Agenda

♢ Last week
  - Propositional logic
  - Logical equivalences

♢ This week
  - Predicate logic & rules of inference
Predicate Logic

Define:

\[ \text{UGA}(x) = "x \text{ is a UGA student}". \]

**Universe of Discourse** - all people

- \( x \) is a variable that represents an arbitrary individual in the Universe of Discourse

A *predicate* \( P \), or propositional function, is a function that maps objects in the universe of discourse to propositions

- \( \text{UGA}(\text{Paris Hilton}) \) is a **proposition**.
- \( \text{UGA}(x) \) is **not a proposition**.

\( \text{UGA}(x) \) is like an English predicate template

- \( \text{___________ is a UGA student} \)
By convention,

- Uppercase Roman letters, P, Q, R..., are used to denote predicates
- Lowercase Roman letters, x, y, z, are used to denote variables for objects.
Predicate Logic

- P(x): x > 0
  - Universe of Discourse (UD): Integers \( \mathbb{Z} \)
  - P(1): 1 > 0 = T; P(50): 50 > 0 = T; P(-10): -10 > 0 = F
  - P(x) is not a proposition

- F(x): x is Female
  - UD: all people
  - F(Maria) = T, F(Edward) = F
  - F(x) is not a proposition
Aside

- $\mathbb{Z}$ -- set of integers \{... -2, -1, 0, 1, 2, ...\}
- $\mathbb{Z}^+$ -- set of positive integers \{1, 2, 3, ...\}
- $\mathbb{R}$ -- set of real numbers
- $\mathbb{N}$ -- set of natural numbers \{0, 1, 2, ...\}
- $\mathbb{Q}$ -- set of rational numbers \{\(p/q \mid p, q \in \mathbb{Z}, q \neq 0\}\}
Predicate Logic

A predicate that states a property about one object is called monadic predicate.

- UGA(x)
- P(x)
- F(x)

A monadic predicate takes one argument.

- UD(x): all people
Predicate Logic

A predicate of the form $P(x_1,\ldots,x_n)$, $n>1$ that states the relationships among the objects $x_1,\ldots,x_n$ is called polyadic. Also, $n$-place predicate or $n$-ary predicate.

- It takes $n>1$ arguments
- $UD(x_1,\ldots,x_n) = UD(x_1) \times UD(x_2) \times \ldots \times UD(x_n)$

$L(x,y): x$ loves $y$

$UD(x) = UD(y) = \text{all people}$

$L(\text{Adam},\text{Mary}) = T \quad L(\text{Mike},\text{Jill}) = F$

$Q(x,y,z): x+y = z$

$UD(x) = UD(y) = UD(z): \text{Real numbers}$

$Q(1,1,2) = T \quad Q(1,2,5) = F$
Predicate Logic: Universal Quantifier

Suppose that $P(x)$ is a predicate on some universe of discourse.

The universal quantification of $P(x)$ ($\forall x \ P(x)$) is the proposition:

"$P(x)$ is true for all $x$ in the universe of discourse."

$\forall x \ P(x)$ reads "for all $x$, $P(x)$ is True"

$\forall x \ P(x)$ is TRUE means $P(x)$ is true for all $x$ in UD($x$).

$\forall x \ P(x)$ is FALSE means there is an $x$ in UD($x$) for which $P(x)$ is false.
Predicate Logic: Universal Quantifier

- ID(x) means x has a student ID.
  UD(x): UGA students
  \( \forall x \text{ ID}(x) \) means
  for all UGA student x, x has a student ID
  every UGA student has a student ID

- IsFemale(x) means x is a Female
  UD(x): UGA students
  \( \forall x \text{ IsFemale}(x) \) means
  for all UGA student x, x is a female
  Every UGA student is a female!

Mike is a UGA student who is a male
Mike is called a counterexample for \( \forall x \text{ IsFemale}(x) \)
Predicate Logic: Universal Quantifier

In the special case that the universe of discourse $U$, is finite, \((U = \{a_1, a_2, a_3, \ldots, a_n\})\)

\[ \forall x \ P(x) \]

corresponds to the proposition:

\[ P(a_1) \land P(a_2) \land \ldots \land P(a_n) \]

We can write a program to loop through the elements in the universe and check each for truthfulness.

If all are true, then the proposition is true.

Otherwise it is false!
Predicate Logic: Existential Quantifier

Suppose $P(x)$ is a predicate on some universe of discourse.

The existential quantification of $P(x)$ is the proposition:

“There exists at least one $x$ in the universe of discourse such that $P(x)$ is true.”

$\exists x P(x)$ reads “for some $x$, $P(x)$” or “There exists $x$, $P(x)$ is True”

$\exists x P(x)$ is \textbf{TRUE} means

there is an $x$ in $UD(x)$ for which $P(x)$ is true.

$\exists x P(x)$ is \textbf{FALSE} means:

for all $x$ in $UD(x)$ is $P(x)$ false
Predicate Logic: Existential Quantifier

Examples:

- F(x) means \( x \) is female.
  - UoD(x): UGA students
  - \( \exists x \ F(x) \) means
    - For some UGA student \( x \), \( x \) is a female.
    - There exists a UGA student \( x \) who is a female.

- Y(x) means \( x \) is less than 5 years old
  - UoD(x): UGA students
  - \( \exists x \ Y(x) \) means
    - for some UGA student \( x \), \( x \) is less than 5 years old
    - there exists a UGA student \( x \) such that \( x \) is less than 5 years old

What will be the truth-value of \( \exists x \ Y(x) \) if the UoD is all people?
Predicate Logic: Existential Quantifier

In the special case that the universe of discourse, \( U \), is finite, \( (U = \{x_1, x_2, x_3, ..., x_n\}) \)

\[ \exists x \ P(x) \]

corresponds to the proposition:

\[ P(x_1) \lor P(x_2) \lor ... \lor P(x_n) \]

We can write a program to loop through the elements in the universe and check each for truthfulness. If all are false, then the proposition is false. Otherwise, it is true!
Predicates - Quantifier negation

$\exists x \ P(x)$ means "$P(x)$ is true for some $x$."]

What about $\neg \exists x \ P(x)$?

It is not the case that ["$P(x)$ is true for some $x$."]

"$P(x)$ is not true for all $x$."

$\forall x \ \neg P(x)$

Existential negation:

$\neg \exists x \ P(x) \equiv \forall x \ \neg P(x)$. 
Predicates - Quantifier negation

\( \forall x \ P(x) \) means “\( P(x) \) is true for every \( x \).”

What about \( \neg \forall x \ P(x) \) ?

It is not the case that [“\( P(x) \) is true for every \( x \).”]

“There exists an \( x \) for which \( P(x) \) is not true.”

\[ \exists x \ \neg P(x) \]

Universal negation:

\[ \neg \forall x \ P(x) \equiv \exists x \ \neg P(x). \]
Predicate Logic - Binding

The scope of a quantifier is the part of the logical expression to which the quantifier is applied.

Example:

$$\forall x (P(x) \land \neg Q(x)) \land \exists y R(y) \lor S(z)$$
Predicate Logic - Binding

- An occurrence of a variable is *bound* if it is known or in the scope of a quantifier. Otherwise, it is *free*.

Examples:
- \( P(x) \)
- \( P(5) \)
- \( \forall x \, P(x) \)
- \( \forall x \, (P(x) \land \neg Q(x)) \land \exists y \, R(y) \lor S(z) \)
Predicate Logic - Binding

- An expression with zero free variables is an actual proposition

Ex. \( Q(x) : x > 0 \), \( R(y) : y < 10 \)

\[ \exists x \ Q(x) \land \exists y \ R(y) \]