Two Problems in Asymptotic Combinatorics

Rod Canfield

erc@cs.uga.edu

Department of Computer Science

University of Georgia

Ga Tech Combinatorics Seminar

April 2, 2010

Thanks to Coauthors

Part I, Stirling numbers: Carl Pomerance

Part II, Integer matrices: Brendan McKay

Stirling Numbers

$S(n,k)$ (a Stirling number of the second kind) equals the number of partitions of an n-set into k (nonempty, pairwise disjoint) blocks.

Recursion:

$$S(n+1,k) \;=\; kS(n,k) \;+\; S(n,k-1)$$

Implies strict log-concavity:

$$S(n,k)^2 \;\geq\; \left(1+\frac{3}{k}\right) S(n,k+1)\, S(n,k-1)$$

Location of Maximum

There is a unique index K_n

$$S(n,1) < \cdots < S(n,K_n) \geq S(n,K_n+1) > \cdots > S(n,n).$$

$n\backslash k$	1	2	3	4	5
1	1				
2	1	1			
3	1	3	1		
4	1	7	6	1	
5	1	15	25	10	1

Exceptional Integers

Define the *exceptional set* E:

$$E \quad = \quad \{n : S(n, K_n) = S(n, K_n + 1)\}$$

Problem.

$$E \quad \stackrel{?}{=} \quad \{2\}$$

First Upper Bound

Let $E(x)$ be the associated counting function:

$$E(x) \quad = \quad \{n : n \leq x \text{ and } n \in E\}$$

Using the asymptotic formula (Harper)

$$K_n \quad \sim \quad n/\log n$$

we can prove

$$E(x) \quad = \quad O(x/\log x)$$

Proof Sketch

Recursion + log-concavity imply

$$K_{n+1} \in \{K_n, K_n + 1\}$$

$$n \in E \Longrightarrow K_{n+1} = K_n + 1$$

Harper's Formula

Roots real and negative:

$$\sum_{k=1}^{n} S(n,k)x^k \;=\; x\prod_{j=1}^{n-1}(x + r_j^{(n)})$$

Let $B_n := \sum_k S(n,k)$, the nth Bell number.

$X_1, \ldots, X_n$ independent Bernoulli r.v.'s

$$B_n^{-1} S(n,k) \;=\; \mathrm{Prob}\left\{\sum_i X_i = k\right\}$$

By Darroch (1964) the mean and mode differ by at most 1!

The Mean

What is it

$$\text{mean} = \mu_n = B_n^{-1} \sum_{k=1}^{n} kS(n,k) \quad = \quad ??$$

The recursion

$$S(n+1,k) \quad = \quad kS(n,k) \; + \; S(n,k-1)$$

implies (summing)

$$B_{n+1} \quad = \quad \sum_{k=1}^{n} kS(n,k) \; + \; B_n$$

$$\mu_n \quad = \quad B_{n+1}/B_n \; - \; 1$$

Estimating Bell

Exponential generating function:

$$\sum_{n=1}^{\infty} \frac{B_n}{n!} z^n \quad = \quad \exp(e^x - 1)$$

Cauchy integral formula:

$$[x^n]F(x) \quad = \quad \frac{1}{2\pi i} \oint_{|z|=r} \frac{F(z)}{z^{n+1}} dz$$

Moser and Wyman (1955): Let $re^r = n$; then,

$$B_n \quad \sim \quad \frac{1}{e\sqrt{\log n}} \exp\left(nr - n + n/r\right)$$

Theorem on Mode

Let r be defined (again) by $re^r = n$; then,

$$K_n \in \{\lfloor e^r - 1 \rfloor, \lceil e^r - 1 \rceil\}$$

for all sufficiently large n,

and for $1 \leq n \leq 1200$.

Stirling Second Asymptotic

$$\sum_{n=k}^{\infty} S(n,k)\frac{x^n}{n!} = \frac{(e^x-1)^k}{k!}$$

$$\frac{e^{re^{i\theta}}-1}{e^r-1} = \exp\left(Ai\theta - (1/2)B\theta^2 + O(r|\theta|^3)\right),$$

$$A = r + O(e^{-r}), \quad B = r + O(e^{-r})$$

Property of E

$$k = e^r + O(1)$$

$$S(n,k) = \frac{(e^r - 1)^k}{k!} \frac{n!}{r^n} (2\pi k B)^{-1/2} (1 + \cdots)$$

For $n \in E$

$$e^r = \mathrm{INTEGER} + 1/2 + \frac{1/2}{r+1} + R(r)e^{-r} + O_*(e^{-2r})$$

Three Together

Suppose X is large, and

$$\left|[X, X + X^{1/3-\epsilon}] \cap E\right| \quad \geq \quad 3$$

Say $n + \ell_i$, $i = 0, 1, 2$;

$$0 = \ell_0 < \ell_1 < \ell_2$$

Have also r_i

$$e^{r_i} = e^r + \frac{\ell_i}{r+1} - \frac{\ell_i^2}{2(r+1)^3 e^r} + O_*(\ell_i^3 e^{-2r})$$

Likewise, $\frac{1/2}{r_i+1}$, $\ldots$

Consequence

$$e^{r_i} = \mathrm{INTEGER} + 1/2 + \frac{1/2}{r_i + 1} + R(r_i)e^{-r_i} + O_*(e^{-2r})$$

$(\ell_1 - \ell_2)$, $+\ell_2$, $-\ell_1$ combination:

$$\frac{\ell_1\ell_2^2 - \ell_2\ell_1^2}{2(r+1)^3 e^r} + O_*(\frac{\ell_2\ell_1^3 + \ell_1\ell_2^3}{e^{2r}})$$

$$= \mathrm{INTEGER} + O_*(\frac{\ell_1\ell_2^2 + \ell_2\ell_1^2}{e^{2r}})$$

Contradiction

Theorems, 2001

$$E(x) = O(x^{2/3+\epsilon})$$

$$E(10^6) = 1$$

$$E(x) = O(x^{3/5+\epsilon})$$

Huxley, Integer points close to a curve (1999)

Better Theorem

Via $S(x,y)$ for complex x,y

Kemkes, Merlini, and Richmond (2008)

$$E(x) \quad = \quad O(x^{1/2+\epsilon})$$

Bombieri & Pila, The number of integral points on arcs and ovals (1989)

Erdős' Theorem

$\Pi_{n,s} =$ product 1 to n, taken s at time

for prime p with $n/(k+1) < p \leq n/k$ we have
$\Pi_{n,n-k} \not\equiv 0$ mod p, but
$\Pi_{n,n-r} \equiv 0$ mod p for $r < k$

P. Erdős, On a conjecture of Hammersley, (1953).

Part II, Integer Matrices

Definition. Let m, s, n, t be integers with $ms = nt$

$$M(m, s; n, t) = \#\{A : \quad A \text{ is an } m \times n \text{ matrix over } \{0, 1, \ldots\},$$

$$\sum_{j=1}^{n} A_{ij} = s \text{ for all } i,$$

$$\sum_{i=1}^{m} A_{ij} = t \text{ for all } j$$

$$\}.$$

Example

Let $m = 4$, $s = 15$, $n = 6$, $t = 10$

$$\begin{bmatrix} 2 & 3 & 0 & 0 & 10 & 0 \\ 1 & 2 & 3 & 4 & 0 & 5 \\ 5 & 3 & 1 & 1 & 0 & 5 \\ 2 & 2 & 6 & 5 & 0 & 0 \end{bmatrix}$$

Estimated count $= 2.36e + 11$

99 % confidence interval $= \quad [2.01e + 11, \; 2.71e + 11]$

Exact value: $M(4, 15; 6, 10) = 234, 673, 404, 860$

Three Items

Exact value
Confidence interval
Estimated value

Exact Value

$$M(m,s;n,t) = [(x_1 \cdots x_m)^s (y_1 \cdots y_n)^t] \prod_{j=1}^{m} \prod_{k=1}^{n} (1 - x_j y_k)^{-1}$$

Make right side poly: $(1 + x_j y_k + \cdots + (x_j y_k)^{\min(s,t)})$

$$[\vec{x}^{\vec{s}} \vec{y}^{\vec{t}}] Poly(\vec{x}, \vec{y}) = \frac{1}{q_1^m q_2^n} \sum_{\vec{x} \in \langle \alpha \rangle^m} \sum_{\vec{y} \in \langle \beta \rangle^n} \frac{Poly(\vec{x}, \vec{y})}{\vec{x}^{\vec{s}} \vec{y}^{\vec{t}}}$$

$$\langle \alpha \rangle = \{1, \alpha, \ldots, \alpha^{q_1 - 1}\}$$

Use symmetry

Confidence Interval

X a random variable with mean μ variance σ^2

sample mean $X_N = N^{-1}(x_1 + \cdots x_N)$

normal approximation: $[X_N \pm 3\sigma/N^{1/2}]$ is a 99% confidence interval for μ

approximate σ^2 by the sample variance

A Trick

Population: $\Omega = \{\omega_1, \ldots, \omega_\ell\}$, with probabilites p_i

$$X(\omega_i) = \frac{1}{p_i}$$

$$E(X) = |\Omega|$$

Application to Enumeration of contingency tables: Chen, Diaconis, Holmes, Liu *JASA*(2005)

Sampling Algorithm

Number columns $0, \ldots, n-1$; choose one at the time

When choosing $x[i,j], 0 \leq i < m$

- j columns are complete
- target row sums are $r_i =$ target

Set $x[i,j]$ as if it were the first part in a random composition of r_i with $n-j$ parts

If $\sum_i x[i,1] = t$, accept and proceed to next column; else, try again

Probability p_ω of obtaining $\omega = x[i,j]$ equals ?

Three Estimates

Define the *density* λ by

$$\frac{s}{n} \; = \; \lambda \; = \; \frac{t}{m}.$$

(It is the average entry in the matrix.)

$$M(m,n;s,t) \;\approx\; \binom{mn(1+\lambda)-1}{\lambda mn}$$

$$M(m,n;s,t) \;\approx\; \binom{n+s-1}{s}^{m}$$

$$M(m,n;s,t) \;\approx\; \binom{m+t-1}{t}^{n}$$

A Good Estimate

$$M(m,n;s,t) \approx \frac{\binom{n+s-1}{s}^m \binom{m+t-1}{t}^n}{\binom{mn(1+\lambda)-1}{\lambda mn}}$$

A Good Estimate

$$M(m,n;s,t) \approx \frac{\binom{n+s-1}{s}^{m} \binom{m+t-1}{t}^{n}}{\binom{mn(1+\lambda)-1}{\lambda mn}}$$

I. J. Good, *Probability and the Weighing of Evidence*, Charles Griffin, London, 1950.

A Good Estimate

$$M(m,n;s,t) \approx \frac{\binom{n+s-1}{s}^m \binom{m+t-1}{t}^n}{\binom{mn(1+\lambda)-1}{\lambda mn}}$$

I. J. Good, *Probability and the Weighing of Evidence*, Charles Griffin, London, 1950.

I. J. Good, On the application of symmetric Dirichlet distributions and their mixtures to contingency tables, *Ann. Statist.* **4** (1976) 1159–1189.

A Good Estimate

$$M(m,n;s,t) \approx \frac{\binom{n+s-1}{s}^m \binom{m+t-1}{t}^n}{\binom{mn(1+\lambda)-1}{\lambda mn}}$$

I. J. Good, *Probability and the Weighing of Evidence*, Charles Griffin, London, 1950.

I. J. Good, On the application of symmetric Dirichlet distributions and their mixtures to contingency tables, *Ann. Statist.* **4** (1976) 1159–1189.

I. J. Good and J. F. Crook, The enumeration of arrays and a generalization related to contingency tables, *Discrete Math.* **19** (1977) 23–45.

Conjecture 1

If $m, n \to \infty$,

$$M(m,n;s,t) = \frac{\binom{n+s-1}{s}^m \binom{m+t-1}{t}^n}{\binom{mn(1+\lambda)-1}{\lambda mn}} \times \exp\left(\frac{1}{2} + o(1)\right)$$

Conjecture 2

Define $\Delta(m,s;n,t)$ by

$$M(m,n;s,t) = \frac{\binom{n+s-1}{s}^m \binom{m+t-1}{t}^n}{\binom{mn(1+\lambda)-1}{\lambda mn}} \times \left(\frac{m+1}{m}\right)^{(m-1)/2} \left(\frac{n+1}{n}\right)^{(n-1)/2} \times \exp\left(-\frac{1}{2} + \frac{\Delta(m,s;n,t)}{m+n}\right).$$

Then, $0 < \Delta(m,s;n,t) < 2$.

Evidence

Theorem. Conjecture 1 is correct provided that $m, n \to \infty$ in such a way that

$$\frac{(1+2\lambda)^2}{4\lambda(1+\lambda)}\left(1+\frac{5m}{6n}+\frac{5n}{6m}\right) \quad \le \quad a\log n, \qquad a < 1/2.$$

Proof begins with

$$M(m,s;n,t) = \frac{1}{(2\pi i)^{m+n}}$$

$$\times \oint \cdots \oint \frac{\prod_{j,k}(1-x_j y_k)^{-1}}{x_1^{s+1}\cdots x_m^{s+1} y_1^{t+1}\cdots y_n^{t+1}}\, dx_1\cdots dx_m\, dy_1\cdots dy_n$$

More Evidence

Exact Calculations

Several thousands of $(m, s; n, t)$ with $m, n \leq 30$

Statistical Calculations

High degree of confidence for many larger values