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Asymptotic Normality in Binomial Type Enumeration

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by

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The dissertation of Earl Rodney Canfield, Jr. is approved, and it is acceptable in quality and form for publication on microfilm:

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ABSTRACT OF THE DISSERTATION

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In this thesis we are concerned with central and local limit theorems for the coefficients $s(n,k)$ of sequences of polynomials $P_n(x)$ which are of binomial type, that is, defined by the equation $\sum_{n=0}^{\infty} P_n(x) \cdot \frac{u^n}{n!} = \exp\{x\,g(u)\}$, with $g(u)$ a (formal) power-series lacking constant term. In the first section we give a uniform big-O error estimate for the local limit theorem in the case where $P_n(x)$ have only non-positive, real roots. In the second section we give various criteria on $g(u)$ that assure a central limit theorem for the $s(n,k)$. In the third section we prove local limit theorems for

the $s(n, k)$, using log concavity, also giving an elementary sufficient condition on $g(u)$ which yields log concave $s(n, k)$. Some specific combinatorial examples are given, as well as some general examples covering a wide range of $g(u)$ arising as combinatorial generating functions.

0. Introduction

In this section we describe the subject matter which follows, give it a little background, and establish basic terminology and notational conventions that will be adhered to throughout the thesis. We say that a doubly-indexed sequence of non-negative numbers is <u>asymptotically normal</u>, or equivalently, satisfies a <u>central limit theorem</u>, if for the normalized probabilities $p(n,k) = \frac{s(n,k)}{\sum_k s(n,k)}$ we can find sequences μ_n and σ_n such that

(0.1)
$$\lim_{n\to\infty} \sup_x \left| \sum_{k \le \mu_n + x\sigma_n} p(n,k) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt \right| = 0.$$

We also say that $s(n,k)$ is asymptotically normal with <u>mean μ_n</u> and <u>variance σ_n^2</u>. Since the limit distribution function is continuous, (0.1) is equivalent to pointwise convergence

(0.1′) for each real x, $$\sum_{k \le \mu_n + x\sigma_n} p(n,k) \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt \quad .$$

If for some set S of real numbers it is the case that

(0.2) $$\lim_{n\to\infty} \sup_{x\in S} \left| \sigma_n \cdot p(n, [\mu_n + x\sigma_n]) - \frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2} \right| = 0 \ ,$$

then we say that $s(n,k)$ satisfies a <u>local limit theorem on</u> S. (Brackets denote greatest integer.)

Asymptotic normality is a common occurrence for sequences $s(n,k)$ arising in combinatorial enumeration. When $s(n,k) = \binom{n}{k}$, the binomial coefficient, (0.1) and (0.2) are the classical central and local limit theorems for sums of identically distributed, independent random variables taking the values 0 and 1 each with probability 1/2. Harper [10] proves (0.1) for $s(n,k)$ = Stirling numbers of the second kind = the number of partitions of an n element set into k non-empty blocks. Carlitz et al. [3] established relation (0.1) for $s(n,k)$ = Eulerian numbers = the number of permutations of $\{1,2,\ldots,n\}$ having k rises. In both of these results it is a crucial fact that $P_n(x) = \sum_k s(n,k)x^k$ are polynomials having all their roots real and non-positive. For then it is possible to define independent 0-1 random variables $X_{n,\ell}$ such that

$$\text{Prob}\left\{\sum_{\ell} X_{n,\ell} = k\right\} = \frac{s(n,k)}{\sum_k s(n,k)} .$$

It is shown that these random variables satisfy the Lindeberg condition for an increasing number of summands of independent random variables to approach the normal distribution. Another example in this same class is $s(n,k)$ = signless Stirling numbers of the first kind = the number of permutations of an n element set having k cycles. In this case the explicit computation $P_n(x) = \sum_k s(n,k)x^k = x(x+1)\ldots(x+n-1)$ is possible. The asymptotic normality of this distribution, as well as the number of permutations of an n element set having k inversions, are considered by Feller [6], p. 241.

Carlitz [3] also gives asymptotic estimates for the size of the Eulerian numbers when k is near the mean, employing a Berry-Esséen inequality to obtain a big-O bound on the error of the estimate. In Section I we shall see how this inequality may be used to give explicit error estimates for local theorems in the general situation of polynomials having all roots real and non-positive (Theorem I).

In Section II we search for a general class of <u>triangular arrays</u> ($s(n,k) = 0$ if $k < 0$ or $k > n$) which are asymptotically normal, without relying on knowing the location of the roots of $P_n(x) = \sum_k s(n,k)x^k$. Bender [2] has found conditions on the two variable generating function $f(z,w) = \sum_{k,n} s(n,k)z^n w^k$ which guarantee (0.1); also he shows how certain smoothness conditions on the $s(n,k)$ allow passage from (0.1) to (0.2). While ultimate interest is in asymptotics for $s(n,k)$, the local limit theorem (0.2) provides such information only for k within $O(\sigma_n)$ of the mean μ_n; Bender points out that since the mean of $r^k s(n,k)$ varies with r (μ_n corresponding to $r = 1$), it is possible to extend this range by "shifting" the mean. His technique will be of much use to us; however, we focus on a class of triangular arrays $s(n,k)$ which might, but only coincidentally, yield generating functions $f(z,w)$ of the type which he studies.

When $s(n,k)$ = Stirling numbers of either first or second kind, the resulting sequences of polynomials $P_n(x) = \sum_k s(n,k)x^k$ are of <u>binomial type</u>. The theory of polynomials of binomial type is a rich one for combinatorial enumeration, and the analytic properties have been much developed - see Rota-Mullin [18], Fillmore-Williamson [7], Garsia [8]. A sequence of polynomials $P_n(x)$ is of binomial type if

(0.3) a) $P_0(x) \equiv 1$

b) $\deg P_n(x) = n$, and $P_n(0) = 0$ for $n > 0$

c) $P_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} P_k(x) P_{n-k}(y)$.

In an obvious sense polynomials of binomial type generalize the binomial formula, the simplest example being $P_n(x) = x^n$ so that (0.3c) is precisely the binomial formula. In view of the asymptotic normality of both kinds of Stirling numbers, we look for further cases in which the coefficients of polynomials of binomial type are asymptotically normal. For our use a most important fact about these polynomials is that conditions (0.3) are equivalent to the existence of a (formal) power series $g(u)$ lacking constant term with $g'(0) \neq 0$ such that

$$\sum_{n=0}^{\infty} \frac{P_n(x)}{n!} u^n = \exp\{x g(u)\} . \tag{0.4}$$

We will say that $g(u)$ generates the polynomials $P_n(x)$.

The combinatorial significance of these polynomials is this: let g_n be the number of ways to construct objects of some sort on a labeled n-set; form the exponential generating function $g(u) = \sum_{n=1}^{\infty} \frac{g_n}{n!} u^n$; use $g(u)$ to generate a sequence of polynomials $P_n(x)$ by (0.4); then $s(n,k)$, the coefficient of x^k in $P_n(x)$, is the number of ways to partition a labeled n-set into k non-empty blocks and construct one of the objects on each block. This may be seen by actual expansion of (0.4) to determine

$$s(n,k) = \frac{1}{k!} \sum \binom{n}{\nu_1 \cdots \nu_k} g_{\nu_1} \cdots g_{\nu_k} \quad . \tag{0.5}$$

In (0.5) the summation extends over all ordered k-tuples $(\nu_1, \ldots, \nu_k)$ such that $\nu_i \geq 1$ and $\sum \nu_i = n$. The distribution $s(n,k)$ is then the distribution of "connected components" of our objects.

For example, when $g_n \equiv 1$ (the number of ways to construct a set on a labeled n-set) then $s(n,k)$ is the number of partitions of an n-set into k blocks. When $g_n = (n-1)!$ (the number of n-cycles on a labeled n-set), $s(n,k)$ is the number of permutations of an n-set having k cycles. When $g_n = n^{n-2}$ (the number of trees on a labeled n-set), $s(n,k) =$ the number of k-forests.

The $s(n,k)$ are completely determined by the function $g(u)$;

whether or not $s(n,k)$ are asymptotically normal depends somehow on $g(u)$. Theorem II exhibits a class of functions $g(u)$ for which the $s(n,k)$ do satisfy a central limit theorem.

For a power series with real positive coefficients having positive radius of convergence, $g(z) = \sum c_n z^n$, $|z| < R > 0$, it is possible to consider asymptotic normality of the coefficients c_n. Namely, for each $r < R$ we have a random variable X_r given by

$$\text{(0.6)} \qquad \operatorname{Prob}\{X_r = n\} = \frac{c_n r^n}{g(r)} \quad .$$

Then

$$E(X_r) = a(r) = r \cdot \frac{g'(r)}{g(r)}$$

and

$$\operatorname{Var}(X_r) = b(r) = r\,a'(r) \quad .$$

If $\dfrac{X_r - a(r)}{\sqrt{b(r)}}$ tends as $r \to R$ to a normal distribution, that is,

$$\text{(0.7)} \quad \text{for each real } x, \quad \sum_{n \le a(r) + x\sqrt{b(r)}} \frac{c_n r^n}{g(r)} \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt \ ,$$

then we say that the coefficients $\{c_n\}$ are asymptotically normal. Hayman [11] discovered an extensive class of functions for which not only (0.7) but also a local theorem is valid. Theorem III shows that this same class of functions, when used in (0.4), yield $s(n,k)$

which are asymptotically normal. Theorem IV shows that when $g(u)$ is a polynomial, essentially no restrictions need be imposed on $g(u)$.

Finally, in Section III we consider local theorems for the $s(n,k)$. Theorem V will show that if $g(u)$ satisfies (0.7) and the coefficients are <u>log concave</u> ($c_n^2 \geq c_{n-1} c_{n+1}$), then the $s(n,k)$ are asymptotically normal and also log concave ($(s(n,k))^2 \geq s(n,k-1)\, s(n,k+1)$.) This permits, via Bender's mean shift, asymptotic formulae for $s(n,k)$.

I. A Local Error Estimate using a Berry-Esséen Inequality

In this section we prove a theorem which under favorable conditions gives an estimate of the coefficients of a polynomial having all its roots real and negative, most importantly with a bound on the error.

The Berry-Esséen Inequality

This theorem is found in Feller [5], p. 521: Let X_j, $1 \le j \le m$, be independent random variables with zero expectation. With $X = \sum_j X_j$, $\sigma^2 = \mathrm{Var}(X)$, $F(x)$ the distribution of $\frac{1}{\sigma} \cdot \sum_j X_j$, $\eta(x)$ the normal distribution $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt$, and λ a constant such that $E(|X_j|^3) < \lambda \,\mathrm{Var}(X_j)$ for each j, we have

$$(I.1) \qquad |F(x) - \eta(x)| \le \frac{33\lambda}{4\sigma} .$$

We shall now apply this theorem. Let $P(x)$ be a polynomial with negative roots:

$$P(x) = \sum_{j=1}^{m} s(j)\, x^j = c \prod_{j=1}^{m} (x + r_j), \quad r_j > 0 .$$

Denote $\frac{s(j)}{\sum_j s(j)}$ by $p(j)$. Define independent random variables $\hat{X}_j$ by

$$\hat{X}_j = 0, \text{ with probability } r_j/1 + r_j$$

$$= 1, \text{ with probability } 1/1 + r_j .$$

Let $X = \sum_j \hat{X}_j$. Considering the convolutions of probability generating functions, we see

$$\text{Prob}\{X = k\} = p(k) .$$

We use the notations

$$\mu = \mu(1) = E(X) = \sum_k k\, p(k) = \frac{P'(1)}{P(1)} ,$$

and

$$\sigma^2 = \sigma^2(1) = \text{Var}(X) = \sum_k (k(k-1) + k)\, p(k) - E(X)^2$$

$$= \frac{P''(1)}{P(1)} + \frac{P'(1)}{P(1)} - \left(\frac{P'(1)}{P(1)}\right)^2$$

$$= x \left[\frac{x P'(x)}{P(x)}\right]' , \text{ at } x = 1 .$$

For use in the mean shifting technique, as discussed in the introduction, let us say for $\alpha > 0$,

$$\mu(\alpha) = \frac{\alpha P'(\alpha)}{P(\alpha)}$$

$$\sigma^2(\alpha) = \alpha \frac{d}{d\alpha} \left[\frac{\alpha P'(\alpha)}{P(\alpha)}\right] .$$

Just as we calculated above for $\alpha = 1$, $\mu(\alpha)$ and $\sigma^2(\alpha)$ are the mean and variance of the random variable whose probability generating function is $\frac{P(\alpha x)}{P(\alpha)}$. Continuing with our independent 0-1 random variables $\hat{X}_j$, introduce $X_j = \hat{X}_j - \frac{1}{1 + r_j}$. By choice $E(X_j) = 0$; also, with some calculation,

$$\mathrm{Var}(X_j) = \frac{r_j}{(1 + r_j)^2}$$

$$E(|X_j|^3) = \frac{r_j(1 + r_j^2)}{(1 + r_j)^4} .$$

Since $\frac{1 + r_j^2}{(1 + r_j)^2} < 1$ for all j $(r_j > 0)$, we may apply the previously quoted theorem with $\lambda = 1$:

$$\text{(I.2)} \qquad \left| \sum_{k \le \mu + x\sigma} p(k) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt \right| \le \frac{33}{4\sigma} .$$

Before stating the theorem let us introduce two additional notations, that

ℓ denotes the integer $[\mu + x\sigma]$, (greatest integer),

and

$\sum_{y}^{x} p(k)$ is to denote $\sum_{\mu + y\sigma < k \le \mu + x\sigma} p(k)$.

<u>Theorem I.</u> Let δ be such that

(1) $\delta \leq 1$

(2) $(0.0875)\,\delta^3 > \frac{33\sqrt{2\pi}}{\sigma}$

If $|x| \geq \delta + \sigma^{-1/2}$, then

$$\left| \sigma\, p(\ell) - \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \right| \leq \frac{17.74}{\sigma^{1/2}} .$$

(If δ cannot be so chosen, then σ is too small to make an estimate by this technique.)

<u>Proof:</u> We shall consider positive x, as the argument for negative x is quite similar. First we examine the significance of δ. Two facts are needed:

If $f''(x) < 0$ on (α, β), it is not difficult to show

$$\text{(I. 3)} \qquad (\beta - \alpha)\, f\left(\frac{\alpha+\beta}{2}\right) > \int_\alpha^\beta f(t)\, dt > (\beta - \alpha)\, \frac{f(\alpha) + f(\beta)}{2} .$$

Also we use

$$\text{(I. 4)} \qquad 1 - \tau < e^{-\tau} < 1 - \tau + \tau^2/2 \quad \text{for} \quad 0 < \tau < 1 .$$

Consider $g(a, \varepsilon) = \int_0^{a-\varepsilon} e^{-t^2/2}\, dt - \int_{a-\varepsilon}^{2a} e^{-t^2/2}\, dt.$ We assume that

$\epsilon < a$, and also that $2a \le 1$, so that $\frac{d^2}{dt^2} e^{-1/2\, t^2} < 0$ on $(0, 2a)$.

Then we may apply (I. 3) and (I. 4) to get:

$$g(a,\epsilon) > (a-\epsilon)\frac{1+1-\frac{1}{2}(a-\epsilon)^2}{2} - (a+\epsilon)\left(1-\frac{1}{2}\left(\frac{3a-\epsilon}{2}\right)^2+\frac{1}{8}\left(\frac{3a-\epsilon}{2}\right)^4\right)$$

$$= -2\epsilon - \frac{1}{4}(a-\epsilon)^3 + \frac{1}{8}(a+\epsilon)(3a-\epsilon)^2 - \frac{1}{128}(a+\epsilon)(3a-\epsilon)^4 .$$

Let $\epsilon = c\,a^3$. Then $3a-\epsilon \ge a(3-c/4)$, $(a+\epsilon) \le a(1+c/4)$, $a^3 \le 1/4\, a$, and $a^5 \le 1/4\, a^3$ - the last two since $2a \le 1$. Then,

$$g(a,\epsilon) > -2\epsilon - \frac{a^3}{4} + \frac{1}{8}\,a \cdot a^2(3-c/4)^2 - \frac{1}{128}\cdot a(1+c/4)(3a)^4$$

$$> a^3[-2c - 1/4 + 1/8(3-c/4)^2 - 81/512\,(1+c/4)] .$$

$$> (0.7)a^3 \text{ if } c = 0.005.$$

Let $2a = \delta$, so that $g(a,\epsilon) > (0.0875)\,\delta^3$. Then by (2),

$$\frac{1}{\sqrt{2\pi}}\int_0^{\delta/2-\epsilon} e^{-t^2/2}\,dt - \frac{33}{2\sigma} > \frac{1}{\sqrt{2\pi}}\int_{\delta/2-\epsilon}^{\delta} e^{-t^2/2}\,dt + \frac{33}{2\sigma} .$$

The left side of the last inequality is smaller than $\sum_0^{\delta/2-\epsilon} p(k)$ by (I.2), while the right side is larger than $\sum_{\delta/2-\epsilon}^{\delta} p(k)$ by the same estimate.

Now, the $p(k)$ are log concave $(p(k)^2 \geq p(k-1)\,p(k+1))$, and hence unimodal. (See Lieb [16].) Since $\epsilon = \frac{(0.005)\cdot\delta^3}{8}$, it is easy to prove from (2) that $\epsilon > \frac{1}{2\sigma}$. Then $(\delta/2 + \epsilon) - (\delta/2 - \epsilon) = 2\epsilon > 1/\sigma$ tells us that the range over which k varies in $\sum_{\delta/2-\epsilon}^{\delta} p(k)$ is at least one unit longer than the range over which k varies in $\sum_{0}^{\delta/2-\epsilon} p(k)$. For this reason, the latter sum has at most as many terms as the former. Hence the $p(k)$ are not still monotone increasing before $\mu + \delta\sigma$, for if they were it would be the case that every term in the latter sum is smaller than every term in the former sum, and since there are fewer terms in the latter sum it could not be the larger of the two, as we know it is. Therefore the $p(k)$ are monotone decreasing after $\mu + \delta\sigma$. In an entirely analogous fashion we may see that the $p(k)$ are still monotone increasing before $\mu - \delta\sigma$. This is the significance of δ. Again we derive from (I.2),

$$\sum_{y}^{x} p(k) \leq \frac{1}{\sqrt{2\pi}} \int_{y}^{x} e^{-t^2/2}\, dt + \frac{33}{2\sigma} \quad .$$

Since x is positive and larger than $\delta + \sigma^{-1/2}$, with y set equal to $x - \sigma^{-1/2}$ there follows

$$((x-y)\sigma - 1)\; p(\ell) \leq \frac{1}{\sqrt{2\pi}} \int_{y}^{x} e^{-t^2/2}\, dt + \frac{33}{2\sigma} \; .$$

For it is at $\ell = [\mu + x\sigma]$ that the smallest term in the sum appears (since $y \geq \delta$), and $((x - y)\sigma - 1)$ is the fewest possible number of terms in the sum. Dividing by $x - y = \sigma^{-1/2}$,

$$(\sigma - \sigma^{1/2})\, p(\ell) \leq \frac{1}{\sqrt{2\pi}} e^{-y^2/2} + \frac{33}{2\sigma^{1/2}}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-x^2/2} + \frac{33}{2\sigma^{1/2}} + \frac{1}{\sqrt{2\pi}} \left(e^{-y^2/2} - e^{-x^2/2}\right)$$

$$\leq \frac{1}{\sqrt{2\pi}} e^{-x^2/2} + \frac{\frac{33}{2} + \frac{1}{\sqrt{2\pi e}}}{\sigma^{1/2}} ,$$

since $\left| e^{-y^2/2} - e^{-x^2/2} \right| \leq e^{-1/2} |y - x|$. This leads to

$$\sigma\, p(\ell) \leq \left(\frac{1}{\sqrt{2\pi}} e^{-x^2/2} + \frac{\frac{33}{2} + \frac{1}{\sqrt{2\pi e}}}{\sigma^{1/2}} \right) \left(\frac{1}{1 - \sigma^{-1/2}} \right) .$$

Now from (1) and (2), $\sqrt{\sigma} > 30$, so

$$\frac{1}{1 - \sigma^{-1/2}} < \frac{30}{29} , \quad \text{and} \quad \frac{1}{1 - \sigma^{-1/2}} = 1 + \frac{\sigma^{-1/2}}{1 - \sigma^{-1/2}} < 1 + \frac{30/29}{\sigma^{1/2}} .$$

Finally then,

$$\sigma\, p(\ell) \leq \frac{1}{\sqrt{2\pi}} e^{-x^2/2} + \frac{\left(1 + \frac{1}{29}\right)\left(\frac{33}{2} + \frac{1}{\sqrt{2\pi e}}\right) + \frac{30}{29\sqrt{2\pi}}}{\sigma^{1/2}}$$

$$\leq \frac{1}{\sqrt{2\pi}} e^{-x^2/2} + \frac{17.74}{\sigma^{1/2}} .$$

Choosing $y = x + \sigma^{-1/2}$ and considering $\sum_{x}^{y} p(k) \geq \frac{1}{\sqrt{2\pi}} \int_{x}^{y} e^{-t^2/2} dt - \frac{33}{2\sigma}$,

we find in the same way that $(\sigma + \sqrt{\sigma})\, p(\ell) \geq \frac{1}{\sqrt{2\pi}} e^{-x^2/2} - \frac{\frac{33}{2} + \frac{1}{\sqrt{2\pi e}}}{\sigma^{1/2}}$.

This time we use the inequality $\frac{1}{1 + \sigma^{-1/2}} > 1 - \sigma^{-1/2}$, and find a constant in the numerator over $\sigma^{1/2}$ which is smaller than the previous. This completes the proof of Theorem I.

To use Theorem I for estimating $p(k)$, we determine x by $k = \mu + x\sigma$. If x is large, the theorem will not be too useful. To avoid this difficulty we use the mean shift:

Theorem I′. Let δ be such that

(1) $\delta \leq 1$

(2) $(0.0875)\,\delta^3 > \frac{33\sqrt{2\pi}}{\sigma(\alpha)}$, some positive α.

With $\ell = [\mu(\alpha) + x\sigma(\alpha)]$, if $|x| > \delta + 1/\sigma(\alpha)^{1/2}$, then

$$\left| \sigma(\alpha) \cdot p(\ell) \cdot \alpha^{\ell} - \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \right| \leq \frac{17.74}{\sigma(\alpha)^{1/2}} .$$

Proof: It is only necessary to apply Theorem I to the polynomial $P(\alpha x)$. The idea, of course, is to choose α so that x is small. We are tacitly assuming that somehow the function $\sigma(\alpha)$ is understood. This can be a difficulty. To convert information about $p(k)$

into information about $s(k)$, we also must be able to estimate $P(1)$. To establish asymptotic normality of coefficients $s(n,k)$ of a polynomial sequence $P_n(x)$ by this method, we must somehow prove all roots negative, and that the variance tends to infinity. Harper and Carlitz (see discussion in Introduction) use a recurrence to prove negative roots. The first and last examples that follow illustrate this procedure. For the first, we can estimate μ_n and σ_n by our next theorem in Section II. For the latter, however, the behavior of the variance has not been determined, although almost surely it does become infinite. The second example is a situation in which the variance does not go to infinity. It leads to a question which we will answer later.

Remark: Concerning Theorem I, we assumed that $P(x) = c\prod (x+r_j)$ with $r_j > 0$, so that $\frac{1+r^2}{(1+r)^2} < 1$. However, if $r_j = 0$ for any number of j the proof is not violated. This merely means that some of the random variables $\hat{X}_j$ are identically 1.

Example. The signless Laguerre polynomials.

The classical Laguerre polynomials $Q_n(x)$ (an orthogonal system) are defined by the equation

$$\sum_{n=0}^{\infty} \frac{Q_n(x)}{n!} u^n = \exp\left\{x \frac{u}{u-1}\right\}.$$

These polynomials alternate in sign, and we consider $L_n(x) \equiv Q_n(-x)$, having positive coefficients. Since

$$\sum_{n=0}^{\infty} \frac{L_n(x)}{n!} u^n = \exp\left\{x \frac{u}{1-u}\right\} ,$$

we have $g(u) = u + u^2 + \cdots$, so that $g_n = n!$ According to our combinatorial interpretation the coefficients $s(n,k)$ of $L_n(x)$ count the number of partitions of an n-element set into k ordered blocks. This permits the following recursion: to partition an $(n+1)$-element set into k ordered blocks, we may place the element $n+1$ by itself as a block and partition the remaining n elements into $k-1$ blocks; else, if $n+1$ is not to be a singleton block we may place it in any one of the $n+k$ possible positions in a partition of n elements into k ordered blocks, namely to the left of any one of the n elements or at the end of any one of the k blocks. Hence,

$$s(n+1, k) = s(n, k-1) + (n+k)\, s(n,k) \; .$$

This leads us to the formula

$$L_{n+1}(x) = (n+x)\, L_n(x) + x\, L_n'(x) \; .$$

Assuming that $L_n(x)$ has n distinct non-positive roots $x_1 = 0 > x_2 > \cdots > x_n$, then $L_{n+1}(x_j) = x_j\, L_n'(x_j)$. We consider $j = 2, 3, \ldots, n$. Since $L_n(x)$ is clearly negative for $x_2 < x < x_1 = 0$, and since $L_n(x)$ has no repeated roots, it must be the case that

$L_n'(x_2) < 0$, $L_n'(x_3) > 0, \ldots, L_n'(x_n) > 0$ if n is odd, or $L_n'(x_n) < 0$ if n is even. Consequently $L_{n+1}(x_2) > 0$, $L_{n+1}(x_3) < 0, \ldots,$ $L_{n+1}(x_n) > 0$ if n is even or $L_{n+1}(x_n) < 0$ if n is odd. Since $L_{n+1}(x)$ is negative for x negative and near zero, it follows that $L_{n+1}(x)$ has a root in each of the open intervals $(x_2, 0)$, $(x_3, x_2), \ldots,$ (x_n, x_{n-1}). When n is even $L_{n+1}(x)$ is negative as $x \to -\infty$, yet $L_{n+1}(x_n) > 0$; and vice-versa when n is odd. Hence $L_{n+1}(x)$ has another root beyond x_n; with the root at zero this makes $n+1$ distinct non-positive roots for $L_{n+1}(x)$. Inductively then the $L_n(x)$ satisfy the hypothesis needed to apply the previous results. However, we must address the question of how σ_n behaves. In terms of the roots,

$$\sigma_n^2 = \sum_{j=1}^{n} \frac{-x_j}{(1 - x_j)^2} .$$

The question of $\sigma_n \to \infty$ depends on how quickly (or should we say slowly) the roots tend to $-\infty$. Along with the initial problem in knowing that a given sequence of polynomials has non-positive roots, this is the other drawback of this method. After Theorem II we shall see that indeed $\sigma_n \to \infty$, give approximations of μ_n and σ_n, and so establish asymptotic normality of the $s(n,k)$, and a means of using Theorem I.

<u>Example</u>. The Abel polynomials. To illustrate a situation in which the variance does not become infinite, consider the Abel polynomials

$$A_n(x) = x(x+n)^{n-1} \quad .$$

We have that $\sum_{n=0}^{\infty} A_n(x) \cdot \frac{u^n}{n!} = \exp\{x g(u)\}$, where $g(u) = \sum_{n=1}^{\infty} n^{n-1} \frac{u^n}{n!}$.

See Rota [18] for proofs and discussion. Since n^{n-1} counts the number of labeled, rooted trees on n elements, $s(n,k)$, the coefficients of $A_n(x)$, count the number of forests on n elements containing k such trees. Given the nice form of $A_n(x)$, it is not difficult to calculate

$$\mu_n = \frac{2n}{n+1}$$

$$\sigma_n^2 = \frac{n^2 - n}{(n+1)^2} .$$

As $n \to \infty$, $\mu_n \to 2$ and $\sigma_n^2 \to 1$. Bounding $\sum_{j=4}^{n} \binom{n-1}{j-1} n^{n-j}$ by a geometric series, one may see that there are asymptotically at least twelve times as many forests with one, two, or three trees as there are forests of all other sizes. These small forests contain large trees, of course. We wonder how it might change the distribution if the trees were "pruned" as they developed, that is no trees were allowed to contain more than M nodes. This question will be answered by Theorem IV.

<u>Example</u>. The derangement polynomials. We close with another sequence of polynomials having negative roots, but for which the behavior of σ_n is not known for sure. It is suspected that $\sigma_n \to \infty$; however, none of the later theorems apply. A derangement is a permutation having no fixed points. The derangement polynomials $P_n(x) = \sum_k s(n,k)x^k$ have coefficients $s(n,k)$ = the number of derangements of n objects having k cycles. To define a derangement of $n+1$ elements into k cycles, we may place the element $n+1$ with any one of the other n elements, forming a 2-cycle, and then derange the remaining $n-1$ elements into $k-1$ cycles; else, if $n+1$ is not to be in a 2-cycle, it may be inserted into any one of n positions of a derangement on n objects having k-cycles. Hence,

$$s(n+1, k) = n\,s(n-1, k-1) + n\,s(n, k) \quad .$$

This leads to the equation

$$P_{n+1}(x) = nx\,P_{n-1}(x) + n\,P_n(x) \quad .$$

We give the first seven derangement polynomials:

$$P_1(x) = 0$$

$$P_2(x) = x$$

$$P_3(x) = 2x$$

$$P_4(x) = 3x^2 + 6x$$

$$P_5(x) = 20x^2 + 24x$$

$$P_6(x) = 15x^3 + 130x^2 + 120x$$

$$P_7(x) = 210x^3 + 924x^2 + 720x \ .$$

These $P_n(x)$ are not of binomial type in the strict sense given in the Introduction, as $\deg P_n(x) \neq n$. In fact, given the combinatorial meaning of the coefficients, it is clear that $\deg P_n(x) = \left[\frac{n}{2}\right]$. This lowered degree is due to the fact that the generating function $g(u)$ has no linear term: $g(u) = \frac{u^2}{2} + \frac{u^3}{3} + \cdots$, for a derangement is made up of cycles whose size is at least two, so that $g_n = (n-1)!$ for $n \geq 2$, and $g_1 = 0$. However, the binomial identity (0. 3c) is still valid. Reasoning very much as in the first example, and with only slightly more care, one may prove inductively:

(1) If n is even $= 2k$, and $P_n(x)$ has roots at $x_1 = 0 > x_2 > \cdots > x_k$, then $P_{n+1}(x)$ has roots at $y_1 = 0$, and one root in each open interval $(x_2, 0), (x_3, x_2), \ldots, (x_k, x_{k-1})$.

(2) If n is odd $= 2k-1$, and $P_n(x)$ has roots at $x_1 = 0 > x_2 > \cdots > x_{k-1}$, then $P_{n+1}(x)$ has roots at $y_1 = 0$, one root in each open interval $(x_2, 0), (x_3, x_2), \ldots, (x_{k-1}, x_{k-2})$, and one root beyond x_{k-1}.

We now abandon the technique of root location as a means of establishing central and local limit theorems, studying instead properties of the generating function $g(u)$.

II. Central Limit Theorems for Polynomials of Binomial Type

Throughout this section $P_n(x)$ will denote a sequence of polynomials of binomial type, generated by the power series $g(u)$ with non-zero radius of convergence R. We still use $s(n,k)$ for the coefficients of $P_n(x)$:

$$g(u) = \sum_1^\infty \frac{g_n}{n!} u^n = \sum_1^\infty c_n u^n, \quad |u| < R$$

$$\sum_0^\infty P_n(x) \cdot \frac{u^n}{n!} = \exp\{x g(u)\}$$

$$P_n(x) = \sum_k s(n,k) x^k .$$

The g_n are non-negative integers, so that the $s(n,k)$ have the combinatorial significance as discussed in the Introduction.

1. Admissible functions

Hayman [11] finds a class of functions, which he terms admissible, whose coefficients are asymptotically normal in the sense previously described. He shows that if $f(z)$ and $g(z)$ are admissible in $|z| < R$, then so are $\exp\{f(z)\}$; $f(z) \cdot g(z)$; $f(z) + P(z)$, where $P(z)$ is a real polynomial; $P(f(z))$, provided the leading coefficient of P is positive; $f(x) \cdot P(z)$, provided the leading coefficient of P is positive if $R = +\infty$, or $P(R) > 0$ if $R < +\infty$;

and $\exp\{P(z)\}$ provided its power series has almost all coefficients non-zero, and $P(z)$ has non-negative coefficients. Admissible functions are given by the

<u>Definition.</u> Let $g(z) = \sum c_n z^n$ be analytic for $|z| < R$, real for real z, and positive for sufficiently large real argument. Let $a(r) = r \frac{g'(r)}{g(r)}$ ("mean") and $b(r) = r\,a'(r)$ ("variance"). If there is a function $\delta(r)$, $0 < \delta(r) < \pi$ such that

(II. 1) $g(re^{i\theta}) \sim g(r) \cdot \exp\{i\theta\, a(r) - \frac{1}{2}\theta^2 b(r)\}$ as $r \to R$,

uniformly for $|\theta| \leq \delta(r)$,

(II. 2) $g(re^{i\theta}) = \frac{o(g(r))}{\sqrt{b(r)}}$ as $r \to R$, uniformly for $\delta(r) \leq |\theta| \leq \pi$,

(II. 3) $b(r) \to +\infty$ as $r \to R$,

then $g(z)$ is admissible.

<u>Remark.</u> Concerning (II. 1), the Taylor expansion for $\log(g(re^{i\theta}))$ as a function of θ begins as

$$\log(g(re^{i\theta})) = \log g(r) + i\theta\, a(r) - \frac{1}{2}\theta^2 b(r) + \cdots$$

For us the c_n are real and positive, so $a(r)$, $b(r)$ are precisely the mean and variance in our probabilistic interpretation of the coefficients. In Hayman's generality this may not be the case, hence the use of quotation marks in the definition.

In addition to asymptotic normality,

(II. 4) $$\sum_{n \le a(r) + x\sqrt{b(r)}} \frac{c_n r^n}{g(r)} \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt \quad \text{as} \quad r \to R \ ,$$

one also has a local theorem for admissible functions:

(II. 5) $$c_n r^n = \frac{g(r)}{\sqrt{2\pi\, b(r)}} \left\{ \exp\left[- \frac{(a(r) - n)^2}{2\, b(r)} \right] + o(1) \right\} , \quad \text{as} \quad r \to R \ ,$$

uniformly in n.

Hayman also establishes a number of interesting inequalities and asymptotic formulae concerning admissible functions. Those needed in our development are listed below. For proofs of (II. 4-10) we refer to the original paper. However, we do mention that given $b(r) \to \infty$, the central limit theorem (II. 4) is sufficient to establish (II. 6-9); for (II. 10) the local theorem (II. 5) appears necessary.

(II. 6) $b(r) = o(a(r)^2)$, $a(r)$ is positive, increasing to $+\infty$ as $r \to R$.

(II. 7) On derivatives: $g^{(k)}(r) \sim g(r) \left[\frac{a(r)}{r} \right]^k$, as $r \to R$.

(II. 8) For any positive ϵ , $a(r) = O(g(r)^{\epsilon})$, as $r \to R$.

(II. 9) $$g(r e^{i\theta}) = g(r) + i\theta\, r g'(r) - \frac{1}{2}\, \theta^2 [r g'(r) + r^2 g''(r)]$$

$$+ O(\theta^3 g(r)\, a(r)^3) \quad \text{as} \quad r \to R, \text{ uniformly for}$$

$$|\theta| \le [a(r)]^{-1} \ .$$

(II. 10) $\exists\, R_1 < R$ such that $|g(re^{i\theta})| \le g(r) - g(r)^{1/7}$ for

$$R_1 < r < R \text{ and } [g(r)]^{-2/5} \le |\theta| \le \pi .$$

2. Moment Generating Functions

The following proposition is a standard argument; however no explicit reference was found, so it is included. Let X be a random variable; associated with X is the probability measure m: $m(A) = \text{Prob}\{X \in A\}$. If $\varphi(t) = \int e^{tx}\, m(dx)$ exists for all real t, then $\varphi(t)$ is called the moment generating function. Such a φ is C^∞, and $\varphi^{(k)}(0) = k^{th}$ moment of X. A distribution need not be uniquely determined by its moments; however, if $\varphi(t) = e^{t^2/2}$ - the moment generating function of the normal distribution - then m does equal η, the measure associated with the normal distribution, for only η has that particular set of moments (see Feller [5], p. 224).

<u>Proposition</u>. Let X_n be a sequence of random variables whose moment generating functions exist and converge to $e^{t^2/2}$. Then

$$\text{Prob}\{X_n \le x\} \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt .$$

(The conclusion is the statement that m_n, the measures associated with X_n, converge weakly to η.)

Proof:

$$\text{(II. 11)} \quad \int_{\delta}^{+\infty} e^{tx}\, m_n(dx) \le \int_{\delta}^{+\infty} e^{tx} \cdot e^{x-\delta}\, m_n(dx) \le e^{-\delta} \int e^{x(t+1)}\, m_n(dx)$$

$$\to e^{-\delta} \cdot e^{\frac{(t+1)^2}{2}}$$

$$\int_{-\infty}^{-\delta} e^{tx}\, m_n(dx) \le \int_{-\infty}^{-\delta} e^{tx}\, e^{-x-\delta}\, m_n(dx) \le e^{-\delta} \int e^{x(t-1)}\, m_n(dx)$$

$$\to e^{-\delta} \cdot e^{\frac{(t-1)^2}{2}}$$

Setting $t = 0$ in the equations we see that given $\epsilon > 0$ there is a δ large enough that $\left| \int_{|x| > \delta} m_n(dx) \right| < \epsilon \ \forall\ n$. This is the statement that the sequence $\{m_n\}$ is tight; by the Helly selection theorem there is a subsequence $\{m_{n_k}\}$ converging weakly to a probability measure m. We claim that $\int e^{tx}\, m(dx)$ exists for all t and equals $e^{t^2/2}$, that is, $m = \eta$. Since

$$\int_{|x| \le \delta} e^{tx}\, m(dx) = \lim_k \int_{|x| \le \delta} e^{tx}\, m_{n_k}(dx) \le e^{t^2/2} ,$$

independent of δ, $\int e^{tx}\, m(dx)$ exists for all t. Let $\epsilon > 0$ be given. Using (II. 11) we find a large enough δ so that (t is fixed now)

$$\left| \int e^{tx}\, m_{n_k}(dx) - \int_{|x| \le \delta} e^{tx}\, m_{n_k}(dx) \right| < \frac{\epsilon}{2} \ \forall\ k$$

and

$$\left| \int e^{tx}\, m(dx) - \int_{|x| \le \delta} e^{tx}\, m(dx) \right| < \frac{\epsilon}{2} .$$

Let $k \to \infty$ in the first;

$$\int e^{tx} m_{n_k}(dx) \to e^{t^2/2} \quad \text{and} \quad \int_{|x| \le \delta} e^{tx} m_{n_k}(dx) \to \int_{|x| \le \delta} e^{tx}\, m(dx).$$

Hence

$$\left| \int e^{tx}\, m(dx) - e^{t^2/2} \right| < \epsilon$$

and the claim is justified. Now we have weak convergence $m_n \to \eta$ for the full sequence; else by tightness there is a subsequence converging weakly to something different than η. But the last argument shows that any weakly convergent subsequence has η as its limit.

Corollary. To show that $s(n, k)$, the coefficients of $P_n(x)$, are asymptotically normal it suffices to find sequences μ_n and σ_n such that

$$\text{(II. 12)} \qquad e^{-\frac{t\mu_n}{\sigma_n}} \frac{P_n(e^{t/\sigma_n})}{P_n(1)} \to e^{t^2/2} \quad \text{as } n \to \infty .$$

For the left side is the moment generating function of $\dfrac{X_n - \mu_n}{\sigma_n}$, where $\operatorname{Prob}\{X_n = k\} = s(n,k) \Big/ \sum_k s(n,k)$.

3. A Central Limit Theorem

Let $g(u)$, the generating function for $P_n(x)$, with positive radius of convergence R be given. Define $A(r) = r g'(r)$ and $B(r) = r A'(r)$. We shall assume that $\exp\{x g(u)\}$ is a "uniformly admissible" function of u, $|u| < R$, for $x \in N$, a compact set of positive numbers bounded away from zero and containing 1 in its interior. By this we mean that there is a function $\delta(r)$, $0 < \delta(r) < \pi$, such that

$$\text{(II.13)} \quad \exp\{x g(re^{i\theta})\} \sim \exp\{x g(r)\} \cdot e^{i\theta x A(r) - \frac{1}{2}\theta^2 x B(r)},$$

as $r \to R$, uniformly for $|\theta| \le \delta(r)$ and $x \in N$, and

$$\text{(II.14)} \quad |\exp\{x(g(re^{i\theta}))\}| = \frac{o(\exp\{x g(r)\})}{\sqrt{x B(r)}} \quad \text{as } r \to R,$$

uniformly for $\delta(r) \le |\theta| \le \pi$ and $x \in N$. Of course, we still require $B(r) \to \infty$ as $r \to R$. In particular, $A(r)$ is positive increasing to $+\infty$, and we shall employ the inverse function $r(\beta)$ defined by

$$\text{(II.15)} \qquad A(r(\beta)) = \beta .$$

This is well defined for β large and positive.

If $g = \sum c_n z^n$, so that $A = \sum n c_n z^n$ and $B = \sum n^2 c_n z^n$, then

$$\text{(II.16)} \quad gB - A^2 = \sum_n \left\{ \sum_{i+j=n} (i^2 - ij) c_i c_j \right\} u^n = \sum_n \left\{ \sum_{\substack{i+j=n \\ i>j}} (i-j)^2 c_i c_j \right\} u^n .$$

Since in $gB - A^2$ the coefficient of u^n is always non-negative we may define a non-negative real function $\sigma(r)$ by

(II.17) $$\sigma(r)^2 = g(r) - \frac{A(r)^2}{B(r)} .$$

Also,

$$\sigma_n \equiv \sigma(r(n)) .$$

In the following proof we shall be showing that a certain function of t converges to $e^{t^2/2}$ as $n \to \infty$. We shall show that the convergence is uniform for t in a compact set. This compact set, K, is given and fixed; t denotes an arbitrary element of K. For notational convenience, we set $\alpha = e^{t/\sigma_n}$, the dependence of α on n and t being suppressed.

<u>Theorem II.</u> Let $g(u)$, $|u| < R$, be such that

(1) $\exp\{x g(u)\}$ is uniformly admissible for $|u| < R$ and $x \in N$, a compact set of positive numbers containing 1 in its interior, and bounded away from zero.

(2) $\sigma_n \to \infty$ as $n \to \infty$. Equivalently, $\sigma(r) \to \infty$ as $r \to R$.

(3) $\dfrac{B(r(n))}{B\left(r\left(\frac{n}{\alpha}\right)\right)} \to 1$ as $n \to \infty$, uniformly for $t \in K$.

(4) $g = o(\sigma^3)$ as $r \to R$.

(5) $A^2/B = o(\sigma^3)$ as $r \to R$.

(6) $\dfrac{d^2}{d\beta^2}\, g\left(r\left(\frac{n}{\beta}\right)\right) = o(\sigma_n^3)$ as $n \to \infty$, uniformly for β between 1 and α.

Then $s(n,k)$, the coefficients of $P_n(x)$ are asymptotically normal with mean $\mu_n = g(r(n))$ and variance σ_n^2.

Proof: We establish the convergence of (II. 12), uniformly for $t \in K$. By (2), $\alpha \in N$ for large n, and so (1) is applicable with $x = \alpha$. By the same proof which yields (II. 5), keeping track of the extra variable α and using uniform admissibility, we obtain

$$\frac{P_n(\alpha)}{n!} r^n = \frac{\exp\{\alpha g(r)\}}{\sqrt{2\pi\alpha B(r)}} \left\{ \exp\left[- \frac{(\alpha A(r) - n)^2}{2\alpha B(r)} \right] + o(1) \right\}, \quad \text{as } r \to R$$

where $o(1)$ is uniform in α and n: by Cauchy's formula

$$\frac{P_n(\alpha)}{n!} = \frac{1}{2\pi i} \oint \frac{\exp\{\alpha g(z)\}\, dz}{z^{n+1}} \quad ,$$

where the contour is $z = re^{i\theta}$, any $r < R$.

$$\frac{P_n(\alpha)}{n!} \cdot r^n = \frac{1}{2\pi} \int_{-\delta}^{+\delta} \exp\{\alpha g(re^{i\theta})\}\, e^{-in\theta}\, d\theta + \frac{1}{2\pi} \int_{+\delta}^{2\pi-\delta} \exp\{\alpha g(re^{i\theta})\} e^{-in\theta}\, d\theta$$

$$= \frac{\exp\{\alpha g(r)\}}{2\pi} \int_{-\delta}^{+\delta} [1 + o(1)]\, e^{i\theta(\alpha A - n) - \frac{1}{2}\alpha B \theta^2}\, d\theta + \frac{o(\exp\{\alpha g(r)\})}{\sqrt{\alpha B(r)}}$$

where both little-o's $\to 0$ as $r \to R$ uniformly in α and n, by (II. 13, 14) respectively.

$$\int_{-\delta}^{+\delta} [1+o(1)] \exp\{i\theta(\alpha A - n) - \tfrac{1}{2}\alpha B \theta^2\}\, d\theta$$

$$= \int_{-\delta}^{+\delta} \exp\{i\theta(\alpha A - n) - \tfrac{1}{2}\alpha B \theta^2\}\, d\theta + o\left[\int_{-\infty}^{+\infty} \exp\{-\tfrac{1}{2}\alpha B \theta^2\}\, d\theta\right]$$

$$= \int_{-\delta}^{+\delta} \exp\{i\theta(\alpha A - n) - \tfrac{1}{2}\alpha B \theta^2\} d\theta + o[\alpha B]^{-1/2} .$$

Since $\dfrac{|\exp\{g(r e^{i\delta})\}|}{\exp\{g(r)\}} \sim e^{-\frac{1}{2}\delta^2 B} = o(B^{-1/2})$ by (II. 13, 14) with $x = 1$, we must have $\delta^2 B \to \infty$, because $B^{-1/2} \to 0$. With the change of variables $\theta\left(\frac{\alpha B}{2}\right)^{1/2} = y$, $(\alpha A - n)\left(\frac{2}{\alpha B}\right)^{1/2} = c$ we find

$$\frac{P_n(\alpha)}{n!} r^n = \frac{\exp\{\alpha g(r)\}}{2\pi} \left\{ \int_{-\delta\left(\frac{\alpha B}{2}\right)^{1/2}}^{+\delta\left(\frac{\alpha B}{2}\right)^{1/2}} \exp\{-y^2 + icy\} \left(\frac{\alpha B}{2}\right)^{-1/2} dy + o(\alpha B)^{-1/2} \right\} + \frac{o(\exp\{\alpha g(r)\}}{\sqrt{\alpha B(r)}}$$

$$= \frac{\exp\{\alpha g(r)\}}{\pi\sqrt{2\alpha B}} \left\{ \int_{-\delta\left(\frac{\alpha B}{2}\right)^{1/2}}^{+\delta\left(\frac{\alpha B}{2}\right)^{1/2}} \exp\{-y^2 + icy\}\, dy + o(1) \right\}$$

$$= \frac{\exp\{\alpha g(r)\}}{\pi\sqrt{2\alpha B}} \left\{ \int_{-\infty}^{+\infty} \exp\{-y^2 + icy\}\, dy + o(1) \right\}, \text{ since } \delta^2 B \to +\infty,$$

$$= \frac{\exp\{\alpha\, g(r)\}}{\sqrt{2\pi\alpha B}} \left\{ \exp\left[-\frac{(\alpha A - n)^2}{2\alpha B}\right] + o(1) \right\}$$

since $\int \exp\{-y^2 + icy\} = \sqrt{\pi}\, e^{-1/4\, c^2}$. All little-o's are uniform in n, α. This is Hayman's proof of (II.5), with the embellishment of an α and the appropriate definition of uniform admissibility.

Since $A(r(n/\alpha)) = n/\alpha$,

$$\frac{P_n(\alpha)}{n!} \sim \frac{\exp\left\{\alpha \cdot g\left(r\left(\frac{n}{\alpha}\right)\right)\right\}}{\left[r\left(\frac{n}{\alpha}\right)\right]^n \sqrt{2\pi\alpha\, B\left(r\left(\frac{n}{\alpha}\right)\right)}}, \quad \text{as } n \to \infty \;;$$

and

$$\frac{P_n(\alpha)}{P_n(1)} \sim \sqrt{\frac{B(r(n))}{\alpha B\left(r\left(\frac{n}{\alpha}\right)\right)}} \cdot \frac{\exp\left\{\alpha \cdot g\left(r\left(\frac{n}{\alpha}\right)\right)\right\}}{\exp\{g(r(n))\}} \cdot \left(\frac{r(n)}{r\left(\frac{n}{\alpha}\right)}\right)^n$$

Both of these estimates are uniform for $t \in K$. However,

$$\frac{d}{d\beta}\left[\beta \cdot g\left(r\left(\frac{n}{\beta}\right)\right) - n \log r\left(\frac{n}{\beta}\right)\right] = \beta \cdot g'\left(r\left(\frac{n}{\beta}\right)\right) \cdot r'\left(\frac{n}{\beta}\right) \cdot \frac{-n}{\beta^2} + g\left(r\left(\frac{n}{\beta}\right)\right)$$

$$+ \frac{n^2}{\beta^2} \frac{r'\left(\frac{n}{\beta}\right)}{r\left(\frac{n}{\beta}\right)}$$

$$= \beta \cdot A\left(r\left(\frac{n}{\beta}\right)\right) \cdot \frac{r'\left(\frac{n}{\beta}\right)}{r\left(\frac{n}{\beta}\right)} \cdot \frac{-n}{\beta^2} + g\left(r\left(\frac{n}{\beta}\right)\right) + \frac{n^2}{\beta^2} \frac{r'\left(\frac{n}{\beta}\right)}{r\left(\frac{n}{\beta}\right)}$$

$$= g\left(r\left(\frac{n}{\beta}\right)\right).$$

Using this, plus assumption (3), and the fact that $\alpha \sim 1$:

$$\text{(II. 18)} \qquad \frac{P_n(\alpha)}{P_n(1)} \sim \exp\left\{ \int_1^{\alpha} g\left(r\left(\frac{n}{\beta}\right)\right) d\beta \right\} .$$

For use in Theorem IV we note that so far only (1) - (3) have been used.

Expanding $\int_1^{\alpha} g\left(r\left(\frac{n}{\beta}\right)\right) d\beta$ as a function of α about $\alpha = 1$,

$$\int_1^{\alpha} g\left(r\left(\frac{n}{\beta}\right)\right) d\beta = g(r(n)) \, [\alpha - 1] - g'(r(n)) \cdot r'(n) \cdot n \cdot \frac{[\alpha - 1]^2}{2}$$

$$+ \frac{d^2}{d\beta^2} \, g\left(r\left(\frac{n}{\beta}\right)\right)\Bigg|_{\beta = \zeta} \cdot \frac{[\alpha - 1]^3}{6} ,$$

with ζ between 1 and α. Note that

$$g'(r(n)) \cdot r'(n) \cdot n = A(r(n)) \cdot n \cdot \frac{r'(n)}{r(n)} = \frac{A(r(n))^2}{B(r(n))} ,$$

since $A(r(x)) = x$

$$\Rightarrow \quad A'(r(x)) \cdot r'(x) = 1$$

$$\Rightarrow \quad \frac{r'(x)}{r(x)} = \frac{1}{r(x) \cdot A'(r(x))} = \frac{1}{B(r(x))} .$$

Since

$$\alpha - 1 = \frac{t}{\sigma_n} + \frac{t^2}{2\sigma_n^2} + O\left(\frac{1}{\sigma_n^3}\right)$$

we find

$$\frac{e^{-t\mu_n/\sigma_n} P_n(\alpha)}{P_n(1)} \sim \exp\left\{\frac{t}{\sigma_n}(g(r(n)) - \mu_n) + \frac{t^2}{2\sigma_n^2}\left(g(r(n)) - \frac{A(r(n))^2}{B(r(n))}\right)\right.$$

$$+ O\left(\frac{1}{\sigma_n^3}\right) \cdot g(r(n)) + O\left(\frac{1}{\sigma_n^3}\right) \cdot \frac{A(r(n))^2}{B(r(n))}$$

$$\left. + O\left(\frac{1}{\sigma_n^3}\right) \cdot \frac{d^2}{d\beta^2} g\left(r\left(\frac{n}{\beta}\right)\right)\Big|_{\beta = \zeta}\right\}$$

$$\sim e^{t^2/2}, \text{ by the choice of } \mu_n, \sigma_n, (4), (5), (6).$$

All asymptotics are uniform for $t \in K$.

<u>Example</u>. The signless Laguerre polynomials, continued. In the next theorem we shall see that the conditions of Theorem II are satisfied by functions $g(u)$ which are themselves admissible. For now we consider a $g(u)$ which is <u>not</u> admissible, namely $g(u) = \frac{u}{1-u}$, but for which the previous theorem is applicable. This completes the estimate of μ_n and σ_n for the coefficients $s(n,k)$ of the signless Laguerre polynomials, introduced in the first example of Section I.

First we show that $\exp\left\{\frac{u}{1-u}\right\}$ is an admissible function; the modifications needed to prove $\exp\left\{x \frac{u}{1-u}\right\}$ uniformly admissible for $x \in N$ will be apparent. After a few calculations

$$g(r) = \frac{r}{1-r}$$

$$A(r) = \frac{r}{(1-r)^2}$$

$$B(r) = \frac{r+r^2}{(1-r)^3}$$

$$C(r) = r\,B'(r) = \frac{r+4r^2+r^3}{(1-r)^4} .$$

Expanding in a Taylor series about $\theta = 0$,

$$g(r e^{i\theta}) = g(r) + i\theta\, A(r) - \frac{1}{2}\theta^2\, B(r) - \frac{1}{6}\, i\theta^3\, C(r e^{i\zeta}) ,$$

where ζ is between 0 and θ.

$$|C(r e^{i\zeta})| \le C(r) ,$$

so if $\delta(r)$ is chosen such that $\delta^3 \cdot C \to 0$ as $r \to 1$, then we have

$$\exp\{g(r e^{i\theta})\} \sim \exp\{g(r)\}\, e^{i\theta A(r) - \frac{1}{2}\theta^2 B(r)} \qquad \text{as } r \to 1 ,$$

uniformly for $|\theta| \le \delta(r)$. Let us say $\delta = (1-r)^{\epsilon}$, where we will determine ϵ; so far $3\epsilon > 4$ assures us that $\delta^3 C \to 0$ as $r \to 1$. Since

$$\mathrm{Re}\left\{\frac{r e^{i\theta}}{1 - r e^{i\theta}}\right\} = \frac{r\cos\theta - r^2}{1 - 2r\cos\theta + r^2} ,$$

$$g(r) - \mathrm{Re}\left\{\frac{re^{i\theta}}{1-re^{i\theta}}\right\} = \frac{r}{1-r}\left[1 - \frac{(\cos\theta - r)(1-r)}{1-2r\cos\theta + r^2}\right]$$

$$= \frac{r}{1-r}\left[\frac{(1+r)(1-\cos\theta)}{(1-r)^2 + 2r(1-\cos\theta)}\right]$$

$$= \frac{r(1+r)}{\frac{(1-r)^3}{1-\cos\theta} + 2r - 2r^2}$$

$$\geq \frac{1}{\frac{\pi^2}{2}(1-r)^{3-2\epsilon} + 2r - 2r^2}, \quad \text{as } r \to 1,$$

for $\delta(r) \leq |\theta| \leq \pi$, since $1 - \cos\theta \geq \frac{2}{\pi^2}\theta^2$ for all $|\theta| \leq \pi$.

Finally then,

$$\sqrt{B(r)}\left|\frac{\exp\{g(re^{i\theta})\}}{\exp\{g(r)\}}\right| \leq \sqrt{B(r)}\,\exp\left\{\frac{-1}{\frac{\pi^2}{2}(1-r)^{3-2\epsilon} + 2r - 2r^2}\right\},$$

as $r \to 1$, for all $\delta(r) \leq |\theta| \leq \pi$. The right side goes to zero provided $3-2\epsilon > 0$. So we may take $\delta(r) = (1-r)^{\epsilon}$ where $3-2\epsilon > 0$ and $3\epsilon > 4$. Noting that $B(r) \to \infty$ as $r \to 1$, the admissibility of $\exp\left\{\frac{u}{1-u}\right\}$ is completed. See (II. 1-3). In this case the explicit computation

$$r(x) = 1 + \frac{1}{2x} - \sqrt{\frac{1}{4x^2} + \frac{1}{x}}$$

is possible by solving for r in the equation

$$A(r) \equiv \frac{r}{(1-r)^2} = x .$$

μ_n may then be found from $g(r(n))$. Also,

$$\sigma^2(r) = g(r) - \frac{A(r)^2}{B(r)} = \frac{r^2}{1-r^2} \to \infty \quad \text{as} \quad r \to 1 .$$

σ_n may be computed by $\sigma_n = \sigma(r(n))$. Since $B(r) \sim \frac{2}{(1-r)^3}$ as $r \to 1$, to prove that $\frac{B(r(n))}{B\left(r\left(\frac{n}{\alpha}\right)\right)} \to 1$ we need that $\frac{1-r(n/\alpha)}{1-r(n)} \sim 1$ as $n \to \infty$.

But

$$1 - r(x) = \sqrt{\frac{1}{4x^2} + \frac{1}{x}} - \frac{1}{2x} \sim \frac{1}{\sqrt{x}}, \quad \text{as} \quad x \to \infty .$$

Hence what we need follows from the fact that $n/\alpha \sim n$ as $n \to \infty$.

Finally we verify (4), (5), and (6).

$$\sigma(r)^3 = \frac{r^3}{(1-r^2)^{3/2}} \sim \frac{1}{(1-r^2)^{3/2}} \quad \text{as} \quad r \to 1 .$$

Since $g(r) = \frac{r}{1-r} \sim \frac{1}{1-r}$ as $r \to 1$, we see that $g = o(\sigma^3)$. Likewise for $\frac{A^2}{B} = \frac{r}{1-r^2}$.

Finally we consider $\frac{d^2}{d\beta^2}\, g\left(r\left(\frac{n}{\beta}\right)\right)$. For later use we make the general calculation before considering this case:

$$\frac{d}{d\beta}\, g\left(r\left(\frac{n}{\beta}\right)\right) = g'\left(r\left(\frac{n}{\beta}\right)\right) \cdot r'\left(\frac{n}{\beta}\right) \cdot \frac{-n}{\beta^2}$$

$$= \frac{A(r(n/\beta))}{B(r(n/\beta))} \cdot \frac{-n}{\beta^2}$$

$$= \frac{-n^2}{\beta^3\, B(r(n/\beta))} ,$$

the previous equality following from the note made during the expansion of the integral (II. 18). It then follows

$$\frac{d^2}{d\beta^2}\, g\left(r\left(\frac{n}{\beta}\right)\right) = \frac{3n^2}{\beta^4\, B\left(r\left(\frac{n}{\beta}\right)\right)} - \frac{n^3 B'\left(r\left(\frac{n}{\beta}\right)\right) \cdot r\left(\frac{n}{\beta}\right)}{\beta^5\, B\left(r\left(\frac{n}{\beta}\right)\right)^3} ,$$

after some calculation, using the relation $\frac{r'(x)}{r(x)} = \frac{1}{B(r(x))}$ again.

$$\frac{n^2}{\beta^4\, B\left(r\left(\frac{n}{\beta}\right)\right)} \sim \frac{A(r(n))^2}{B(r(n))} = o(\sigma_n^3) \quad \text{from above.}$$

For the second term,

$$\frac{n^3 B'\left(r\left(\frac{n}{\beta}\right)\right) \cdot r\left(\frac{n}{\beta}\right)}{\beta^5\, B\left(r\left(\frac{n}{\beta}\right)\right)^3} = \frac{n^3\left[r\left(\frac{n}{\beta}\right) + 4r\left(\frac{n}{\beta}\right)^2 + r\left(\frac{n}{\beta}\right)^3\right]}{\beta^5\left[1 - r\left(\frac{n}{\beta}\right)\right]^4} \cdot \frac{\left(1 - r\left(\frac{n}{\beta}\right)\right)^9}{\left(r\left(\frac{n}{\beta}\right) + r\left(\frac{n}{\beta}\right)^2\right)^3}$$

$$\sim \frac{n^3 \cdot 6(1 - r(n/\beta))^5}{8}$$

$$\sim \frac{6(1 - r(n))^5}{8(1 - r(n))^6}$$

$$= o(\sigma_n^3) .$$

This completes the discussion of the Laguerre polynomials.

Next we see that the conditions of Theorem II are met by functions $g(u)$ which are themselves admissible.

Theorem III. If $g(u)$ is admissible for $|u| < R$, then the coefficients $s(n,k)$ of the polynomials $P_n(x)$ generated by $g(u)$ are asymptotically normal with mean μ_n and variance σ_n^2, as defined in Theorem II.

Proof: We will verify the hypotheses of Theorem II.

(1) $\exp\{xg(u)\}$ is uniformly admissible for $|u| < R$ when $x \in N$, a compact set of positive numbers containing 1 in its interior and bounded away from zero. As before let $A(r)$ be $rg'(r)$ and $B(r)$ be $rA'(r)$. (II. 13, 14) will be satisfied by $\delta(r) = [g(r)]^{-2/5}$. From (II. 9),

$$xg(re^{i\theta}) = xg(r) + ix\theta A(r) - \frac{1}{2}x\theta^2 (B(r)) + O(\theta^3 g(r) a(r)^3)$$

uniformly for $|\theta| \leq \delta(r)$, as $r \to R$, since $x = O(1)$ and $(g(r))^{-2/5} < (a(r))^{-1}$ by (II. 8). By our choice of δ, $\theta^3 g a^3 \leq g^{-1/5} \cdot a^3 \to 0$ by (II. 8) again, so (II. 13) is valid. Also, by (II. 10),

$$\frac{|\exp\{x g(re^{i\theta})\}|}{\exp\{x g(r)\}} \sqrt{xB(r)} \leq \exp\{x|g(re^{i\theta})| - xg(r)\}\sqrt{xB(r)}$$

$$\leq e^{-xg(r)^{1/7}} \sqrt{xB(r)}$$

for $g(r)^{-2/5} \le |\theta| \le \pi$, provided r is large. The right side goes to 0 as $r \to R$, without reference to θ and uniformly in $x \in N$, since $x\,B(r) = o(x\,g(r))$. This takes care of (II. 14).

Recall that $a(r) = r\,\dfrac{g'(r)}{g(r)}$ and $b(r) = r\,a'(r)$. Then

(II. 19) $$A(r) = g(r) \cdot a(r)$$

$$B(r) = g(r) \cdot b(r) + g(r) \cdot a(r)^2 \sim g(r) \cdot a(r)^2$$

by (II. 6). Clearly then $x\,B(r) \to +\infty$ as $r \to R$ for $x \in N$. This completes the uniform admissibility of $\exp\{x\,g(u)\}$.

(2) $\sigma(r) \to \infty$ as $r \to R$.

$$\sigma^2 = g - \frac{A^2}{B} = g\left(1 - \frac{A^2}{Bg}\right) = g\left(1 - \frac{(A/g)^2}{B/g}\right) = g\left(1 - \frac{a^2}{b + a^2}\right) = g \cdot \frac{b}{a^2}.$$

Now $\dfrac{g}{a^2} \to \infty$ as $r \to R$ by (II. 8) and by (II. 3), $b \to \infty$.

(3) $$\frac{B(r(n))}{B\left(r\left(\frac{n}{\alpha}\right)\right)} \to 1 \quad \text{as } n \to \infty, \text{ uniformly for } t \in K \quad (\alpha = e^{t/\sigma_n}).$$

Before proving (3) we need two lemmas.

<u>Lemma.</u> Let $[\beta, \gamma] \subseteq (0, \infty)$. Then $r\left(\frac{n}{\beta}\right) \sim r\left(\frac{n}{\gamma}\right)$ as $n \to \infty$.

Since $r\left(\frac{n}{\tau}\right)$ is a decreasing function of τ, it follows that $r\left(\frac{n}{\zeta}\right) \sim r\left(\frac{n}{\tau}\right)$ as $n \to \infty$, uniformly for $\zeta, \tau \in [\beta, \gamma]$.

<u>Proof</u>: Since g is admissible, $b = o(a^2)$ as $r \to R$ (II.6). Also $b \to \infty$ as $r \to R$ (II.3), so $a \to \infty$ as $r \to R$. From (II.19), $A/B \sim 1/a \to 0$ as $r \to R$. By the mean value theorem

$$A\left(r\left(\frac{n}{\beta}\right)\right) - A\left(r\left(\frac{n}{\gamma}\right)\right) = \left[r\left(\frac{n}{\beta}\right) - r\left(\frac{n}{\gamma}\right)\right] A'\left(r\left(\frac{n}{\zeta}\right)\right) ,$$

ζ between β and γ. Recalling that $A(r(x)) = x$,

$$\frac{n\left[\frac{1}{\beta} - \frac{1}{\gamma}\right]}{r\left(\frac{n}{\zeta}\right) A'\left(r\left(\frac{n}{\zeta}\right)\right)} = \frac{r\left(\frac{n}{\beta}\right) - r\left(\frac{n}{\gamma}\right)}{r\left(\frac{n}{\zeta}\right)} ,$$

and

$$\text{(II.20)} \qquad \frac{A\left(r\left(\frac{n}{\zeta}\right)\right) \cdot \zeta\left[\frac{1}{\beta} - \frac{1}{\gamma}\right]}{B\left(r\left(\frac{n}{\zeta}\right)\right)} = \frac{r\left(\frac{n}{\beta}\right) - r\left(\frac{n}{\gamma}\right)}{r\left(\frac{n}{\zeta}\right)} .$$

Since $A/B \to 0$ as $r \to R$, the left side of (II.20) goes to zero as $n \to \infty$. The right side is bigger than the positive

$$\frac{r\left(\frac{n}{\beta}\right) - r\left(\frac{n}{\gamma}\right)}{r\left(\frac{n}{\beta}\right)} ,$$

so the lemma is proved.

<u>Lemma.</u> Let $[\beta, \gamma]$ be a compact set $\subseteq (0, \infty)$ which shrinks to $\{1\}$ as $n \to \infty$; that is, $\beta \uparrow 1$ and $\gamma \downarrow 1$. Then $g\left(r\left(\frac{n}{\beta}\right)\right) \sim g\left(r\left(\frac{n}{\gamma}\right)\right)$ as $n \to \infty$. Since $g\left(r\left(\frac{n}{\tau}\right)\right)$ is a decreasing function of τ, it follows

that $g\left(r\left(\frac{n}{\tau}\right)\right) \sim g\left(r\left(\frac{n}{\zeta}\right)\right)$ as $n \to \infty$ uniformly for $\tau, \zeta \in$ such a $[\beta, \gamma]$.

<u>Proof:</u> Again starting with the mean value theorem,

$$g\left(r\left(\frac{n}{\beta}\right)\right) - g\left(r\left(\frac{n}{\gamma}\right)\right) = \left[r\left(\frac{n}{\beta}\right) - r\left(\frac{n}{\gamma}\right)\right] \cdot g'\left(r\left(\frac{n}{\tau}\right)\right), \text{ some } \tau \in [\beta, \gamma]$$

$$= \frac{r\left(\frac{n}{\beta}\right) - r\left(\frac{n}{\gamma}\right)}{r\left(\frac{n}{\zeta}\right)} \cdot \frac{r\left(\frac{n}{\zeta}\right)}{r\left(\frac{n}{\tau}\right)} \cdot A\left(r\left(\frac{n}{\tau}\right)\right)$$

where $\zeta \in [\beta, \gamma]$ is given by (II.20).

$$= \frac{\zeta\left[\frac{1}{\beta} - \frac{1}{\gamma}\right] \cdot A\left(r\left(\frac{n}{\zeta}\right)\right) \cdot A\left(r\left(\frac{n}{\tau}\right)\right)}{B\left(r\left(\frac{n}{\zeta}\right)\right)} \cdot \frac{r\left(\frac{n}{\zeta}\right)}{r\left(\frac{n}{\tau}\right)}$$

Now,

$A\left(r\left(\frac{n}{\zeta}\right)\right) = \frac{n}{\zeta} \sim \frac{n}{\tau} = A\left(r\left(\frac{n}{\tau}\right)\right)$, since $|\beta - \gamma| \to 0$, and $\dfrac{r\left(\frac{n}{\zeta}\right)}{r\left(\frac{n}{\tau}\right)} \sim 1$ by the last lemma. Thus we have

$$\frac{g\left(r\left(\frac{n}{\beta}\right)\right) - g\left(r\left(\frac{n}{\gamma}\right)\right)}{g\left(r\left(\frac{n}{\zeta}\right)\right)} \sim \frac{\zeta\left[\frac{1}{\beta} - \frac{1}{\gamma}\right] \cdot A\left(r\left(\frac{n}{\zeta}\right)\right)^2}{g\left(r\left(\frac{n}{\zeta}\right)\right) \cdot B\left(r\left(\frac{n}{\zeta}\right)\right)}$$

Now as $n \to \infty$ the right side goes to 0 since $\frac{A^2}{B} \sim g$ by (II.19).

Since the left side is $\geq$ the positive $\dfrac{g\left(r\left(\frac{n}{\beta}\right)\right) - g\left(r\left(\frac{n}{\gamma}\right)\right)}{g\left(r\left(\frac{n}{\beta}\right)\right)}$, the lemma is finished.

To complete the proof of (3), note that $A(r(n))^2 \sim A\left(r\left(\frac{n}{\alpha}\right)\right)^2$ uniformly for $t \in K$. Since $B \sim \frac{A^2}{g}$, the second lemma is all that is needed.

(4) $g = o(\sigma^3)$ as $r \to R$.

By the proof of (2), $\sigma^2 = g \cdot b/a^2$. So $\frac{g}{\sigma^3} = \frac{a^3}{g^{1/2} b^{3/2}}$, with $a^3 = o(g^{1/2})$ by (II. 8), and $b \to \infty$ by (II. 3).

(5) $\frac{A^2}{B} = o(\sigma^3)$ as $r \to R$.

Immediate from (4) above, since $\frac{A^2}{B} \sim g$ (II. 19).

(6) $\frac{d^2}{d\beta^2}\, g\left(r\left(\frac{n}{\beta}\right)\right) = o(\sigma_n^3)$ as $n \to \infty$, uniformly for β between 1 and $\alpha = e^{t/\sigma_n}$.

From the calculation of the example after Theorem II,

$$g\left(r\left(\frac{n}{\beta}\right)\right)'' = \frac{3n^2}{\beta^4 B\left(r\left(\frac{n}{\beta}\right)\right)} - \frac{n^3 B'\left(r\left(\frac{n}{\beta}\right)\right) r\left(\frac{n}{\beta}\right)}{\beta^5 B\left(r\left(\frac{n}{\beta}\right)\right)^3}$$

Now,

$$\frac{n^2}{\beta^4 \cdot B\left(r\left(\frac{n}{\beta}\right)\right)} \sim \frac{A(r(n))^2}{B(r(n))} = o(\sigma_n^3) \text{ settles the first term.}$$

For the second, $B = u^2 g'' + ug'$

$$uB' = u^3 g''' + 3u^2 g'' + ug'.$$

As $u \to R$, $\frac{uB'(u)}{B(u)}$ is either bounded or

$$\sim \frac{u^3 g'''(u)}{B(u)} \sim \frac{g(u) \cdot a(u)^3}{B(u)} \qquad (II.7)$$

In the bounded case,

$$\frac{n^3}{\beta^5 B\left(r\left(\frac{n}{\beta}\right)\right)^2} \sim \frac{A(r(n))^3}{B(r(n))^2} = o(g(r(n)) = o(\sigma_n^3) ,$$

since $\frac{A^2}{B} \sim g$ by (II.19) and $\frac{A}{B} \to 0$, as noted in the first Lemma.

While in the second case,

$$\frac{n^3 B'\left(r\left(\frac{n}{\beta}\right)\right) r\left(\frac{n}{\beta}\right)}{\beta^5 B\left(r\left(\frac{n}{\beta}\right)\right)^3} \sim \frac{n^3 g\left(r\left(\frac{n}{\beta}\right)\right) a\left(r\left(\frac{n}{\beta}\right)\right)^3}{B\left(r\left(\frac{n}{\beta}\right)\right)^3}$$

$$= \frac{n^3 g\left(r\left(\frac{n}{\beta}\right)\right) \cdot A\left(r\left(\frac{n}{\beta}\right)\right)^3}{B\left(r\left(\frac{n}{\beta}\right)\right)^3 \quad g\left(r\left(\frac{n}{\beta}\right)\right)^3}$$

$$\sim \left(\frac{A(r(n))^2}{B(r(n))}\right)^3 \cdot \frac{1}{g(r(n))^2}$$

$$\sim g(r(n))$$

$$= o(\sigma_n^3) .$$

This completes the proof of Theorem III.

We now show how some typical admissible functions may arise in combinatorial enumeration.

1. $g(u) = Q(u)\, e^u$, $Q(u)$ a polynomial with positive leading coefficient. Notice that if $g_n = n_j \equiv n(n-1) \cdots (n-j+1)$, ($j$ fixed), then

$g(u) = \sum \frac{g_n}{n!} u^n = u^j e^u$. Considering linear combinations, we see that $g(u) = Q(u)\, e^u$ if and only if g_n is expressed as a (fixed) polynomial in n. Consequently, when the number g_n of connected objects constructable on an n-element set is given by a polynomial formula, then the distribution $s(n,k)$ of components is asymptotically normal. For instance,

$g_n = \binom{n}{j}$, j fixed; $s(n,k)$ enumerates partitions with k blocks, each with a distinguished j-set.

2. Products of admissible functions. Notice that for the convolution of two exponential generating functions,

$$\left(\sum a_n \frac{u^n}{n!}\right)\left(\sum b_n \frac{u^n}{n!}\right) = \sum c_n \frac{u^n}{n!} \quad ,$$

we have that

$$c_n = \sum_{j=0}^{n} \binom{n}{j} a_j b_{n-j} \quad .$$

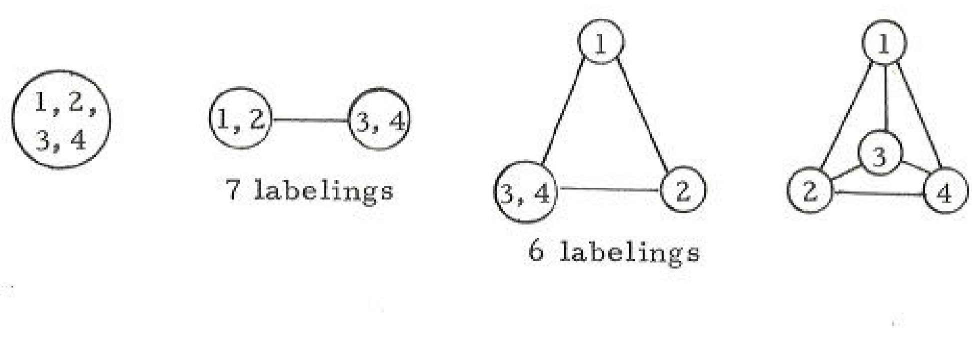
This has the combinatorial interpretation that to construct a "c-type" object on n elements one must choose j of the elements, construct an "a-type" object on them, and construct a "b-type" object on the remaining n-j.

3. Example of a product. What is the distribution of components of graphs on labeled n-sets, all of whose components are complete bipartite graphs? To simplify matters distinguish between left and right parts of a bipartite graph. Then $g_n = 2^n - 1$, since we need only select a non-empty subset for the left part. Then,

$$g(u) = \sum_{1}^{\infty} \frac{2^n - 1}{n!} u^n = (e^{2u} - 1) - (e^u - 1) = e^u(e^u - 1) .$$

4. Exponentials of admissible functions. It will be the case that the generating function $g(u)$ is itself of the form $\exp\{h(u)\}$ when the connected components which g_n enumerates are themselves built up from smaller blocks. To illustrate:

Graphs with Multi-labels. Let g_n be the number of ways to label the vertices of a complete graph of some size with subsets of $\{1, 2, \ldots, n\}$ in such a way that each element $1, 2, \ldots, n$ is used once and only once, and each vertex receives at least one label. The connected objects on 4 vertices are:



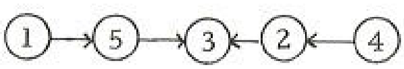
Then the distribution of components is asymptotically normal. For $g_n = B_n$ = Bell numbers, total number of partitions of an n-set. Hence,

$$g(u) = \sum_{1}^{\infty} \frac{B_n}{n!} u^n = \exp\{e^u - 1\} - 1 .$$

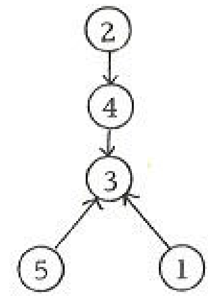
Star-like Graphs. Here is a curious use of the signless Laguerre polynomials $L_n(x)$. We call a graph on a labeled n-set star-like if it is a rooted tree all of whose vertices have degree ≤ 2, except possibly the root. The root may be indicated by directing the edges. Here are the distinct star-graphs on five vertices:

120 labelings

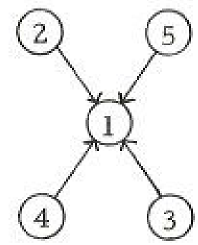
120 labelings



60 labelings



60 labelings



5 labelings

We consider the distribution of $s(n,k)$ = the number of graphs with k star-like components on n labeled vertices. Note that g_n, the number of connected stars on n vertices, is $nL_{n-1}(1)$. For we may choose the root in n ways, and adding the tails involves partitioning the remaining $n-1$ vertices into various ordered blocks. (See first example in Section I.) A "closed form" for $s(n,k)$ is possible. Summing over all ordered k-tuples of strictly positive integers which add to n:

$$s(n,k) = \frac{1}{k!} \sum_{\vec{\nu}} \binom{n}{\nu_1 \dots \nu_k} g_{\nu_1} \dots g_{\nu_k}$$

$$= \frac{1}{k!} \sum_{\vec{\nu}} \binom{n}{\nu_1 \dots \nu_k} \nu_1 \dots \nu_k \, L_{\nu_1 - 1}(1) \dots L_{\nu_k - 1}(1)$$

$$= \binom{n}{k} \sum_{\vec{\nu}} \binom{n-k}{\nu_1 - 1, \dots, \nu_k - 1} L_{\nu_1 - 1}(1) \dots L_{\nu_k - 1}(1)$$

$$= \binom{n}{k} L_{n-k}(k).$$

The last identity is common to all polynomials of binomial type. From the identity (0.3c)

$$P_n(x+y) = \sum_{j=0}^{n} \binom{n}{j} P_j(x) \, P_{n-j}(y) \quad ,$$

there follows rather naturally

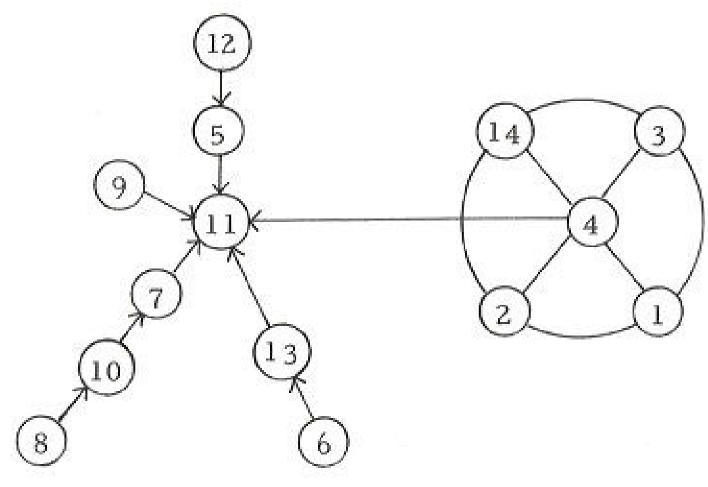
$$P_n(x_1 + \dots + x_k) = \sum_{\substack{(\nu_1, \dots, \nu_k) \\ \nu_i \geq 0 \\ \sum \nu_i = n}} \binom{n}{\nu_1 \dots \nu_k} P_{\nu_1}(x_1) \dots P_{\nu_k}(x_k)$$

The generating function for the $s(n,k)$ is

$$g(u) = \sum_{1}^{\infty} g_n \cdot \frac{u^n}{n!} = u \sum_{1}^{\infty} L_{n-1}(1) \cdot \frac{u^{n-1}}{(n-1)!} = u \cdot \exp\left\{\frac{u}{1-u}\right\}.$$

This is not the exponential of an admissible function. However, we have previously verified that $\exp\left\{\frac{u}{1-u}\right\}$ is admissible, hence so is $g(u)$, by the remarks in Section II, part 1.

5. $[Q(u)] \cdot$ [admissible function], where $Q(u)$ is a polynomial. Hayman's proof that a product of a polynomial and admissible function is admissible involves showing that multiplication by a polynomial does not perturb the original function too greatly. This is reflected in the combinatorics of the situation. Considering the star-like graphs of the last example, suppose we alter it slightly to say that acceptable components are stars again, except that one of the tails need not be a straight segment, but may be a wheel of bounded size M whose center is attached to the root of the star. For example (if $M \geq 5$)



is now an acceptable connected graph on 14 vertices. The number of wheels on n vertices is $n \cdot (n-2)!$, since one may choose the hub in n ways, and then form a cycle on the remaining $n-1$ elements. Form the exponential generating function

$$Q(u) = \sum_{j=2}^{M} \frac{j(j-2)!}{j!} u^j = \sum_{j=2}^{M} \frac{u^j}{j-1} .$$

If g_n denotes the number of components consisting of a rooted star with attached wheel of bounded size, we have (recalling the generating function of the previous example, and the interpretation of convolutions given in the second example):

$$g(u) = \sum_{n=1}^{\infty} g_n \frac{u^n}{n!} = u \cdot Q(u) \cdot \exp\left\{\frac{u}{1-u}\right\} ,$$

the product of a polynomial and an admissible function.

We now turn our attention to a situation for which neither of the two previous theorems is generally applicable; namely, $g(u) =$ a polynomial. This means that components are of bounded size. The case in which $g(u)$ is a polynomial is of special interest. For such $g(u)$, Moser and Wyman [17] give an asymptotic development of the numbers B_n defined by $\sum \frac{B_n}{n!} u^n = \exp\{g(u)\}$. Our result is

<u>Theorem IV.</u> Let $g(u)$ be a polynomial with real non-negative coefficients: $g(u) = \sum_{1}^{m} c_n u^n$ $(c_m \neq 0)$. Assume that g.c.d. $\{i: c_i \neq 0\} = 1$.

Then the coefficients $s(n,k)$ of the polynomials $P_n(x)$ generated by $g(u)$ are asymptotically normal with mean μ_n and variance σ_n^2 defined as in the previous two theorems.

<u>Remarks</u>.

1. The assumption about the g.c.d. is not needed. In fact, if g.c.d. $\{i: c_i \neq 0\} = d > 1$, then $s(n,k) = 0$ when $d \nmid n$. In such a case define a polynomial $h(u) = g(u^{1/d})$. Then $h(u)$ satisfies the hypothesis, and the $s_h(n,k)$ which it generates are precisely the non-trivial $s(nd,k)$ generated by the original $g(u)$.

2. $g(u) = c_1 u$ satisfies the hypothesis of the theorem. In this case $\sigma \equiv 0$, and $\mu_n = n$. While in some sense the $s(n,k)$ are indeed "asymptotically normal with mean n and variance 0," basically this is a degenerate situation which we will ignore so that the integer $J = \max\{i < m: c_i \neq 0\} > 0$ may be used in what follows.

<u>Proof</u>: First we establish the uniform admissibility of $\exp\{x\, g(u)\}$. Again N denotes a compact set of positive numbers containing 1 in its interior and bounded away from zero. $A(r)$ and $B(r)$ are the polynomials $r g'(r)$ and $r A'(r)$. Of course $B(r) \to +\infty$ as $r \to \infty$. We shall verify (II.13, 14) with $\delta(r) = r^{-m/2 + 1/8}$ (m denotes the degree of $g(u)$). By Taylor's theorem about $\theta = 0$,

$$x\, g(re^{i\theta}) = x g(r) + i x A(r)\theta - \frac{1}{2} x B(r)\theta^2 + O(r^m \theta^3), \text{ as } r \to \infty$$

uniformly for $|\theta| \leq \pi$ and $x \in N$, and so (II.13) is immediate by our choice of $\delta(r)$. To establish (II.14) we need

<u>Lemma.</u> $\exists$ a positive constant c and an R_1 such that

$$g(r) - \operatorname{Re} g(re^{i\theta}) \geq c \cdot r^{1/4} \;\forall\; \delta(r) \leq |\theta| \leq \pi \quad \text{and} \quad r \geq R_1 .$$

<u>Proof</u>: Let $c = \frac{1}{8} \min \{c_k : c_k \neq 0\}$. Note that

$$g(r) - \operatorname{Re} g(re^{i\theta}) = \sum_{k=1}^{m} c_k r^k (1 - \cos k\theta) \geq c_k r^k (1 - \cos k\theta) .$$

If $1 - \cos k\theta \geq \frac{1}{8} r^{-3/4}$ for some k with $c_k \neq 0$, then the conclusion holds. Suppose $1 - \cos k\theta \leq \frac{1}{8} r^{-3/4}$ for all k with $c_k \neq 0$. Notice that if ϵ is a small positive number and $1 - \cos x < \frac{1}{8} \epsilon$, then e^{ix} has to be within ϵ of 1, distance measured as arc length along the unit circle. All distances and neighborhoods we speak of now are for points on the unit circle, measured in arc length. So if $1 - \cos k\theta < \frac{1}{8} r^{-3/4}$ for all k with $c_k \neq 0$ (and r is large), $e^{ik\theta}$ is within a distance $r^{-3/4}$ of 1, $\forall$ k with $c_k \neq 0$. Therefore $e^{i\theta}$ is within a distance $r^{-3/4}$ of a k^{th}-root of unity $\forall$ k with $c_k \neq 0$. Consider k^{th} roots of 1 for all k with $c_k \neq 0$. Some of these may coincide, but for large r the $r^{-3/4}$ neighborhoods of the distinct ones are all disjoint. Hence $e^{i\theta}$ is within $r^{-3/4}$ of some root of unity which is a k^{th} root of unity for all k with $c_k \neq 0$. Since g.c.d.$\{k : c_k \neq 0\} = 1$, this root of unity can only be 1. Thus $|\theta| < r^{-3/4}$. For large r such a θ will be small enough that

$$(1 - \cos m\theta) \geq 1 - \cos \theta \geq \frac{1}{8} \theta^2 \geq \frac{1}{8} r^{-m + 1/4} .$$

Thus $g(r) - \operatorname{Re} g(re^{i\theta}) \geq c_m r^m \cdot \frac{1}{8} r^{-m+1/4} \geq c\, r^{1/4}$. This completes the lemma. By the lemma and $B(r) = O(r^m)$,

$$\left| \frac{\exp\{x g(re^{i\theta})\}}{\exp\{x\, g(r)\}} \right| \sqrt{x\, B(r)} = O(e^{-cxr^{1/4}} r^{m/2}), \quad r > R_1 ,$$

$$\delta(r) \leq |\theta| \leq \pi, \; x \in N$$

$$= o(1).$$

This establishes (II. 14).

Next, from (II. 16) we see that $\sigma^2 \sim c \cdot r^J$ as $r \to \infty$, some positive c. (J is defined in the second remark.) Also,

$$B(r) \sim m^2 c_m r^m \text{ as } r \to \infty \quad \text{and} \quad r(\beta) \sim \sqrt[m]{\frac{\beta}{mc_m}} \text{ as } \beta \to \infty .$$

Hence,

$$\frac{B(r(n))}{B\left(r\left(\frac{n}{\alpha}\right)\right)} \to 1 \quad \text{as } n \to \infty \quad ,$$

uniformly for $t \in K$, where $\alpha = e^{t/\sigma_n}$ We have checked (1), (2), and (3) of Theorem II, so that (II. 18) holds:

$$\frac{P_n(\alpha)}{P_n(1)} \sim \exp\left\{ \int_1^{\alpha} g\left(r\left(\frac{n}{\beta}\right)\right) d\beta \right\} \quad \text{as } n \to \infty \quad .$$

If it were the case that $g = o(\sigma^3)$, then (4), (5), (6) of Theorem II would follow as in the proof of Theorem III. However, $g = o(\sigma^3)$ $\Leftrightarrow$ $J > \frac{2}{3} m$, and since we wish to avoid imposing this additional constraint, it is necessary to develop some algebraic tools suitable for understanding the higher derivatives of the function $\beta \to g\left(r\left(\frac{n}{\beta}\right)\right)$. Let T_i, $i = 1, 2, \ldots$ be a countable collection of variables. O denotes the operator $O\,T_i = T_{i+1}$. Extend O to polynomials in the T_i by using linearity with respect to addition and scalar multiplication, and the Liebnitz product rule:

$$O(T_{i_1} \cdots T_{i_\ell}) = T_{i_1+1}\, T_{i_2} \cdots T_{i_\ell} + T_{i_1}\, T_{i_2+1} \cdots T_{i_\ell} + \cdots + T_{i_1}\, T_{i_2} \cdots T_{i_\ell+1}.$$

Now define polynomials P_k in the variables T_i as follows

$$P_1(T) = -T_1^2$$

$$P_{k+1}(T) = -T_1 T_2\, O P_k - \{ k T_2^2 - (2k-1) T_1 T_3 \} P_k .$$

The recursion shows us that P_k is a homogeneous polynomial of degree $2k$. Hence,

$$P_{k+1}(1) = -2k\, P_k(1) - \{-k+1\}\, P_k(1)$$

$$= (-k-1)\, P_k(1) .$$

Therefore $P_k(1) = (-1)^k \cdot k!$

Note also (easily seen inductively) that if $T_{i_1} T_{i_2} \cdots T_{i_{2k}}$ appears in $P_k(T)$, then $i_1 + i_2 + \cdots + i_{2k} = 4k - 2$. The purpose of these polynomials is

Proposition. Let $F(\beta) = g\left(r\left(\frac{n}{\beta}\right)\right)$. Then for $k \geq 1$,

$$F^{(k)}(\beta) = \frac{P_k(T)}{\beta^k B^{2k-1}} ,$$

where T_i is set equal to the polynomial $r\left(\frac{d}{dr}\right)^i g$, evaluated at $r\left(\frac{n}{\beta}\right)$. Specifically, $T_i(r) = \sum_{j=1}^{m} j^i c_j r^j$. In particular, $T_1(r) = A(r)$, and $T_2(r) = B(r)$. From our observations above about the P_k there follows

Corollary.

$$F^{(k)}(\beta = \frac{(-1)^k k!\, c_m^{2k} \cdot r\left(\frac{n}{\beta}\right)^{2km} \cdot m^{4k-2} + \text{lower powers of } r\left(\frac{n}{\beta}\right)}{\beta^k \left\{ c_m^{2k-1} \cdot r\left(\frac{n}{\beta}\right)^{(2k-1)m} \cdot m^{4k-2} + \text{lower powers of } r\left(\frac{n}{\beta}\right) \right\}} .$$

For each term (constant) $\cdot\ T_{i_1} \ldots T_{i_{2k}}$ appearing in $P_k(T)$ has leading term

$$(\text{constant})\ c_m^{2k} \cdot r\left(\frac{n}{\beta}\right)^{2km} \cdot m^{i_1 + \cdots + i_{2k}}$$

$$= (\text{constant}) \cdot c_m^{2k}\, r\left(\frac{n}{\beta}\right)^{2km} \cdot m^{4k-2} ,$$

and the sum of the constants is $P_k(1) = (-1)^k k!$ This accounts for the numerator in the corollary. As for the denominator, it is a matter of determining the leading coefficient of B^{2k-1}, evaluated at $r\left(\frac{n}{\beta}\right)$.

<u>Proof of the Proposition</u>: In the note following (II. 18) we saw that

$\frac{r'(x)}{r(x)} = \frac{1}{B(r(x))}$. Also, recall that $A(r(x)) \equiv x$. In what follows all polynomials T_i, A, B are understood to be evaluated at $r\left(\frac{n}{\beta}\right)$.

First,

$$\frac{d}{d\beta} T_i = T_i' \cdot r'\left(\frac{n}{\beta}\right) \cdot \frac{-n}{\beta^2}$$

$$= T_{i+1} \cdot \frac{r'(n/\beta)}{r(n/\beta)} \cdot \frac{-n}{\beta^2}$$

$$= T_{i+1} \cdot \frac{-n}{\beta^2} \cdot \frac{1}{B}$$

$$= -\frac{A}{\beta B} \cdot T_{i+1} \quad .$$

In the example following Theorem I we calculated that

$$\frac{d}{d\beta} g\left(r\left(\frac{n}{\beta}\right)\right) = \frac{-A^2}{\beta B} \quad .$$

Hence the proposition is valid for $k=1$. Proceeding by induction, using the above formula for $\frac{d}{d\beta} T_i$, we have

$$\frac{d}{d\beta} \frac{P_k(T)}{\beta^k B^{2k-1}} = \frac{\beta^k B^{2k-1} \cdot \frac{-A}{\beta B} \, O \, P_k(T) - \left\{k\beta^{k-1} B^{2k-1} + \beta^k (2k-1) B^{2k-2} \cdot \frac{-T_3 A}{\beta B}\right\} P_k(T)}{\beta^{2k} B^{4k-2}}$$

$$= \frac{-AB \, O \, P_k(T) - \{k B^2 - (2k-1) A \, T_3\} \, P_k(T)}{\beta^{k+1} B^{2k+1}}$$

$$= \frac{P_{k+1}(T)}{\beta^{k+1} B^{2k+1}} \quad .$$

This completes the proof of the proposition. Before returning to the theorem, we note the expansion:

$$\frac{(e^u - 1)^k}{k!} = \sum_{n \le K} \frac{S(n,k)\, u^n}{n!} + \sum_{n > K} \frac{S(n,k)\, u^n}{n!} ,$$

where $S(n,k)$ denote Stirling numbers of the second kind. Indeed, $g(u) = e^u - 1$ generates these Stirling numbers, and

$$\frac{g(u)^k}{k!} \quad \text{is} \quad \sum_n s(n,k) \frac{u^n}{n!}$$

in general when $g(u)$ generates $s(n,k)$. Now we have

$$\sum_{n > K} \frac{S(n,k)\, u^n}{n!} \le \sum_{n > K} 3^n |u|^n = \frac{(3|u|)^{K+1}}{1 - 3|u|} ,$$

provided that $|u| < 1/3$, since $S(n,k) \le n^n < 3^n \cdot n!$ Recalling that $\alpha = e^{t/\sigma_n}$, and $\sigma_n \to \infty$, we have

$$\text{(II.21)} \quad \frac{(\alpha - 1)^{j+1}}{(j+1)!} = \sum_{\ell \le K} \frac{S(\ell, j+1)}{\sigma_n^{\ell}} \cdot \frac{t^{\ell}}{\ell !} + O\left(\frac{1}{\sigma_n^{K+1}}\right) .$$

Continuing with the main theorem, choose K so that $\frac{(K+1)J}{2} > m$, and let $F(\beta) = g\left(r\left(\frac{n}{\beta}\right)\right)$. Since

$$\frac{e^{-\mu_n t/\sigma_n} P_n(\alpha)}{P_n(1)} \sim \exp\left\{ \int_1^{\alpha} F(\beta)\, d\beta - \frac{\mu_n t}{\sigma_n} \right\} ,$$

we may expand the integral in a Taylor series about $\alpha = 1$ to $K+1$ terms:

$$\text{(II.22)} \quad \frac{-\mu_n t}{\sigma_n} + \sum_{j=0}^{K-1} F^{(j)}(1) \frac{(\alpha-1)^{j+1}}{(j+1)!} + F^{(K)}(\beta) \frac{(\alpha-1)^{K+1}}{(K+1)!} \quad ,$$

where β is between 1 and α, and show that this expression converges to $t^2/2$ as $n \to \infty$. By the corollary,

$$F^{(K)}(\beta) = O\left(r\left(\frac{n}{\beta}\right)^m\right) = O(r(n)^m) \quad ,$$

because $r\left(\frac{n}{\beta}\right) \sim r(n)$. Also, $(\alpha-1)^{K+1} = O\left(\frac{1}{\sigma_n^{K+1}}\right)$. Recalling that $\sigma_n^2 \sim c \cdot r(n)^J$, by our choice of K the last term of (II.22) is negligible. Similarly, by (II.21), our choice of K says that we may concentrate on the finite sum

$$\frac{-\mu_n t}{\sigma_n} + \sum_{j=0}^{K-1} F^{(j)}(1) \left\{ \sum_{\ell=j+1}^{K} \frac{S(\ell, j+1)}{\sigma_n^{\ell}} \cdot \frac{t^{\ell}}{\ell!} \right\} .$$

In this expression, we know that the coefficient of t is zero, and the coefficient of $t^2/2$ is 1 by our choice of μ_n, σ_n. The coefficient of $t^{\ell}/\ell!$ for $3 \le \ell \le K$ is

$$\frac{1}{\sigma_n^{\ell}} \cdot \sum_{j=0}^{\ell-1} F^{(j)}(1)\ S(\ell, j+1)$$

$$= \frac{1}{\sigma_n^{\ell}} \frac{g \cdot B^{2\ell-3} S(\ell,1) + P_1(T) B^{2\ell-4} S(\ell,2) + P_2(T) B^{2\ell-6} \cdot S(\ell,3) + \cdots + P_{\ell-1}(T)\ S(\ell,\ell)}{B^{2\ell-3}}$$

Every term in the numerator starts with (coefficient)$\cdot\ c_m^{2\ell-2}\ r(n)^{m(2\ell-2)}$ $\cdot\ m^{4\ell-6}$, and then has smaller powers of $r(n)$. Using the corollary, we find that the total coefficient of $c_m^{2\ell-2}\ r(n)^{m(2\ell-2)}\ m^{4\ell-6}$ is

$$\sum_{j=1}^{\ell} (-1)^{j-1} (j-1)!\ S(\ell, j) \equiv 0 .$$

(See, for example, Jordan [12], p. 183.) Hence the numerator does not have the maximum possible $= m(2\ell-2)$ degree in $r(n)$, so that degree of the numerator $\leq m(2\ell-3) + J$. Therefore the coefficient of $\frac{t^{\ell}}{\ell!}$ is $\frac{1}{\sigma_n^{\ell}} \cdot O(r(n)^J) \to 0$ since $\ell \geq 3$. This completes the proof of the theorem.

Theorem IV is quite general and so applicable to any situation in which components are of bounded size. We list here a few specific cases of interest.

(1) $g(u) = \sum_{1}^{M} \frac{u^n}{n!}$; here, $g_n \equiv 1$ for $n \leq M$ and $g_n = 0$ if $n > M$.

Therefore $s(n, k)$ = the number of partitions of an n element set into k blocks, no block containing more than M elements. This problem may also be interpreted as the distribution of n labeled balls into unlabeled cells of limited capacity.

$$(2) \qquad g(u) = \sum_{1}^{M} \frac{u^n}{n} \text{; here, } g_n = (n-1)! \text{ for } n \le M$$

and $g_n = 0$ if $n > M$. Therefore $s(n, k)$ = the number of permutations on an n element set having k cycles, none of which has length greater than M.

$$(3) \qquad g(u) = \sum_{d \mid M} \frac{u^d}{d} .$$

Similar to (2), now $s(n, k)$ is the number of permutations, whose M^{th} power is the identity, having k cycles.

III. Local Limit Theorems for Polynomials of Binomial Type

1. Log concavity (LC). As mentioned in the Introduction, LC is a useful smoothness condition for passage from a central to local limit theorem. Very explicit details are included in part 2 of this section. LC is a property which has been studied and verified for several familiar combinatorial sequences: binomial coefficients, Stirling numbers of the first and second kind, Eulerian numbers. In his thesis Kurtz [15] considers conditions on the coefficient functions which yield LC for triangular arrays defined by recurrence. In this section we give a criterion on $g(u)$ that gives strict LC for the coefficients of $P_n(x)$. We mention again that a sequence c_n is <u>log concave</u> if $c_n^2 \geq c_{n-1}c_{n+1}$. It is not difficult to derive as a consequence of this that $c_i c_j \geq c_{i+\ell} \cdot c_{j-\ell}$ whenever $i \geq j$ and $\ell \geq 0$.

<u>Proposition</u>. If the coefficients in $g(u)$ are LC, then the coefficients of $P_n(x)$ are strictly LC. In our standard notation:

$$\left(\frac{g_n}{n!}\right)^2 \geq \frac{g_{n-1}}{(n-1)!}\,\frac{g_{n+1}}{(n+1)!}\ \forall\ n \Rightarrow s(n,k)^2 > s(n,k-1)\,s(n,k+1)\ \forall\ n,k.$$

<u>Proof</u>: Let us define: $c_n = g_n/n!$,

$$\mathfrak{A}(n,k) = \{(\nu_1,\ldots,\nu_k) \mid \nu_i \geq 1 \text{ and } \sum \nu_i = n\} \ ,$$

and for $\vec{\nu} \in \mathfrak{A}(n,k)$, $c_{\vec{\nu}} = c_{\nu_1} \cdots c_{\nu_k}$ where $\vec{\nu} = (\nu_1,\ldots,\nu_k)$.

With this notation,

$$\text{(III. 1)} \qquad [g(u)]^k = \sum_n u^n \left\{ \sum_{\vec{\nu} \in \mathfrak{G}(n,k)} c_{\vec{\nu}} \right\}$$

$$s(n,k) = \frac{n!}{k!} \sum_{\vec{\nu} \in \mathfrak{G}(n,k)}$$

Strict LC of $s(n,k)$ follows easily from LC of $\sum_{\vec{\nu} \in \mathfrak{G}(n,k)} c_{\vec{\nu}}$, since $\left(\frac{1}{k!}\right)^2 > \frac{1}{(k-1)!} \cdot \frac{1}{(k+1)!}$.

Let

$$\mathfrak{G}_\beta^\alpha(n,k+1) = \left\{ (\nu_1, \ldots, \nu_{k+1}) \mid \nu_i \geq 1, \sum \nu_i = n, \nu_k = \alpha, \nu_{k+1} = \beta \right\}$$

$$\mathfrak{G}_\gamma(n,k) = \left\{ (\nu_1, \ldots, \nu_k) \mid \nu_i \geq 1, \sum \nu_i = n, \nu_k = \gamma \right\}.$$

Since $\mathfrak{G}(n,k+1)$ is the disjoint union of $\mathfrak{G}_\beta^\alpha(n,k+1)$ over all (α, β), and $\mathfrak{G}(n,k)$ is the disjoint union of $\mathfrak{G}_\gamma(n,k)$ over all γ, the desired result

$$\left(\sum_{\vec{\nu} \in \mathfrak{G}(n,k+1)} c_{\vec{\nu}} \right) \left(\sum_{\vec{\nu} \in \mathfrak{G}(n,k-1)} c_{\vec{\nu}} \right) \leq \left(\sum_{\vec{\nu} \in \mathfrak{G}(n,k)} c_{\vec{\nu}} \right)^2$$

may be established by verifying that

$$\left(\sum_{\vec{\nu} \in \mathfrak{G}_\beta^\alpha(n,k+1)} c_{\vec{\nu}} \right) \left(\sum_{\vec{\nu} \in \mathfrak{G}(n,k-1)} c_{\vec{\nu}} \right) \leq \left(\sum_{\vec{\nu} \in \mathfrak{G}_\alpha(n,k)} c_{\vec{\nu}} \right) \left(\sum_{\vec{\nu} \in \mathfrak{G}_\beta(n,k)} c_{\vec{\nu}} \right)$$

for all ordered pairs (α, β). However, the left side of this expression is the same as

$$c_\alpha c_\beta \left(\sum_{\vec{\nu} \in \mathfrak{G}(n-\alpha-\beta,k-1)} c_{\vec{\nu}} \right) \left(\sum_{\vec{\nu} \in \mathfrak{G}(n,k-1)} c_{\vec{\nu}} \right),$$

while the right side is

$$c_\alpha c_\beta \left(\sum_{\vec{\nu} \in \mathcal{A}(n-\alpha, k-1)} c_{\vec{\nu}} \right) \left(\sum_{\vec{\nu} \in \mathcal{A}(n-\beta, k-1)} c_{\vec{\nu}} \right) .$$

In view of the first equation of (III. 1), we need only know that the coefficients in $[g(u)]^{k-1}$ are LC, and then use the fact that $c_i c_j \geq c_{i+\ell} c_{j-\ell}$ when $i \geq j$ and $\ell \geq 0$, provided the c_i are LC. However, it is well known that the convolution of two LC sequences is itself LC, (see for example Karlin [13], chapt. 8). A simple proof of this fact may be based on the identity:

$$\left(\sum_{j=0}^{k} a_j b_{k-j} \right)^2 - \left(\sum_{j=0}^{k-1} a_j b_{k-1-j} \right) \left(\sum_{j=0}^{k+1} a_j b_{k+1-j} \right)$$

$$= \sum_{\alpha < \beta} (a_\alpha a_{\beta-1} - a_\beta a_{\alpha-1})(b_{k-\alpha} b_{k+1-\beta} - b_{k-\beta} b_{k+1-\alpha}).$$

If the a's and b's are LC, then every term in the right sum is positive.

2. A local limit theorem. In this section the $s(n,k)$ are assumed strictly LC and asymptotically normal with mean μ_n and variance σ_n^2, which we continue to assume becomes infinite as $n \to \infty$. We use $p(n,k)$ for the probability $s(n,k) / \sum_k s(n,k)$, and abbreviate

$$\sum_{\mu_n + y\sigma_n < k \le \mu_n + x\sigma_n} p(n,k) \quad \text{to} \quad \sum_{y}^{x} p(n,k) \ ,$$

and

$$\frac{1}{\sqrt{2\pi}} \int_y^x e^{-t^2/2} dt \quad \text{to} \quad \eta(y,x) \ .$$

Since $s(n,k)$ (hence $p(n,k)$) are strictly LC, they are unimodal and have at most two consecutive maxima. We let $k = m_n$ denote the position of the left-most.

<u>Lemma</u>. Given $\delta > 0$ there is N such that $\mu_n - \delta\sigma_n < m_n < \mu_n + \delta\sigma_n$ for all $n \ge N$.

<u>Proof</u>: Since $\eta(0, \delta/2) > \eta(\delta/2, \delta)$, we may choose $\bar{\delta}$ positive and smaller than $\delta/2$ so that $\eta\left(0, \frac{\delta}{2} - \bar{\delta}\right) > \eta\left(\frac{\delta}{2} - \bar{\delta}, \delta\right)$. As we are assuming a central limit theorem for the $s(n,k)$, this means that $\sum_{0}^{\frac{\delta}{2} - \bar{\delta}} p(n,k) > \sum_{\frac{\delta}{2} - \bar{\delta}}^{\delta} p(n,k)$ for $n \ge N_1$. Also, since $\sigma_n \to \infty$, we have that $\frac{1}{2\sigma_n} \le \bar{\delta}$ for $n \ge N_2$. We claim that $m_n < \mu_n + \delta\sigma_n$ for $n \ge \max\{N_1, N_2\}$. Otherwise each term in $\sum_{0}^{\frac{\delta}{2} - \bar{\delta}}$ is smaller than each term in $\sum_{\frac{\delta}{2} - \bar{\delta}}^{\delta} p(n,k)$; also with $\frac{1}{2\sigma_n} \le \bar{\delta}$, the range over which k varies in the second sum is at least one unit longer than

that in the first, so that the first sum contains at most as many terms as the second. These two observations contradict the fact that the first sum is the larger of the two, so indeed $m_n < \mu_n + \delta\sigma_n$ for $n \geq \max\{N_1, N_2\}$. Similarly $m_n > \mu_n - \delta\sigma_n$ for sufficiently large n. The idea here was used previously in Theorem I. In that theorem we had explicit estimates for the rate of convergence in the central limit theorem, and δ was determined as a function of σ_n. In this lemma δ is fixed. In principle, the rate of convergence in the central limit theorem, and the rate at which $\sigma_n \to \infty$ will tell how to choose $\overline{\delta}$ so that N is minimized.

Bender [2] shows that given a central limit theorem for $s(n, k)$, LC is sufficient to derive a local limit theorem for $S = \mathbb{R}$. (See (0.2)) In the following proposition we make estimates which, though overly generous, are explicit enough to indicate how the rate of convergence in (0.2) — local limit theorem — depends on the rate of convergence in (0.1) — central limit theorem — and the rate at which $\sigma_n \to \infty$.

<u>Proposition.</u> Let $\epsilon > 0$ be given; we shall assume $\epsilon < 1/9$. There exist positive numbers

(1) $\rho < 1$ such that $|x| < \rho \Rightarrow |1 - e^x| < \epsilon$,

(2) δ such that $36\,\delta < \rho$,

(3) $\eta < \delta$ such that $|x - y| < 3\eta \Rightarrow |e^{-x^2/2} - e^{-y^2/2}| < \frac{\varepsilon}{4\sqrt{2\pi}}$.

(By uniform continuity of $e^{-t^2/2}$.)

With ρ, δ, and η determined, let N be so large that for all $n \geq N$ we have

(4) $$\left| \sum_{y}^{x} p(n,k) - \eta(y,x) \right| < \frac{\varepsilon\eta}{4\sqrt{2\pi}}$$

(5) $1/\sigma_n < \eta$

(6) $\mu_n - \delta\sigma_n < m_n < \mu_n + \delta\sigma_n$ (previous lemma)

Then for all $n \geq N$ and for all x, we have

$$\left| \sigma_n p(n, [\mu_n + x\sigma_n]) - \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \right| < 5\varepsilon .$$

(Brackets denote greatest integer.)

Proof: We consider $x \geq 0$. Negative x are handled similarly. All inequalities are for $n \geq N$. First we suppose that $x > \delta + 2\eta$. Let $\bar{x}$ be such that $[\mu_n + x\sigma_n] = \mu_n + \bar{x}\sigma_n$. Choose y, depending on n, such that $\mu_n + y\sigma_n$ is an integer and $3\eta > x - y > \bar{x} - y > \eta$; this is possible by (5), and it will be the case that $y > \delta$. From (4),

$$\left| \sum_{y}^{\bar{x}} p(n,k) - \eta(y,\bar{x}) \right| < \frac{\varepsilon\eta}{4\sqrt{2\pi}} ,$$

it follows that

$$(\bar{x} - y)\, \sigma_n\, p(n, [\mu_n + x\sigma_n]) < \frac{1}{\sqrt{2\pi}} \int_y^{\bar{x}} e^{-t^2/2} dt + \frac{\epsilon\eta}{4\sqrt{2\pi}} \quad ,$$

$(\bar{x} - y)\sigma_n$ being the number of terms in $\sum_y^{\bar{x}} p(n, k)$, all of which are at least $p(n, [\mu_n + x\sigma_n])$ by (6). Then

$$\sigma_n\, p(n, [\mu_n + x\sigma_n]) < \frac{1}{\sqrt{2\pi}} e^{-y^2/2} + \frac{\epsilon}{4\sqrt{2\pi}}, \text{ since } \bar{x} - y > \eta ,$$

$$< \frac{1}{\sqrt{2\pi}} e^{-x^2/2} + \frac{\epsilon}{2\sqrt{2\pi}} \quad \text{by (3).}$$

Likewise we may see that $\sigma_n\, p(n, [\mu_n + x\sigma_n]) > \frac{1}{\sqrt{2\pi}} e^{-x^2/2} - \frac{\epsilon}{2\sqrt{2\pi}}$ by letting $\bar{x}$ be such that $[\mu_n + x\sigma_n] - 1 = \mu_n + \bar{x}\sigma_n$, and y be such that $3\eta > y - \bar{x} > y - x > \eta$, then considering $\sum_x^y p(n, k)$ much as above. Next we suppose that $0 \le x \le \delta + 2\eta$. Let $k = [\mu_n + x\sigma_n]$, and choose ℓ so that $\frac{k + \ell - \mu}{\sigma} > 3\delta$. This may be done with $\ell/\sigma < 5\delta$. Let $y = \frac{k + \ell - \mu}{\sigma}$ and $z = \frac{k + 2\ell - \mu}{\sigma}$. Noting that $3\delta > \delta + 2\eta$ (3), our previous estimates are valid for $p(n, \mu_n + y\sigma_n)$ and $p(n, \mu_n + z\sigma_n)$. Hence, by LC,

$$\sigma_n p_n(k) \le \sigma_n p_n(k+\ell) \cdot \frac{\sigma_n p_n(k+\ell)}{\sigma_n p_n(k+2\ell)}$$

$$\le \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \frac{e^{-y^2/2}}{e^{-z^2/2}} \frac{(1+\epsilon)^2}{(1-\epsilon)} ,$$

as

$$\frac{1}{\sqrt{2\pi}} e^{-z^2/2} - \frac{\epsilon}{2\sqrt{2\pi}} = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\left[1 - \frac{\epsilon}{2} e^{z^2/2}\right] \ge \frac{1}{\sqrt{2\pi}} e^{-z^2/2}[1-\epsilon],$$

since $|z| \le x + 2\ell/\sigma \le 13\delta < \rho$, so that $e^{z^2/2} < 1+\epsilon < 2$ by (1).

Similarly,

$$\frac{1}{\sqrt{2\pi}} e^{-y^2/2} + \frac{\epsilon}{2\sqrt{2\pi}} \le \frac{1}{\sqrt{2\pi}} e^{-y^2/2} [1+\epsilon] .$$

Now,

$$\frac{z^2}{2} - y^2 = -\frac{1}{2}\left(\frac{k-\mu}{\sigma}\right)^2 + \frac{\ell^2}{\sigma^2} < -\frac{1}{2} x^2 + \frac{x}{\sigma} - \frac{1}{2\sigma^2} + \frac{\ell^2}{\sigma^2} ,$$

since $\frac{k-\mu}{\sigma} > x - 1/\sigma$. So,

$$p(n,k) \le \frac{1}{\sqrt{2\pi}} e^{-x^2/2} e^{x/\sigma + \frac{\ell^2 - 1/2}{\sigma^2}} \cdot \frac{(1+\epsilon)^2}{1-\epsilon}$$

$$\le \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \cdot \frac{(1+\epsilon)^3}{1-\epsilon} ,$$

since $x/\sigma + \ell^2/\sigma^2 < 3\delta + 25\delta^2 < \rho$ and (1) is applicable,

$$< \frac{1}{\sqrt{2\pi}} e^{-x^2/2} + 5\varepsilon \text{ , using } \varepsilon < 1/9.$$

Finally, $\sigma_n p(n,k) \geq \sigma_n p(n, k \pm \ell)$, one or the other, where we assume $\left|\frac{k \pm \ell - \mu}{\sigma}\right| > 3\delta$, so that earlier estimates are applicable. This can be done with $\ell/\sigma < 6\delta$. Then,

$$\sigma_n p(n,k) \geq \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\cdot\left(\frac{k \pm \ell - \mu}{\sigma}\right)^2} - \varepsilon$$

$$\geq \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} - \varepsilon + \frac{e^{-x^2/2}}{\sqrt{2\pi}} \left\{ e^{\mp\frac{\ell}{\sigma}\cdot\frac{k-\mu}{\sigma} - \frac{\ell^2}{2\sigma^2}} - 1 \right\},$$

as $\left(\frac{k \pm \ell - \mu}{\sigma}\right)^2 \leq x^2 \pm 2\frac{\ell}{\sigma}\cdot\frac{k-\mu}{\sigma} + \frac{\ell^2}{\sigma^2}$, since $\frac{k-\mu}{\sigma} \leq x$.

Therefore $\sigma_n p_n(k) \geq \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} - 2\varepsilon$, since

$$\left|\frac{\ell}{\sigma}\right| \cdot \left|\frac{k-\mu}{\sigma}\right| + \left|\frac{\ell^2}{2\sigma^2}\right| < 6\delta \cdot 3\delta + 18\delta^2 < \rho ,$$

so (1) is again applicable. This completes the proof of the proposition.

3. Asymptotic formulae. Given $g(u)$, we have employed $A(u) = ug'(u)$, $B(u) = uA'(u)$, the inverse function $r(x)$ where $A(r(x)) \equiv x$, $\sigma^2 = g - A^2/B$, $\sigma_n = \sigma(r(n))$, and $\mu_n = g(r(n))$. More generally, let $\sigma_n(\tau) = \sigma\left(r\left(\frac{n}{\tau}\right)\right)$ and $\mu_n(\tau) = g\left(r\left(\frac{n}{\tau}\right)\right)$, where $\tau > 0$.

<u>Theorem V</u>. Let $g(u)$ have coefficients which satisfy the central limit theorem (II. 4), and which are log concave. Let k_n be a sequence such that $k_n - \mu_n(\tau) = O(\sigma_n(\tau))$ as $n \to \infty$. Then

$$\text{(III. 2)} \qquad s(n, k_n) \sim \frac{P_n(\tau)\, e^{-x^2/2}}{\sigma_n(\tau)\, \tau^{k_n} \sqrt{2\pi}}, \text{ as } n \to \infty,$$

where $k_n = \mu_n(\tau) + x\sigma_n(\tau)$. Alternatively, let k_n determine numbers τ_n by

$$k_n = g\left(r\left(\frac{n}{\tau_n}\right)\right).$$

If $\tau_n \to \tau$, then

$$\text{(III. 3)} \qquad s(n, k_n) \sim \frac{P_n(\tau_n)}{\sigma_n(\tau_n)\, \tau_n^{k_n} \sqrt{2\pi}}, \text{ as } n \to \infty.$$

Proof: It is readily seen that like $g(u)$, $\tau g(u)$ will satisfy (II. 4). Since the coefficients of $\tau g(u)$ are LC, $\tau g(u)$ will satisfy the local limit theorem (II. 5). As remarked in Section II, part 1, this yields equations (II. 6-10). Hence Theorem III applies, with mean $g\left(r\left(\frac{n}{\tau}\right)\right)$

and variance $\sigma\left(r\left(\frac{n}{\tau}\right)\right)^2$. The results of parts 1 and 2 apply to give a local theorem for $s(n,k)\,\tau^k$ — the coefficients generated by $\tau\, g(u)$ — and formula (III. 2) follows from the local theorem by the $k_n - \mu_n(\tau) = O(\sigma_n(\tau))$ assumption.

Version (III. 3) also follows by Theorem III. Letting $\alpha = e^{t/\sigma_n(\tau_n)}$, we show that

$$\frac{e^{-tk_n/\sigma_n(\tau_n)}\, P_n(\tau_n \alpha)}{P_n(\tau_n)} \to e^{t^2/2} \quad ,$$

as in Theorem III. For the assumption that $\tau_n \to \tau$ permits the second lemma in the proof of that theorem to hold and show $g\left(r\left(\frac{n}{\alpha\tau_n}\right)\right) \sim g\left(r\left(\frac{n}{\tau_n}\right)\right)$. Then,

$$\frac{P_n(\tau_n \alpha)}{P_n(\tau_n)} \sim \exp\left\{ \int_{\tau_n}^{\tau_n \alpha} g\left(r\left(\frac{n}{\beta}\right)\right) d\beta \right\} ,$$

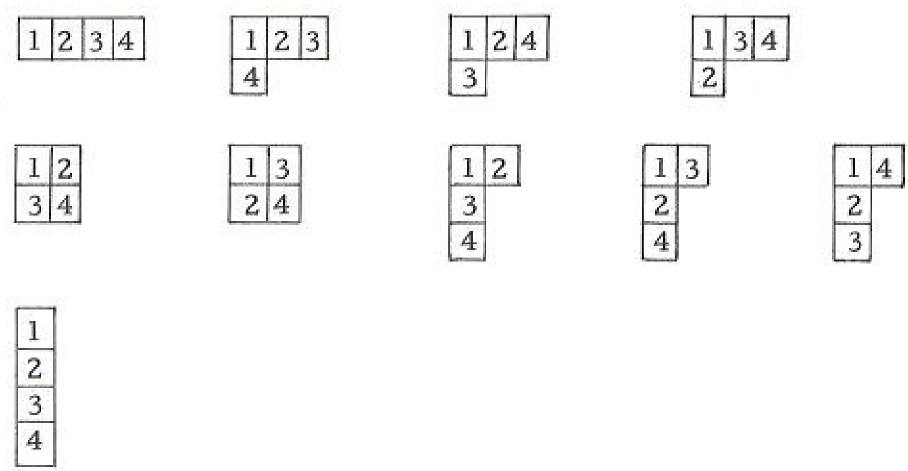
and the rest of the proof goes as before. Hence $s(n,k)\,\tau_n^k$ are asymptotically normal with mean k_n and variance $\sigma\left(r\left(\frac{n}{\tau_n}\right)\right)^2$. Formula (III. 3) is then the local limit theorem used to estimate $s(n,k)$ for k = mean.

<u>Remark</u>. We note that formulae (III. 2, 3) allow us to estimate $s(n,k)$ purely in terms of expressions derived from the generating function $g(u)$. For in (III. 2) only $P_n(\tau)$, and in (III. 3) $P_n(\tau_n)$, are not so

expressed already. However, both of these quantities may be estimated by the formula of page 32:

$$\frac{P_n(\alpha)}{n!} \sim \frac{\exp\left\{\alpha \cdot g\left(r\left(\frac{n}{\alpha}\right)\right)\right\}}{\left[r\left(\frac{n}{\alpha}\right)\right]^n \sqrt{2\pi\,\alpha\;B\left(r\left(\frac{n}{\alpha}\right)\right)}} \quad .$$

Example. Tableaux. A tableaux on the elements $\{1, 2, \ldots, n\}$ is an arrangement of these elements in rows of lengths $p_1 \geq p_2 \geq \cdots \geq p_k > 0$, where $p_1 + \cdots + p_k = n$, in such a way that all rows and columns are in increasing order from left to right and top to bottom respectively. Here, for example, are the tableaux on 4 elements:



Of course the number of tableaux on n elments does not depend on the fact that the elements are $\{1, 2, \ldots, n\}$, only that they are distinct. We take for our connected objects on an n-element set the n-celled labeled tableaux. Knuth [14], chapt. 5, shows that the number g_n of tableaux on $\{1, 2, \ldots, n\}$ is the same as the number of involutions on $\{1, 2, \ldots, n\}$, using a remarkable correspondence. (An involution is a permutation whose second power is the identity.) Consequently,

$$g(u) = \sum_{1}^{\infty} \frac{g_n}{u!} u^n = \exp\{u + u^2/2\} - 1 \quad .$$

This is an admissible function (see Section II, part 1) so that Theorem III applies to give a central limit theorem for $s(n, k)$ = the number of ways to partition $\{1, 2, \ldots, n\}$ into k non-empty blocks and construct a tableau on each block. To use the results of this section, we establish the LC of the coefficients $c_n = \frac{g_n}{n!}$. For this

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we take $g_0 = 1$ for convenience, as this does not affect the LC property. Then, since $(1 + u) e^{u+u^2/2} = \frac{d}{du} e^{u+u^2/2}$, we have the relation

$$nc_n = c_{n-1} + c_{n-2}, \quad c_0 = 1 = c_1 \quad .$$

After checking the first few values of c_n for LC, we proceed by induction:

$$c_n^2 = \frac{c_{n-1} + 2c_{n-1}c_{n-2} + c_{n-2}^2}{n^2} \geq c_{n-1}\left(\frac{c_{n-1} + 2c_{n-2} + c_{n-3}}{n^2}\right)$$

as we may assume that $c_{n-2}^2 \geq c_{n-1}c_{n-3}$. The verification that $c_n^2 \geq c_{n-1}c_{n+1}$ is then complete provided that

$$c_{n-1} + 2c_{n-2} + c_{n-3} \geq n^2 c_{n+1} \quad .$$

But,

$$n^2 c_{n+1} = \frac{n^2(c_n + c_{n-1})}{n+1}$$

$$= \frac{n(nc_n + (n-1)c_{n-1} + c_{n-1})}{n+1}$$

$$= \frac{n(2c_{n-1} + 2c_{n-2} + c_{n-3})}{n+1} \quad ,$$

and the statement that $(n+1)(c_{n-1} + 2c_{n-2} + c_{n-3}) \geq n(2c_{n-1} + 2c_{n-2} + c_{n-3})$ is algebraically equivalent to $2c_{n-2} + c_{n-3} \geq (n-1)c_{n-1}$, which we know is true since $c_{n-2} + c_{n-3} = (n-1)c_{n-1}$.

<u>A general example</u>. This is a technique which may be useful when the $g_n/n!$ are known to be LC, but no explicit form of $g(u) = \sum_1^\infty \frac{g_n}{n!} u^n$ is available. Such is the case when $g_n = \binom{2n}{n}$, or $g_n = \frac{1}{n+1}\binom{2n}{n}$ - the Catalan numbers. Although simple expressions for the ordinary generating functions of these two sequences are known, it is the exponential generating function which is our $g(u)$. However, for both of these sequences we do have asymptotic estimates of the form $g_n \sim c \cdot n^p \cdot \gamma^n$, where c, p, and γ are fixed. We take this as our starting point in this general example:

(1) $g_n = c \cdot n^p \cdot \gamma^n \left[1 + O\left(\frac{1}{n}\right)\right]$,

and $\frac{g_n}{n!}$ are LC.

We then demonstrate that the function $g(u)$ has coefficients which are asymptotically normal (see 0.7). With the LC assumption this is sufficient information to apply Theorem V. Notice that the asymptotic formula in (1) implies that $\frac{g_{n+1}/(n+1)}{g_n/n!} \to 0$ as $n \to \infty$, so that $g(u)$ is entire. The verification which we desire is not difficult, and uses mainly the known asumptotic normality of the coefficients of e^u:

$$(2) \qquad \sum_{r + y\sqrt{r} < n \leq r + x\sqrt{r}} \frac{r^n}{n!\,e^r} \to \frac{1}{\sqrt{2\pi}} \int_y^x e^{-t^2/2}\,dt, \text{ as } r \to \infty .$$

(For e^u is admissible and the results of Section II, part 1 apply.) First, let x depend on r in such a way that $x \to \infty$ and $x = o(r^{1/6})$, as $r \to \infty$. Then

$$\frac{g_n r^n}{n!} \sim c \cdot (\gamma r)^p \cdot \frac{(\gamma r)^n}{n!}, \text{ uniformly for } \left|\frac{\gamma r - n}{\sqrt{\gamma r}}\right| \leq x .$$

Hence, by (2), since $x \to \infty$,

$$\sum_{\left|\frac{\gamma r - n}{\sqrt{\gamma r}}\right| \leq x} \frac{g_n r^n}{n!} \sim c \cdot (\gamma r)^p \cdot e^{\gamma r}, \text{ as } r \to \infty .$$

We claim that $g(r) \sim c \cdot (\gamma r)^p e^{\gamma r}$ as $r \to \infty$, that is both the left and right "tails" $\left(\left|\frac{\gamma r - n}{\sqrt{\gamma r}}\right| > x\right)$ of the sum $\sum \frac{g_n r^n}{n!}$ are $o((\gamma r)^p e^{\gamma r})$.

By LC the terms $\frac{g_n r^n}{n!}$ are first increasing, then decreasing. At $n = \gamma r - \sqrt{\gamma r}$ they are still increasing, as we may estimate the following ratio and see that it becomes infinite as $r \to \infty$:

$$\frac{\dfrac{g_{\gamma r}\, r^{\gamma r}}{(\gamma r)!}}{\dfrac{g_{\gamma r - x\sqrt{\gamma r}} \cdot r^{\gamma r - x\sqrt{\gamma r}}}{(\gamma r - x\sqrt{\gamma r})!}} \sim \frac{(\gamma r - x\sqrt{\gamma r})!}{(\gamma r)!} \cdot (\gamma r)^{x\sqrt{\gamma r}}, \text{ by (1).}$$

Using Stirling's formula on the factorials, the ratio of the square-root terms being asymptotic to 1,

$$\frac{(\gamma r - x\sqrt{\gamma r})!}{(\gamma r)!}(\gamma r)^{x\sqrt{\gamma r}} \sim \frac{(\gamma r - x\sqrt{\gamma r})^{\gamma r - x\sqrt{\gamma r}}}{(\gamma r)^{\gamma r}} \cdot e^{x\sqrt{\gamma r}} \cdot (\gamma r)^{x\sqrt{\gamma r}}$$

$$= \left(1 - \frac{x}{\sqrt{\gamma r}}\right)^{\gamma r - x\sqrt{\gamma r}} \cdot e^{x\sqrt{\gamma r}}$$

$$\sim e^{x^2/2} \quad ,$$

since

$$\ln\left(1 - \frac{x}{\sqrt{\gamma r}}\right) = \frac{-x}{\sqrt{\gamma r}} - \frac{x^2}{2\gamma r} + O\left(\frac{x^3}{r^{3/2}}\right) \quad ,$$

and

$$(\gamma r - x\sqrt{\gamma r})\left(\frac{-x}{\sqrt{\gamma r}} - \frac{x^2}{2\gamma r} + O\left(\frac{x^3}{r^{3/2}}\right)\right) = -x\sqrt{\gamma r} + \frac{x^2}{2} + o(1),$$

using $x = o(r^{1/6})$. The same argument holds when $-x$ is replaced by $+x$, so the terms $\frac{g_n r^n}{n!}$ are decreasing at $n = \gamma r + x\sqrt{\gamma r}$. Hence we may bound the left-tail by $n \cdot \frac{g_n r^n}{n!}$, where $n = \gamma r - x\sqrt{\gamma r}$. However,

$$n \cdot \frac{g_n r^n}{n!} \sim c \cdot (\gamma r)^{p+1} \frac{(\gamma r)^{\gamma r - x\sqrt{\gamma r}}}{n!}$$

$$\sim \frac{c}{\sqrt{2\pi}} (\gamma r)^{p+\frac{1}{2}} \cdot e^{\gamma r - x^2/2},$$

estimating the $n!$ and arguing as before by expanding a logarithm and using $x = o(r^{1/6})$. Of course $(\gamma r)^{p+\frac{1}{2}} e^{\gamma r - x^2/2} = o((\gamma r)^p e^{\gamma r})$ as $r \to \infty$, so we now turn our attention to the right tail. Here we find, now with $n = \gamma r + x\sqrt{\gamma r}$,

$$\frac{g_n r^n}{n!} \sim \frac{c}{\sqrt{2\pi}} (\gamma r)^{p-\frac{1}{2}} e^{\gamma r - x^2/2} ,$$

exactly as we handled $n = \gamma r - x\sqrt{\gamma r}$. Now the right tail may be bounded by the sum of a geometric series, $\frac{g_n r^n}{n!} \cdot \frac{1}{1-R}$, where

$$R = \frac{g_n r^n / n!}{g_{n-1} r^{n-1}/(n-1)!} = \frac{\gamma r}{n}\left[1 + O\left(\frac{1}{n}\right)\right].$$

Using $n = \gamma r + x\sqrt{\gamma r}$, we check that $n \cdot \left(\frac{1}{R} - 1\right) \to \infty$. Then,

$$\frac{g_n}{n!} r^n \cdot \frac{1}{1-R} \sim \frac{c}{\sqrt{2\pi}} e^{\gamma r - x^2/2} (\gamma r)^{p+\frac{1}{2}} \cdot \frac{1/R}{n\left(\frac{1}{R}-1\right)} = o((\gamma r)^p e^{\gamma r}) ,$$

as $r \to \infty$.

Finally, proceeding as in the estimate of

$$\sum_{\left|\frac{\gamma r - n}{\sqrt{\gamma r}}\right| \leq x} \frac{g_n r^n}{n!} ,$$

with x now a fixed real number, so that there is only a left tail to contend with, we may use (2) to show that

$$\sum_{n \leq \gamma r + x\sqrt{\gamma r}} \frac{g_n r^n}{n!} \sim c \cdot (\gamma r)^p e^{\gamma r} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt ,$$

as $r \to \infty$.

With our estimate of $g(r)$ this establishes (II. 4), the asymptotic normality of the coefficients in $g(u)$. Theorem V is now applicable.

IV. Conclusion

The preceding results may be considered a start in the study of asymptotic properties of the coefficients of polynomials of binomial type; however, numerous areas in need of further research are apparent. We mention a few of the immediate possibilities here.

The estimate of Section I is quite precise, but provides satisfactory numerical information only when σ is large. The Berry-Esséen inequality used there is a general theorem. It may be that for the special case where all summands are 0-1 random variables an improvement is possible. Also we have stated the theorem with x bounded away from 0 by $\delta + \sigma^{-1/2}$. By the technique of Section III, part 2, this restriction may be removed, although the resulting constant is several times larger than 17.74. Again, because of the special nature of the situation, it might be possible to strengthen such a result.

The location of the roots of polynomials of binomial type is a rather broad subject, although some special results in terms of conditions on the generating function $g(u)$ may exist. Log concavity is also referred to as 2-positivity, and Karlin [13] considers more general r-positivity. In particular, chapter 8 deals with locating roots of polynomials whose coefficients are r-positive. The recurrence method of root location as in [3], [11], and two of our Section I examples is definitely of limited use, requiring such a special form of recurrence relation for the $s(n,k)$. The technique of Section I is applicable, however, provided that all roots of $P_n(x)$ lie in the left half-plane, or more generally when $P_n(x)$ factors as a product of sufficiently many polynomials whose coefficients are real and non-negative.

We mention that it is desirable to study $s(n,k)$ defined by such a recurrence as

$$s(n,k) = \sum_{i=1}^{N} \sum_{j=-M}^{+M} f(i,j,n,k)\, s(n-i, k+j) \quad ,$$

with N and M fixed. Suitable restrictions on the coefficient function f may yield asymptotically normal $s(n,k)$, so that generating functions may be by-passed completely.

For central limit theorems for polynomials of binomial type, only the case when $g(u)$ is a polynomial is completely resolved. For power-series $g(u)$ our results typically require asymptotic normality of the coefficients of $g(u)$. Such $g(u)$ does not include, for example, signless Stirling numbers of the first kind, for which $g(u) = \ln\left(\frac{1}{1-u}\right)$. The basic saddle point method of estimating $P_n(\alpha)$ by a Cauchy contour integral in which the main contribution comes from one or few small lengths of arc could very well be used to prove Theorem III for functions $g(u)$ having analytic properties less restrictive than admissibility.

Finally in Section III there is an important desirable feature missing: namely, equation (III. 3) ought to be uniform for $\tau_n \in \left[\varepsilon, \frac{1}{\varepsilon}\right]$, $\varepsilon > 0$. (This would include uniformity for $\tau \in \left[\varepsilon, \frac{1}{\varepsilon}\right]$ of (III. 2).) Although the uniform convergence of moment generating functions

$$\frac{e^{\frac{-t\mu_n(\tau_n)}{\sigma_n(\tau_n)}} P_n(e^{t/\sigma_n(\tau_n)})}{P_n(\tau_n)} \to e^{t^2/2}$$

might follow from our method, still, the proposition on moment generating functions (II. 12) is non-constructive in nature and reveals nothing about the rate of weak convergence $m_n(\tau_n) \to \eta$.

Glossary of Frequently Used Notations

$P_n(x)$	A sequence of polynomials of binomial type
$s(n, k)$	The coefficients of $P_n(x) = \sum_k s(n,k)x^k$
$g(u)$	The generating function for $P_n(x)$; $\sum_0^\infty P_n(x)\frac{u^n}{n!} = \exp\{xg(u)\}$
g_n	The number of "connected" objects constructable on a labeled n-set. $g(u)$ is the exponential generating function of $\{g_n\}$: $g(u) = \sum_1^\infty g_n \cdot \frac{u^n}{n!}$
$c_n = \frac{g_n}{n!}$	The coefficients of $g(u)$
$a(r) = r \cdot \frac{g'(r)}{g(r)}$	The expected value of X_r, where $\text{Prob}\{X_r = n\} = \frac{c_n r^n}{g(r)}$
$b(r) = r \cdot a'(r)$	The variance of X_r (see previous)
$A(r) = r\,g'(r)$	This has the same meaning for $\exp\{g(r)\}$ that $a(r)$ has for $g(r)$
$B(r) = rA'(r)$	This has the same meaning for $\exp\{g(r)\}$ that $b(r)$ has for $g(r)$
$r(x)$	Inverse of $A(r)$: $A(r(x)) \equiv x$

$$\sigma^2(r) = g(r) - \frac{A(r)^2}{B(r)}$$

$$\sigma_n = \sigma(r(n))$$

$$\mu_n = g(r(n))$$

$$\alpha = e^{t/\sigma_n}$$

$$F(\beta) = g\left(r\left(\frac{n}{\beta}\right)\right)$$

References

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