Instructions

Closed book, no calculators. You may leave answers in unevaluated form, in terms of binomial coefficients, factorials, falling factorials, summations, etc.

1. State and prove a formula for the number of \( k \)-tuples of non-negative integers, 
\[(x_1, \ldots, x_k), \quad x_i \geq 0,\]
such that \( \sum_{i=1}^{k} x_i = n \).

2. Twenty different books are to be put on five book shelves, each of which holds at least twenty books.
   (a) How many different arrangements are there if you only care about the number of books on the shelves (and not which book is where)?
   (b) How many different arrangements are there if you care about which books are where, but the order of the books on the shelves doesn’t matter?
   (c) How many different arrangements are there if the order of the books on the shelves does matter?

3. Give a combinatorial proof of the identity
\[\sum_{k=2}^{n} (k-1)(n+1-k) = \binom{n+1}{3}.\]

4. Suppose that \( \{A_1, A_2, \ldots, A_b\} \) is a Steiner triple system with parameter \( \lambda = 1 \), based on the ground set \( G = \{1, 2, \ldots, v\} \). This means that each \( A_i \) is a 3-element subset of \( G \), and every pair \( i, j \ (i \neq j; i, j \in G) \) is contained in exactly one of the \( A_i \).
   (a) Prove that \( v \) must be odd.
   (b) Give a formula for \( b \), the number of blocks.
   (c) Prove that \( \binom{v}{2} \) must be divisible by 3.
   (d) True or False: there exists such a system when \( |G| = v = 600 \).

5. (a) Define a sequence of numbers \( C_n \) using the recursion
\[C_{n+1} = \sum_{j=0}^{n} C_j C_{n-j}, \quad C_0 = 1\]
The first few of these numbers, the Catalan numbers, are 1, 1, 2, 5, 14. Give a simple closed formula for these numbers.
   (b) Given: 2\( n \) (labeled) equally spaced points on a circle. Prove that the number of ways to connect them in pairs by chords with no two chords intersecting is \( C_n \).

6. By “inspection” write down ordinary generating functions for the following sequences:
   (a) \( a_n = \) the number of partitions of \( n \) into odd parts.
(b) $a_n$ = the number of partitions of $n$ into distinct parts.
(c) $a_n$ = the number of partitions of $n$ in which no part is repeated more than three times.

**Graduate Students**

**G1.** (a) How many integers in the set \{1, 2, \ldots, 1000\} are divisible by at least one of 2, 3, or 5?
   (b) Determine the number of permutations of the string

   $AABBCCCDDEEEE$.

**G2.** Determine the number of strings of $n$ digits (a digit is a number 0 through 9) having exactly three 0s, none of which are consecutive. (So, for example, you cannot have 134005609.)