Third week summary

Tue 8/28. End of Discrete Fourier Transform. The set of polynomials whose degree is less than $N$, all those of the form

$$p(x) = a_{N-1}x^{N-1} + a_{N-2}x^{N-2} + \cdots + a_1x + a_0,$$

form an $N$-dimensional vector space, as seen by associating the polynomial $p(x)$ with the column vector of its coefficients

$$\begin{bmatrix} a_0 \\ \vdots \\ a_{N-1} \end{bmatrix}.$$

The map which sends the polynomial $p(x)$ to the column of its values at the $N$th roots of unity,

$$p(x) \mapsto \begin{bmatrix} p(1) \\ p(e^{2\pi i/N}) \\ p(e^{4\pi i/N}) \\ \vdots \\ p(e^{2(N-1)\pi i/N}) \end{bmatrix},$$

is a linear transformation. The matrix for this transformation is, up to a constant factor, the Fourier transform, $F_N$. A polynomial whose degree is less than $N$ is uniquely determined by its value at $N$ distinct points.
There is an algorithm called the Fast Fourier Transform, generally ascribed to John Tukey but said to appear in work of Gauss, which computes the discrete Fourier transform applied to a given vector in $O(N \log N)$ amount of arithmetic. (As opposed to $O(N^2)$ which would be the case for standard matrix multiplication of a vector.) Since the inverse of the Fourier transform is very similar to the transform itself, the algorithm applies as well to the inverse. This leads to a $O(N \log N)$ algorithm for multiplying two polynomials of degree less than $N$.

**Wed 8/29.** End of linear algebra review. We proved that $\lambda$ is an eigenvalue for matrix $A$ iff it is a root of the characteristic polynomial of $A$. (The latter, by definition, is $\det(A - xI)$.) We also proved that eigenvalues of hermetian matrices (aka self-adjoint matrices) are always real. Moreover, eigenvectors belonging to distinct eigenvalues (of a hermetian matrix) are orthogonal.

Finite dimensional complex Hilbert space was defined. The spectral decomposition theorem was pointed out in the text. The following theorem was stated:

**Theorem.** The following are equivalent for a finite matrix $A$ over the complex numbers:

(i) $A$ has a left-inverse.

(ii) $A$ has a right-inverse.

(iii) There is a matrix $B$ such that $AB = BA = I$

(iv) The columns of $A$ are linearly independent

(v) The determinant $A$ is nonzero.

**Thu 8/30.** We started into Chapter 3 of the text. This has to do with interpreting matrices as describing dynamics of a directed network. We looked at the case where each column contains a unique 1, and then the case where each column contains non-negative real numbers which sum to 1. For the latter, we confirmed that 1 is a eigenvalue. Any eigenvector with non-negative entries corresponds to a stable distribution.

**Fourth week topics**

**Tues 9/4.** Return HW #1 – any questions? Collect HW #2. Comment
on Möbius transformations. Comment on Perron-Frobenius theory. Return to Chapter 3: Section 3.2, probabilistic systems.

Wed 9/5. Section 3.3: quantum systems.

Thu 9/6. Section 3.4: composite systems.