# Decidability (What, stuff is unsolvable?)

CSCI 2670

University of Georgia

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# Outline

- Decidability
- Decidable Problems for Regular Languages
- Decidable Problems for Context Free Languages
- The Halting Problem
- Countable and Uncountable Sets
- Diagonalization
- Unrecognizable Problems

# Turing Machines

## Definition

A **Turing machine** is a 7-tuple $(Q, \Sigma, \Gamma, \delta, q_0, q_{accept}, q_{reject})$ such that

1. Q is a finite set of states;
2. $\Sigma$ is the input alphabet, and $\sqcup \notin \Sigma$;
3. $\Gamma$ is the tape alphabet, where $\sqcup \in \Gamma$ and $\Sigma \subseteq \Gamma$;
4. $\delta : (Q \times \Gamma) \rightarrow (Q \times \Gamma \times \{L, R\})$ is the transition function;
5. $q_0 \in Q$ is the start state;
6. $q_{accept} \in Q$ is the accept state;
7. $q_{reject} \in Q$ is the reject state, and $q_{reject} \neq q_{accept}$.

# Configurations

**Definition**

The state, tape contents, and tape head position constitute a **configuration**. An **accepting configuration** (**rejecting configuration**) is one in which the machine is in state $q_{accept}$ ($q_{reject}$). The **start configuration** has the tape head in the leftmost position, and the input string is written on it.

- A string $01010q_7010111$ may be used to indicate a configuration.
- The transition function maps one configuration to another.

**Definition**

A Turing Machine M accepts a string $w$ if there exists a sequence of configurations $C_1, C_2, \ldots C_n$ such that

1. $C_1$ is the starting configuration of $M$ on input $w$,
2. $C_n$ is an accepting configuration,
3. for each $1 \leq i < n$, $C_i$ yields $C_{i+1}$.

$L(M)$, **language of** $M$ is the set of strings accepted by $M$.

# Turing Recognizable and Decidable Languages

**Definition**

A language $L$ is **Turing–recognizable** (**recursively enumerable**) if there is a Turing machine $M$ such that $L(M) = L$.

- It is possible for a TM to never reach a halting configuration. On any given input $w$, it might instead **loop**.
  - These are the three options for any input: **accept**, **reject**, **loop**.
- A TM $M$ that halts on every input is called a **decider**.
- If $L(M) = L$, then $M$ is said to **decide** $L$.

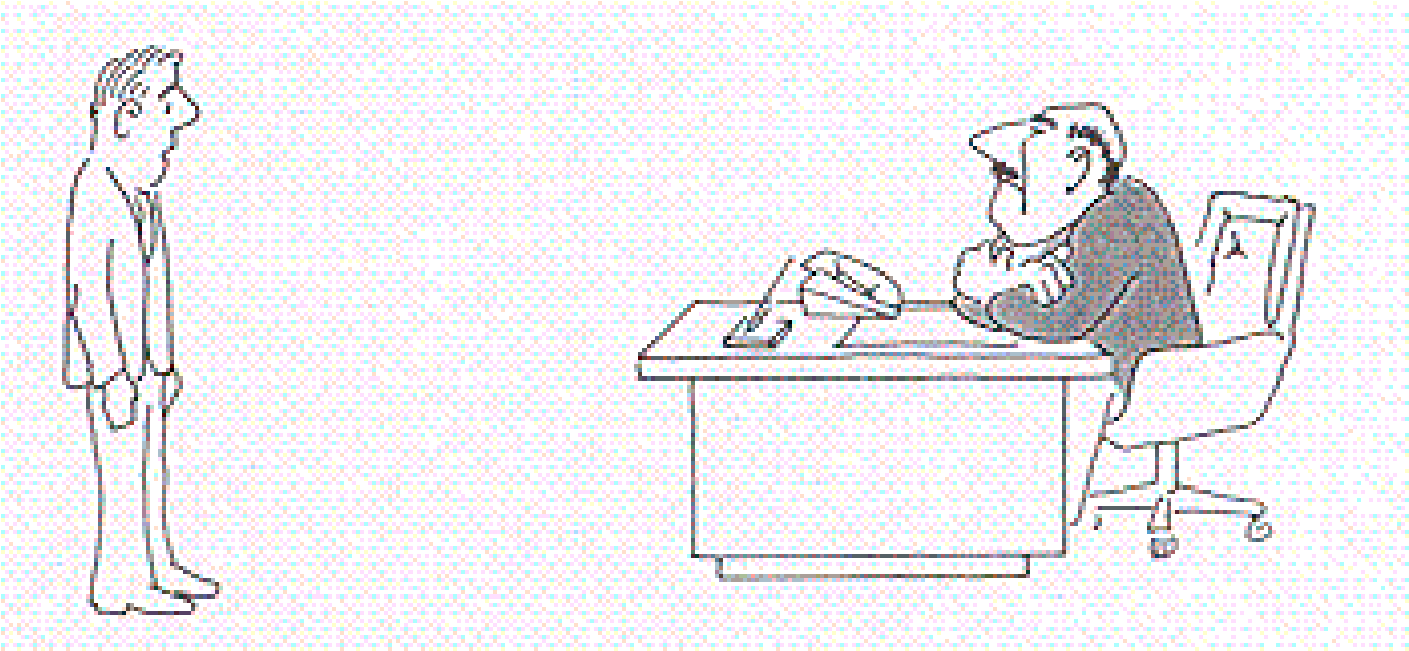
**Definition**

A language $L$ is **Turing–decidable** (also called **recursive**) if there is a Turing machine $M$ that decides $L$.

# Decidability and Undecidability

- The definitions indicate that not all languages are decidable.
- The Church-Turing Thesis equates "algorithmically solvable" with solvable by Turing machines.
  - A yes/no problem is algorithmically decidable if there is some Turing machine that will decide it.
  - Given any instance of the problem (input), the machine will always halt in finite time with the correct answer (yes or no).
  - A problem is not algorithmically solvable if every TM loops on at least one input.
- Chapter 4 discusses problems that are solvable/unsolvable by TMs.
- Knowing which problems are unsolvable is important (and pretty cool).
  - Why try to write a program to solve a problem when someone's already told you it can't be done.

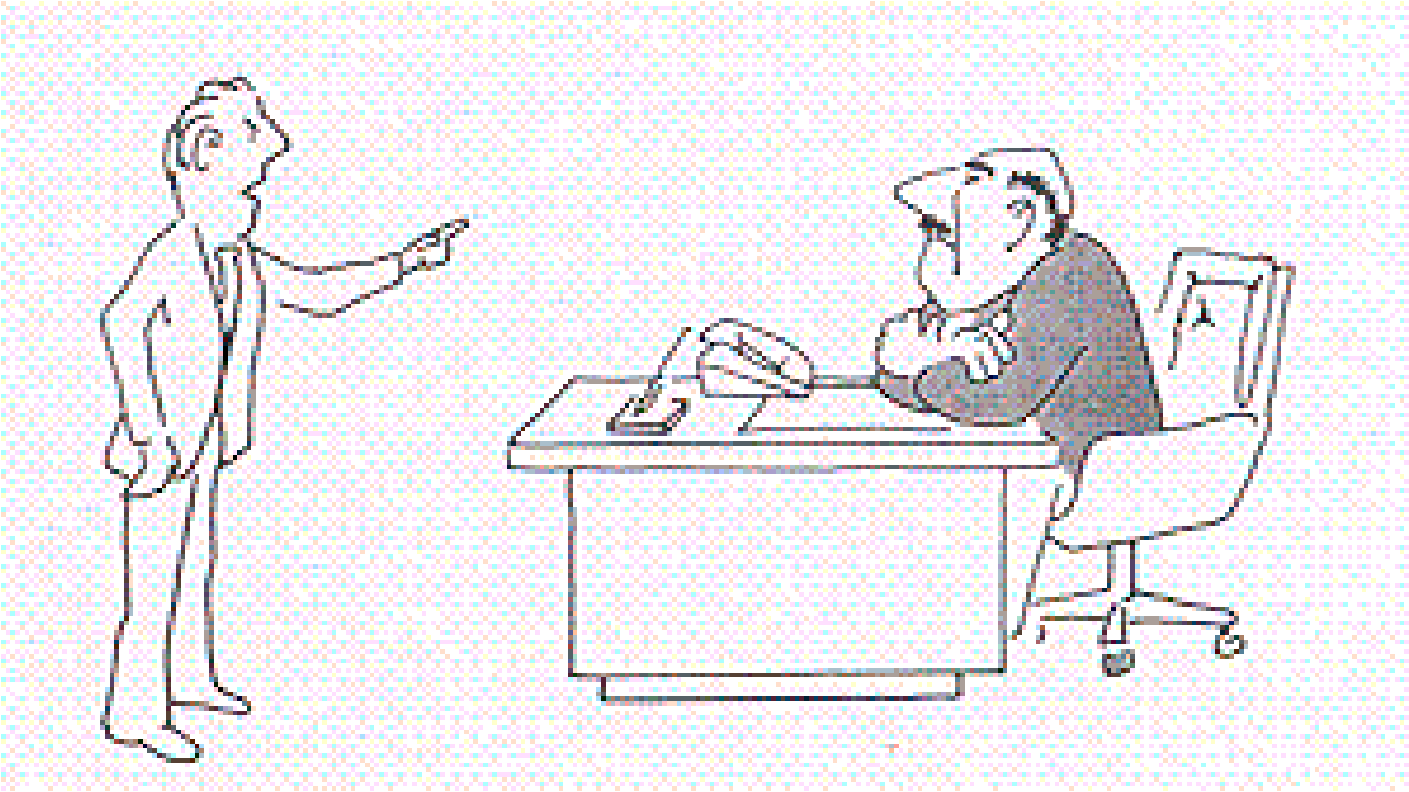
# Decidability and Undecidability



"I can't find an ... algorithm, I guess I'm just too lame."

- **source:** MR Garey and DS Johnson. Computers and Intractability. W. H. Freeman, 1970.
- **image:** `http://max.cs.kzoo.edu/~kschultz/CS510/ClassPresentations/NPCartoons.html`

# Decidability and Undecidability



"I can't find an ... algorithm, because no such algorithm is possible."

- **source:** MR Garey and DS Johnson. Computers and Intractability. W. H. Freeman, 1970.
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# Decidable Problems for Regular Languages

The following languages/problems are all decidable.

- $A_{DFA} = \{\langle B, w\rangle | B \text{ is a DFA and } w \in L(B)\}$
- For any DFA $B$, $L(B)$ is decidable.
- $A_{NFA} = \{\langle B, w\rangle | B \text{ is an NFA and } w \in L(B)\}$
- $A_{REX} = \{\langle R, w\rangle | R \text{ is a regular expression that generates } w\}$
- $E_{DFA} = \{\langle B\rangle | B \text{ is a DFA and } L(B) = \varnothing\}$
- $EQ_{DFA} = \{\langle A, B\rangle | A, B \text{ are DFAs and } L(A) = L(B)\}$

- In each case, the input is a string representing, for instance, a DFA and an input string $w$.
- We design a TM to decide the problem (typically by simulating the machines described in the input).
- A reasonable representation of the DFA is needed. E.g.,

$$q_0, \ldots, q_n \# a_1, \ldots, a_m \# (q_0, a_0, q_i), \ldots \# q_0 \# q_i, q_j, \ldots \# w_0 \ldots w_k$$

# Decidable Problems for Regular Languages

**Theorem**

*$A_{DFA}$ is decidable, where $A_{DFA} = \{\langle B, w\rangle | B$ is a DFA and $w \in L(B)\}$*

# Decidable Problems for Regular Languages

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**(Proof idea)**

- We construct a Turing machine $M$ to decide the problem.
- Scan the input string $\langle B, w\rangle$, determining whether the input constitutes a valid DFA and string $w$. If not, then reject.
- Scan the input string repeatedly.
  - Keep track of the current DFA state and position of $w$.
  - Find the appropriate transition to update the state appropriately.
- After processing $w_k$, check whether $B$ is in an accept state.
  - If so, then accept $\langle B, w\rangle$. Otherwise reject $\langle B, w\rangle$.

# Decidable Problems for Regular Languages

**Theorem**

*If $B$ is a DFA, then $L(B)$ is decidable.*

- This is clearly true, as every DFA halts on every input.
- We can easily create a TM $M_B$ to recognize $L(B)$ by modifying the transition function of $B$ to fit the form for TMs.
  - If $\delta_B(q, a) = q'$, then $\delta_{M_B}(q, a) = (q', a, R)$.

Alternatively, we can use $M$ from the previous proof in $M_B$.

- On input $w$, run $M$ on $\langle B, w \rangle$.
- If $M$ accepts $\langle B, w \rangle$, accept $w$.
- Otherwise reject $w$.

# Decidable Problems for Regular Languages

**Theorem**

*$A_{NFA}$ is decidable, where $A_{NFA} = \{\langle B, w\rangle | B$ is an NFA and $w \in L(B)\}$*

# Decidable Problems for Regular Languages

**Theorem**

*$A_{NFA}$ is decidable, where $A_{NFA} = \{\langle B, w\rangle | B$ is an NFA and $w \in L(B)\}$*

**(Proof idea)**

- We construct another Turing machine $N$ to decide $A_{NFA}$:
- Scan the input string $\langle B, w\rangle$, determining whether the input constitutes a valid NFA and string $w$. If not, then reject.
- Convert $B$ to a DFA $C$ (using the construction described in Ch 1).
- Run $M$ on $\langle C, w\rangle$.
- Accept $\langle B, w\rangle$ if $M$ accepts $\langle C, w\rangle$. Otherwise reject $\langle B, w\rangle$.

# Decidable Problems for Regular Languages

**Theorem**

*$A_{REX}$ is decidable, where*

$$A_{REX} = \{\langle R, w\rangle | R \text{ is a regular expression that generates } w\}$$

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**(Proof idea)**

- We construct another Turing machine $P$ to decide $A_{REX}$:
- Scan the input string $\langle R, w\rangle$, determining whether the input constitutes a valid reg-exp and string $w$. If not, then reject.
- Convert $R$ to an NFA $A$ (using the construction described in Ch 1).
- Run $N$ on $\langle A, w\rangle$.
- Accept $\langle R, w\rangle$ if $N$ accepts $\langle A, w\rangle$. Otherwise reject $\langle R, w\rangle$.

# Decidable Problems for Regular Languages

**Theorem**

*$E_{DFA}$ is decidable, where*

$$E_{DFA} = \{\langle A \rangle | A \text{ is a DFA and } L(A) = \varnothing\}$$

Idea: Check whether an accept state is reachable from $A$'s start state.

# Decidable Problems for Regular Languages

**Theorem**

*$E_{DFA}$ is decidable, where*

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Idea: Check whether an accept state is reachable from $A$'s start state.

**(Proof idea)**

- We construct Turing machine $T$ to decide $E_{DFA}$:
- Scan the input string $\langle A \rangle$, determining whether the input constitutes a valid DFA. If not, then reject.
- We assume states of $A$ are listed left to right, with $q_0$ being leftmost.
- Mark $q_0$. Using the transition function of $A$, mark any unmarked state $q'$ such that $\delta(q, a) = q'$ and $q$ is already marked.
- Continue until no new nodes can be marked.
- If an accept state is marked, then reject $\langle A \rangle$. Otherwise accept it.

# Decidable Problems for Regular Languages

**Theorem**

*$EQ_{DFA}$ is decidable, where $EQ_{DFA} = \{\langle A, B\rangle | A, B$ are DFAs and $L(A) = L(B)\}$*

- Regular languages are closed under complementation, union, intersection.
- Given $A$ and $B$, we can construct a DFA for the **symmetric difference**:

$$Diff = (L(A) \cap \overline{L(B)}) \cup (L(B) \cap \overline{L(A)})$$

- If $Diff$ is nonempty, then $L(A) \neq L(B)$.

# Decidable Problems for Regular Languages

**Theorem**

*$EQ_{DFA}$ is decidable, where $EQ_{DFA} = \{\langle A, B\rangle | A, B$ are DFAs and $L(A) = L(B)\}$*

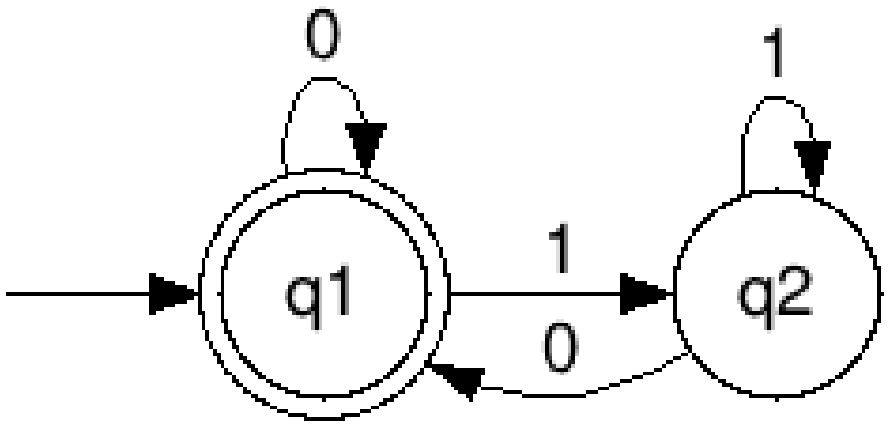
- Regular languages are closed under complementation, union, intersection.
- Given $A$ and $B$, we can construct a DFA for the **symmetric difference**:
$$Diff = (L(A) \cap \overline{L(B)}) \cup (L(B) \cap \overline{L(A)})$$
- If $Diff$ is nonempty, then $L(A) \neq L(B)$.

**(Proof idea)**

- We construct a Turing machine $F$ to decide $EQ_{DFA}$:
- Scan the input string $\langle A, B\rangle$ determining whether the input constitutes two valid DFAs $A$ and $B$. If not, then reject.
- Construct a string representing the DFA $C$, where $L(C)$ is the symmetric difference of $L(A)$ and $L(B)$.
- Run Turing Machine $T$ on $\langle C\rangle$. (Recall that $T$ decides $E_{DFA}$.)
- If $T$ accepts $\langle C\rangle$, accept $\langle A, B\rangle$. Otherwise reject it.

# Decidable Problems for Regular Languages

Answer all questions for the following DFA M and give reasons for your answers.




1. Is $\langle M, 0110 \rangle \in A_{DFA}$?
2. Is $\langle M, 001 \rangle \in A_{DFA}$?
3. Is $\langle M \rangle \in A_{DFA}$?
4. Is $\langle M, 0110 \rangle \in A_{REX}$?
5. Is $\langle M \rangle \in E_{DFA}$?
6. Is $\langle M, M \rangle \in EQ_{DFA}$?

# Decidable Problems for Context Free Languages

The following languages/problems are decidable.

- $A_{CFG} = \{\langle G, w\rangle | G \text{ is a CFG generating } w\}$
- $E_{CFG} = \{\langle G\rangle | G \text{ is a CFG and } L(A) = \varnothing\}$
- If $G$ is a CFG, then $L(G)$ is decidable.

The following language/problem is **NOT** decidable.

- $EQ_{CFG} = \{\langle G, H\rangle | G, H \text{ are CFGs and } L(G) = L(H)\}$

Why won't using $(L(G) \cap \overline{L(H)}) \cup (L(H) \cap \overline{L(G)})$ work in this case?

# An Undecidable Problem for Context Free Languages

The following language/problem is **NOT** decidable.

- $EQ_{CFG} = \{\langle G, H\rangle | G, H \text{ are CFGs and } L(G) = L(H)\}$

Context free grammars are not closed under complementation or intersection, and so we cannot use

$$(L(G) \cap \overline{L(H)}) \cup (L(H) \cap \overline{L(G)})$$

as was done for $EQ_{DFA}$.

# Decidable Problems for Context Free Languages

**Theorem**

*$A_{CFG}$ is decidable, where $A_{CFG} = \{\langle G, w\rangle | G$ is a CFG generating $w\}$.*

# Decidable Problems for Context Free Languages

**Theorem**

*$A_{CFG}$ is decidable, where $A_{CFG} = \{\langle G, w\rangle | G$ is a CFG generating $w\}$.*

Idea: If $G$ is in CNF, then it takes at most $2n - 1$ steps to generate $w$.

**(Proof idea)**

- We construct a Turing machine $S$ to decide the problem.
- Scan the input string $\langle G, w\rangle$, determining whether the input constitutes a valid CFG and string $w$. If not, then reject.
- Convert $G$ to CNF, using the procedure described in Ch 2.
- If $w = \varepsilon$, look for grammar rule $S \to \varepsilon$. If present, then accept. Otherwise reject.
- If $w \neq \varepsilon$, systematically generate all derivations of at most $2n - 1$ steps, where $|w| = n$.
- If any of these derivations produce $w$, then accept. If not, reject.

# Decidable Problems for Context Free Languages

**Theorem**

*$E_{CFG}$ is decidable, where $E_{CFG} = \{\langle G \rangle | G$ is a CFG and $L(G) = \varnothing\}$.*

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Idea: Work backwards from terminal symbols to start symbol $S$.

**(Proof idea)**

- We construct a Turing machine $R$ to decide the problem.
- Scan the input string $\langle G \rangle$, determining whether the input constitutes a valid CFG. If not, then reject.
- Convert $G$ to Chomsky Normal Form, as described in Ch 2.
- For each rule $A \to a$, mark every occurrence of $A$ in the rules.
- For each rule of the form $A \to BC$, if both $B$ and $C$ are marked, then mark every occurrence of $A$ in the rules.
- Continue until no new variables are marked.
- If $S$ is NOT marked, then accept $\langle G \rangle$. Otherwise Reject.

# Decidable Problems for Context Free Languages

**Theorem**

*$E_{CFG}$ is decidable, where $E_{CFG} = \{\langle G \rangle | G$ is a CFG and $L(G) = \varnothing\}$.*

An easier alternative.

**(Proof idea)**

- We construct a Turing machine $R$ to decide the problem.
- Scan the input string $\langle G \rangle$, determining whether the input constitutes a valid CFG. If not, then reject.
- Mark every terminal in every rule.
- For each rule of the form $A \to w$, if every symbol of $w$ is marked, mark every occurrence of $A$ in the rules.
- Continue until no new variables are marked.
- If $S$ is NOT marked, then accept $\langle G \rangle$. Otherwise Reject.

# Decidable Problems for Context Free Languages

**Theorem**

*If $G$ is a CFG, then $L(G)$ is decidable.*

# Decidable Problems for Context Free Languages

**Theorem**

*If $G$ is a CFG, then $L(G)$ is decidable.*

We can use the machine $S$, which decides $A_{CFG} = \{\langle H, w\rangle | H$ is a CFG generating $w\}$, to construct a TM $M_G$ to decide $L(G)$.

- On input $w$, run $S$ on $\langle G, w\rangle$.
- If $S$ accepts $\langle G, w\rangle$, accept $w$.
- Otherwise reject $w$.

# Closure Property

| ?Is it closed under | Set of Languages | | | |
|---|---|---|---|---|
| Operations | RL | CFL | T-Recog | T-Deci |
| Union | Y | Y | Y | Y |
| Concatenation | Y | Y | Y | Y |
| Star | Y | Y | Y | Y |
| Intersection | Y | N | Y | Y |
| Complementation | Y | N | N | Y |

- RL: Regular Languages; CFL: Context-Free Languages; T-Recog: Turing-Recognizable Languages; T-Deci: Turing Decidable Languages.
- The intersection of a CFL and a RL is a CFL.
- It is recommended by Ian Lindsey Berrigan.

# Undecidable Problems

- There are problems/languages that are undecidable.
- In general, program verification (deciding whether a program runs "correctly") is undecidable.
- **The Halting Problem**: The most famous undecidable problem in CS.

**Theorem**

*$A_{TM}$ is undecidable, where*

$$A_{TM} = \{\langle M, w\rangle | M \text{ is a TM and } M \text{ accepts } w\}.$$

- The proof of this is a proof by contradiction.

# Undecidable Problems

- Before we get to the proof, let us first observe that $A_{TM}$ is Turing-recognizable.
- Thus this theorem shows that recognizers are more powerful than deciders.
- The following TM $U$ recognizes $A_{TM}$.

$U =$ "On input $\langle M, w \rangle$, where $M$ is a TM and $w$ is a string:

1. Simulate $M$ on input $w$.
2. If $M$ ever enters its accept state, accept; if $M$ ever enters its reject state, reject."

Note that this machine loops on $\langle M, w \rangle$ if $M$ loops on $w$. Hence $A_{TM}$ sometimes called the halting problem.

# The Halting Problem

**Theorem**

*$A_{TM}$ is undecidable, where*

$$A_{TM} = \{\langle M, w\rangle | M \text{ is a TM and } M \text{ accepts } w\}.$$

**Proof.**

Suppose $A_{TM}$ is decidable. Then there's a machine $H$ deciding it.

- The input of $H$ is a string $\langle M, w\rangle$.
- $M$ is a TM, and $w$ is a string.
- $H$ runs $M$ on $w$.
  - If $M$ accepts $w$, then $H$ stops and accepts.
  - If $M$ does not accept $w$, then $H$ stops and rejects.

Observe that there is no string on which $H$ does not halt.

# The Halting Problem

**continued.**

Given $H$, we construct a new Turing machine $D$:

- The input of $D$ is a string $\langle M \rangle$.
- $D$ runs $H$ on $\langle M, \langle M \rangle \rangle$.
  - If $H$ accepts $\langle M, \langle M \rangle \rangle$, then $D$ **rejects** $\langle M \rangle$.
  - If $H$ rejects $\langle M, \langle M \rangle \rangle$, then $D$ **accepts** $\langle M \rangle$.

- Given the definition of $H$ and $D$:
- On input $\langle M \rangle$,
  - $D$ **accepts** $\langle M \rangle$ if $M$ does not accept $\langle M \rangle$.
  - $D$ **rejects** $\langle M \rangle$ if $M$ accepts $\langle M \rangle$.

# The Halting Problem

- Given the definition of $H$ and $D$:
- On input $\langle M \rangle$,
  - $D$ **accepts** $\langle M \rangle$ if $M$ does not accept $\langle M \rangle$.
  - $D$ **rejects** $\langle M \rangle$ if $M$ accepts $\langle M \rangle$.

- On input $\langle D \rangle$, we then find
  - $D$ **accepts** $\langle D \rangle$ if $D$ does not accept $\langle D \rangle$.
  - $D$ **rejects** $\langle D \rangle$ if $D$ accepts $\langle D \rangle$.

- Under the assumption that $A_{TM}$ is decidable, there is a machine $D$ which simultaneously accepts and rejects its own specification $\langle D \rangle$.
- This is a contradiction!
- And so the assumption that $A_{TM}$ is decidable is false.
- We conclude that $A_{TM}$ is undecidable.

$\square$

# Countability and Uncountability/Diagonalization

- Implicitly, the proof that $A_{TM}$ is undecidable relies on Cantor's **diagonalization method**.
- This is used to prove that a set is **uncountable**.

  - Recall that a set $S$ is **countable** if either it is finite or there is a one-to-one correspondence between elements of $S$ and those of $\mathbf{N}$.
  - That is, there is a bijective function $f$ from $\mathbf{N}$ to $S$.
    - injective: For all $a, b \in \mathbf{N}$, if $a \neq b$, then $f(a) \neq f(b)$.
    - surjective: For all $b \in S$, there is an $a \in \mathbf{N}$ such that $f(a) = b$.
  - Intuitively, $S$ is countable if we can write the elements of $S$ as a list, and each element of $S$ appears exactly once.

- A set $A$ is countable if either it is finite or it has the same size as $\mathbf{N}$.
- The set of positive even integers is countable, as is the set of positive odd numbers.
- The set of positive rational numbers is countable as well.

# The set of finite strings are countable

The set of finite strings $S$ over any finite alphabet $\Sigma$ is countably infinite.

- Assume an ordering of symbols in A.
- The strings of $S$ can be ordered as follows:
  - For any $n \geq 0$, all string of length $n$ come before all those of length $n + 1$.
  - For any given $n$, arrange the strings of length $n$ in according to the ordering of the alphabet symbols (as in a dictionary).
- Crucially, for any $n$, only a finite number of strings have length $n$.

- As a corollary to this, the set of Turing machines is countably infinite.
- Each TM can be written as a finite string over a given alphabet.

# The set of infinite strings are uncountable

- In contrast, the **set of infinite strings** $S$ over a finite alphabet $\Sigma$ is **not countable**.
- We'll prove this using Cantor's diagonalization method, a type of proof by contradiction. Assume $\Sigma = \{0, 1\}$ as the alphabet.
- An infinite binary sequence is an unending sequence of 0s and 1s.
- Each element of $S$ is an infinitely long string of 0s and 1s.
  - E.g., $0101010101111101010101011100010 1 \ldots$.
  - We will refer to particular bits of a string: $b_1 b_2 b_3 \ldots$.

**Theorem**

*The Set $B$ of infinitely long strings over $\Sigma = \{0, 1\}$ is uncountable.*

# The set of infinite strings are uncountable

**Theorem**

*The Set $B$ of infinite strings over $\Sigma = \{0, 1\}$ is uncountable.*

**Proof.**

- Assume $B$ is countable.
- As such, the elements of $B$ can be listed as a sequence $s_1$, $s_2$, ... (there's a first element, a second element, a third ...).
- Every element of $B$ appears exactly once in the sequence.
  - $s_1 = b_{1,1} b_{1,2} b_{1,3} b_{1,4} b_{1,5} \ldots$
  - $s_2 = b_{2,1} b_{2,2} b_{2,3} b_{2,4} b_{2,5} \ldots$
  - $s_3 = b_{3,1} b_{3,2} b_{3,3} b_{3,4} b_{3,5} \ldots$
  - $\vdots$
- Above, $b_{i,j}$ refers to the $j$th bit of the $i$th element of $B$.

# The set of infinite strings are uncountable

**Theorem**

*The Set $B$ of infinite strings over $\Sigma = \{0, 1\}$ is uncountable.*

**Proof, continued.**

- We construct an infinite bitstring $X = c_1 c_2 c_3 \ldots$ by examining the diagonal:
  - $s_1 = \mathbf{b_{1,1}} b_{1,2} b_{1,3} b_{1,4} b_{1,5} \ldots$
  - $s_2 = b_{2,1} \mathbf{b_{2,2}} b_{2,3} b_{2,4} b_{2,5} \ldots$
  - $s_3 = b_{3,1} b_{3,2} \mathbf{b_{3,3}} b_{3,4} b_{3,5} \ldots$
  - $\vdots$
- For each $i \geq 1$, if $b_{i,i} = 1$, then $c_i = 0$. Otherwise $c_i = 1$.
- For instance:

$$\begin{array}{ll} \text{If } b_{1,1} b_{2,2} b_{3,3} b_{4,4} b_{5,5} \ldots = & 01101 \ldots \\ \text{Then } c_1 c_2 c_3 c_4 c_5 \ldots = & 10010 \ldots \end{array}$$

- Observe that for any $k$, $b_{k,k} \neq c_k$.

# The set of infinite strings are uncountable

**Theorem**

*The Set $B$ of infinite strings over $\Sigma = \{0, 1\}$ is uncountable.*

**Proof, continued.**

- We construct an infinite bitstring $X = c_1 c_2 c_3 \ldots$:

$$\begin{array}{ll} \text{If } b_{1,1} b_{2,2} b_{3,3} b_{4,4} b_{5,5} \ldots = & 01101\ldots \\ \text{Then } c_1 c_2 c_3 c_4 c_5 \ldots = & 10010\ldots \end{array}$$

- Observe that $X$ is an infinite bitstring and so must be some $s_n$ on the list (it must appear somewhere on the list).
- And so $b_{n,n}$ (the nth bit of $s_n$) is the same as $c_n$.
- However, by construction of $X$, $b_{n,n} \neq c_n$.
- A contradiction!
- And so the assumption that $B$ is countable must be false.

$\square$

# Characteristic Sequences

- For any finite $\Sigma$, $\Sigma^*$ is a set of finite strings and so is countable.
- That is, we can list out the strings of $\Sigma^*$: $s_1, s_2, s_3, \ldots$.
- Any language $L$ over $\Sigma$ can thus be represented as an infinite bitstring $b_1 b_2 b_3 \ldots$.
  - $s_i \in L$ if and only if $b_i = 1$.
  - $s_i \notin L$ if and only if $b_i = 0$.
- The string $b_1 b_2 b_3 \ldots$ is called the **characteristic sequence** for $L$.

- It is clear that, for any $\Sigma$, there is a one-to-one correspondence between the languages over $\Sigma$ and the set of infinite bit-strings.
- Given the previous proof, the following may be inferred.

**Theorem**

*For any $\Sigma$, the set of languages over $\Sigma$ is uncountable.*

**Corollary**

*There exist languages that are not Turing Recognizable.*

# Back to the Halting Problem

- The set of Turing Machines is countable, and so we can list them.
- The set of finite strings over $\Sigma$ is also countable.
- We then can construct the following table.

| | $\langle M_1 \rangle$ | $\langle M_2 \rangle$ | $\langle M_3 \rangle$ | $\langle M_4 \rangle$ | $\ldots$ |
|---|---|---|---|---|---|
| $M_1$ | 1 | 0 | 1 | 1 | $\ldots$ |
| $M_2$ | 0 | 1 | 1 | 0 | $\ldots$ |
| $M_3$ | 0 | 1 | 0 | 1 | $\ldots$ |
| $M_4$ | 1 | 0 | 1 | 1 | $\ldots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ldots$ |

- A 1 in a cell $i, j$ means that $M_i$ accepts $\langle M_j \rangle$.
- A 0 in a cell $i, j$ means that $M_i$ does not accept $\langle M_j \rangle$.
- And so the table gives a partial description of $H$.

# Back to the Halting Problem

- Recall $D$: On input $\langle M_i \rangle$,
  - $D$ **accepts** $\langle M_i \rangle$ if $M_i$ does not accept $\langle M_i \rangle$.
  - $D$ **rejects** $\langle M_i \rangle$ if $M_i$ accepts $\langle M_i \rangle$.

| | $\langle M_1 \rangle$ | $\langle M_2 \rangle$ | $\langle M_3 \rangle$ | $\langle M_4 \rangle$ | $\ldots$ |
|---|---|---|---|---|---|
| $M_1$ | **1** | 0 | 1 | 1 | $\ldots$ |
| $M_2$ | 0 | **1** | 1 | 0 | $\ldots$ |
| $M_3$ | 0 | 1 | **0** | 1 | $\ldots$ |
| $M_4$ | 1 | 0 | 1 | **1** | $\ldots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ldots$ |

- So, the output of $D$ can be found by flipping bits in the diagonal.

# Back to the Halting Problem

- Recall $D$: On input $\langle M_i \rangle$,
  - $D$ **accepts** $\langle M_i \rangle$ if $M_i$ does not accept $\langle M_i \rangle$.
  - $D$ **rejects** $\langle M_i \rangle$ if $M_i$ accepts $\langle M_i \rangle$.

| | $\langle M_1 \rangle$ | $\langle M_2 \rangle$ | $\langle M_3 \rangle$ | $\langle M_4 \rangle$ | $\dots$ | $\langle D \rangle$ | $\dots$ |
|---|---|---|---|---|---|---|---|
| $M_1$ | $\underline{\mathbf{1}}$ | 0 | 1 | 1 | $\dots$ | 1 | $\dots$ |
| $M_2$ | 0 | $\underline{\mathbf{1}}$ | 1 | 0 | $\dots$ | 0 | $\dots$ |
| $M_3$ | 0 | 1 | $\underline{\mathbf{0}}$ | 1 | $\dots$ | 1 | $\dots$ |
| $M_4$ | 1 | 0 | 1 | $\underline{\mathbf{1}}$ | $\dots$ | 1 | $\dots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\dots$ | $\vdots$ | $\dots$ |
| $D$ | 0 | 0 | 1 | 0 | $\dots$ | $\underline{\mathbf{???}}$ | $\dots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\dots$ | $\dots$ | $\dots$ |

- So, the output of $D$ can be found by flipping bits in the diagonal.
- But $D$ must be equal to some $M_n$ and so appear on the list.
- At cell $(n, n)$, the contents is both 1 and 0. A contradiction!

# Unrecognizable Languages

**Definition**

Language $L$ is **co-Turing recognizable** (**co-recursively enumerable**) if it is the complement of a Turing recognizable language.

**Theorem**

*$L$ is Turing decidable if and only if $L$ is recognizable and co-recognizable.*

**Proof.**

- If $L$ is decidable, then it is clearly recognizable. Also, if $M$ is the machine deciding $L$, we construct a new machine $M'$ to decide $\overline{L}$ by simply doing the opposite of $M$. So $M'$ recognizes $\overline{L}$. As such, $L$ is co-recognizable.
- Now assume that $L$ is recognizable and co-recognizable. Then $\overline{L}$ is recognizable, and TMs $M_1$ and $M_2$ recognize $L$ and $\overline{L}$. We construct TM $M$ deciding $L$. On input $w$, $M$ runs $M_1$ and $M_2$ in parallel. If $M_1$ accepts $w$, then $M$ accepts. If $M_2$ accepts $w$, then $M$ rejects. It is clear that it accepts all strings of $L$ and reject all those not in $L$.

$\square$

# Unrecognizable Languages

**Theorem**

*$L$ is Turing decidable if and only if $L$ is recognizable and co-recognizable.*

- $A_{TM} = \{\langle M, w\rangle | M$ is a TM and $M$ accepts $w\}$ is undecidable.
- $A_{TM}$ is clearly recognizable, however. $H$ runs $M$ on $w$ and accepts if and only if $M$ accepts $w$.
- From this, the below holds.

**Theorem**

$\overline{A_{TM}}$ *is* **not recognizable**.

**Proof.**

If $\overline{A_{TM}}$ is recognizable, then a machine $M$ recognizes it. We can construct a machine $N$ to decide $A_{TM}$ by running $M$ and $H$ simultaneously. On any input $w$, one of them must accept $w$. If $H$ accepts $w$, then $N$ accepts. If $M$ accepts, then $N$ rejects. But $A_{TM}$ is not decidable, and so no such machine $M$ exists. □

# More Decidable Problems

Show the following languages/problems are decidable.

- $EQ_{DFA,REX} = \{\langle A, R\rangle | A$ is a DFA, $R$ is a regular expression and $L(A) = L(R)\}$
- $A = \{\langle M\rangle | M$ is a DFA which does not accept any string containing an odd number of 1s $\}$
- $A_{\varepsilon-CFG} = \{\langle G\rangle | G$ is a CFG that generates $\varepsilon\}$

# More Decidable Problems

Show the following languages/problems are decidable.

- $S = \{\langle M \rangle | M$ is a DFA that accepts $w^R$ whenever it accepts $w\}$
- $ALL_{DFA} = \{\langle A \rangle | A$ is a DFA and $L(A) = \Sigma^*\}$
- $A = \{\langle R, S \rangle | R$ and $S$ are regular expressions and $L(R) \subseteq L(S)\}$
- $A = \{\langle R \rangle | R$ is a regular expression describing a language containing at least one string $w$ that has 111 as a substring(i.e., $w = x111y$ for some $x$ and $y$ $\}$

# More Decidable Problems

Show the following languages/problems are decidable.

- $INFINITE_{DFA} = \{\langle A \rangle | A$ is a DFA and L(A) is an infinite language$\}$
- $INFINITE_{PDA} = \{\langle B \rangle | B$ is a PDA and L(B) is an infinite language$\}$

# More Decidable Problems

**Theorem**

*$INFINITE_{DFA}$ is decidable, where*

$$INFINITE_{DFA} = \{\langle A \rangle | A \text{ is a DFA and } L(A) \text{ is an infinite language}\}.$$

**Proof.**

The following TM $X$ decides $INFINITE_{DFA}$.
$X =$ "On input $\langle A \rangle$ where $A$ is a DFA:

1. Let $k$ be the number of states of $A$.
2. Construct a DFA $D$ that accepts all strings of length $k$ or more.
3. construct a DFA $M$ such that $L(M) = L(A) \cap L(D)$.
4. Test $L(M) = \emptyset$, using the $E_{DFA}$ decider $T$.
5. If $T$ accepts, reject; if $T$ rejects, accept."

# More Decidable Problems

**Theorem**

*$INFINITE_{PDA}$ is decidable, where*

$$INFINITE_{PDA} = \{\langle B \rangle | B \text{ is a PDA and } L(B) \text{ is an infinite language}\}.$$

**Proof.**

The following TM $Y$ decides $INFINITE_{PDA}$.
$Y$ = "On input $\langle B \rangle$ where $B$ is a PDA:

1. Convert $B$ to a CFG $G$ and compute $G$'s pumping length $p$.
2. Construct a regular expression $E$ that contains all strings of length $p$ or more.
3. construct a CFG $H$ such that $L(H) = L(G) \cap L(E)$.
4. Test $L(H) = \emptyset$, using the $E_{CFG}$ decider $R$.
5. If $R$ accepts, reject; if $R$ rejects, accept."