Undecidable Problems and Reducibility

CSCI 2670

University of Georgia

Fall 2014

Reducibility

- We show a problem decidable/undecidable by **reducing** it to another problem. One type of reduction: **mapping reduction**.

Definition

- Let A, B be languages over Σ. A is **mapping reducible** to B, written $A \leq_m B$, if there is a **computable function** $f : \Sigma^* \to \Sigma^*$ such that

$$w \in A \text{ if and only if } f(w) \in B.$$

- Function f is called the **reduction** of A to B.

Definition

A function $f : \Sigma^* \to \Sigma^*$ is a **computable function** if some Turing machine M, on every input w, halts with just $f(w)$ on its tape.

- A TM computes a function by starting with the input to the function on the tape and halting with the output of the function on the tape.

Reducibility

Definition

- Let A, B be languages over Σ. A is **mapping reducible** to B, written $A \leq_m B$, if there is a **computable function** $f : \Sigma^* \to \Sigma^*$ such that

$$w \in A \text{ if and only if } f(w) \in B.$$

- Function f is a **reduction** from A to B.

- The idea here is that if B is decidable, then A must be decidable, too.
  - (The proof is shown in the next slide).
- By contraposition, if A is **not** decidable, then B is not decidable.

Note that A could be decidable and B undecidable (consider what happens when f is not surjective).

Mapping Reducibility

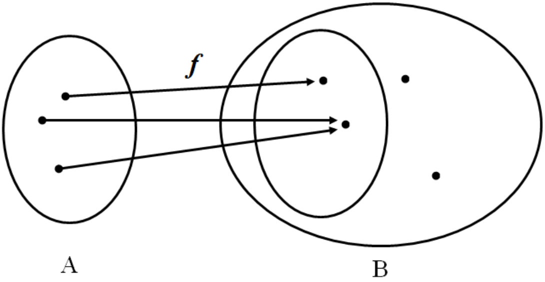
Theorem

If $A \leq_m B$ and B is decidable, then A is decidable.

Proof.

- Suppose $A \leq_m B$ and B is decidable. Then there exists a TM M to decide B, and there is a computable function f such that $w \in A$ if and only if $f(w) \in B$.
- We construct a decider N for A that acts as follows:
    - On input w, compute $f(w)$.
    - Run M on $f(w)$.
    - If M accepts $f(w)$, then accept. Otherwise, reject.

Reducibility



- Note that A could be decidable and B undecidable.
- Consider what happens when f is not surjective.

Undecidable Problems from Language Theory: $HALT_{TM}$

Recall that the following problem is undecidable.

$$A_{TM} = \{\langle M, w\rangle | M \text{ is a TM and } M \text{ accepts } w\}.$$

- Before, we called this the "halting problem".
- Really, we should call it the "acceptance problem" for TMs.
- And we should call the language $HALT_{TM}$ below the halting problem.

$$HALT_{TM} = \{\langle M, w\rangle | M \text{ is a TM and } M \text{ halts on input } w\} \text{ is undecidable.}$$

- We use the acceptance problem to prove $HALT_{TM}$ undecidable.

Theorem

$HALT_{TM}$ is undecidable.

Undecidable Problems from Language Theory: $HALT_{TM}$

Theorem

$HALT_{TM} = \{\langle M, w\rangle | M \text{ is a TM and } M \text{ halts on input } w\}$ *is undecidable.*

- Suppose we want to decide A_{TM}
- On input $\langle M, w\rangle$, if M halts on w, then it's "safe" to run M on w.
- If M accepts w, then we accept $\langle M, w\rangle$.
- If M rejects w, then we reject $\langle M, w\rangle$.
- So...

  if we could decide whether a TM halts on its input, we could decide A_{TM}.
- That's the idea in the proof. We reduce A_{TM} to $HALT_{TM}$.
  - We assume $HALT_{TM}$ decidable.
  - We then show that a decider for $HALT_{TM}$ can be used to decide A_{TM}.
  - Since A_{TM} is undecidable, $HALT_{TM}$ must be undecidable.

Undecidable Problems from Language Theory: $HALT_{TM}$

Theorem

$HALT_{TM} = \{\langle M, w\rangle | M \textit{ is a TM and } M \textit{ halts on input } w\}$ *is undecidable.*

Proof.

- Suppose for a proof by contradiction that $HALT_{TM}$ is decidable. Then it has a TM R that decides it.
- We construct a TM S to decide A_{TM}:
- On input $\langle M, w\rangle$:
  - Run R on input $\langle M, w\rangle$.
  - If R rejects, then reject.
  - If R accepts, then run M on input w.
  - <u>Note that M must halt on w.</u>
  - If M accepts w, then accept. Otherwise reject.
- S clearly decides A_{TM}. But A_{TM} is undecidable....
- A contradiction, and so $HALT_{TM}$ is not decidable.

Undecidable Problems from Language Theory: E_{TM}

- In the previous problem, we reduced A_{TM} to $HALT_{TM}$.
- Since A_{TM} is undecidable, $HALT_{TM}$ must be undecidable, too.
- This sort of reduction is a standard technique.
- We use it again below.

Theorem

$$E_{TM} = \{\langle M\rangle | M \textit{ is a TM and } L(M) = \varnothing\} \textit{ is undecidable.}$$

- The idea is to assume E_{TM} is decidable and then use that to decide A_{TM}.
- Given a decider R for E_{TM}, we use it in a decider S for A_{TM}.
- Note that if the input to S is $\langle M, w\rangle$, we can create a new TM M' that only accepts w or else nothing.
- This is the secret to constructing S.

Undecidable Problems from Language Theory: E_{TM}

Theorem

$E_{TM} = \{\langle M \rangle | M \textit{ is a TM and } L(M) = \varnothing\}$ *is undecidable.*

Proof.

- Suppose E_{TM} is decidable and let R be a decider for it.
- From R, we construct a decider S for A_{TM}, which works as follows.
- On input $\langle M, w \rangle$:
  1. Construct TM M': On any input v, if $v \neq w$, then reject. Otherwise, run M on v. If M accepts, then accept. **Note:** $v \in L(M')$ if and only if $v = w$ and $w \in L(M)$.
  2. Run R on $\langle M' \rangle$.
  3. If R accepts $\langle M' \rangle$, then $L(M') = \varnothing$ (meaning $w \notin L(M)$), and so reject.
  4. If R rejects $\langle M' \rangle$, then $L(M') = \{w\}$ (meaning $w \in L(M)$), and so accept.
- S decides A_{TM}. A contradiction!
- And so E_{TM} must be undecidable.

Rice's Theorem

- Rice's theorem asserts that all "nontrivial" properties of Turing machines are undecidable. (To determine whether a given Turing machine's language has property P is undecidable.)

Theorem

- *Let P be a language consisting of TM descriptions such that*
  1. *P is nontrivial-it contains some, but not all, TM descriptions.*
  2. *P is a property of the TM's language (Here, M_1 and M_2 are any TMs.)*

     Whenever $L(M_1) = L(M_2)$, we have $\langle M_1 \rangle \in P$ iff $\langle M_2 \rangle \in P$.
- *Then P is undecidable.*

Undecidable Problems from Language Theory: $REGULAR_{TM}$

- For instance, determining whether the language of a TM is regular is undecidable.

Theorem

$REGULAR_{TM} = \{\langle M\rangle | M$ is a TM and $L(M)$ is regular$\}$ is undecidable.

- We can prove this by reduction from A_{TM}.
- We assume $REGULAR_{TM}$ has decider R and then use it to decide A_{TM}.
- On input $\langle M, w\rangle$ We construct a new machine M_2 such that $L(M_2)$ is regular iff $w \in L(M)$.
- We then run R on $\langle M_2\rangle$.
- Note that we never actually run M_2. We instead use R to decide a property of M_2.
- The difficult part is knowing how to construct M_2 from M and w.

Undecidable Problems from Language Theory: $REGULAR_{TM}$

Theorem

$REGULAR_{TM} = \{\langle M \rangle | M \text{ is a TM and } L(M) \text{ is regular}\}$ *is undecidable.*

- Given TM M and string w, construct M_2 which operates as follows:
  - On input x:
  - If x has form 1^n0^n, then accept.
  - If not, then run M <u>on w</u> (not x) and accept x if M accepts w.
- $L(M_2) = \Sigma^*$ if $w \in L(M)$,
- Observe that 1^n0^n is nonregular and Σ^* is regular.
- M_2 recognizes a regular language if and only if M accepts w.

Undecidable Problems from Language Theory: $REGULAR_{TM}$

Theorem

$REGULAR_{TM} = \{\langle M \rangle | M \text{ is a TM and } L(M) \text{ is regular}\}$ *is undecidable.*

Proof.

Let R be a TM that decides $REGULAR_{TM}$ and construct TM S to decide A_{TM}.
$S =$ "On input $\langle M, w \rangle$, where M is a TM and w is a string:

1. Construct the following TM M_2.
2. $M_2 =$ "On input x:
   - If x has the form 1^n0^n, accept.
   - If x does not have this form, run M on input w and accept if M accepts w."
3. Run R on input $\langle M_2 \rangle$.
4. If R accepts, accept; if R rejects, reject."

Undecidable Problems from Language Theory

Similarly, determining the following properties of TMs is undecidable.

- $CF_{TM} = \{\langle M \rangle | M \text{ is a TM and } L(M) \text{ is context free}\}$
- $FINITE_{TM} = \{\langle M \rangle | M \text{ is a TM and } L(M) \text{ is finite}\}$
- $DECIDABLE_{TM} = \{\langle M \rangle | M \text{ is a TM and } L(M) \text{ is decidable}\}$

Undecidable Problems from Language Theory: EQ_{TM}

The idea is simple: if EQ_{TM} were decidable, E_{TM} also would be decidable, by giving a reduction from E_{TM} to EQ_{TM}.

Theorem

The following language is undecidable.

$$EQ_{TM} = \{\langle M_1, M_2\rangle | M_1, M_2 \text{ are TMs and } L(M_1) = L(M_2)\}.$$

- The E_{TM} problem is a special case of the EQ_{TM} problem wherein one of the machines is fixed to recognize the empty language.
- The idea here is to construct a TM M_2 such that $L(M_2) = \varnothing$.
- Then use this to determine whether $L(M_1) = \varnothing$.

Undecidable Problems from Language Theory: EQ_{TM}

- Recall that EQ_{DFA} is decidable.
- EQ_{TM} isn't, as can be proved via reduction from E_{TM}.

Theorem

The following language is undecidable.

$$EQ_{TM} = \{\langle M_1, M_2 \rangle | M_1, M_2 \text{ are TMs and } L(M_1) = L(M_2)\}.$$

Proof.

- Suppose that EQ_{TM} is decidable and let R be a decider for it.
- We construct a TM S to decide E_{TM} as follows.
- $S =$ "On Input $\langle M \rangle$, where M is a TM:
  1. Run R on input $\langle M, M_1 \rangle$, where M_1 is a TM that rejects all inputs.
  2. If R accepts $\langle M, M_1 \rangle$, then accept ($L(M) = L(M_1) = \varnothing$);
  3. if R rejects $\langle M, M_1 \rangle$, then reject ($L(M) \neq \varnothing$."

Configuration Histories

- Recall that the state, tape contents, and tape head position of a TM constitute a **configuration**.
- Using configurations, we can define **computation histories**.

Definition

An **accepting computation history** of TM M on string w is a finite sequence of configurations $C_1, \ldots, C_n$, where

- C_1 is the start configuration of M on w,
- C_n is an accepting configuration, and
- for each $1 \leq i < n$ C_{i+1} follows from C_i via M's transition function.

A **rejecting computation history** is defined similarly, save that C_n is a rejecting configuration.

- Deterministic TMs have ≤ 1 computation history for an input w.
- Computation histories are useful in proving properties of a restricted type of TM called a **linear bounded automaton**.

Linear Bounded Automata

Definition

A **Linear Bounded Automaton** is a Turing machine that may only use the portion of tape originally occupied by the input string w. Its tapehead cannot move beyond the left- or rightmost tape cells. Hence we say that for an input of length n, the amount of memory available is linear in n.

- LBAs are restricted TMs, but they are powerful.
- The languages accepted by LBAs are called **context sensitive**.
- These are called Type-1 languages in the Chomsky Hierarchy.

Type	Description	Accepting Machine	Grammar
0	Turing Recognizable	Turing Machine	Unrestricted
1	Context Sensitive	LBA	Context Sensitive
2	Context Free	PDA	Context Free
3	Regular	DFAs	Regular

- Each class n is a proper subset of class $n-1$ (there are some caveats).

Linear Bounded Automata: A_{LBA} is decidable

- Recall that the following were decidable: A_{DFA}, A_{CFG}. (In fact, they are decidable by LBAs).
- The language A_{TM} was not decidable.
- The language A_{LBA} is, however.

Theorem

$A_{LBA} = \{\langle M, w\rangle | M$ is an LBA and $w \in L(M)\}$ is decidable.

To prove this we need the following lemma, which asserts that there is a finite number of configurations for an LBA with input length n.

Lemma

- *Let M be an LBA with $|Q| = q$, $|\Gamma| = g$, and let $w \in \Sigma^*$.*
- *For a tape of length n, there are qng^n possible configurations of M.*

So, if M is a LBA and $w \in L(M)$ w will be accepted within qng^n steps.

Linear Bounded Automata: A_{LBA} is decidable

Theorem

$A_{LBA} = \{\langle M, w\rangle | M \text{ is an LBA and } w \in L(M)\}$ *is decidable.*

Proof.

We construct a TM L to decide A_{LBA}. On input $\langle M, w\rangle$:

- Run M on w, counting how many steps have been taken.
- If M accepts (rejects) w before qng^n steps, then accept (reject).
- If M has not halted within qng^n steps, then reject.

- If the M has not halted within qng^n steps, it never will.
- Instead, it will begin to repeat steps it's previously entered.

Linear Bounded Automata: E_{LBA} is undecidable

- Not all problems involving LBAs are decidable.
- Although E_{DFA} and E_{CFG} are decidable, E_{LBA} is not.

Theorem

$E_{LBA} = \{\langle M\rangle | M$ is an LBA and $L(M) = \varnothing\}$ is undecidable.

- The proof is via reduction from A_{TM} to E_{LBA}.
- From input $\langle M, w\rangle$, an LBA B is constructed to accept all and only accepting computation histories of M on w.

1. On input x, B checks that x is $C_1\#C_2\#\ldots\#C_n$, where C_1 is the start configuration of M on w and C_n is an accepting configuration. If not, it rejects.
2. B then checks to see that each C_{i+1} follows from C_i. If so, it accepts. Otherwise, it rejects.
3. Observe this can all be done by marking symbols on B's tape, and <u>without exceeding the tape boundaries</u>.

Linear Bounded Automata: E_{LBA} is undecidable

Theorem

$E_{LBA} = \{\langle M\rangle | M \text{ is an LBA and } L(M) = \varnothing\}$ *is undecidable.*

- Note that B is constructed from TM M and string w.
- B accepts all and only accepting computation histories of M on w.
- So, if we had a decider R for E_{LBA} we can run it on $\langle B\rangle$.
- This could be used to decide A_{TM}.

Linear Bounded Automata: E_{LBA} is undecidable

Theorem

$E_{LBA} = \{\langle M \rangle | M \text{ is an LBA and } L(M) = \varnothing\}$ *is undecidable.*

Proof.

- Suppose E_{LBA} is decidable, and let R be a decider for it.
- We construct a decider S for A_{TM} as follows.
- On input $\langle M, w \rangle$,
- Construct B as in the previous slide.
- Run R on $\langle B \rangle$.
  1. If R accepts, then reject (there are no accepting computation histories of M on w).
  2. If R rejects, then accept (there is an accepting computation history of M on w).
- Observe that B is never actually executed. It's just input into R.

Undecidable Languages: ALL_{CFG}

- Computation histories can be used to show properties of other languages.
- For instance, ALL_{CFG} is undecidable.

Theorem

$ALL_{CFG} = \{\langle G \rangle | G$ is a CFG and $L(G) = \Sigma^*\}$ is undecidable.

- To prove this, we show that for a TM M and string w, we can construct a special PDA D.
- D accepts all and only strings that are **not valid accepting computation histories** $C_1, C_2, \ldots, C_n$ of M on w.
- For the PDA (for technical reasons), we encode a history with every other configuration reversed:

$$C_1 \# C_2^{\mathcal{R}} \# C_3 \# C_4^{\mathcal{R}} \ldots \# C_n.$$

Undecidable Languages: ALL_{CFG}

Theorem

$ALL_{CFG} = \{\langle G \rangle | G \text{ is a CFG and } L(G) = \Sigma^*\}$ *is undecidable.*

- From a TM M and an input w, we construct a PDA D that generates all strings if and only if M does not accept w.
- So, if M does accept w, D does not generate some particular string.
- This particular string is the accepting computation history for M on w.
- That is, D is designed to generate all strings that are not accepting computation histories for M on w.

Undecidable Languages: ALL_{CFG}

- A computation history fails to be an accepting one if:
  1. C_1 is not the start configuration;
  2. C_n is not an accepting configuration;
  3. some C_{i+1} doesn't follow from C_i.

  PDA D nondeterministically chooses a failure to check.
- To check C_1, D reads through C_1, accepting if the first symbol is not q_{start} or if the string between q_{start} and the first $\#$ is not w..
- To check C_n, D reads through C_n, accepting if a state other than q_{accept} appears (or more than one state appears).

Undecidable Languages: ALL_{CFG}

- To check for a failure in δ, D selects a C_i to compare to C_{i+1}.
  - It pushes C_i onto the stack.
  - It then reads through C_{i+1}, comparing symbols to C_i.
  - C_i is reversed relative to C_{i+1}. The symbols popped from the stack should match those read from C_{i+1} (except for changes due to δ).
- D accepts if the transition is invalid.
- Observe that D only rejects accepting computation histories of M on w.
- And so if D accepts all strings, then there are no such histories (and so M does not accept w).
- As such, we could use a decider R for ALL_{CFG} to decide A_{TM}.

Undecidable Languages: ALL_{CFG}

Theorem

$ALL_{CFG} = \{\langle G\rangle | G \text{ is a CFG and } L(G) = \Sigma^*\}$ *is undecidable.*

Proof.

Suppose ALL_{CFG} is decidable and let R be a decider for it. We use it to construct a TM S deciding A_{TM}.

On input $\langle M, w\rangle$:

- Construct a PDA D from M and w as described in the previous slide.
- Convert D into a CFG G.
- Run R on $\langle G\rangle$.
  - If R accepts $\langle G\rangle$, then reject (there are no accepting histories).
  - If R rejects $\langle G\rangle$, then accept (an accepting history exists).

The Post Correspondence Problem: PCP

The previous problems all dealt with automata. Below is an undecidable problem involving string manipulation.

(The Post Correspondence Problem (PCP))

Given a set of dominoes $P = \{[\frac{t_1}{b_1}], [\frac{t_2}{b_2}], \ldots, [\frac{t_n}{b_n}]\}$, where each t_i and b_i is a string, a **match** for P is a sequence $i_1, i_2, \ldots i_m$ such that

$$t_{i_1} t_{i_2} \ldots t_{i_m} = b_{i_1} b_{i_2} \ldots b_{i_m}.$$

$PCP = \{\langle P \rangle | P \text{ is a collection of dominoes with a match}\}$

Example

If $P = \{[\frac{b}{ca}], [\frac{a}{ab}], [\frac{ca}{a}], [\frac{abc}{c}]\}$, then the following is a match.

$$[\frac{a}{ab}][\frac{b}{ca}][\frac{ca}{a}][\frac{a}{ab}][\frac{abc}{c}]$$

Note that duplicates are possible.

In-class Questions???

Find a match in the following instance of the Post Correspondence Problem.

$$\{[\frac{ab}{abab}], [\frac{b}{a}], [\frac{aba}{b}], [\frac{aa}{a}]\}$$

The Post Correspondence Problem: PCP

Theorem

$PCP = \{\langle P \rangle | P$ is a collection of dominoes with a match$\}$ is undecidable.

- The idea behind showing it undecidable is to reduce A_{TM} to it.
- TM configurations are converted into domino sequences.
- For a given M and w, $w \in L(M)$ iff a match exists for the dominoes.
- Since A_{TM} is undecidable, PCP must be, too.

(Some restrictions)

For the sake of the problem, we will assume:

- The TM M never attempts to move beyond the left of the tape.
- If $w = \varepsilon$, string $\sqcup$ is used for w.
- The match always begins with $[\frac{t_1}{b_1}]$.

These restrictions can all be done away with.

The Post Correspondence Problem: PCP

Theorem

$PCP = \{\langle P \rangle | P$ is a collection of dominoes with a match$\}$ is undecidable.

The construction of the problem proceeds in stages.

Part 1: Domino $[\frac{t_1}{b_1}] = [\frac{\#}{\#q_0 w_1 \ldots w_n \#}]$, where $w = w_1 \ldots w_n$.

- t_1 must be extended using more dominoes to match b_1.
- We add dominoes to simulate moves (parts 2 and 3).
- We also add dominoes for strings unaffected by moves (part 4).

Part 2: For every $\delta(q, a) = (r, b, R)$, $q \neq q_{reject}$, construct $[\frac{qa}{br}]$

Part 3: For every $c \in \Gamma$ and $\delta(q, a) = (r, b, L)$, $q \neq q_{reject}$, construct $[\frac{cqa}{rcb}]$.

Part 4: For every $a \in \Gamma$, construct $[\frac{a}{a}]$.

The Post Correspondence Problem: PCP

Theorem

$PCP = \{\langle P\rangle | P$ is a collection of dominoes with a match$\}$ is undecidable.

- Extending t_1 forces beginning of a new configuration on the bottom.
- That is, we are forced to simulate the execution of M on w.
- Dominoes are added to fill out the part of the configuration not affected by a transition.
- To mark configuration ends, we need more dominoes:

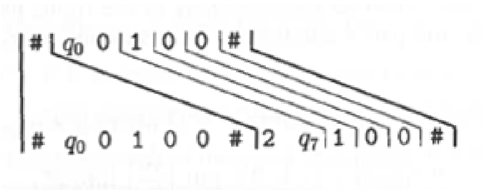
Part 5: Add dominoes $[\frac{\#}{\#}]$ and $[\frac{\#}{\sqcup\#}]$. The latter allows us to represent the empty space at the right of the tape.

The Post Correspondence Problem: PCP

Suppose a TM M with

- $Q = \{q_0, q_1, \ldots, q_7, q_{acc}\}$
- $\Gamma = \{0, 1, 2, 3, \sqcup\}$
- δ includes the following: $\delta(q_0, 0) = (q_7, 2, R)$,

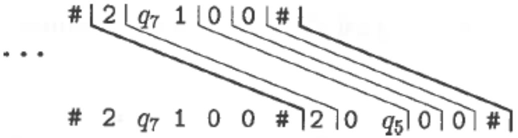
Using steps 1-5, we can construct the following, representing a partial computation history of M on $w = 0100$



The first domino is from Part 1, the second from Part 2, and the rest from part 4 and 5.

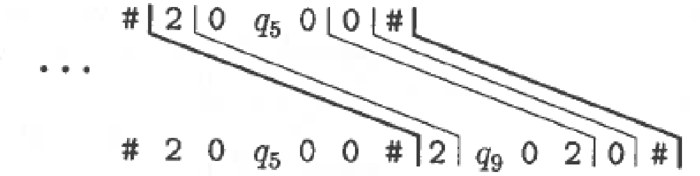
The Post Correspondence Problem: PCP

Suppose $\delta(q_7, 1) = (q_5, 0, R)$. Then we can form...



The Post Correspondence Problem: PCP

Suppose $\delta(q_5, 0) = (q_9, 2, L)$. Then we can form...



The Post Correspondence Problem: PCP

Theorem

$PCP = \{\langle P\rangle | P$ is a collection of dominoes with a match$\}$ is undecidable.

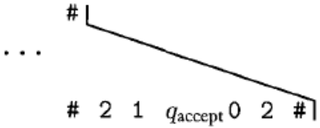
- Special dominoes are needed to ensure that the end sequence of dominoes match.

Part 6: For each $a \in \Gamma$, add dominoes $[\frac{aq_{accept}}{q_{accept}}]$ and $[\frac{q_{accept}a}{q_{accept}}]$.
(This is a technical step, intended to remove symbols around q_{accept} until it is adjacent only to $\#$).

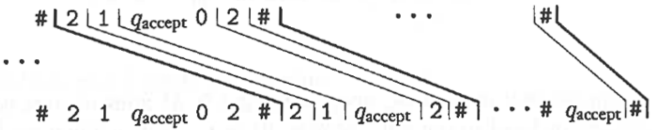
Part 7: add dominoes $[\frac{q_{accept}\#\#}{\#}]$.

The Post Correspondence Problem: PCP

Suppose we have arrived at the following:



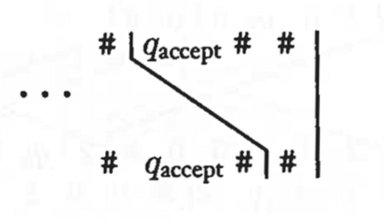
Then we can form the following:



Observe that the 0 to the right of q_{accept} has been "eaten".

The Post Correspondence Problem: PCP

$\left[\frac{q_{accept}\#\#}{\#}\right]$ is needed at the very end.



In-class Questions???

Find a match in the following instance of the Post Correspondence Problem.

$$\{[\frac{ab}{abab}], [\frac{b}{a}], [\frac{aba}{b}], [\frac{aa}{a}]\}$$

Mapping Reducibility

Definition

- Let A and B be languages over Σ.
- A is **mapping reducible** to B, written $A \leq_m B$, if there is a **computable function** $f : \Sigma^* \to \Sigma^*$ such that

$$w \in A \text{ if and only if } f(w) \in B.$$

- Function f is a **reduction** from A to B.

- Reducibility here hinges on there being a computable function.

Definition

A function $f : \Sigma^* \to \Sigma^*$ is a **computable function** if there exists a Turing machine M such that on every input $w \in \Sigma^*$, M halts with just $f(w)$ on its tape.

- From the definition, one sees that a function is computable if and only if there is some algorithm that computes it.

Mapping Reducibility

Show that $\leq_m$ is a transitive relation.

Proof.

Suppose that $A \leq_m B$ and $B \leq_m C$. Then there are computable functions f and g such that $x \in A \iff f(x) \in B$ and $y \in B \iff g(y) \in C$.

Consider that composition function $h(x) = g(f(x))$. We can build a TM that computes h as follows:

1. Simulate a TM for f (such a TM exists because we assumed that f is computable) on input x and call the output y.
2. Simulate a TM for g on y. The output is $h(x) = g(f(x))$.

Therefore h is a computable function. Moreover, $x \in A \iff h(x) \in C$. Hence $A \leq_m C$ via the reduction function h.

Mapping Reducibility

- In previous examples, we reduced one problem A to another B and then leveraged a property of one to conclude something about the other.
- We can formalize what we've been doing in a theorem.

Theorem

If $A \leq_m B$ and B is decidable, then A is decidable.

Proof.

- Suppose $A \leq_m B$ and B is decidable. Then there exists a TM M to decide B, and there is a computable function f such that $w \in A$ if and only if $f(w) \in B$.
- We construct a decider N for A that acts as follows:
    - On input w, compute $f(w)$.
    - Run M on $f(w)$.
    - If M accepts $f(w)$, then accept. Otherwise, reject.

Mapping Reducibility

- A corollary to the previous theorem exists.
- Similarly, The theorem can be reformed for recognizability.

Corollary

If $A \leq_m B$ and A is undecidable, then B is undecidable.

Theorem

If $A \leq_m B$ and B is Turing-recognizable, then A is Turing-recognizable.

- Suppose we have a recognizer R for B. A recognizer S for A would work as follows: On input w, compute $f(w)$ and run R on $f(w)$. Accept if R accepts; Reject if R rejects.

Corollary

If $A \leq_m B$ and A is not Turing-recognizable, then B is not Turing-recognizable.

Mapping Reducibility

- If $A \leq_m B$, then there's a computable function $f : \Sigma^* \to \Sigma^*$ with
$$w \in A \text{ if and only if } f(w) \in B.$$
- Observe that this implies
  - $w \notin A$ if and only if $f(w) \notin B$.
  - $w \in \overline{A}$ if and only if $f(w) \in \overline{B}$.
- Thus, if $A \leq_m B$, then $\overline{A} \leq_m \overline{B}$.
- The converse is also true.
- And so the following proposition is true.

Proposition

$$A \leq_m B \text{ if and only if } \overline{A} \leq_m \overline{B}.$$

Mapping Reducibility

Show that if A is Turing-recognizable and $A \leq_m \overline{A}$, then A is decidable.

Proof.

- Suppose that $A \leq_m \overline{A}$, then $\overline{A} \leq_m A$ via the same mapping reduction.
- Because A is Turing-recognizable, it implies that $\overline{A}$ is Turing-recognizable.
- Then A is Turing-recognizable and co-Turing-recognizable.
- Hence it implies that A is decidable.

Mapping Reducibility

We define a function F showing $A_{TM} \leq_m \overline{EQ_{TM}}$.

F:

- On input $\langle M, w \rangle$ construct TMs M_1 and M_2.
  - M_1: On any input x, **reject**.
  - M_2: On any input x, run M on w. Accept x if M accepts w.
- Output $\langle M_1, M_2 \rangle$.

- F is really a Turing machine, one that computes a function.
- M_1 is trivial to create, and given M, M_2 is also easy to create.
- Observe that if $w \notin L(M)$, then M_2 doesn't accept any strings.
- If $w \in L(M)$, then $L(M_2)$ is the set of all strings.
- Thus $w \in L(M)$ iff $\langle M_1, M_2 \rangle \in \overline{EQ_{TM}}$ (i.e., $L(M_1) \neq L(M_2)$).
- This shows $A_{TM} \leq_m \overline{EQ_{TM}}$.
- F is a mapping reduction from A_{TM} to $\overline{EQ_{TM}}$.

Mapping Reducibility

We define a function G showing $A_{TM} \leq_m EQ_{TM}$.

G:

- On input $\langle M, w \rangle$ construct TMs M_1 and M_2.
  - M_1: On any input x, **accept**.
  - M_2: On any input x, run M on w. Accept x if M accepts w.
- Output $\langle M_1, M_2 \rangle$.

- F and G are very similar, save that M_1 accepts all inputs.
- If $w \in L(M)$, then M_2 accepts all strings, too.
- If $w \notin L(M)$, then M_2 accepts no strings.
- Thus, $w \in L(M)$ iff $\langle M_1, M_2 \rangle \in EQ_{TM}$ (i.e., $L(M_1) = L(M_2)$).
- Hence, G is a mapping reduction from A_{TM} to EQ_{TM}.

Mapping Reducibility

Theorem

EQ_{TM} is neither Turing recognizable nor co-Turing recognizable.

Proof.

- We have $A_{TM} \leq_m \overline{EQ_{TM}}$ and $A_{TM} \leq_m EQ_{TM}$ (earlier slides).
- From this, $\overline{A_{TM}} \leq_m EQ_{TM}$ and $\overline{A_{TM}} \leq_m \overline{EQ_{TM}}$.
- Since $\overline{A_{TM}}$ is not Turing recognizable, neither EQ_{TM} nor $\overline{EQ_{TM}}$ is.
- Since $\overline{EQ_{TM}}$ is not recognizable, then EQ_{TM} is not co-recognizable.
    - This follows by definition of co-recognizability.

- What does this mean?
- We can't reliably recognize when pairs of Turing machines have the same language (EQ_{TM} is not Turing recognizable).
- Nor can we reliably recognize when pairs of Turing machines have different languages (EQ_{TM} is not co-Turing recognizable).

Mapping Reducibility

Show that EQ_{CFG} is co-Turing-recognizable.

Mapping Reducibility

Show that EQ_{CFG} is co-Turing-recognizable.

Proof.

We can construct a TM M which recognizes the complement of EQ_{CFG}:
$M =$ "On input $\langle G, H \rangle$:

1. For each string $x \in \Sigma^*$ in lexicographic order:
2. Test whether $x \in L(G)$ and whether $x \in L(H)$, using the algorithm for A_{CFG}.
3. If one of the tests accepts and the other rejects, accept; otherwise, continue."

Mapping Reducibility

Show that EQ_{CFG} is undecidable.

Mapping Reducibility

Show that EQ_{CFG} is undecidable.

Proof.

Suppose that EQ_{CFG} were decidable. We can construct a decider M for $ALL_{CFG} = \{\langle G \rangle | G \text{ is a CFG and } L(G) = \Sigma^*\}$ as follows:
M = "On input $\langle G \rangle$:

1. Construct a CFG H such that $L(H) = \Sigma^*$.
2. Run the decider for EQ_{CFG} on $\langle G, H \rangle$.
3. If it accepts, accept. If it rejects, reject."

In-class Questions???

Let $J = \{w | w = 0x$ for some $x \in A_{TM}$, or $w = 1y$ for some $y \in \overline{A_{TM}}\}$.
Show that neither J nor $\overline{J}$ is Turing recognizable.

Questions for Group Discussion

1. A state in an automaton is **useless** if it is never entered on any input string. Consider the language $U_{PDA} = \{\langle P \rangle | P$ is a PDA with useless states$\}$. Show that it is **decidable**. Hint: If given a PDA P with state q, consider modifying P so that q is the only accept state of P.
2. Consider the problem of determining whether a Turing machine M on input w ever attempts to move its tapehead left **at any point** while processing w. Let $L = \{\langle M, w \rangle | M$ attempts moves left at some point when processing $w\}$. Show that L is **decidable**.

Questions for Group Discussion

A state in an automaton is **useless** if it is never entered on any input string. Consider the language $U_{PDA} = \{\langle P\rangle | P \text{ is a PDA with useless states}\}$. Show that it is **decidable**. Hint: If given a PDA P with state q, consider modifying P so that q is the only accept state of P.

Proof.

E_{CFG} is decidable and so has decider N. We create a decider M for U_{PDA}.
$M =$ "On input w:

- Scan w. Reject if it is not a valid representation of a PDA P.
- Identify the states $q_1, q_2, \ldots, q_n$ of P, and for each q_i do the following:
  - Modify P so that q_i is its only accept state (call the modified PDA P_{q_i}).
  - Convert P_{q_i} to an equivalent CFG G_{q_i} using techniques from Chapter 2.
  - Run N on $\langle G_{q_i}\rangle$.
  - If N accepts, then **accept** w.
  - If N rejects, then continue.
- If each q_i has been processed without accepting, **reject** w."

Questions for Group Discussion

A state in an automaton is **useless** if it is never entered on any input string. Consider the language $U_{PDA} = \{\langle P \rangle | P \text{ is a PDA with useless states}\}$. Show that it is **decidable**. Hint: If given a PDA P with state q, consider modifying P so that q is the only accept state of P.

Proof.

N accepts $\langle G_{q_i} \rangle$ iff $L(G_{q_i})$ is empty. However, since $L(P_{q_i}) = L(G_{q_i})$, it must be that q_i is never entered (because q_i) is the only accept state of P_{q_i}. So if N accepts on some P_{q_i}, then P has a useless state, and so $w = \langle P \rangle$ should be accepted. Above, M rejects only if N rejects on every $\langle P_{q_i} \rangle$.

Questions for Group Discussion

Let $L = \{\langle M, w\rangle | M$ attempts moves left at some point when processing $w\}$. Show that L is **decidable**.

Proof.

We construct a decider R for L. It works as follows:
$R =$ "On input x:

1. Scan x, checking whether it is of the form $\langle M, w\rangle$, where M is a TM and w a string. If not, reject. Otherwise continue.
2. Run M on w for $|w| + |Q| + 1$ steps. If M ever moves left within that number of steps, then accept. Otherwise reject."

Questions for Group Discussion

Let $L = \{\langle M, w\rangle | M$ attempts moves left at some point when processing $w\}$. Show that L is **decidable**.

Proof.

The idea here is that if M does not move left within w steps, then it must have moved right, moving passed input string w. At that point, it will read only blank spaces. It can move from one state to another, reading blanks, but at some point, it must return to a state it has previously been in (that point is $|w| + |Q| + 1$ steps). And so, if it hasn't moved left within $|w| + |Q| + 1$ steps, the machine will simply repeat states (reading blanks forever). It will never move left.