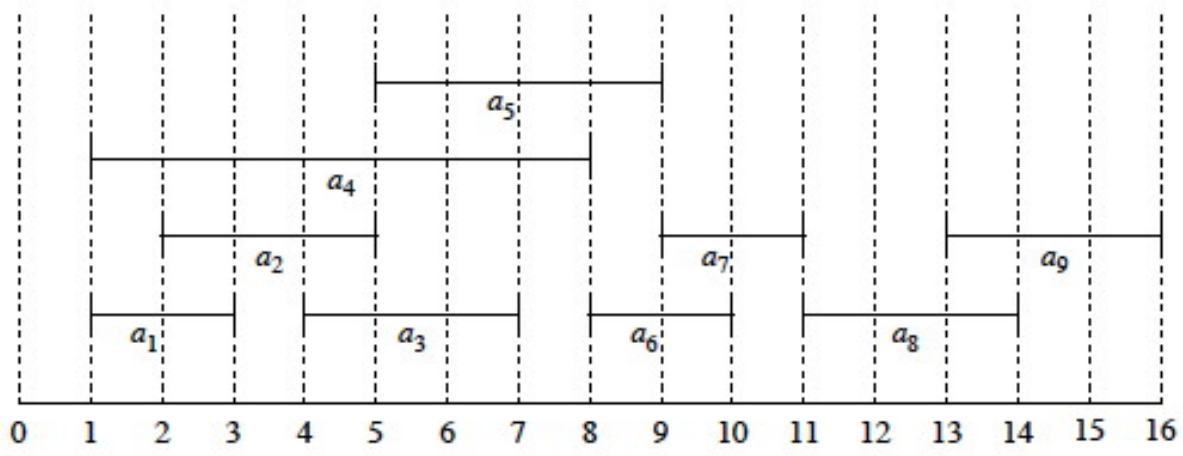

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Chapter 16. Greedy Algorithms

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We define $B_{i,j} = A_{i,j} - \{a_p\} \cup \{a_k\}$. Then

- because $f_p \geq f_k$, a_k is compatible with other activities in $B_{i,j}$.
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Since $a_k \in B_{i,j}$, we prove the theorem.

Chapter 16. Greedy Algorithms

Using the Theorem 16.1 and the recurrence

$$|A_{i,j}| = \max_{a_k \in S_{ij}} \{|A_{i,k} \cup \{a_k\} \cup A_{k,j}|\}$$

we can derive greedy algorithms for the Activity Selection problem.

Chapter 16. Greedy Algorithms

Assume that the activities are sorted according to their finish times.
$f_1 \leq f_2 \leq \cdots \leq f_n$.

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Adding hypothetical activity a_0 (with $s_0 = f_0 = s_1$)

Chapter 16. Greedy Algorithms

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Chapter 16. Greedy Algorithms

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Chapter 16. Greedy Algorithms

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Chapter 16. Greedy Algorithms

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Chapter 16. Greedy Algorithms

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Chapter 16. Greedy Algorithms

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Chapter 16. Greedy Algorithms

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Time complexity: $O(n)$.

Chapter 16. Greedy Algorithms

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(This is left as an exercise)

Chapter 16. Greedy Algorithms

Huffman Code

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compressing data using binary bits

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frequency	.45	.13	.12	.16	.09	.05
fixed length code	000	001	010	011	100	101
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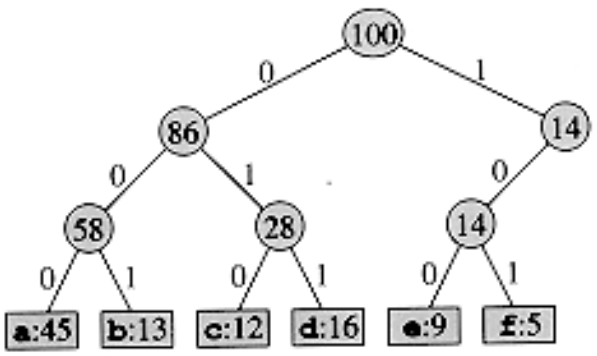
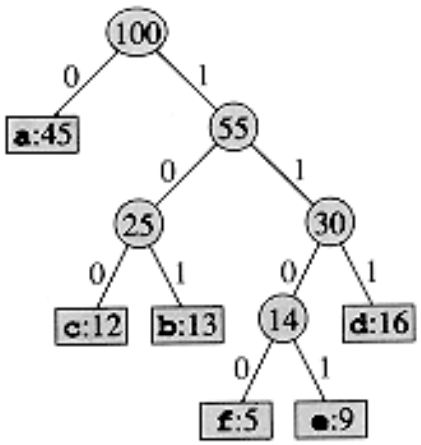


(a)



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Chapter 16. Greedy Algorithms

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- $d_T(c)$ is the length of codeword for c under the code scheme T.
- $B(T) \times n$ is the number of bits required to code a file (of n characters).

Chapter 16. Greedy Algorithms

Algorithm $\textsc{Huffman}(C, f)$

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1. $n = |C|$

Chapter 16. Greedy Algorithms

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Chapter 16. Greedy Algorithms

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    x = EXTRACT-MIN(Q)        extract two least frequent characters
```

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n = |C|
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    z.leftchild = x
```

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    INSERT(Q, z)
return (EXTRACT-MIN(Q))
```
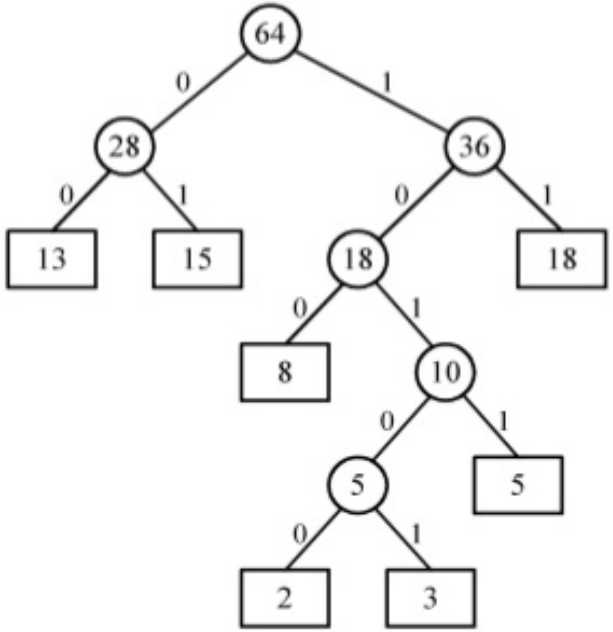

An example of Huffman algorithm

- Symbols: A, B, C, D, E, F, G
  freq. : 2, 3, 5, 8, 13, 15, 18
- Huffman codes:
  A: 10100 B: 10101 C: 1011
  D: 100 E: 00 F: 01
  G: 11



A Huffman code Tree

Correctness of Huffman's algorithm

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Lemma 16.2 (Greedy choice property)

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Let C be an alphabet and f be the frequency function for characters in C. Let x and y be two characters in C having the lowest frequencies. Then there exists an optimal prefix code for C in which the codewords for x and y have the same length and differ in the last bit.

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Proof:
Assume C, f, x and y are as described in the lemma condition.

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(b) let $a, b \in C$ be two characters such that their codewords have the same length and differ in the last bit. Also assume that their codeword length

$$d_T(a) = d_T(b) \geq d_T(c), \text{ for every } c \in C$$

We note that such a and b can be found from T without difficulty.

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$$-f(a)d_{T'}(a) - f(b)d_{T'}(b) - f(x)d_{T'}(x) - f(y)d_{T'}(y)$$

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$$
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&= f(a)d_T(a) + f(b)d_T(b) + f(x)d_T(x) + f(y)d_T(y) \\
&\qquad - f(a)d_{T'}(a) - f(b)d_{T'}(b) - f(x)d_{T'}(x) - f(y)d_{T'}(y) \\
&= (f(a) - f(x))d_T(a) + (f(x) - f(a))d_T(x) + (f(b) - f(y))d_T(b)
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Chapter 16. Greedy Algorithms

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Chapter 16. Greedy Algorithms

We use the following swapping strategy to obtain another prefix code T' for C.

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Chapter 16. Greedy Algorithms

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Theorem 16.4 Huffman's algorithm produces an optimal prefix code.

Chapter 16. Greedy Algorithms

Coin change problem

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Chapter 16. Greedy Algorithms

A dynamic programming solution

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analysis:

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Chapter 16. Greedy Algorithms

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A greedy algorithm

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  (1) all coins in A are one-cent coins, let $B = A - \{1, 1, 1, 1, 1, 1, 1, 1, 1, 1\} \cup \{10\}$, then $|B| < |A|$, which is impossible. So $10 \in A$;

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Review for Parts I, II, and III

Summaries for Parts I, II, and III

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Summaries for Parts I, II, and III

Part I: Fundamentals of Analysis of algorithms

Review for Parts I, II, and III

Summaries for Parts I, II, and III

Part I: Fundamentals of Analysis of algorithms

- time/space complexity functions

Review for Parts I, II, and III

Summaries for Parts I, II, and III

Part I: Fundamentals of Analysis of algorithms

- time/space complexity functions
  input size

Review for Parts I, II, and III

Summaries for Parts I, II, and III

Part I: Fundamentals of Analysis of algorithms

- time/space complexity functions
  input size
  upper bound

Review for Parts I, II, and III

Summaries for Parts I, II, and III

Part I: Fundamentals of Analysis of algorithms

- time/space complexity functions
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Review for Parts I, II, and III

Summaries for Parts I, II, and III

Part I: Fundamentals of Analysis of algorithms

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Review for Parts I, II, and III

Summaries for Parts I, II, and III

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  big-O

Review for Parts I, II, and III

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Review for Parts I, II, and III

Part II: Sorting and order statistics

Review for Parts I, II, and III

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- sorting algorithms: heap, insertion, merge, quick

Review for Parts I, II, and III

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- linear time to find kth smallest element

Review for Parts I, II, and III

Part III. Exhaustive search, DP and greedy algorithms

Review for Parts I, II, and III

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Part III. Exhaustive search, DP and greedy algorithms

- exhaustive search by
  simple enumeration of all possible solutions

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Part III. Exhaustive search, DP and greedy algorithms

- exhaustive search by
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  search tree method

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  search tree method resulting in $T(n) = T(n-1) + T(n-1-m)$, $m \geq 1$.
- dynamic programming
  recursive solution (with optimal substructure)
  recurrence for numerical objective function
  iterative, bottom-up table-filling
  traceback of solution.

  'forward DP', 'backward DP', and 'inside-out DP'

Review for Parts I, II, and III

Part III. Exhaustive search, DP and greedy algorithms

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  DP solution
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  efficient algorithms