Theorem 1. Let $G = (V, E, \phi)$ be a graph. Then,

$$\sum_{v \in V} d_G(v) = 2|E|.$$ 

Definition. A partition of a set $S$ is a set $\Pi$ of subsets of $S$, called blocks, with these properties:

- $X \in \Pi \Rightarrow X \neq \emptyset$
- $X, Y \in \Pi \Rightarrow X = Y$ or $X \cap Y = \emptyset$
- $S = \bigcup_{X \in \Pi} X$.

Theorem 2. Let $\sim$ be an equivalence relation on set $S$; then the equivalence classes form a partition of $S$. Conversely, given a partition $\Pi$ of the set $S$, there is an equivalence relation for which the blocks of $\Pi$ are the equivalence classes.

Definition. A simple graph $G'$ is said to be a minor of a simple graph $G$, denoted $G' \prec_m G$, provided there is a partition $\Pi$ of $V(G)$ such that $V(G') = \Pi$, and two blocks $X, Y \in \Pi$ are joined by an edge of $G'$ iff there is an edge $e \in E(G)$ connecting some vertex $v \in X$ with some vertex $w \in Y$.

Theorem 3. The relation $\prec_m$ is a partial order.

Theorem 4. Among any infinite set of simple graphs at least two are related by $\prec_m$.

Theorem 5. Let $G = (V, E, \phi)$ be a graph. Then there is a unique partition $\Pi$ of $V$, $\Pi = \{V_1, \ldots, V_k\}$, with these properties:

- $V = \bigcup_{i=1}^k V_i$, $E = \bigcup_{i=1}^k E(G[V_i])$
- each graph $G[V_i]$ is connected
- if $G'$ is a connected subgraph of $G$, then $G'$ is a subgraph of one of the $G[V_i]$
Theorem 6. Let $G = (V, E, \phi)$ be a graph having $k$ connected components. Then
$$|E| \leq \binom{|V| - k + 1}{2}$$

Theorem 7. See Text, page 70, for six equivalent definitions of tree.

Theorem 8. The number of labeled trees on $n$ vertices is $n^{n-2}$. (first published by Arthur Cayley in 1889)

Theorem 9. Let $G = (V, E, \phi)$ be a graph. Then the number of distinct vertex-labelings with the integers $\{1, 2, \ldots, |V|\}$ times the number of automorphisms of $G$ equals $|V|!$.

Theorem 10. (Erdős-Szekeres) Let $x_1, \ldots, x_{hk+1}$ be a sequence of $hk+1$ distinct numbers, $h, k \geq 1$ being integers. Then there must be either an increasing subsequence of length $h + 1$ or a decreasing subsequence of length $k + 1$.

Homework 4, due Tues 2/8/2011

1. T or F: Every minor of a path is a path; T or F: Every minor of a tree is a tree; T or F: If there are two or more paths joining two vertices $u, v$ in a graph $G$, then $G$ is not a tree; T or F: If a (general) graph $G$ with $n$ vertices has $n$ edges, then $G$ contains a cycle.

Exercises 3.18*, 3.22, 3.29, 4.12*, 4.13*.

Graduate/bonus exercises 3.8, 4.11.

Notes

* 3.18 Prove for all $k \geq 0$.
* 4.12 Show the relevant data for each step of the tree reconstruction.
* 4.13 Show the relevant data for each step of the Prüfer code construction.