Homework 5. Due Tue 10/2.

1. 5.13 (some Poisson estimates)

2. 5.17. Here is a hint leading to a four-line proof, which is modeled on the proof of Theorem 5.17. Let $\mathcal{E}$ be an event, and by $\text{Pr}(\mathcal{E})$ we mean the probability in the model $G_{n,p}$. First line: Write $\text{Pr}(\mathcal{E})$ as a sum for $0 \leq k \leq \binom{n}{2}$, where we condition on $k =$ number of edges. Second line: extract one term from the sum; now we have a lower bound. Third line: replace $\text{Pr}(\mathcal{E} \mid N$ edges) by its equivalent, which is the probability of the event $\mathcal{E}$ in the model $G_{n,N}$; also, replace $\text{Pr}(N$ edges) by its equivalent,

$$\binom{n}{2} p^N (1 - p)^{\binom{n}{2} - N}. \tag{*}$$

Fourth line: lower bound $(*)$ by $\binom{n}{2}^{-1}$.

3. 5.18. The graph is disconnected iff there is a set $S$ such that all the edges between $S$ and its complement are missing. Since both $S$ and its complement are involved, there is no loss in saying there is an $S$ whose size is at most $n/2$.

Step 1. Let $S$ be a particular set of size $k$, $1 \leq k \leq n/2$. The number of edges between $S$ and its complement is $k(n-k) \geq kn/2$. The probability that all these edges are missing from $G_{n,N}$ is bounded above by

$$\left(1 - \frac{k(n-k)}{\binom{n}{2}}\right)^N \leq \left(1 - \frac{k}{n-1}\right)^N.$$

By considering all possible ways to partition the vertices, argue that the probability of disconnectedness is bounded above by

$$\sum_{1 \leq k \leq n/2} \binom{n}{k} \left(1 - \frac{k}{n-1}\right)^N.$$
Why does the latter go to zero with \( n \) for \( N = cn \log n \)?

**Midterm 1.** Exercise 5.20. You can work in teams of size two if you like. Turn in source, plus runs \( n = 100(100)1000, 100 \) trials, for each of \( f(n), g(n) \).

**Sixth week recap**

Tue 9/18. Talked about threshold phenomena. For example, \( \log n/n \) is a sharp threshold for connectivity. This means, for any \( \epsilon > 0 \), the probability that \( G_{n,(1-\epsilon)\log n/n} \) is connected goes to zero as \( n \) goes to infinity; while the probability that \( G_{n,(1+\epsilon)\log n/n} \) is connected goes to one as \( n \) goes to infinity. We mentioned that threshold phenomena are of interest in SAT, too, although much less is known. It is conjectured, but not proven, that for each \( k \) there is a constant \( C_k \) such that the probability that a random \( k \)-SAT problem on \( n \) variables with \( (C_k - \epsilon)n \) clauses is solvable goes to 1; while the probability that a random \( k \)-SAT problem on \( n \) variables with \( (C_k + \epsilon)n \) clauses is solvable goes to 0. We made ourselves familiar with the randomized algorithms given in the text for finding a Hamilton cycle in a random graph.

Wed 9/19. We went over the proof that the randomized Hamilton cycle finder works properly. We did not prove the key lemma: that at each iteration, assuming the unused-list to be nonempty, the probability that any given vertex ends up at the head of the path is \( 1/n \).

Thu 9/20. Started Chapter 6, on the Probabilistic Method. Talked about Shannon’s contribution for Boolean formulas, defined \( R(k,k) \), proved \( R(3,3) = 6 \), and mentioned that \( R(4,4) = 18 \). We proved this formula from the book: if

\[
\binom{n}{k} 2^{-\left(\frac{k}{2}\right) + 1} < 1
\]

then there exists a graph on \( n \) vertices which has neither a clique of size \( k \) nor an independent set of size \( k \). Beware of a typo in the statement of the theorem, where \( \binom{n}{k} \) is miswritten as \( \binom{n}{2} \).