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Spring, 2013
Tuesday, April 9. Notes, Homework 13

Handout 13
CSCI/MATH 4690/6690

Reading
April 9 : Section 7.4
April 11 : Sections 7.5 and 7.6
April 16 : Section 7.7
April 18 : Sections 7.8 and 7.9

Homework #13. due Thurs, 4/18/2013: Chapter 7:
Exercises 31, 32, 34*, 36, 37.
Graduate exercise 28*.

Notes
* 28 This is 20 points, but can be divided in two, with 10 points for showing that 1 \iff 2, and 10 points for showing 1 \iff 3 in Thm. 7.59.

* 34 Should read "... in a simple graph with ...". Also, it should read "Find an upper bound for the number of edges ...", since finding the actual maximum is unreasonably difficult. (we should discuss in class)

Twelfth week summary

Tue 4/2. Review day. We passed out handout 12. We discussed the Petersen graph: is it planar; does it contain a Hamiltonian cycle; does it contain a Hamiltonian path. Audience request topics included: observation 7.2, problem 6.33, the \infty-face. It was pointed out that we have a condition for proving that a graph is not Hamiltonian: if you can delete \( k \) vertices and get more than \( k \) components, then the graph is not Hamiltonian.

Thu 4/4. We took Test 2. Here are the solutions:
\[
\begin{array}{ccc}
1 & c & 6 \ b \\
2 & d & 7 \ b \\
3 & c & 8 \ c \\
4 & T & 9 \ c \\
5 & d & 10 \ c \\
\end{array}
\]

**Part II. Short Answers**

1. For any set \( S \) which includes \( s \) and excludes \( t \) we have

\[
\text{val}(f) = f^+(S) - f^-(S).
\]  

(1)

Let \( e \) be an edge which originates in the given \( S \) and terminates in \( \bar{S} \). We must have \( f(e) = c(e) \), else the endpoint of \( e \) would have been part of the set \( S \). So,

\[
f^+(S) = c^+(S).
\]  

(2)

Now let \( e \) be an edge which originates in the complement \( \bar{S} \) and terminates in \( S \). We must have \( f(e) = 0 \), else the origin of \( e \) would have been part of the set \( S \). So,

\[
f^-(S) = 0.
\]  

(3)

Relations (1), (2), and (3) together prove

\[
\text{val}(f) = c^+(S).
\]

2. The assertion is clearly true for \( n = 1, 2 \), so we proceed by induction. Let \( G \) be a tournament digraph with \( n \geq 3 \) vertices \( u_1, u_2, \ldots, u_n \). By induction, we have

\[
P: \quad u_1 \to u_2 \to \cdots \to u_{n-1},
\]

a directed Hamilton path in the smaller digraph obtained by omitting vertex \( u_n \). If \((u_n, u_1) \in E\), then \( u_n \) may be attached to the beginning of
$P$ to obtain the desired path in $\tilde{G}$. Similarly, if $(u_{n-1}, u_n) \in E$, then $u_n$ may be attached to the end of $P$ to obtain the desired path in $\tilde{G}$. So, let us assume the worst, that $(u_1, u_n) \in E$ and $(u_n, u_{n-1}) \in E$. If we take the smallest $i$ such that $(u_n, u_i) \in E$, we must have $1 < i \leq n - 1$. Then, the edge $(u_{i-1}, u_i)$ in $P$ can be replaced by $(u_{i-1}, u_n, u_i)$ to produce in this final case the desired path in $\tilde{G}$.

3. Here is the solution,

4. Since we always have the inequalities

$$\kappa(G) \leq \kappa'(G) \leq \delta(G),$$

it suffices to find $G$ such that $\kappa(G) = \delta(G)$. An example of such a graph is $H_{n,k}$, the Harary graph. This graph has $n$ vertices arranged in a circlce. Each vertex is joined by an edge to its nearest $k/2$ neighbors in both the clockwise and counterclockwise direction.

5. Let $f_i$ be the number of faces having degree $i$. Since the graph has $n \geq 3$ vertices and is simple, we have $f_1 = f_2 = 0$. Then,

$$e = \frac{1}{2} \sum_{i \geq 3} if_i \geq \frac{3}{2} \sum_{i \geq 3} f_i = \frac{3f}{2}.$$
and then \[ 3v - 3e + 2e \geq 6 \]

which gives \[ e \leq 3v - 6, \]

as was to be shown.