Asymptotic Enumeration of Integer Matrices with Large Equal Row and Column Sums

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Abstract

Let s, t, m, n be positive integers such that $sm = tn$. Let $M(m, s; n, t)$ be the number of $m \times n$ matrices over $\{0, 1, 2, \ldots\}$ with each row summing to s and each column summing to t. Equivalently, $M(m, s; n, t)$ counts 2-way contingency tables of order $m \times n$ such that the row marginal sums are all s and the column marginal sums are all t. A third equivalent description is that $M(m, s; n, t)$ is the number of semiregular labelled bipartite multigraphs with m vertices of degree s and n vertices of degree t. When $m = n$ and $s = t$ such matrices are also referred to as $n \times n$ magic or semimagic squares with line sums equal to t. We prove a precise asymptotic formula for $M(m, s; n, t)$ which is valid over a range of $(m, s; n, t)$ in which $m, n \to \infty$ while remaining approximately equal and the average entry is not too small. This range includes the case where m/n, n/m, s/n and t/m are bounded from below.

1 Introduction

Let m, s, n, t be positive integers such that $ms = nt$. Let $M(m, s; n, t)$ be the number of $m \times n$ matrices over $\{0, 1, 2, \ldots\}$ with each row summing to s and each column summing to t.

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Lemma 1. *Let $\varepsilon', \varepsilon'', \varepsilon''', \bar\varepsilon, \Delta$ be constants such that $0 < \varepsilon' < \varepsilon'' < \varepsilon'''$, $\bar\varepsilon \ge 0$, and $0 < \Delta < 1$. The following is true if ε''' and $\bar\varepsilon$ are sufficiently small.*

Let $\hat A = \hat A(N)$ be a real-valued function such that $\hat A(N) = \Omega(N^{-\varepsilon'})$. Let $\hat a_j = \hat a_j(N)$, $\hat B_j = \hat B_j(N)$, $\hat C_{jk} = \hat C_{jk}(N)$, $\hat E_j = \hat E_j(N)$, $\hat F_{jk} = \hat F_{jk}(N)$ and $\hat J_j = \hat J_j(N)$ be complex-valued functions $(1 \le j,k \le N)$ such that $\hat B_j, \hat C_{jk}, \hat E_j, \hat F_{jk} = O(N^{\bar\varepsilon})$, $\hat a_j = O(N^{1/2+\bar\varepsilon})$, and $\hat J_j = O(N^{-1/2+\bar\varepsilon})$, uniformly over $1 \le j,k \le N$. Suppose that

$$f(\boldsymbol z) = \exp\biggl(-\hat A N \sum_{j=1}^N z_j^2 + \sum_{j=1}^N \hat a_j z_j^2 + N \sum_{j=1}^N \hat B_j z_j^3 + \sum_{j,k=1}^N \hat C_{jk} z_j z_k^2 + N \sum_{j=1}^N \hat E_j z_j^4 + \sum_{j,k=1}^N \hat F_{jk} z_j^2 z_k^2 + \sum_{j=1}^N \hat J_j z_j + \delta(\boldsymbol z)\biggr)$$

is integrable for $\boldsymbol z = (z_1, z_2, \ldots, z_N) \in U_N$ and $\delta(N) = \max_{\boldsymbol z \in U_N} |\delta(\boldsymbol z)| = o(1)$, where

$$U_N = \bigl\{ \boldsymbol z \bigm| |z_j| \le N^{-1/2+\hat\varepsilon} \text{ for } 1 \le j \le N \bigr\},$$

where $\hat\varepsilon = \hat\varepsilon(N)$ satisfies $\varepsilon'' \le 2\hat\varepsilon \le \varepsilon'''$. Then, provided the $O(\,)$ term in the following converges to zero,

$$\int_{U_N} f(\boldsymbol z)\, d\boldsymbol z = \left(\frac{\pi}{\hat A N}\right)^{N/2} \exp\bigl(\Theta_1 + \Theta_2 + O\bigl((N^{-\Delta} + \delta(N))\hat Z\bigr)\bigr),$$

where

$$\begin{aligned}
\Theta_1 &= \frac{1}{2\hat A N} \sum_{j=1}^N \hat a_j + \frac{1}{4\hat A^2 N^2} \sum_{j=1}^N \hat a_j^2 + \frac{15}{16\hat A^3 N} \sum_{j=1}^N \hat B_j^2 + \frac{3}{8\hat A^3 N^2} \sum_{j,k=1}^N \hat B_j \hat C_{jk} \\
&\quad + \frac{1}{16\hat A^3 N^3} \sum_{j,k,\ell=1}^N \hat C_{jk} \hat C_{j\ell} + \frac{3}{4\hat A^2 N} \sum_{j=1}^N \hat E_j + \frac{1}{4\hat A^2 N^2} \sum_{j,k=1}^N \hat F_{jk} \\
\Theta_2 &= \frac{1}{6\hat A^3 N^3} \sum_{j=1}^N \hat a_j^3 + \frac{3}{2\hat A^3 N^2} \sum_{j=1}^N \hat a_j \hat E_j + \frac{45}{16\hat A^4 N^2} \sum_{j=1}^N \hat a_j \hat B_j^2 \\
&\quad + \frac{1}{4\hat A^3 N^3} \sum_{j,k=1}^N (\hat a_j + \hat a_k) \hat F_{jk} + \frac{3}{4\hat A^2 N} \sum_{j=1}^N \hat B_j \hat J_j + \frac{1}{4\hat A^2 N^2} \sum_{j,k=1}^N \hat C_{jk} \hat J_j \\
&\quad + \frac{1}{16\hat A^4 N^4} \sum_{j,k,\ell=1}^N (\hat a_j + 2\hat a_k) \hat C_{jk} \hat C_{j\ell} + \frac{3}{8\hat A^4 N^3} \sum_{j,k=1}^N (2\hat a_j + \hat a_k) \hat B_j \hat C_{jk} \\
\hat Z &= \exp\biggl(\frac{1}{4\hat A^2 N^2} \sum_{j=1}^N \operatorname{Im}(\hat a_j)^2 + \frac{15}{16\hat A^3 N} \sum_{j=1}^N \operatorname{Im}(\hat B_j)^2 \\
&\quad + \frac{3}{8\hat A^3 N^2} \sum_{j,k=1}^N \operatorname{Im}(\hat B_j) \operatorname{Im}(\hat C_{jk}) + \frac{1}{16\hat A^3 N^3} \sum_{j,k,\ell=1}^N \operatorname{Im}(\hat C_{jk}) \operatorname{Im}(\hat C_{j\ell})\biggr). \quad \square
\end{aligned}$$