From Thurs 10/20. Let \( t_n \) be the number of rooted, unlabelled trees with \( n \) vertices. We found by hand the first four values: 1, 1, 2, 4. Then by combinatorial reasoning we found

\[
\sum_{n=0}^{\infty} t_n x^n = x \prod_{j=1}^{\infty} \left( \frac{1}{1 - x^j} \right)^{t_j}.
\]

We observed that the latter equation allows one to determine \( t_{n+1} \) once \( t_1, t_2, \ldots, t_n \) are known. We illustrated this by computing \( t_5 = 9 \), and then checking by finding nine trees with 5 vertices.

From Tues 10/25. By putting the product into exponential format, taking a derivative, and algebraic manipulation, we derived a recursion for \( t_n \). The auxiliary sequence

\[
T_n \overset{\text{def}}{=} \sum_{d|n} d t_d
\]

is needed, and the recursion is

\[
t_{n+1} = \frac{1}{n} \sum_{j=1}^{n} T_j t_{n+1-j}.
\]

NOTE. This \( T_n \) is slightly different from the \( T_n \) which showed up in class Tuesday. Such a recursion allows one to perform daring mathematical stunts such as:

\[
\begin{array}{c|cccccccc}
 n & 1 & 2 & 3 & 4 & 5 & 6 & 10 & 15 & 20 \\
 t_n & 1 & 1 & 2 & 4 & 9 & ? & 719 & 87811 & 12826228 \\
\end{array}
\]

New Reading Rest of Chapter 7, Sections 4 through 6. Problems to know for the final (you do not have to turn them in): 17–30, 48.

Homework Due Thursday, Nov 3.

#1. Determine the missing value \( t_6 \) in the above table.

#2. A certain sequence \( r_n, n \geq 1 \) has an EGF

\[
R(x) \overset{\text{def}}{=} \sum_{n=1}^{\infty} \frac{r_n x^n}{n!}
\]

which satisfies the equation

\[
R(x) = x \exp(R(x)).
\]
Take a derivative with respect to $x$, and perform algebra to derive:

$$(n - 1)r_n = \sum_{j=1}^{n-1} \binom{n}{j} jr_j r_{n-j}, \quad n \geq 2$$  \hspace{1cm} (2)

**#3.** Use equation (2) above to obtain a recursion expressing $r_{n+1}$ in terms of $r_1, r_2, \ldots, r_n$. Use your recursion to supply the three missing numbers in this table:

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_n$</td>
<td>1</td>
<td>2</td>
<td>9</td>
<td>?</td>
<td>?</td>
<td>?</td>
</tr>
</tbody>
</table>

Guess the general formula for $r_n$ by looking at the data.

**#4.** Let $\pi_n, n \geq 0$ be defined by the ogf

$$\sum_{n=0}^{\infty} \pi_n x^n = \prod_{j=1}^{\infty} \left( \frac{1}{1 - x^j} \right)^j.$$

Follow the process used on Tuesday with $t_n$ — replace product w/ an exponential; take the derivative w/ respect to $x$, multiply by $x$, extract the coefficient of $x^{n+1}$ from both sides – to get a recursion of the form:

$$(n + 1)\pi_{n+1} = \sum_{j=1}^{n+1} \pi_{n+1-j} P_j.$$

Supply the missing part in the defn of $P_j$:

$$P_j = \sum_{d|j} \ldots.$$

Use the recursion to fill in the missing values:

<table>
<thead>
<tr>
<th>$n$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi_n$</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>6</td>
<td>?</td>
<td>?</td>
<td>?</td>
</tr>
</tbody>
</table>

**#5.** 3 can be written as a 2-dimensional monotone decreasing sum in 6 ways:

$$\{ 3 \}, \{ 2 1 \}, \{ 1 1 1 \}, \{ 2 \}, \{ 1 1 \}, \{ 1 \}.$$

By “monotone decreasing” we mean that in each matrix the summands are weakly decreasing along rows and columns. (Treat missing summands as 0.) In how many ways may 4 be so written?

**Grad 1.** p. 260, #26

**Grad 2.** p. 261, #28