Rod Canfield
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Test #1, solutions

1. Let $(A_1, \ldots, A_n)$ be an n-tuple of sets.

(a) What is meant by *a system of distinct representatives* (SDR) for the sets A_i ?

SOLUTION An SDR is an n-tuple $(x_1, \ldots, x_n)$ satisfying two constraints:

- the x_i are pairwise disjoint
- $x_i \in A_i$ for $1 \leq i \leq n$

(b) What is the necessary and sufficient condition for $(A_1, \ldots, A_n)$ to have an SDR, as given by Phillip Hall ?

SOLUTION Phillip Hall's Condition: for all $I \subseteq [n]$,

$$|I| \leq \left| \bigcup_{i \in I} A_i \right|.$$

In words, the union of any k of the sets must have at least k elements.

2. Twenty different books are to be put on five book shelves, each of which can hold as many as twenty books. Throughout, the shelves are considered distinguishable. It is OK not to show work on this problem, but some reasoning/justification will help in earning partial credit for an incorrect answer.

(a) How many different arrangements are there if you only care about the number of books on each shelf (and not the exact location of each book) ?

SOLUTION We must choose nonnegative integers $x_1, x_2, \ldots, x_5$ whose sum is 20. (x_i is how many books are on shelf i) This may be done in

$$\binom{20+5-1}{5-1} = \binom{24}{4}$$

ways.

(b) How many different arrangements are there if you care about which shelf each book is are on, but the order of the books on the shelves doesn't matter ?

SOLUTION Assign each book to a shelf, 5^{20}.

(c) How many different arrangements are there if the order of the books on the shelves does matter ?

SOLUTION Start as in part (a), but then choose also a permutation of the books and place them on the shelves in that order, using the x_i as a guide to how many books to put on each shelf. Hence,

$$\binom{24}{4} \times 20!$$

3. Let $S(n, k)$ be the number of partitions of $[n] = \{1, 2, \ldots, n\}$ into k nonempty, pairwise disjoint subsets. Prove the following two identities combinatorially:

(a) $S(n+1,k) = S(n,k-1) + kS(n,k)$.

SOLUTION First term on right counts partitions of $[n+1]$ in which $n+1$ is in a block by itself, and the other n elements are in $k-1$ blocks. Second term on right counts partitions of $[n]$ into k blocks, then with a factor of k for inserting $n+1$ into one of the blocks.

(b) $S(n+1,k) = \sum_{j=0}^{n} \binom{n}{j} S(n-j, k-1)$.

SOLUTION Choose $j \geq 0$ elements to be in a block with $n+1$; then partition the remaining $n-j$ elements into $k-1$ blocks. (When $j > n-k+1$, $S(n-j,k-1) = 0$.)

4. For positive integers $n \geq k \geq 1$ let $p(n,k)$ be the number of solutions, in whole numbers, to the system of requirements:

$$x_k \geq x_{k-1} \geq \cdots x_1 \geq 1, \quad \sum_{i=1}^{k} x_i = n.$$

Derive a recursion for $p(n,k)$.

SOLUTION Suppose $x_k = 1$. Then, omitting x_k from the list, we have a partition of $n-1$ into $k-1$ parts. These are $p(n-1,k-1)$ in number. If, on the other hand, all of the x_i are strictly larger than one, then each may be reduced by 1 to find

$$x_k - 1 \geq x_{k-1} - 1 \geq \cdots x_1 - 1 \geq 1, \quad \sum_{i=1}^{k} (x_i - 1) = n - k.$$

These are $p(n-k,k)$ in number. Hence,

$$p(n,k) = p(n-1,k-1) + p(n-k,k).$$

5. Let $C(x) = \sum_{n=0}^{\infty} c_n x^n$ be the ordinary generating function for the sequence c_n defined by the recursion

$$c_{n+1} = \sum_{j=0}^{n} c_j c_{n-j}, \quad n \geq 0, \qquad (*)$$

with initial value $c_0 = 1$. Find a quadratic equation satisfied by $C(x)$.

SOLUTION Since

$$C(x)^2 = \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} c_j c_{n-j} \right) x^n,$$

we have

$$\sum_{j=0}^{n} c_j c_{n-j} \quad = \quad [x^n] C(x)^2.$$

Multiply both sides of $(*)$ above by x^{n+1} and sum for $n \geq 0$. We find

$$C(x) - 1 = xC(x)^2.$$

Altogether, the quadratic is

$$1 - C(x) + xC(x)^2 = 0.$$

For Grad Students and Undergraduate Extra Credit

GUEC1. How many unordered pairs $\{x, y\}$ are there where x and y are subsets of $[19]$ of size three which satisfy

$$|x \cap y| = 2.$$

SOLUTION $\binom{19}{2} \times \binom{17}{2}$. Proof: Choose two elements a, b to comprise the intersection. Then choose two other elements c, d to form the unordered pair

$$\{\{a, b, c\}, \{a, b, d\}\}.$$