Computational Learning Theory

What general laws constrain inductive learning?

We seek theory to relate:

- Probability of successful learning
- Number of training examples
- Complexity of hypothesis space
- Accuracy to which target concept is approximated
- Manner in which training examples presented
Prototypical Concept Learning Task

• Given:
  – Instances $X$: Possible days, each described by the attributes $Sky$, $AirTemp$, $Humidity$, $Wind$, $Water$, $Forecast$
  – Target function $c$: $EnjoySport : X \rightarrow \{0, 1\}$
  – Training examples $D$: Positive and negative examples of the target function
    $\langle x_1, c(x_1) \rangle, \ldots \langle x_m, c(x_m) \rangle$

• Determine:
  – A hypothesis $h$ in $H$ such that $h(x) = c(x)$ for all $x$ in $D$?
  – A hypothesis $h$ in $H$ such that $h(x) = c(x)$ for all $x$ in $X$?
Sample Complexity

How many training examples are sufficient to learn the target concept?

1. If learner proposes instances, as queries to teacher
   • Learner proposes instance $x$, teacher provides $c(x)$

2. If teacher (who knows $c$) provides training examples
   • teacher provides sequence of examples of form $(x, c(x))$

3. If some random process (e.g., nature) proposes instances
   • instance $x$ generated randomly, teacher provides $c(x)$
Sample Complexity: 1

Learner proposes instance $x$, teacher provides $c(x)$
(assume $c$ is in learner’s hypothesis space $H$)

Optimal query strategy: play 20 questions

- pick instance $x$ such that half of hypotheses in $VS$ classify $x$ positive, half classify $x$ negative
- When this is possible, need $\lceil \log_2 |H| \rceil$ queries to learn $c$
- when not possible, need even more
Sample Complexity: 2

Teacher (who knows $c$) provides training examples (assume $c$ is in learner’s hypothesis space $H$)

Optimal teaching strategy: depends on $H$ used by learner

Consider the case $H =$ conjunctions of up to $n$ boolean literals and their negations

- e.g., $(\text{AirTemp} = \text{Warm}) \land (\text{Wind} = \text{Strong})$, where $\text{AirTemp}, \text{Wind}, \ldots$ each have 2 possible values.

- if $n$ possible boolean attributes in $H$, $n + 1$ examples suffice

- why?
Sample Complexity: 3

Given:

- set of instances $X$
- set of hypotheses $H$
- set of possible target concepts $C$
- training instances generated by a fixed, unknown probability distribution $\mathcal{D}$ over $X$

Learner observes a sequence $\mathcal{D}$ of training examples of form $\langle x, c(x) \rangle$, for some target concept $c \in C$

- instances $x$ are drawn from distribution $\mathcal{D}$
- teacher provides target value $c(x)$ for each

Learner must output a hypothesis $h$ estimating $c$

- $h$ is evaluated by its performance on subsequent instances drawn according to $\mathcal{D}$

Note: probabilistic instances, noise-free classifications
True Error of a Hypothesis

Definition: The true error \( \text{error}_D(h) \) of hypothesis \( h \) with respect to target concept \( c \) and distribution \( D \) is the probability that \( h \) will misclassify an instance drawn at random according to \( D \).

\[
\text{error}_D(h) \equiv \Pr_{x \in D}[c(x) \neq h(x)]
\]
Two Notions of Error

*Training error* of hypothesis $h$ with respect to target concept $c$

- How often $h(x) \neq c(x)$ over training instances

*True error* of hypothesis $h$ with respect to $c$

- How often $h(x) \neq c(x)$ over future random instances

Our concern:

- Can we bound the true error of $h$ given the training error of $h$?

- First consider when training error of $h$ is zero (i.e., $h \in VS_{H,D}$)
Exhausting the Version Space

Hypothesis space $H$

$(r = \text{training error}, \text{error} = \text{true error})$

**Definition:** The version space $V S_{H,D}$ is said to be $\epsilon$-exhausted with respect to $c$ and $\mathcal{D}$, if every hypothesis $h$ in $V S_{H,D}$ has error less than $\epsilon$ with respect to $c$ and $\mathcal{D}$.

$$(\forall h \in V S_{H,D}) \text{ error}_\mathcal{D}(h) < \epsilon$$
How many examples will $\epsilon$-exhaust the VS?

**Theorem:** [Haussler, 1988].

If the hypothesis space $H$ is finite, and $D$ is a sequence of $m \geq 1$ independent random examples of some target concept $c$, then for any $0 \leq \epsilon \leq 1$, the probability that the version space with respect to $H$ and $D$ is not $\epsilon$-exhausted (with respect to $c$) is less than

$$|H|e^{-\epsilon m}$$

Interesting! This bounds the probability that any consistent learner will output a hypothesis $h$ with $\text{error}(h) \geq \epsilon$

If we want to this probability to be below $\delta$

$$|H|e^{-\epsilon m} \leq \delta$$

then

$$m \geq \frac{1}{\epsilon} (\ln |H| + \ln(1/\delta))$$
Learning Conjunctions of Boolean Literals

How many examples are sufficient to assure with probability at least $(1 - \delta)$ that

$$\text{every } h \text{ in } \mathcal{V}S_{H,D} \text{ satisfies } \text{error}_D(h) \leq \epsilon$$

Use our theorem:

$$m \geq \frac{1}{\epsilon}(\ln |H| + \ln(1/\delta))$$

Suppose $H$ contains conjunctions of constraints on up to $n$ boolean attributes (i.e., $n$ boolean literals). Then $|H| = 3^n$, and

$$m \geq \frac{1}{\epsilon}(\ln 3^n + \ln(1/\delta))$$

or

$$m \geq \frac{1}{\epsilon}(n \ln 3 + \ln(1/\delta))$$
How About \textit{EnjoySport}?

\[
m \geq \frac{1}{\varepsilon} (\ln |H| + \ln(1/\delta))
\]

If \( H \) is as given in \textit{EnjoySport} then \( |H| = 973 \), and

\[
m \geq \frac{1}{\varepsilon} (\ln 973 + \ln(1/\delta))
\]

... if want to assure that with probability 95\%, \( VS \)
contains only hypotheses with \( \text{error}_D(h) \leq .1 \), then it is sufficient to have \( m \) examples, where

\[
m \geq \frac{1}{.1} (\ln 973 + \ln(1/.05))
\]
\[
m \geq 10(\ln 973 + \ln 20)
\]
\[
m \geq 10(6.88 + 3.00)
\]
\[
m \geq 98.8
\]
PAC Learning

Consider a class $C$ of possible target concepts defined over a set of instances $X$ of length $n$, and a learner $L$ using hypothesis space $H$.

Definition: $C$ is **PAC-learnable** by $L$ using $H$ if for all $c \in C$, distributions $\mathcal{D}$ over $X$, $\epsilon$ such that $0 < \epsilon < 1/2$, and $\delta$ such that $0 < \delta < 1/2$, learner $L$ will with probability at least $(1 - \delta)$ output a hypothesis $h \in H$ such that $error^D(h) \leq \epsilon$, in time that is polynomial in $1/\epsilon$, $1/\delta$, $n$ and $size(c)$.
Agnostic Learning

So far, assumed $c \in H$

Agnostic learning setting: don’t assume $c \in H$

• What do we want then?
  – The hypothesis $h$ that makes fewest errors on training data

• What is sample complexity in this case?

$$m \geq \frac{1}{2\epsilon^2}(\ln |H| + \ln(1/\delta))$$

derived from Hoeffding bounds:

$$Pr[error_D(h) > error_D(h) + \epsilon] \leq e^{-2m\epsilon^2}$$
Shattering a Set of Instances

*Definition:* a **dichotomy** of a set $S$ is a partition of $S$ into two disjoint subsets.

*Definition:* a set of instances $S$ is **shattered** by hypothesis space $H$ if and only if for every dichotomy of $S$ there exists some hypothesis in $H$ consistent with this dichotomy.
Three Instances Shattered

Instance space $X$
The Vapnik-Chervonenkis Dimension

Definition: The Vapnik-Chervonenkis dimension, $VC(H)$, of hypothesis space $H$ defined over instance space $X$ is the size of the largest finite subset of $X$ shattered by $H$. If arbitrarily large finite sets of $X$ can be shattered by $H$, then $VC(H) \equiv \infty$. 
VC Dim. of Linear Decision Surfaces

(a)

(b)
Sample Complexity from VC Dimension

How many randomly drawn examples suffice to $\epsilon$-exhaust $V S_{H,D}$ with probability at least $(1 - \delta)$?

$$m \geq \frac{1}{\epsilon}(4 \log_2(2/\delta) + 8 VC(H) \log_2(13/\epsilon))$$
Mistake Bounds

So far: how many examples needed to learn?
What about: how many mistakes before convergence?

Let’s consider similar setting to PAC learning:

- Instances drawn at random from $X$ according to distribution $\mathcal{D}$
- Learner must classify each instance before receiving correct classification from teacher
- Can we bound the number of mistakes learner makes before converging?
Mistake Bounds: Find-S

Consider Find-S when $H = \text{conjuntion of boolean literals}$

**FIND-S:**

- Initialize $h$ to the most specific hypothesis $l_1 \land \neg l_1 \land l_2 \land \neg l_2 \ldots l_n \land \neg l_n$
- For each positive training instance $x$
  - Remove from $h$ any literal that is not satisfied by $x$
- Output hypothesis $h$.

How many mistakes before converging to correct $h$?
Mistake Bounds: Halving Algorithm

Consider the Halving Algorithm:

- Learn concept using version space
  \textsc{Candidate-Elimination} algorithm

- Classify new instances by majority vote of version space members

How many mistakes before converging to correct $h$?
- ... in worst case?
- ... in best case?
Optimal Mistake Bounds

Let $M_A(C)$ be the max number of mistakes made by algorithm $A$ to learn concepts in $C$. (maximum over all possible $c \in C$, and all possible training sequences)

$$M_A(C) \equiv \max_{c \in C} M_A(c)$$

Definition: Let $C$ be an arbitrary non-empty concept class. The optimal mistake bound for $C$, denoted $Opt(C)$, is the minimum over all possible learning algorithms $A$ of $M_A(C)$.

$$Opt(C) \equiv \min_{A \in \text{learning algorithms}} M_A(C)$$

$$VC(C) \leq Opt(C) \leq M_{\text{Halving}}(C) \leq \log_2(|C|).$$