I. Agenda for the rest of the semester
Thur 4/7 Thm 6.57==Menger, vertex version; Phillip Hall’s Theorem
Theorem. Let \( u \) and \( v \) be non-adjacent vertices in a [connected] graph \( G \). Then the maximum number of internally disjoint \( u - v \) paths in \( G \) equals the minimum number of vertices needed to separate \( u \) and \( v \).

Tue 4/12 Thm 7.14==Euler’s Formula; Corollary 7.15; Platonic solids
Thur 4/14 Thm 7.26==Kuratowski
Tue 4/19 Thm 8.19, Thm 8.22==Brooks
Thur 4/21 Thm 8.28 (Five Color Theorem), Thm 8.29 (Four Color Theorem)
Tue 4/26 Thm 8.43==Vizing
Thur 4/28 Review for final exam

II. Reading for the rest of the semester
Tue 4/12 Sections 7.1 through 7.3
Thur 4/14 Section 7.4
Tue 4/19 Sections 8.1 and 8.3
Thur 4/21 Section 8.4
Tue 4/26 Section 8.5

III. Homework #10
Due date: Thur 4/14

1. (Menger in the Abstract) Let \( S \) be a set, and let
\[
\mathcal{A} = \{A_1, \ldots, A_\ell\}
\]
be a collection of nonempty subsets of \( S \). That is,
\[
\emptyset \neq A_i \subseteq S, \text{ for } 1 \leq i \leq \ell.
\]
A subcollection \( \mathcal{B} \) of \( \mathcal{A} \) is called non-overlapping if
\[
A_i, A_j \in \mathcal{B} \text{ and } A_i \neq A_j \implies A_i \cap A_j = \emptyset.
\]
In words, no two distinct sets in \( \mathcal{B} \) have any elements in common. Let \( M(\mathcal{A}) \) be the maximum size of a non-overlapping subcollection of \( \mathcal{A} \).
Determine \( M(\mathcal{A}) \) when \( S = \{a, b, c, d, e, f\} \) and \( \mathcal{A} \) consists of these six subsets:
\[
\{a, b, c\}, \{c, d, e\}, \{b, d, f\}, \{a, e, f\}, \{c, f\}, \{b, e\}, \{a, d\}
\]

2. (Blocker) Again \( \mathcal{A} \) is a collection of nonempty sets. A set \( B \) is a blocker for \( \mathcal{A} \) if
\[
A_i \in \mathcal{A} \implies A_i \cap B \neq \emptyset.
\]
In words, $B$ has nonempty intersection with every set in the collection $A$. Let $m(A)$ be the minimum size of a blocker for $A$.

Determine $m(A)$ when $A$ is the family listed in (*) above.

3. **(A Relation)** What inequality holds between $M(A)$ and $m(A)$ whenever $S$ and $A$ are finite? Prove your answer.

4. **(Strict)** We want to find an example demonstrating that the inequality you stated in #3 can be strict. (a) Show that there is no such example when $|A| \leq 2$. (b) Find such an example with $|A| = 3$.

5. **(Menger in the Concrete)** Explain how the version of Menger’s theorem presented in class 4/7 involves an $(S,A)$ pair. Specifically, let $G$ be a simple connected graph, and $u,v$ two nonadjacent vertices in $G$. Identify the set $S$ and the family $A$ which Menger’s theorem treats. Since the theorem asserts that for $(S,A)$ arising from $(G,u,v)$ we DO have

$$M(A) = m(A),$$

it must be the case that your example in #4 could never arise from the graphical situation; confirm this.

6. **(Graduate or bonus #1)** Exercise 6.15. According to the errata, we are to assume $\Delta \geq 2$ and $n \geq \Delta + 1$. Is there any graph that fails to satisfy the second of these?

7. **(Graduate or bonus #2)** Exercise 6.16.

**IV. Test 2, solutions**

1. **(a)** Apply Kruskal’s algorithm (for finding the minimum cost spanning tree) to the weighted graph pictured later.

ANSWER: All three edges of weight 1, three additional edges of weight 2. Several solutions valid.

(b) Let $G$ be connected and $F \subseteq E(G)$ a subset of edges which is cycle-free. Prove there is a spanning tree of $G$ that includes all the edges of $F$.

PROOF: Assign the small weight to each edge belonging to $F$, and the large weight to the other edges. Since $F$ is cycle free, Kruskal will greedily keep all the low-weight edges, and then extend them to a spanning tree.

2. **(a)** Let $d_1^+ \leq d_2^+ \leq \cdots \leq d_n^+$ be the sorted sequence of outdegrees of a tournament on $n$ players. Prove that for $1 \leq k \leq n$

$$\sum_{i=1}^{k} d_i^+ \geq \binom{k}{2}.$$

PROOF: Let $v_1, \ldots, v_k$ be distinct vertices such that $d^+(v_i) = d_i^+$. The players $v_1, \ldots, v_k$ play $\binom{k}{2}$ matches among themselves, and these matches alone will contribute $\binom{k}{2}$ to $\sum_{i=1}^{k} d_i^+$

(b) Let $A \subseteq \{1, 2, \ldots, n\}$ be a subset of the integers 1 to $n$. Give a formula for the number of tournaments on players $\{1, 2, \ldots, n\}$ that are transitive on the subset $A$.

ANSWER: $k! 2^{\binom{n}{2} - \binom{k}{2}}$, where $k = |A|$.
(c) State, or describe as best you remember, the inequality that was proven using part (b). (Hint: Try to include in your answer the definition of \(\nu(n)\).)

**ANSWER:** Let \(\nu(n)\) equal the largest integer such that every tournament on \(n\) players contains a transitive sub-tournament on \(\nu(n)\) players. Then,

\[
\nu(n) \leq 1 + \lceil \log_2 n \rceil.
\]

3. (a) Complete the statement of the following theorem, which gives a necessary condition for a graph \(G\) to be Hamiltonian:

**Theorem.** If \(G\) is a simple Hamiltonian graph, then for each \(S \subseteq V(G)\), the number of components of \(G - S\) is at most \(\_ceil\rceil\).

**ANSWER:** fill in blank with \(|S|\).

(b) Show that the bipartite graph \(K_{5,6}\) is not Hamiltonian. (Hint: use part (a).)

**ANSWER:** Remove the 5 vertices in the 5-vertex part of \(K_{5,6}\); what is left behind is six isolated points.

(c) State a necessary and sufficient condition (involving the degrees of the vertices) for a graph \(G\) to be Eulerian, and use it to show that \(K_{5,6}\) is not Eulerian.

**ANSWER:** The graph is connected, and all degrees are even. \(K_{5,6}\) fails this condition since it has six vertices of odd degree 5.

4. Complete the statement of the following theorem, due to Dirac, which gives a sufficient condition for a graph \(G\) to be Hamiltonian:

**Theorem.** Let \(G\) be a simple graph with \(n \geq 3\) vertices. If for every \(u \in V(G)\) we have \(d_G(u) \geq \_ceil\rceil\), then \(G\) is Hamiltonian.

**ANSWER:** fill in blank with \(n/2\).

(b) Let \(G\) be a simple graph with \(n \geq 4\) vertices. Suppose that for all triples of distinct vertices \(u, v, w \in V(G)\), the induced subgraph \(G[\{u, v, w\}]\) has at least two edges. Show that \(G\) is Hamiltonian.

**PROOF:** Let \(v\) be a vertex, and let \(u_1, u_2, \ldots, u_{n-1}\) be the other vertices. If there are two different vertices \(u_i, u_j\) such that \(v\) is not jointed to either of them, then the triple \((v, u_i, u_j)\) spans at most one edge, contradiction. Hence, every vertex is joined to at least \(n - 2\) others. Since \(n \geq 4\), we have \(n - 2 \geq n/2\) and Dirac’s condition for a Hamilton cycle is met.

5. Suppose that \(G\) is \(k\)-connected. (There is a review of the definition of connectivity later.) Suppose that \(H\) is obtained from \(G\) by adding a new vertex \(v\) and attaching it to \(k\) distinct vertices in \(G\). Show that \(H\) is \(k\)-connected.

**PROOF:** We must show that if \(k - 1\) vertices are removed from \(H\), the resulting graph is still connected. Let \(u_1, u_2\) be two vertices remaining after \(k - 1\) vertices have been removed. If neither of \(u_1\) nor \(u_2\) equals the special vertex \(v\), then there is a path from \(u_1\) to \(u_2\) using edges in \(E(G)\) because \(G\) is \(k\)-connected. On the other hand, if one of \(u_1\) or \(u_2\) equals the special vertex \(v\), say \(u_1 = v\) for sake of argument, then at least one of the \(k\) vertices which \(v\) is attached to will remain after the deletion. Call this vertex \(u\). Then \(v\) is joined to \(u\) by an edge, and \(u\) is joined to \(u_2\) by a path using edges from \(E(G)\) because \(G\) is \(k\)-connected. Together, these constitute a path from \(v = u_1\) to \(u_2\).

6. (Graduate and/or bonus.) Prove that the product grid graph \(5 \times 5\) (see picture) is not Hamiltonian. (Hint: identity the vertices as pairs \((x, y)\) with \(x\) and \(y\) integers; consider
the parity of $x + y$.)

PROOF: Let the twenty-five points in the graph be denoted $(i, j)$ where $1 \leq i, j \leq 5$, as suggested by the hint. If $(i, j)$ and $(i', j')$ are joined by an edge, then either $|i - i'| = 1$ or $|j - j'| = 1$, but not both. Hence, traveling along an edge from $(i, j)$ to $(i', j')$ causes the parity of $i + j$ to change by 1. Since the graph has twenty-five points, a would-be Hamilton cycle will have twenty-five edges. Traveling across these twenty-five (odd number) edges will end at a vertex whose parity is different from the starting point. Hence, it is impossible to traverse twenty-five edges and have a Hamilton cycle.

Generalize your finding to formulate a conjecture or theorem about which product grid graphs $m \times n$ are Hamiltonian.

THEOREM: The product grid graph $m \times n$ is Hamiltonian iff $m, n \geq 2$, and the product $mn$ is even.

PROOF: If either $m$ or $n$ is one, it is easy to see there is no Hamilton cycle. If the product $mn$ is odd, the proof from part (a) generalizes. Let $m$, say, be even. Picture the graphs as $m$ rows, each with $n$ vertices. Starting at the lower left corner, proceed to the right through all $n$ vertices, and then up to the second row. Now proceed left on the second row, but through only $n - 1$ vertices. Now up to the third row. Etc. Zigzagging back and forth. When you reach the top row, since $m$ is even, you will return to the left. Now return through all $n$ vertices, and use the leftmost column, which we have been avoiding, to come straight back down to the starting point.