An Asymptotic Approach to the Hadamard Conjecture

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Collaborators: Warwick de Launey & David A. Levin

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HAPPY BIRTHDAY, HERB!
Main Definitions

An $n \times t$ matrix over $\{\pm 1\}$ is a partial Hadamard matrix provided the rows are pairwise orthogonal.

Definition: $H_{nt} :=$ the # of $n \times t$ partial Hadamard matrices

Example. One of the matrices counted by $H_{58}$:

\[
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+ & + & - & - & + & + & - & - \\
+ & - & + & - & + & - & + & - \\
- & - & + & + & + & + & - & - \\
\end{array}
\]

Example illustrates: $n \geq 3 \& H_{nt} \neq 0 \implies 4 | t$
The Problem

Find an asymptotic formula for $H_{nt}$ valid along certain infinite sequences of pairs $(n, t)$.
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Theorem (de Launey & Levin, 2010) Let $(n, t)$ be an infinite sequence of pairs satisfying $4|t$ and $t \geq n^{12+\epsilon}$. Then along this sequence

$$H_{nt} \sim \frac{2^{nt+(n-1)^2}}{(2\pi t)^{d/2}}, \quad d = \binom{n}{2}.$$
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A Fourier-analytic approach to counting partial Hadamard matrices

The Hadamard Conjecture

The HADAMARD CONJECTURE states that there exist square Hadamard matrices of size $t \times t$ for $t \in \{1, 2, 4, 8, 12, 16, 20, \ldots \}$.

Various constructions have been found

$t = 668$ is the first undecided value

$t = 428$ was decided in 2004
Outline of Talk

Circle Method Estimates
Progress on Latin rectangles
The Generating Function
The primary and secondary regions
The Circle Method

\[ a_n = [z^n] f(z) \]

\[ = \frac{1}{2\pi i} \oint_{|z|=r} \frac{f(z)}{z^{n+1}} \, dz \]

\[ = \frac{1}{2\pi r^n} \int_{-\pi}^{+\pi} \frac{f(re^{i\theta})}{en^i\theta} \, d\theta \]

\[ = \frac{1}{2\pi r^n} \left[ \int_{-\delta}^{+\delta} \cdots + \int_{\delta \leq |\theta| \leq \pi} \cdots \right] \]
Latin Rectangles

Another two-parameter asymptotic counting problem
How many $k \times n$ Latin rectangles are there?

Erdos & Kaplansky 1946 \quad k = O(\log n)^{3/2-\epsilon})

Yamamoto 1951 \quad k = o(n^{1/3})

Stein 1978 \quad k = o(n^{1/2})

Godsil & McKay 1990 \quad k = o(n^{6/7})

$$\left(\frac{n!}{n^k}\right)^n \left(1 - \frac{k}{n}\right)^{-n/2} e^{-k/2}$$
Main Symbols

\( n \) the height of the pHm
\( t \) the width
\( d = \binom{n}{2} \) the dimension of an integral
\( \delta \) defines primary/secondary regions.
Given vector $y$ of height $n$

$$y = \begin{bmatrix} y_j \\ \vdots \end{bmatrix} \in \{\pm 1\}^n,$$

define the vector of inner products, $Z(y)$, by

$$Z(y) = \begin{bmatrix} y_j y_k \\ \vdots \end{bmatrix} \in \{\pm 1\}^d.$$
Example of $Z(y)$

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$H_{nt}$ as a constant term

Define $M = M_n = \{ Z(y) : y \in \{\pm 1\}^n \}; \ |M| = 2^{n-1}$

\[
H_{nt} = 2^t \times \# \{ (\vec{m}_1, \ldots, \vec{m}_t) : \vec{m}_k \in M \& \sum_k \vec{m}_k = \vec{0} \}
\]

\[
= 2^t \times \left[ x_{12}^0 \cdots x_{n-1n}^0 \right] \left( \sum_{\vec{m} \in M} \prod_{jk} x_{jk}^{m_{jk}} \right)^t
\]
$H_{nt}$ as an Integral

Let $x_{jk} = e^{i\lambda_{jk}}$ and define

$$\psi(\lambda) = \frac{1}{|M|} \sum_{\vec{m} \in M} e^{i\lambda \cdot \vec{m}}.$$ 

Then,

$$H_{nt} = 2^t \times [x_{12}^0 \cdots x_{n-1n}^0] \left( \sum_{\vec{m} \in M} \prod_{jk} x_{jk}^{m_{jk}} \right)^t$$ 

$$= \frac{2^{nt}}{(2\pi)^d} \times \int_{-\pi}^{+\pi} \cdots \int_{-\pi}^{+\pi} \psi(\lambda)^t d\lambda.$$
Let \( \Lambda = \{ \lambda : |\psi(\lambda)| = 1 \} \), and 4\(|t|\).

\[
H_{nt} = \frac{2^{nt}}{(2\pi)^d} \left[ 2^{(n-1)^2} \int_{B_\delta} \psi(\lambda)^t d\lambda + \int_{R_\delta} \psi(\lambda)^t d\lambda \right]
\]

\(B_\delta = \{ \lambda : |\lambda_{jk}| \leq \delta \}\)

\(R_\delta = [-\pi, +\pi]^d \cap \{ \text{dist}(\lambda, \Lambda) \geq \delta \}\).
Primary Integral

Assuming $n\delta \to 0$,

$$\psi(\lambda)^t = \exp \left( -(t/2) \| \lambda \|^2 + O(tn^3\delta^3) \right).$$

$$\int_{-\delta}^{+\delta} \exp \left( -(t/2)x^2 \right) = \sqrt{\frac{2\pi}{t}} \left( 1 + O(e^{-t\delta^2/2}) \right).$$

Provided $tn^3\delta^3 \to 0$, $de^{-t\delta^2/2} \to 0$,

$$\int_{B_\delta} \psi(\lambda)^t = \left( \frac{2\pi}{t} \right)^{d/2} \left( 1 + o(1) \right).$$
Secondary Integral

de Launey and Levin prove that for any $k$, $1 \leq k \leq n$,

$$|\psi(\lambda)|^2 \leq \frac{1}{2} + \frac{1}{2} \prod_{\substack{j=1 \\ j \neq k}}^{n} \cos 2\lambda_{jk}$$

This gives

$$\left| \int_{R_\delta} \psi(\lambda)^t \right| \leq (2\pi)^d e^{-ct\delta^2}$$
Combining the Two

\[ H_{nt} = \frac{2^{nt}}{(2\pi)^d} \left[ 2^{(n-1)^2} \int_{B_\delta} \psi(\lambda)^t d\lambda + \int_{R_\delta} \psi(\lambda)^t d\lambda \right] \]

\[ H_{nt} = \frac{2^{nt}}{(2\pi)^d} \left[ 2^{(n-1)^2} \left( \frac{2\pi}{t} \right)^{d/2} \left( 1 + o(1) \right) + O(1) (2\pi)^d e^{-ct\delta^2} \right] \]

\[ H_{nt} = 2^{nt} \left[ 2^{(n-1)^2} (2\pi t)^{-d/2} \left( 1 + o(1) \right) + O(1) e^{-ct\delta^2} \right] \]

The assumptions: \( tn^3\delta^3 \to 0, \ de^{-t\delta^2/2} \to 0. \)

For sec. = \( o(\text{prim.}) \), \( t\delta^2 = \Omega(d \log t) \)
The de Launey Levin Theorem

\[ H_{nt} = 2^{nt} \left[ 2^{(n-1)^2} (2\pi t)^{-d/2} (1 + o(1)) + O(1)e^{-ct\delta^2} \right] \]

We arrive at deLauney and Levin’s formula,

\[ H_{nt} \sim \frac{2^{nt+(n-1)^2}}{(2\pi t)^{d/2}}, \]

obtained under the assumption that

\[ \delta = \sqrt{\frac{n^2 \log t}{t}} \]

satisfies \( tn^3\delta^3 \to 0 \) and \( de^{-t\delta^2/2} \to 0 \).