Log-Concavity and Related Properties of the Cycle Index Polynomials

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Abstract

Let A_n denote the n-th cycle index polynomial, in the variables X_j, for the symmetric group on n letters. We show that if the variables X_j are assigned nonnegative real values which are log-concave, then the resulting quantities A_n satisfy the two inequalities $A_{n-1}A_{n+1} \le A_n^2 \le \left(\frac{n+1}{n}\right)A_{n-1}A_{n+1}$. This implies that the coefficients of the formal power series $\exp(g(u))$ are log-concave whenever those of $g(u)$ satisfy a condition slightly weaker than log-concavity. The latter includes many familiar combinatorial sequences, only some of which were previously known to be log-concave. To prove the first inequality we show that in fact the difference $A_n^2 - A_{n-1}A_{n+1}$ can be written as a polynomial with positive coefficients in the expressions X_j and $X_jX_k - X_{j-1}X_{k+1}$, $j \le k$. The second inequality is proven combinatorially, by working with the notion of a *marked* permutation, which we introduce in this paper. The latter is a permutation each of whose cycles is assigned a subset of available markers $\{M_{i,j}\}$. Each marker has a *weight*, $\mathrm{wt}(M_{i,j}) = x_j$, and we relate the second inequality to properties of the *weight enumerator polynomials*. Finally, using asymptotic analysis, we show that the same inequalities hold for n sufficiently large when the X_j are fixed with only finitely many nonzero values, with no additional assumption on the X_j.

Section 1. Introduction

Recall that a sequence of nonnegative real numbers b_n, $n \ge 0$, is *log-convex* provided $b_n^2 \le b_{n-1}b_{n+1}$ for all $n \ge 1$ and that it is *log-concave* provided $b_n^2 \ge b_{n-1}b_{n+1}$ for all $n \ge 1$. *Throughout this paper* we strengthen the definition of log-concavity by also requiring that, if $b_n = 0$ for some integer n, then $b_k = 0$ for all $k > n$. A nonnegative sequence b_n satisfies this strengthened condition of log-concavity if and only if $b_j b_k \ge b_{j-1}b_{k+1}$ for all $j \le k$; such sequences are also known as *one sided Pólya frequency sequences of order* 2 [5, p. 393]. This paper is devoted to the following theorem and related results. For a general introduction to the use of generating functions in combinatorics, as well as to the notions of convexity and concavity, we refer the reader to [10].

Theorem 1. *Let $1, X_1, X_2, \ldots$ be a log-concave sequence of nonnegative real numbers and define the sequences A_n and P_n by*

$$\sum_{n=0}^{\infty} A_n u^n = \sum_{n=0}^{\infty} \frac{P_n u^n}{n!} = \exp\left(\sum_{j=1}^{\infty} \frac{X_j u^j}{j}\right). \tag{1.1}$$

Then the A_n are log-concave and the P_n are log-convex. In other words,

$$A_{n-1}A_{n+1} \le A_n^2 \le \left(\tfrac{n+1}{n}\right) A_{n-1}A_{n+1} \tag{1.2}$$

and

$$P_{n-1}P_{n+1} \ge P_n^2 \ge \left(\tfrac{n}{n+1}\right) P_{n-1}P_{n+1}. \tag{1.3}$$

One easily shows that (1.2) and (1.3) are equivalent. Since $P_n = n!$ when $X_j = 1$ for all j while $P_n = 1$ for all n if $X_j = \delta_{j,1}$, the Kronecker delta, (1.3) is best possible. With $X_j = 1$ or $X_j = 1/(j-1)!$ for $j < k$ and $X_j = 0$ otherwise, one easily obtains the following corollaries.

Corollary 1.1. *Let $\pi_{n,k}$ be the number of permutations of an n-element set such that every cycle has less than k elements. Then*

$$\pi_{n-1,k}\,\pi_{n+1,k} \ge \pi_{n,k}^2 \ge \left(\tfrac{n}{n+1}\right)\pi_{n-1,k}\,\pi_{n+1,k}.$$

Corollary 1.2. *Let $B_{n,k}$ be the number of partitions of an n-element set such that every block has less than k elements. Then*

$$B_{n-1,k}\,B_{n+1,k} \ge B_{n,k}^2 \ge \left(\tfrac{n}{n+1}\right)B_{n-1,k}\,B_{n+1,k}.$$

When $k = \infty$, the first corollary is trivial and the second was stated in [3], which is devoted to inequalities about Bell numbers.

Each A_n is a polynomial in the variables X_j, $1 \le j \le n$, having a well known combinatorial significance: Let Σ_n denote the symmetric group and let $N_j(\sigma)$ be the number of j-cycles in the permutation σ. Then

$$A_n(X_1, \ldots, X_n) = \frac{P_n(X_1, \ldots, X_n)}{n!} = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \mathrm{wt}(\sigma), \tag{1.4}$$

where $\mathrm{wt}(\sigma) = X_1^{N_1(\sigma)} \cdots X_n^{N_n(\sigma)}$. The A_n are the *cycle index polynomials* generally associated with Pólya [7] although in fact appearing in earlier work of Redfield [8]. Theorem 1 will be seen to be a consequence of more general results concerning the form of the cycle index polynomials:

Theorem 2. *Let $X_0 = 1$, let $X_1, X_2, \ldots$ be indeterminates, let*

$$\mathcal{Y} = \{X_1, X_2 \ldots\} \cup \{X_j X_k - X_{j-1} X_{k+1} : 0 < j \le k\}$$

and let

$$\sum_{n=0}^{\infty} \frac{P_n u^n}{n!} = \exp\left(\sum_{j=1}^{\infty} \frac{X_j u^j}{j} \right).$$

Then

$$(n+1) P_m P_n - m P_{m-1} P_{n+1} \in \mathbb{N}[\mathcal{Y}] \quad \text{for } 1 \le m \le n; \tag{1.5}$$

that is, $(n+1)P_m P_n - m P_{m-1} P_{n+1}$ can be expressed as a polynomial in the $\mathcal{Y}$ with nonnegative integer coefficients. Let $v \in \mathbb{N}$ and let $x_1, \ldots, x_v$ be indeterminates. After the substitutions

$$X_j = \prod_{i=1}^{v} (1 + x_i)^{\min(i,j)}, \tag{1.6}$$

we have

$$P_{m-1} P_{n+1} - P_m P_n \in \mathbb{N}[x_1, \ldots, x_v] \quad \text{for } 1 \le m \le n. \tag{1.7}$$

We illustrate (1.5) with the example $m = n = 3$:

$$P_2 = X_2 + X_1^2$$

$$P_3 = 2X_3 + 3X_1 X_2 + X_1^3$$

$$P_4 = 6X_4 + 8X_1 X_3 + 3X_2^2 + 6X_2 X_1^2 + X_1^4$$

$$\begin{aligned} 4P_3^2 - 3P_2 P_4 \quad = \quad & (X_1^2 - X_2)^3 + 6X_1(X_1 X_2 - X_3)(X_1^2 - X_2) \\ & + 8(X_2^2 - X_1 X_3)(X_1^2 - X_2) + 4(X_1 X_2 - X_3)^2 + 6X_2(X_1 X_3 - X_4) \\ & + 6X_1^2(X_1 X_3 - X_4) + 12X_1(X_2 X_3 - X_1 X_4) + 12(X_3^2 - X_2 X_4). \end{aligned}$$

The relationships among these polynomials and log-concavity is given in the next section where we deduce Theorem 1 from Theorem 2. Result (1.5) is proved in Section 3. In Section 4, we give a combinatorial interpretation of the x_i's and use it to prove (1.7). The fact that log-concavity of the X_j's produces both log-concavity and log-convexity seems rather curious. This can be explained somewhat by studying the asymptotic behavior of the A_n's and P_n's when the log-concavity of the X_j's is not required. This is illustrated by the following theorem, which we prove in Section 5.

Theorem 3. *Let $P(u) = \sum_{j=1}^{d} c_j u^j$ be a polynomial with nonnegative coefficients, $c_d \neq 0$, and assume that $\gcd\{j : c_j \neq 0\} = 1$. Then there exists an integer n_0 such that for the sequence P_n defined by the generating function equation*

$$\sum_{n=0}^{\infty} \frac{P_n u^n}{n!} = \exp\Big(P(u)\Big)$$

we have

$$P_{n-1}P_{n+1} \geq P_n^2 \geq \left(\tfrac{n}{n+1}\right)P_{n-1}P_{n+1} \quad \text{for all } n \geq n_0. \tag{1.9}$$

(The gcd hypothesis in Theorem 3 is necessary: without it the sequence P_n contains infinitely many nonzero elements whose two immediate neighbors are zero.)

The literature on log-concavity is vast, and we mention only a few selections; the bibliographies of these will lead the interested reader to many other works. A standard reference is [5], especially Chapter 8. Combinatorial inequalities in particular are the subject of [1] and [9]. In [2] it is shown that if the coefficients of the power series $g(u)$ are log-concave then $s(n,k) = [u^n]g(u)^k$ is log-concave in k for fixed n; as a corollary the coefficients of the polynomial $P_n(x) = [u^n/n!]\exp(xg(u))$ are strictly log-concave. In [6] consideration is given to the question of when the coefficients of a sufficiently high power of a polynomial are log-concave.

Section 2. Theorem 2 Implies Theorem 1

The following lemma provides the connection between Theorems 1 and 2.

Lemma 2.1. *The real sequence X_j, with $X_0 = 1$, is strictly positive and log-concave if and only if there exisit $x_j \geq 0$ such that*

$$X_j = X_1^j \prod_{i=1}^{j-1} (1+x_i)^{-j+i}.$$

Proof of Lemma 2.1. From the inequality $X_1^2 \geq 1X_2$ we have for some $x_1 \geq 0$ that $X_2 = X_1^2(1+x_1)^{-1}$. Similarly, from $X_2^2 \geq X_1X_3$ we have for some $x_2 \geq 0$ that

$$X_3 = (1+x_2)^{-1}X_2^2/X_1 = (1+x_2)^{-1}(1+x_1)^{-2}X_1^3.$$

Continuing in this way, by induction, we obtain Lemma 2.1. ∎

With this preparation, we now show that Theorem 2 implies Theorem 1.

Proof of Theorem 1 from Theorem 2. As pointed out after the statement of Theorem 1, (1.2) is equivalent to (1.3). Thus we may concentrate on proving (1.3). Fix an integer $n \geq 1$ and consider the first inequality in (1.3). Let X_j be a real, strictly positive, log-concave sequence and let x_j be the corresponding nonnegative sequence given by the above Lemma 2.1. (We will remove the restriction of strict positivity in a moment.) We may restate the conclusion of the Lemma thus:

$$X_j = X_1^j \prod_{i=1}^{n}(1+x_i)^{-j+\min(i,j)}, \quad \text{for} \quad 1 \leq j \leq n+1. \tag{2.1}$$

Let $\hat{P}_m$ denote the real number that results when the substitutions (1.6) with $v = n$ are made in the polynomial P_m, and the x_j are given the nonnegative values of the Lemma. Because for each permutation $\sigma \in \Sigma_m$ we have

$$\sum_{j \geq 1} jN_j(\sigma) = m,$$

we see from (1.4) and (2.1) that for $m \leq n+1$

$$P_m = \left(X_1 \Big/ \prod_{i=1}^{n}(1+x_i)\right)^m \times \hat{P}_m.$$

Thus (1.7), with $m = n$, implies the first inequality of (1.3).

Suppose now that X_j vanishes for $j > i$. The preceding argument applies to the positive sequence $X_0, \ldots, X_i, X_i\epsilon,\ X_i\epsilon^2, \ldots$, and we obtain the desired inequality by continuity, letting $\epsilon \to 0$.

We turn now to the second inequality in (1.3). As pointed out in the introduction (it is not hard to prove) our definition of log-concavity implies that $X_jX_k - X_{j-1}X_{k+1}$ is nonnegative for $j \leq k$. Hence, the second inequality of (1.3) is an immediate consequence of (1.5) with $m = n$, and the proof is complete. ∎

Section 3. Proof of (1.5)

Let $X_1, \ldots$ be indeterminates and let $\mathcal{Y} \subset \mathbb{Z}[X_1, \ldots]$. For $P, Q \in \mathbb{Z}[X_1, \ldots]$, we define $P \geq Q$ to mean $P - Q \in \mathbb{N}[\mathcal{Y}]$; that is, $P - Q$ is a polynomial in the polynomials in $\mathcal{Y}$ with nonnegative coefficients. Throughout this section, an inequality involving polynomials will have this interpretation with $\mathcal{Y}$ as in Theorem 2. This notion of inequality is reflexive, antisymmetric, transitive, and has two other algebraic properties familiar from the numerical case:

(a) $(P \geq Q) \;\Rightarrow\; (P + R \geq Q + R)$.

(b) $\big((P \geq Q) \text{ and } (R \in \mathbb{N}[\mathcal{Y}])\big) \;\Rightarrow\; (PR \geq QR)$.

The idea can be extended to rings, but we need only this case.

Proof of (1.5). The proof is by induction on m. When $m = 1$ we must show

$$(n+1)X_1 P_n \geq P_{n+1}. \tag{3.1}$$

For $\sigma \in \Sigma_{n+1}$, let σ' be σ with element $n+1$ deleted from the cycle containing it. If $n+1$ belongs to a j-cycle of σ, then

$$X_{j-1}\,\mathrm{wt}(\sigma) = X_j\,\mathrm{wt}(\sigma').$$

Since $X_1 X_{j-1} \geq X_j$, we conclude

$$X_1\,\mathrm{wt}(\sigma') \geq \mathrm{wt}(\sigma).$$

Summing the latter over all $\sigma \in \Sigma_{n+1}$ yields (3.1) and starts the induction.

Now suppose $1 < \mu$ and that (1.5) holds for $1 \leq m < \mu$. We want to prove (1.5) for $m = \mu$. Let $(t)_k$ denote the falling factorial $t(t-1)\cdots(t-k+1)$. Observe that for $\mu > m \geq 1$, $h \geq 0$, and $n \geq m$

$$(n+h)_h P_m P_n \geq (m)_h P_{m-h} P_{n+h}; \tag{3.2}$$

this is obtained by iterating (1.5) h times:

$$\begin{aligned}(n+h)_h P_m P_n &\geq (n+h)_{h-1}\, m P_{m-1} P_{n+1}\\ &\geq (n+h)_{h-2}\, m(m-1) P_{m-2} P_{n+2}\\ &\geq \;\ldots\; \geq (m)_h P_{m-h} P_{n+h}.\end{aligned}$$

Let $n \geq \mu$. With σ' again denoting σ with its largest element deleted,

$$(n+1)P_\mu P_n - \mu P_{\mu-1} P_{n+1} = \sum_{\sigma_1 \in \Sigma_\mu} \sum_{\sigma_2 \in \Sigma_{n+1}} \Big(\mathrm{wt}(\sigma_1)\,\mathrm{wt}(\sigma_2') - \mathrm{wt}(\sigma_1')\,\mathrm{wt}(\sigma_2)\Big).$$

Partition the sum according to the size j of the cycle of σ_1 containing μ and the size k of the cycle of σ_2 containing $n+1$. For example, the sum of $\mathrm{wt}(\sigma_1)$ over all σ_1 for which μ belongs to a j-cycle is $(\mu-1)_{j-1}X_jP_{\mu-j}$ because $(\mu-1)_{j-1}$ counts the number of ways to construct a j-cycle containing μ, X_j is the weight of this cycle and $P_{\mu-j}$ is the sum of the weights over all ways to complete the permutation. Using this approach we find

$$\begin{aligned}(n+1)&P_\mu P_n-\mu P_{\mu-1}P_{n+1}\\&=\sum_{j,k\geq 1}\Big(X_jX_{k-1}-X_{j-1}X_k\Big)(\mu-1)_{j-1}(n)_{k-1}P_{\mu-j}P_{n+1-k}.\end{aligned}$$

Since the summand in this identity vanishes when $j=k$, the sum may be effected by restricting to $1\leq j<k$ while replacing the summand by itself plus the summand with j and k interchanged. Since interchanging j and k simply negates $X_jX_{k-1}-X_{j-1}X_k$, we find

$$\begin{aligned}(n+1)&P_\mu P_n-\mu P_{\mu-1}P_{n+1}\\&=\sum_{1\leq j<k}\Big(X_jX_{k-1}-X_{j-1}X_k\Big)\Big((\mu-1)_{j-1}(n)_{k-1}P_{\mu-j}P_{n+1-k}\\&\qquad\qquad-(\mu-1)_{k-1}(n)_{j-1}P_{\mu-k}P_{n+1-j}\Big)\\&=\sum_{1\leq j<k}(\mu-1)_{j-1}(n)_{j-1}\Big(X_jX_{k-1}-X_{j-1}X_k\Big)\Omega \qquad (3.3)\end{aligned}$$

where

$$\Omega=(n-j+1)_{k-j}P_{\mu-j}P_{n+1-k}-(\mu-j)_{k-j}P_{\mu-k}P_{n+1-j}. \qquad (3.4)$$

Since $X_jX_{k-1}-X_{j-1}X_k\in\mathcal{Y}$ for $j<k$, to complete the proof we need only show that

$$\Omega\geq 0 \quad \text{for all } 1\leq j<k. \qquad (3.5)$$

There are two cases to consider: $\mu-j\leq n+1-k$ and $n+1-k<\mu-j$. In the first case, $\Omega\geq 0$ by (3.2) with the replacements

$$m\leftarrow\mu-j,\quad n\leftarrow n+1-k,\quad h\leftarrow k-j.$$

In the second case, by (3.2) with the replacements

$$m\leftarrow n+1-k,\quad n\leftarrow\mu-j\ \text{ and }\ h\leftarrow n+1-\mu$$

we find that

$$(n+1-j)_{n+1-\mu}P_{n+1-k}P_{\mu-j}\geq(n+1-k)_{n+1-\mu}P_{\mu-k}P_{n+1-j}. \qquad (3.6)$$

Let $S = (\mu - j)_{\mu-j+k-n-1}$. Since $0 \le n+1-k < \mu - j$, S is a positive integer. Noting that $n+1-\mu \ge 0$ and the two simple relations

$$(n+1-j)_{k-j} \;=\; (n+1-j)_{n+1-\mu} \times S$$

and

$$(\mu - j)_{k-j} \;=\; S \times (n+1-k)_{n+1-\mu},$$

we may multiply both sides of (3.6) by S to obtain $\Omega \ge 0$. Thus the right side of (3.3) is in $\mathbb{N}[\mathcal{Y}]$, and the induction is complete. ∎

Section 4. Interpretation and Proof of (1.7)

We begin with a combinatorial interpretation of the x_j's that appear in (1.6).

Fix an integer $v \ge 0$. The $\binom{v+1}{2}$ objects in $\{M_{i,j} : 1 \le i \le j \le v\}$ will be called *markers.* A *marked permutation* $\hat{\sigma}$ on $[n] = \{1, 2, \ldots, n\}$ is a permutation $\sigma \in \Sigma_n$ each of whose cycles is assigned a subset, possibly empty, of markers subject to the one condition that *marker $M_{i,j}$ can be assigned only to cycles of size i or greater.* The set of marked permutations is denoted by $\mathrm{M}\Sigma_n$.

Let $\{x_j : 1 \le j \le v\}$ be a fixed set of v variables. The *weight* of a marker is $\mathrm{Wt}(M_{i,j}) = x_j$, and the weight of a set $\mathcal{S}$ of markers is the product of the weights of the individual elements of $\mathcal{S}$. For example

$$\mathrm{Wt}\big(\{M_{1,1}, M_{1,3}, M_{3,3}\}\big) = x_1 x_3^2.$$

The weight of the empty set is the empty product and is taken to be 1. $\mathrm{Wt}(\hat{\sigma})$, the weight of the marked permutation $\hat{\sigma}$, is the product of the weights of the individual cycles in $\hat{\sigma}$, and $\mathrm{Wt}(\sigma)$ is the sum of the weights of all marked permutations having σ for their underlying unmarked permutation. We define the *weight enumerator polynomial* $P_{n,v}$ in the variables x_j by

$$P_{n,v}(x_1, \ldots, x_v) \;=\; \sum_{\hat{\sigma} \in \mathrm{M}\Sigma_n} \mathrm{Wt}(\hat{\sigma}) \;=\; \sum_{\sigma \in \Sigma_n} \mathrm{Wt}(\sigma).$$

In the future we will always write $P_{n,v}$, without mention of the arguments $x_1, \ldots, x_v$, since they are implicit in the second subscript of the notation.

To illustrate we take $n = 3$ and $v = 2$. The possible weights of a 1-cycle are $1, x_1, x_2$, and $x_1 x_2$. The sum of the latter is $(1+x_1)(1+x_2)$. The sum of the possible weights for any cycle of size greater than 1 is $(1+x_1)(1+x_2)^2$. Within Σ_3 there are

- 2 permutations consisting of a 3-cycle,
- 1 permutation consisting of three 1-cycles and

- 3 permutations consisting of a 2-cycle and a 1-cycle.

Hence,

$$\begin{aligned}
P_{3,2} &= 2\Big((1+x_1)(1+x_2)^2\Big) + \Big((1+x_1)(1+x_2)\Big)^3 \\
&\quad + 3\Big((1+x_1)(1+x_2)^2\Big)\Big((1+x_1)(1+x_2)\Big) \\
&= 6 + 11x_1 + 16x_2 + 6x_1^2 + 31x_1x_2 + 14x_2^2 + x_1^3 + 18x_1^2x_2 + 29x_1x_2^2 + 4x_2^3 \\
&\quad + 3x_1^3x_2 + 18x_1^2x_2^2 + 9x_1x_2^3 + 3x_1^3x_2^2 + 6x_1^2x_2^3 + x_1^3x_2^3.
\end{aligned}$$

We now generalize this example to prove that $P_{n,v}$ equals P_n with the substitutions (1.6). To see this, first observe that $\mathrm{Wt}(\sigma)$, defined as the sum of $\mathrm{Wt}(\hat{\sigma})$ over all marked permutations $\hat{\sigma}$ with σ as their underlying permutation, is the following product

$$\mathrm{Wt}(\sigma) = \prod_{i=1}^{n} W_i^{N_i(\sigma)},$$

where W_i is the sum of all possible weights legally assignable to an i-cycle in a marked permutation. We may assign to an i-cycle any marker $M_{h,j}$ such that $h \le i$ and $h \le j \le v$. Hence, for a given j, the number of h such that marker $M_{h,j}$ can be assigned to an i-cycle is $\min(i,j)$. Since marker $M_{h,j}$ has weight x_j, an i-cycle has $\min(i,j)$ independent chances to include a factor x_j in its assigned weight; whence,

$$W_i = \prod_{j=1}^{v}(1+x_j)^{\min(i,j)}.$$

Since P_n is the sum over σ of the product $\prod X_i^{N_i}$, in view of the last two equations for $\mathrm{Wt}(\sigma)$ and W_i respectively, we see that as claimed $P_{n,v}$ equals P_n after the substitution (1.6). Furthermore, we may combinatorially interpret x_j in $P_{n,v}$ as keeping up with the number of markers $M_{i,j}$ which have been used in a marked permutation. This dual understanding of $P_{n,v}$ is the key to the proof of (1.7), but before that proof we require one lemma.

Lemma 4.1. *After the substitutions (1.6) we have, for $j \le k$,*

$$X_jX_k - X_{j-1}X_{k+1} \in \mathbb{N}[x_1, \ldots, x_v].$$

Proof of Lemma 4.1. With the usual convention that, when the starting index of a product is greater than the ending index, as in $\prod_{i=3}^{2}$, the product is empty and equals 1, we have for $j \le k$,

$$\begin{aligned}
&X_jX_k - X_{j-1}X_{k+1} \\
&\quad = \left(\prod_{i=1}^{v}(1+x_i)^{\min(i,j-1)}\right)\left(\prod_{i=1}^{v}(1+x_i)^{\min(i,k)}\right)\left(\prod_{i=j}^{v}(1+x_i) - \prod_{i=k+1}^{v}(1+x_i)\right)
\end{aligned}$$

and

$$\left(\prod_{i=j}^{v}(1+x_i) - \prod_{i=k+1}^{v}(1+x_i)\right) = \left(\prod_{i=k+1}^{v}(1+x_i)\right)\left(\prod_{i=j}^{\min(k,v)}(1+x_i) \; - \; 1\right). \quad \blacksquare$$

We are now ready to proceed with the main proof of this section.

Proof of (1.7). The case $m = 1$ requires a separate argument. Since $P_{1,v}$ can be considered the weight enumerator for all permutations of the singleton set $\{n+1\}$, it follows that $P_{n+1,v} - P_{1,v}P_{n,v}$ is the weight enumerator for all permutations in $\mathrm{M}\Sigma_{n+1}$ for which $\{n+1\}$ is not a 1-cycle. To complete the proof of (1.7) for $m = 1$, note that $P_{0,v} = 1$.

Let $\hat{\sigma} \in \mathrm{M}\Sigma_n$ be a marked permutation. We say that $\hat{\sigma}$ is *maximally marked* if the cycle containing n carries one or more of the marks $M_{j,j}, M_{j,j+1}, \ldots, M_{j,v}$, where j is the length of the cycle. Let $\mathrm{M}^*\Sigma_n \subseteq \mathrm{M}\Sigma_n$ be the set of marked permutations $\hat{\sigma}$ which are not maximally marked. If $\hat{\sigma} \in \mathrm{M}^*\Sigma_n$, then removal of n from the cycle containing it produces a marked permutation in $\mathrm{M}\Sigma_{n-1}$ and all elements of $\mathrm{M}\Sigma_{n-1}$ are obtained exactly n times by this procedure. Hence

$$\sum_{\hat{\sigma}\in \mathrm{M}^*\Sigma_n} \mathrm{Wt}(\hat{\sigma}) = nP_{n-1,v} \tag{4.1}$$

and so

$$\left(\sum_{\hat{\sigma}\in \mathrm{M}^*\Sigma_m} \mathrm{Wt}(\hat{\sigma})\right) \times P_{n,v} = P_{m-1,v} \times \left(\sum_{\hat{\sigma}\in \mathrm{M}^*\Sigma_{n+1}} \mathrm{Wt}(\hat{\sigma}) \; - \; (n+1-m)P_{n,v}\right). \tag{4.2}$$

We next find a different formula for the sum on the left of (4.1). Each $\hat{\sigma} \in \mathrm{M}\Sigma_n$ in which element n *does* reside in a maximally marked cycle is created once and only once by the following procedure: (a) choose a length j for the cycle containing n, (b) complete that cycle, (c) choose a maximal marking for that cycle and (d) choose a marked permutation on the remaining $n-j$ elements. A maximal marking for a j-cycle is one that includes at least one mark from the set $\{M_{j,j}, M_{j,j+1}, \ldots, M_{j,v}\}$. Define the polynomial $Q_{j,v}$ to be the sum of all possible maximal markings for a j-cycle. It is not hard to give an explicit formula for $Q_{j,v}$, but we require only the obvious facts that it has positive coefficients and that $Q_{j,v}$ is 0 when $j > v$. By the above construction of marked permutations in which n resides in a maximally marked cycle, we have

$$\sum_{\hat{\sigma}\in \mathrm{M}^*\Sigma_n} \mathrm{Wt}(\hat{\sigma}) \;\; = \;\; P_{n,v} - \sum_{j=0}^{v-1}(n-1)_j Q_{j+1,v}P_{n-1-j,v}. \tag{4.3}$$

By using (4.3) to replace the sums in (4.2) and rearranging, we have proven the following for all integers $n \geq 1$, $m > 1$ and $v \geq 0$:

$$P_{m-1,v}P_{n+1,v} - P_{m,v}P_{n,v} = (n+1-m)P_{m-1,v}P_{n,v} + \sum_{j=0}^{v-1} Q_{j+1,v}\Omega' \tag{4.4}$$

where

$$\Omega' = (n)_j P_{m-1,v} P_{n-j,v} - (m-1)_j P_{m-1-j,v} P_{n,v}.$$

We can use (3.4), (3.5) with n, μ, j, k replaced by $n, m, 1, j+1$ respectively to conclude

$$(n)_j P_{m-1} P_{n-j} - (m-1)_j P_{m-1-j} P_n \in \mathbb{N}[\mathcal{Y}] \quad \text{for } 1 < m \le n.$$

Since Ω' is obtained from the latter by the substitutions (1.6), and since Lemma 4.1 shows that $X_j X_{k-1} - X_{j-1} X_k \in \mathbb{N}[x_1, \ldots, x_v]$ after these same substitutions, it follows that $\Omega' \in \mathbb{N}[x_1, \ldots, x_v]$. From (4.4) we obtain the desired (1.7). ∎

Section 5. Proof of Theorem 3

If $d = 1$ we have $P_n = c_1^n$ and we may take $n_0 = 1$. Henceforth we assume $d > 1$. We shall prove, uniformly for $h = O(1)$ as $n \to \infty$,

$$P_{n+h} = \frac{(n+h)!}{r^{n+h}} \times \frac{\exp\{P(r)\}}{(2\pi B)^{1/2}} \times \left(1 + \frac{R_0 + hR_1 + h^2 R_2}{B} + O(r^{-2d})\right) \tag{5.1}$$

using the familiar circle method as presented by Hayman [4] and described in [10, p. 152]. The positive quantity r in (5.1) is determined by the equation

$$rP'(r) = n, \tag{5.2}$$

B is given by

$$B = rP'(r) + r^2 P''(r), \tag{5.3}$$

and the R_i are rational functions of r, bounded as $r \to \infty$, with $R_2 = -1/2$. Using (5.2) and (5.3), we find $n = dc_d r^d (1 + O(r^{-1}))$ and $B = d^2 c_d r^d (1 + O(r^{-1}))$. It is now easy to compute

$$(n+1)P_n^2 - nP_{n-1}P_{n+1} = \frac{(n+1)!\, n!}{r^{2n}} \; \frac{\exp\{2P(r)\}}{2\pi B^2} \left(1 + O(r^{-d})\right)$$

and

$$P_{n-1}P_{n+1} - P_n^2 = \frac{(n+1)!\, n!}{r^{2n}} \; \frac{\exp\{2P(r)\}}{2\pi B^2} \; \frac{d-1}{n} \left(1 + O(r^{-1})\right)$$

from (5.1). It remains to prove (5.1).

In what follows, the C_i are positive constants which depend only on $P(u)$.

Let $\mathcal{S} = \{j : c_j \ne 0\}$ and let

$$P(re^{i\theta}) = P(r) + Ai\theta - \tfrac{1}{2}B\theta^2 + \cdots$$

be the Taylor series expansion about $\theta = 0$; we find that $A = A(r) = rP'(r)$ and that B is given by (5.3). Choose r by (5.2) to satisfy $A(r) = n$, and apply Cauchy's integral formula with the circle $|z| = r$ to find

$$\frac{P_{n+h} r^{n+h}}{(n+h)!} = \frac{1}{2\pi} \int_{-\pi}^{+\pi} \exp\{P(re^{i\theta}) - i(n+h)\theta\}\, d\theta. \tag{5.4}$$

Let $\delta = r^{(1-d)/2}$ and partition the interval of integration into $|\theta| < \delta$ and $\delta \le |\theta| \le \pi$.

We now show that the integral over $\delta \le |\theta| \le \pi$ in (5.4) is negligible by using

$$\left|\exp\bigl\{P(re^{i\theta})\bigr\}\right| = \exp\bigl\{\operatorname{Re} P(re^{i\theta})\bigr\}.$$

First, if $\delta \le |\theta| \le \pi/d$ then $\cos d\theta - 1 \le -2d^2\delta^2/\pi^2$ and since $r^d\delta^2 = r$

$$\operatorname{Re} P(re^{i\theta}) = P(r) + \sum_{j \in S} c_j r^j (\cos j\theta - 1) \le P(r) - C_1 r. \tag{5.5}$$

To handle $\pi/d \le \theta \le \pi$ we need the gcd condition which implies the existence of integers N_j, $j \in \mathcal{S}$, such that $\sum_{j\in S} jN_j = 1$. Set $M = \sum_{j \in S} |N_j|$ and for $j \in S$ define λ_j by the two conditions $e^{ij\theta} = e^{i\lambda_j}$ and $|\lambda_j| \le \pi$. At least one λ_j, $j \in \mathcal{S}$, satisfies $|\lambda_j| \ge \pi/M(d+1)$ for otherwise

$$e^{i\theta} = \exp\Bigl\{\theta \sum jN_j\Bigr\} = \exp\Bigl\{i \sum \lambda_j N_j\Bigr\} = e^{i\lambda},$$

with $|\lambda| \le (\max_j |\lambda_j|)(\sum_j |N_j|) \le \pi/(d+1)$, a contradiction. Thus, for at least one $j \in \mathcal{S}$ we have $\cos j\theta - 1 \le -2/M^2(d+1)^2$ and so

$$\operatorname{Re} P(re^{i\theta}) = P(r) + \sum_{j \in \mathcal{S}} c_j r^j (\cos j\theta - 1) \le P(r) - C_2 r. \tag{5.6}$$

Together inequalities (5.5) and (5.6) imply

$$\left| \int_{\delta \le |\theta| \le \pi} \exp\bigl\{P(re^{i\theta}) - i(n+h)\theta\bigr\}\, d\theta \right| \le 2\pi \exp\ \{P(r) - C_3 r\}.$$

This concludes the demonstration that this part of the integral (5.4) is negligible.

Now supose $|\theta| \le \delta$. We use Taylor's theorem with remainder to write

$$P(re^{i\theta}) - i(n+h)\theta = P(r) - \frac{1}{2}B\theta^2 + \Bigl[-hi\theta + \ldots + O(r^d\theta^6)\Bigr].$$

For typographical simplicity we omit explicit statement of the terms involving third, fourth, and fifth powers of θ, although of course these are needed for the exact determination of the rational functions R_0, R_1, R_2 in (5.1). We then integrate as follows

$$\int_{-\delta}^{+\delta} \exp\{P(re^{i\theta}) - i(n+h)\theta\}\, d\theta$$
$$= e^{P(r)} \int_{-\delta}^{+\delta} e^{-B\theta^2/2} \Big(1 + [-hi\theta + \ldots] + \frac{1}{2}[-hi\theta + \ldots]^2 + \ldots \Big) d\theta,$$

with a careful analysis of the remainder. Terms up to the fourth power in h are needed, but only up to the second power of the others. To carry out the term-by-term integration, we make the following standard estimate.

Since $\delta \to 0$ and $\sqrt{B}\delta \to \infty$ we have for sufficiently large n

$$\begin{aligned}
\int_{\theta \geq \delta} \theta^{2m} e^{-B\theta^2/2} d\theta &= B^{-m-1/2} \int_{\psi \geq \sqrt{B}\delta} \psi^{2m} e^{-\psi^2/2} d\psi \\
&\leq B^{-m-1/2} \int_{\psi \geq \sqrt{B}\delta} (\psi^{2m+1} - 2m\psi^{2m-1}) e^{-\psi^2/2} d\psi \\
&= -B^{-m-1/2} \psi^{2m} e^{-\psi^2/2} \Big|_{\sqrt{B}\delta}^{\infty} \\
&= -B^{-1/2} \delta^{2m} e^{-B\delta^2/2} = O(e^{-C_4 r}).
\end{aligned}$$

Hence we have

$$\begin{aligned}
\int_{|\theta| \leq \delta} \theta^{2m} e^{-B\theta^2/2} d\theta &= B^{-m-1/2} \int_{-\infty}^{+\infty} \theta^{2m} e^{-\theta^2/2} d\theta + O(e^{-C_4 r}), \\
&= \sqrt{\frac{2\pi}{B}} \left(\frac{(2m-1)\cdots(3)(1)}{B^m} + O(e^{-C_5 r}) \right),
\end{aligned}$$

and this accounts for the various terms appearing in (5.1).

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