Chapter 2. Basic Structures: sets, functions, sequences, and sums

2.1 Sets

*definition*: a set is an unordered collection of objects. e.g., \( A = \{a_1, a_2, \cdots, a_{10}\} \)

elements in the set, \( a \in A \),
elements not in the set: \( b \notin B \)

More examples of sets

Two sets are *equal* if they have the same elements.
\( \{a, b\} = \{b, a, b\} \)

Venn diagram, empty set (null set), \( \phi \)
subset, \( A \subseteq B \) if and only if
\[ \forall x (x \in A \rightarrow x \in B) \]

\( \emptyset \subseteq S \), \( S \subseteq S \)

\( A = B \) if and only
\( A \subseteq B \) and \( B \subseteq A \)

\( A \subset B \)
if \( A \subseteq B \) and \( \exists b (b \in B \land b \notin A) \)

\( |S| \), the cardinality the number of different elements in a set \( S \)
set builder \( A = \{a : Q(a)\} \)
where \( Q \) is a predicate

infinite sets, e.g., \( N = \{x : x \geq 0 \text{ and } x \text{ is integer}\} \)

power set of \( S \) is the set of all subsets of \( S \)
\( P(S) = \{A : A \subseteq S\} \)

examples:
\( P(\{1, 2\}) = \{\phi, \{1\}, \{2\}, \{1, 2\}\} \)
\( P(\phi) = \{\phi\} \)
\( P(\{\phi\}) = \{\phi, \{\phi\}\} \)
\( |P(S)| = ? \) for a finite set \( S \)
\( P(N) = ? \)
Cartesian product

ordered n-tuple: \((a_1, a_2, \cdots, a_n)\),
\((a, b)\) is an ordered pair

\((a_1, a_2, \cdots, a_n) = (b_1, b_2, \cdots, b_n)\)
if and only \(a_i = b_i\) for \(i = 1, 2, \cdots, n\)

\(A \times B = \{(a, b) : a \in A \land b \in B\}\)
examples, records, database tables

set notations with quantifiers
\(\forall x \in S \ P(x)\)

what is \(A \times \phi\)?
2.2 Set operations

various ways to combine two or more sets

union: $A \cup B = \{x : x \in A \lor x \in B\}$

intersection: $A \cap B = \{x : x \in A \land x \in B\}$

$A$ and $B$ are disjoint if $A \cap B = \emptyset$

difference: $A - B = \{x : x \in A \land x \notin B\}$

complement: $\bar{A} = U - A$, where $U$ is the universal set.

Venn diagram representations
Set identities

table 1, p124. all can be proved by the definitions of sets and set operations

generalized unions and intersections

\[ A_1 \cup A_2 \cup \cdots \cup A_n = \bigcup_{i=1}^{n} A_i = \{ x : \exists i \ x \in A_i \} \]

\[ A_1 \cap A_2 \cap \cdots \cap A_n = \bigcap_{i=1}^{n} A_i = \{ x : \forall i \ x \in A_i \} \]

computer representations of sets

binary bits

union and intersection operations
2.3 Functions

A function from set $A$ to set $B$ is an association of exactly one element of $B$ to each element in $A$, $f(a) = b$.

This can be defined using proposition

$$\exists b \in B[f(a) = b \land \forall c (c \neq b \rightarrow f(a) \neq c)]$$

denoted: $f : A \rightarrow B$

*domain, co-domain, image, pre-image, map*

Addition of two functions: $f + g$

$$(f + g)(x) = f(x) + g(x)$$

Product of two functions: $fg$

$$(fg)(x) = f(x)g(x)$$
function on subsets: if $S \subseteq A$
$f(S) = \{ f(x) : x \in S \}$, images of $S$
$f(S) \subseteq B$

1-1 function (or injective): if

$$\forall x \forall y (f(x) = f(y) \rightarrow x = y)$$

Is $f(x) = x^2$ a 1-1 function?

onto function (or surjective): if

$$\forall y \exists x (f(x) = y)$$

1-1 correspondence (or bijective) if
(1) 1-1
(2) onto

increasing, strictly increasing, decreasing, strictly decreasing functions.
inverse function of a function \( f \)

\[ f^{-1}(b) = a \text{ when } f(a) = b \]

conditions?

so \( f^{-1}(f(a)) = a \)

composite function of two functions \( f \) and \( g \)

\[(f \circ g)(a) = f(g(a))\]

conditions?

Other representations: tables, graphs,

Floor function \([x]\)

Ceiling function \([x]\)

Table 1, p144, useful properties of the Floor and Ceiling functions
A sequence is an ordered list of objects.

example:

1, 1, 2, 3, 5, 8, 13, ...

A sequence is a function: \( N \rightarrow S \)

The \( n \)th element in the sequence is \( a_n \) and the sequence is denoted as \( \{a_n\} \).

example: consider sequence \( \{a_n\} \) where

\[ a_n = \frac{1}{n}, \]  the sequence is actually

1, 1/2, 1/3, ... called harmonic sequence.
arithmetic progression:

\[ a_0 = a \]
\[ a_1 = a + d \]
\[ a_2 = a + 2d \]
\[ \ldots \]
\[ a_n = a + nd \]
\[ \ldots \]

where \( a_n - a_{n-1} = d \)

geometric progression:

\[ a_0 = a \]
\[ a_1 = ar \]
\[ a_2 = ar^2 \]
\[ \ldots \]
\[ a_n = ar^n \]
\[ \ldots \]

where \( a_n / a_{n-1} = r \)
summations:

\[ \sum_{i=1}^{n} i = 1 + 2 + \ldots + n = n(n + 1)/2, \text{ but why?} \]

Generally:

\[ \sum_{i=0}^{n} (a + id) \]

\[ = (n + 1)a + \sum_{i=0}^{n} id \]

\[ = (n + 1)a + d \sum_{i=0}^{n} i \]

\[ = (n + 1)a + dn(n + 1)/2 \]
\[
\sum_{i=0}^{n} r^i = 1 + r + r^2 + \ldots + r^n
\]
\[
= (r^{n+1} - 1)/(r - 1), \text{ but why?}
\]

Generally:

\[
\sum_{i=0}^{n} ar^n
\]
\[
= a \sum_{i=0}^{n} r^n
\]
\[
= a(r^{n+1} - 1)/(r - 1).
\]
summations with lower and upper limits

\[ \sum_{i=m}^{n} a_i = \sum_{i=0}^{n} a_i - \sum_{i=0}^{m-1} a_i \]

e.g.,

\[ \sum_{i=11}^{20} i = \sum_{i=1}^{20} i - \sum_{i=1}^{10} i \]

\[ = 20(20+1)/2 - 10(10+1)/2 = \frac{10}{2}(2 \times 21 - 11) \]

summation of infinite sequences

for \( 0 < r < 1 \),

\[ \sum_{i=0}^{\infty} r^n = 1 + r + r^2 + \ldots + r^n \ldots \]

\[ = \lim_{n \to \infty} (r^{n+1} - 1)/(r - 1) = 1/(1-r) \text{ when } r < 1 \]

So for \( r = 1/2 \) the sum = 2.
Cardinality of sets

A set is *countable* if there is a one-to-one correspondence between this set and the set of positive integers (or the natural number set).

e.g., the set $I$ of all integers is countable.

**Proof:** We build a function $f : I \rightarrow N$ from the set of integers to the set of natural numbers such that

\[
\begin{align*}
f(x) &= 0 \text{ if } x = 0 \\
f(x) &= 2x - 1 \text{ if } x > 0 \\
f(x) &= -2x \text{ if } x < 0
\end{align*}
\]

We need to show $f$ is a one-to-one correspondence.

(1) proof of $f$ being 1-1.

(2) proof of $f$ being onto.
(1) proof of $f$ being 1-1:

This is to show that for any $x \in I$ and $y \in I$, $f(x) = f(y) \rightarrow x = y$.

We have 9 combinations of $f(x)$ and $f(y)$:

a) $f(x) = 0, f(y) = 0$: $f(x) = f(y) \rightarrow x = y = 0$

b) $f(x) = 0, f(y) = 2y - 1$: $f(x) = f(y) \rightarrow y = \frac{1}{2}$ impossible case

c) $f(x) = 0, f(y) = -2y$: $f(x) = f(y) \rightarrow y = 0 = x$

d) $f(x) = 2x - 1, f(y) = 0$: same as b)

e) $f(x) = 2x - 1, f(y) = 2y - 1$: $f(x) = f(y) \rightarrow x = y$

f) $f(x) = 2x - 1, f(y) = -2y$: $f(x) = f(y) \rightarrow$ odd = even impossible case.

g) $f(x) = -2x, f(y) = 0$; same as c)

h) $f(x) = -2x, f(y) = 2y - 1$: same as f)

i) $f(x) = -2x, f(y) = -2y$: $f(x) = f(y) \rightarrow x = y$
(2) proof of $f$ being onto:

This is to show, for any $y \in N$, there is an $x \in I$ such that $f(x) = y$.

We have 3 scenarios for $y$:

a) $y = 0$: then we pick $x = 0$ such that $f(x) = y$.

b) $y$ is odd: let $y = 2k + 1$ odd for some $k$; let $2x - 1 = 2k + 1$, then we pick $x = (2k + 2)/2 = k + 1$ such that $f(x) = y$.

c) $y$ is even: let $y = 2k$ even for some $k$; let $-2x = 2k$, then we pick $x = -k$ such that $f(x) = k$.

Done!

e.g., the set of all positive rational numbers is countable.
e.g., the set $R$ of real numbers is not countable.