BALANCING THE n-CUBE: A census of colorings

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Abstract

Weights of 1 or 0 are assigned to the vertices of the n-cube in n-dimensional Euclidean space. Such an n-cube is called *balanced* if its center of mass coincides precisely with its geometric center. The seldom-used n-variable form of Pólya's enumeration theorem is applied to express the number $N_{n,2k}$ of balanced configurations with $2k$ vertices of vertices of weight 1 in terms of certain partitions of $2k$. A system of linear equations of Vandermonde type is obtained, from which recurrence relations are derived which are computationally efficient for fixed k. It is shown how the numbers $N_{n,2k}$ depend on the numbers $A_{n,2k}$ of specially restricted configurations. A table of values of $N_{n,2k}$ and $A_{n,2k}$ is provided for $n = 3, 4, 5$ and 6. The case in which arbitrary, non-negative, integral weights are allowed is also treated. Finally, alternative derivations of the main results are developed from the perspective of superposition.

Key words. n-cube, boolean functions, Pólya enumeration, superposition.

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1 Introduction

The enumeration of various types of Boolean functions has its origins over one hundred years ago in the work of W.K. Clifford [Cℓ82] (see pp. 134-146 of W.S. Jevons [Je92] for a summary). The quaint terminology of these early references is not easily understood, the methods are laborious, and only a few simple cases are considered. The problems of Clifford and Jevons were recast in a more accessible form by Pólya [Po40], who viewed the logical propositions as distributions of marks of two sorts, say T and F for true and false, on the 2^n vertices of an n-cube. Pólya then derived efficient formulas which permitted him to correct Clifford's errors and verify some results of Jevons. For example, a problem of Jevons asks for the number of certain "consistent" logical propositions in 4 variables. Pólya interpreted these as colorings of the vertices of a fixed 4-cube with two colors, say black and white, such that no face has all of its vertices black, and he used the method of inclusion and exclusion to derive a simple formula that applies to any dimension.

Several papers have been written since [P40] which extend and refine Pólya's results. See, for example [Sℓ53], [Ha63a], [HaH68], [PaR73b] and [PaR84].

In this article we are concerned with 2-colorings of the vertices of a fixed, geometric n-cube. We regard the black vertices as having weight 1 while the whites have weight 0, and we seek to determine the number $N_{n,2k}$ of these configurations with $2k$ black vertices whose center of mass is identical to the geometric center of the n-cube. We apply the seldom-used n-variable form of Pólya's theorem for counting combinations and obtain a formula for $N_{n,2k}$ which depends on the partitions of $2k$. This formula then leads us to a system of linear equations of Vandermonde type from which effective recurrence relations can be derived. We also investigate the number $A_{n,2k}$ of antiantipodal colorings, that is, balanced colorings in which no two black vertices are antipodal. These bear a straightforward relationship to the $N_{n,2k}$'s, from which the $A_{n,2k}$'s can be calculated. The numerical results suggest that $A_{n,2k} = 0$ when $n \geq 3$ and $k = 2^{n-2} - 1$, a fact which is then confirmed by a combinatorial argument.

2 Definitions

The set V of vertices of the geometric n-cube, Q_n, consists of the 2^n points in n-dimensional Euclidean space each of whose coordinates is $+1$ or -1, i.e.,

$$V = \{(\varepsilon_1, \ldots, \varepsilon_n) \mid \varepsilon_i = \pm 1,\ i = 1, \ldots, n\}.$$

Two vertices are adjacent if they differ in exactly one coordinate. Thus the distance between them is 2.

A 2-coloring of the vertices of Q_n is a function f from V into the set {black, white}. Thus f assigns colors to the vertices. The *weight of* f, denoted $w(f)$, is the number of black vertices, i.e. the black vertices have weight 1 while the whites have weight zero. The *center of mass of a coloring* f with $w(f) \neq 0$ is the point whose coordinates are given by

$$\frac{1}{w(f)} \sum (\varepsilon_1, \ldots, \varepsilon_n),$$

where the sum is over all black vertices. If $w(f) = 0$, we take the center of mass to be the origin. A coloring is *balanced* if its center of mass is the origin. The balance condition is easily expressed in terms of the faces F_i and $-F_i$, where F_i contains all 2^{n-1} vertices for which $\varepsilon_i = +1$ and $-F_i$ is the complement. To be balanced, a coloring of weight $2k$ must have k black vertices in F_i and k black vertices in $-F_i$, for each $i = 1, \ldots, n$. Since no coloring of odd weight can be balanced, only even weights are considered.

Two vertices of maximum rectilinear distance $2n$, v and $-v$ are said to be *antipodal*. A *coloring is antipodal* if every two antipodal vertices are assigned the same color. It is *antiantipodal* (with respect to black) if it is balanced and contains no antipodal pair of black vertices.

For definitions not included in this paper, we refer to the book [HP73].

3 Counting formulae for 2-colorings.

The antipodal colorings are obviously balanced. Each antipodal pair of vertices may be colored both black or both white. The number of these with $2k$ black vertices is just the coefficient of y^{2k} in $(1+y^2)^{2^{n-1}}$, which we denote by $[y^{2k}](1+y^2)^{2^{n-1}}$. Thus the number of antipodal colorings is

$$\binom{2^{n-1}}{k}.$$

Let $N_{n,2k}$ be the number of balanced colorings of the n-cube with exactly $2k$ black vertices. The partitions of $2k$ are denoted by vectors $\langle j\rangle = (j_1,\ldots,j_{2k})$ where

$$\sum_{i=1}^{2k} i j_i = 2k. \tag{3.1}$$

Finally we let

$$N(\langle j\rangle) = [x^k]\prod_{i=1}^{2k}(1+x^i)^{j_i}. \tag{3.2}$$

Theorem 3.1. The number $N_{n,2k}$ of balanced colorings of the n-cube with $2k$ black vertices is

$$N_{n,2k} = \sum_{\langle j\rangle} N(\langle j\rangle)^n(-1)^{a(\langle j\rangle)}h(\langle j\rangle) \tag{3.3}$$

where the sum is over all partitions $\langle j\rangle$ of $2k$,

$$a(\langle j\rangle) = \sum_i j_{2i} \tag{3.4}$$

and

$$h(\langle j\rangle) = 1/\prod_i j_i!\,i^{j_i}. \tag{3.5}$$

Proof. We use the form of Pólya's enumeration theorem which counts combinations by weight. Our "figure counting series" is the polynomial in the variables $x_1,\ldots,x_n$ defined by

$$c(x_1,\ldots,x_n) = \prod_{i=1}^{n}(1+x_i). \tag{3.6}$$

The 2^n monomials in the expansion of the product of (3.6) correspond precisely to the vertices of the n-cube. For each $i=1$ to n the i-th coordinate of any vertex is $+1$ if and only if x_i is a factor of its monomial. For example, when $n=5$ the vertex $(1,-1,1,1,-1)$ corresponds to the monomial $x_1x_3x_4$.

The colorings of the n-cube with $2k$ black vertices correspond to $2k$-subsets of the 2^n monomials in $c(x_1,\ldots,x_n)$. We call the product of the monomials in such a $2k$-subset the *weight* of the subset. Suppose the weight of a $2k$-subset is $x_1^{t_1}\ldots x_n^{t_n}$. Then for each $i=1$ to n the corresponding coloring has exactly t_i black vertices for which the i-th coordinate is $+1$. Then $N_{n,2k}$ is just the number of $2k$-subsets of weight $x_1^kx_2^k\ldots x_n^k$ because for each $i=1$ to n, k of the black vertices have i-th coordinate $+1$ and the other k have -1.

The counting series for these $2k$-subsets by weight is obtained by applying the form of Pólya's theorem used to count combinations by weight (see page 48 of [HP73]). The desired series takes the following symbolic form

$$Z(A_{2k}-S_{2k})[s_i \leftarrow c(x_1^i,x_2^i,\ldots,x_n^i)], \tag{3.7}$$

which means that each variable s_i in the cycle index difference $Z(A_{2k}-S_{2k}) = Z(A_{2k}) - Z(S_{2k})$ of the alternating and symmetric groups is replaced by $c(x_1^i,\ldots,x_n^i)$.

It remains to determine the coefficients of $x_1^k\ldots x_n^k$ in the expression (3.7) above for each partition $\langle j\rangle = (j_1,\ldots,j_{2k})$ of $2k$. Therefore consider the term

$$\Big(\prod_{i=1}^{2k} s_i^{j_i}\Big)\Big[s_i \leftarrow \prod_{t=1}^{n}(1+x_t^i)\Big]. \tag{3.8}$$

After substitution, we interchange the two products and obtain

$$\prod_{t=1}^{n}\prod_{i=1}^{2k}(1+x_t^i)^{j_i}. \tag{3.9}$$

Thus the coefficient we seek is

$$\prod_{t=1}^{n}\Big([x_t^k]\prod_{i=1}^{2k}(1+x_t^i)^{j_i}\Big) = N(\langle j\rangle)^n. \tag{3.10}$$

Now the formula (3.3) of the theorem follows from the identity

$$Z(A_{2k}) - Z(S_{2k}) = \sum_{\langle j\rangle}(-1)^{a(\langle j\rangle)}h(\langle j\rangle)\prod_{i=1}^{2k} s_i^{j_i}, \tag{3.11}$$

where $\langle j\rangle$ is summed over all partitions of $2k$ (see [HP73] page 36). $\square$

We now show how to find a recurrence relation satisfied by $N_{n,2k}$ for each fixed k. First, note that $N_{n,0}=1$ for all $n\geq 0$. For $n>0$, the sum in formula (3.3) over all partitions $\langle j \rangle$ of $2k$ can be confined to those terms for which $N(\langle j \rangle) > 0$. Let $\alpha_1, \ldots, \alpha_m$ be the distinct, non-zero values of $N(\langle j \rangle)$ over partitions of $2k$. Define the numbers $C_1, \ldots, C_m$ by

$$\prod_{i=1}^{m}(1-\alpha_i x) = 1 - \sum_{i=1}^{m} C_i x^i. \tag{3.12}$$

Then the following corollary gives the recurrence relation.

Corollary 1. For $n > m$

$$N_{n,2k} = \sum_{i=1}^{m} C_i N_{n-i,2k}. \tag{3.13}$$

Proof. It follows from Theorem 3.1 that $N_{n,2k}$ can be expressed as

$$N_{n,2k} = \prod_{i=1}^{m} b_i a_i^n \tag{3.14}$$

for appropriate $b_1, \ldots, b_m$ and $n > 0$. Thus

$$\begin{aligned} \sum_{n=1}^{\infty} N_{n,2k} x^n &= \sum_{i=1}^{m} \frac{b_i a_i x}{1-\alpha_i x} \\ &= \frac{\phi(x)}{1-\sum_{i=1}^{m} C_i x^i} \end{aligned} \tag{3.15}$$

where $\phi(x)$ is a polynomial of degree at most m. The recurrence follows by multiplying (3.15) by the right side of (3.12) and observing that the coefficient of x^n in the resulting convolution is 0 when $n > m$. □

We illustrate the procedure with the example $k = 2$. Consider the partition $\langle j \rangle = (4,0,0,0)$ of $2k = 4$. Since $j_1 = 4$, $N(\langle j \rangle)) = [x^2](1+x)^4 = \binom{4}{2} = 6$. The results for the other partitions are summarized in the following table.

$\langle j \rangle$	1^4	$1^2 2$	13	2^2	4
$N(\langle j \rangle)$	6	2	0	2	0

Arbitrarily we choose $\alpha_1 = 2$ and $\alpha_2 = 6$ and observe that

$$(1-2x)(1-6x) = 1 - \{8x - 12x^2\}.$$

Thus $C_1 = 8$ and $C_2 = -12$ and therefore for $n > 2$

$$N_{n,4} = 8N_{n-1,4} - 12N_{n-2,4}. \tag{3.16}$$

This relation is solved explicitly later in (5.28). For $n = 3$, the number of evenly balanced colorings of Q_3 is $N_{3,4} = 8(1) - 12(0) = 8$. Continuing to use equation (3.16) we find $N_{4,4} = 8(8) - 12(1) = 52$ and $N_{5,4} = 8(52) - 12(8) = 320$.

We emphasize that most of the entries in Table 1 for $N_{n,2k}$ were computed using formula (3.3) of Theorem 3.1. The condition $n > m$ in the hypothesis of Corollary 1 requires $n > 7$ when $2k = 8$. Even if $2k = 6$, there are $m = 4$ different values for $N(\langle j \rangle)$ and so in Table 1 only $N_{5,6}$ and $N_{6,6}$ can be obtained from the recurrence relation for $N_{n,6}$.

The theorem can also be used to determine the number $A_{n,2k}$ of balanced colorings with $2k$ black vertices but no antipodal pair of black vertices, i.e., antiantipodal colorings. Note that $A_{n,0} = 1$ for all $n \geq 1$ and that $A_{n,2k} = 0$ unless $2k \leq 2^{n-1}$.

Corollary 2.

$$A_{n,2k} = N_{n,2k} - \sum_{i=0}^{k-1} \binom{2^{n-1} - 2i}{k - i} A_{n,2i}. \tag{3.17}$$

Proof. For each $i = 0$ to k, an antiantipodal coloring with $2i$ black vertices has $2^{n-1} - 2i$ pairs of antipodal white vertices. On selecting $k - i$ of these pairs and coloring their $2(k - i)$ vertices black, we obtain a balanced coloring with $2i + 2(k - i) = 2k$ black vertices. Therefore the number of these for each i is

$$\binom{2^{n-1} - 2i}{k - i} A_{n,2i}. \qquad \square$$

The numbers $N_{n,2k}$ and $A_{n,2k}$ are displayed in Table 1 for $n = 3, 4, 5$ and 6 and $2k \leq 2^{n-1}$. If the colors of any 2-coloring of two vertices of the n-cube are switched, we obtain the *complementary coloring* or the *complement*. Clearly the complement of a balanced coloring is also balanced. Hence

$$N_{n,2k} = N_{n,2^n - 2k} \tag{3.18}$$

and so the numbers of balanced colorings with $2k > 2^{n-1}$ have been omitted from Table 1.

Note that when $2k = 2^{n-1} - 2$, $A_{n,2k} = 0$ for $3 \leq n \leq 6$. We have found this to be the case in general.

$A_{n,2k}$	$N_{n,2k}$	$2k$	n
1	1	0	3
0	4	2	3
2	8	4	3
1	1	0	4
0	8	2	4
24	52	4	4
0	152	6	4
8	222	8	4
1	1	0	5
0	16	2	5
200	320	4	5
864	3824	6	5
3980	27640	8	5
4512	123600	10	5
3920	353120	12	5
0	657520	14	5
222	807980	16	5
1	1	0	6
0	32	2	6
1440	1936	4	6
38400	83680	6	6
873400	2452080	8	6
11225024	49585440	10	6
94406496	712645616	12	6
505093760	7472934880	14	6
1756793620	58431976800	16	6
3910422720	346729813920	18	6
5526259040	1583599024656	20	6
4765836160	5629885328736	22	6
2428563760	15718244056816	24	6
645500160	34699914166560	26	6
92788160	60880399587440	28	6
0	85186889390304	30	6
807980	95259103924394	32	6

Table 1. Balanced colorings of the n-cube for $n = 3, 4, 5$ and 6.

Theorem 3.2. $A_{n,2k} = 0$ for $n \geq 3$ and $k = 1$ or $k = 2^{n-2} - 1$.

Proof. When $k = 1$ there are only two black vertices, which must be antipodal in order to achieve balance. But such a configuration is not antiantipodal with respect to black.

For $k = 2^{n-2} - 1$, we proceed by contradiction, so fix on a particular antiantipodal balanced coloring of the n-cube which contains exactly $2^{n-1} - 2$ black vertices. Each of the complementary faces F_n and $-F_n$ must contain exactly $2^{n-2} - 1$ black vertices in order to achieve balance with respect to the n'th coordinate ε_n. The opposing vertices must all be white, which accounts for $2^{n-2} - 1$ vertices each in F_n and $-F_n$. Let u and v be the other two vertices in F_n, i.e., the two which are white and for which $-u$ and $-v$ are also white.

Now consider balance with respect to some other coordinate, say ε_i for $1 \leq i < n$. Let r denote the number of black vertices in $F_i \cap F_n$, so that $(-F_i) \cap F_n$ contains exactly $2^{n-2} - 1 - r$ black vertices. Let δ be the cardinality of $\{u, v\} \cap F_i$, which is 2, 1, or 0. Then the number of white vertices in $(F_n - \{u, v\}) \cap F_i$ is $2^{n-2} - r - \delta$, and the number in $(F_n - \{u, v\}) \cap (-F_i)$ is $r - 1 + \delta$. These are precisely the vertices antipodal to black vertices in $-F_n$, so that the total number of black vertices in F_i is $r + (r - 1 + \delta)$, and in $-F_i$ is $(2^{n-2} - 1 - r) + (2^{n-2} - r - \delta)$. The balance condition for ε_i requires these totals to be equal, which gives

$$\delta = 2^{n-2} - 2r.$$

Now, if $n \geq 3$ then δ must be even, so that u and v take the same value at the ith coordinate. Since this is true for all i with $1 \leq i < n$ and since $u, v \in F_n$ we conclude that $u = v$, which is a contradiction. $\Box$

We also observe in Table 1 that $A_{4,8} = N_{3,4} = 8$, $A_{5,16} = N_{4,8} = 222$ and $A_{6,32} = N_{5,16} = 807980$. This turns out to be the case in general.

Theorem 3.3. $A_{n,2k} = N_{n-1,k}$ for $n \geq 2$ and $k = 2^{n-2}$.

Proof. Here is a sketch of a simple combinatorial argument that verifies this fact. Consider a balanced coloring of dimension $n - 1$ and weight 2^{n-2}. If it has $2j$ antipodal pairs of vertices with one vertex black and the other white, then there must be $2^{n-3} - j$ antipodal pairs for which both vertices are black and the same number of pairs in which both vertices are white. Use this to make face F_n of an n-dimensional coloring. The white vertices of face $-F_n$ are antipodal to the blacks of F_n. This guarantees that the coloring being constructed will be antiantipodal. The remaining 2^{n-2} vertices of F_n are colored black.

Now consider the sum of the i-th coordinates of the black vertices of the new configuration. If $i = n$, then the sum is zero because there are 2^{n-2} black vertices in face F_n and the same number in face $-F_n$. If $i < n$, first consider all $2j$ antipodal pairs of different colored vertices in face F_n. Their i-th coordinates sum to zero because they come from the balanced antiantipodal portion of the original configuration. Similarly the corresponding $2j$ black vertices of face $-F_n$ also have i-th coordinate sum zero. Now consider any pair of antipodal black vertices in the original coloring. Each pair has i-th coordinate sum zero in both the original coloring and the new one. Each of these pairs gives rise to a complementary pair in face $-F_n$ which also has i-th coordinate sum zero. Hence the new configuration is balanced. It should be straightforward for the reader to check that this correspondence is a bijection. $\square$

4 Counting formulae for integer weightings

A 2-coloring can be viewed as an assignment of weights in $\{0, 1\}$, with 0 corresponding to white and 1 to black. The weight of a coloring is then the sum of the weights in that assignment. Suppose now that non-negative integral values or weights are assigned to the vertices of the n-cube so that the sum of the weights is $2k$. Thus we allow $b \leq 2k$ black vertices and their weights range from 1 to $2k$ but must sum to $2k$. The other $2^n - b$ white vertices have weight 0. To be a balanced configuration, the sum of the weights of all the black vertices with i-th coordinate $+1$ must be equal to the sum of the weights of the black vertices with i-th coordinate -1 for each $i = 1$ to n. The number of these is denoted by $TN_{n,2k}$. Note that the number b of black vertices is no longer necessarily even. The derivation of the appropriate formulae for computation closely follows the pattern above for $N_{n,2k}$.

Theorem 4.1. The number $TN_{n,2k}$ of balanced colorings of the n-cube with non-negative integral weights of total $2k$ is

$$TN_{n,2k} = \sum_{\langle j \rangle} N(\langle j \rangle)^n h(\langle j \rangle), \tag{4.1}$$

where the sum is over all partitions $\langle j \rangle$ of $2k$.

The proof is similar to that of Theorem 3.1 and so we shall just observe some of the crucial differences. First, we use $Z(S_{2k})$ instead of $Z(A_{2k} - S_{2k})$ in equation (3.7). The reason is that we want to choose $2k$ black vertices with repetition instead of a $2k$-subset. Then (4.1) follows from (3.10) and the fact that $h(\langle j \rangle)$ is the coefficient of $\prod_{i=1}^{2k} s_i^{j_i}$ in $Z(S_{2k})$.

As for the Corollary of Theorem 3.1, one simply replaces N by TN:

Corollary 3. For $n > m$

$$TN_{n,2k} = \sum_{i=1}^{m} C_i TN_{n-i,2k}. \tag{4.2}$$

Note that this means the recurrence relations for $TN_{n,2k}$ are identical to those for $N_{n,2k}$. Only the initial conditions are different. For example, from (3.16) we have for $n \geq 3$

$$TN_{n,4} = 8TN_{n-1,4} - 12TN_{n-2,4} \tag{4.3}$$

but we must use

$$TN_{2,4} = 3 \quad \text{and} \quad TN_{1,4} = 1. \tag{4.4}$$

Hence $TN_{3,4} = 8(3) - 12(1) = 12$. Continuing to use (4.3) we find $TN_{4,4} = 8(12) - 13(3) = 60$, $TN_{5,4} = 8(60) - 12(12) = 336$ and $TN_{6,4} = 8(336) - 12(60) = 1968$.

Thus to the results given in Table 1, we can add those in Table 2, which gives the corresponding numbers of weighted, balanced n-cubes, where non-negative integer weights are allowed for the vertices. It can be shown quickly by combinatorial means that $TN_{2,2k} = k + 1$ and so these values have been omitted from the table.

The theorem can also be used to find the number $TA_{n,2k}$ of balanced colorings with non-negative integral weights but no antipodal pair of black vertices. Note that for all $n \geq 1$, $TA_{n,2k} = A_{n,2k}$ for $k = 0, 1, 2$. Hence there is some duplication in Table 2. The binomial coefficient in the corollary below simply counts the number of ways to select $k - j$ items from a set of 2^{n-1} with repetitions allowed.

Corollary 4.

$$TA_{n,2k} = TN_{n,2k} - \sum_{j=0}^{k-1} TA_{n,2j} \binom{2^{n-1} - 1 + k - j}{k - j}. \tag{4.5}$$

The proof is similar to that of Corollary 2. One observes that any balanced coloring can be uniquely expressed as an antiantipodal coloring of weight $2j$ with the remaining weight of $2k - 2j$ accounted for by a selection with repetition of $k - j$ pairs of antipodal vertices.

The first instance of an antiantipodal coloring whose black vertices do not all have the same weight occurs in Table 2 for $n = 4$ and $k = 3$. Since $A_{4,6} = 0$ and $TA_{4,6} = 16$, each of the 16 configurations must have some black vertices of different weights. In fact there is just one unlabeled antiantipodal configuration of weight 6; we leave its construction to the reader.

$TA_{n,2k}$	$TN_{n,2k}$	$2k$	n
1	1	0	3
0	4	2	3
2	12	4	3
0	28	6	3
2	57	8	3
1	1	0	4
0	8	2	4
24	60	4	4
16	328	6	4
128	1450	8	4
1	1	0	5
0	16	2	5
200	336	4	5
1184	5200	6	5
11972	61992	8	5
1	1	0	6
0	32	2	6
1440	1968	4	6
42560	94624	6	6
1293560	3468160	8	6

Table 2. Balanced colorings of the n-cube with non-negative integral weights for $n = 3, 4, 5$ and 6.

5 The superposition approach

There is another way to count balanced colorings of the n-cube. One can use the superposition approach to enumeration that was pioneered by Redfield [R27] and Read [Re59]. See Chapter 7 of the book [HP73] for another description of this method.

Let G_1 and G_2 be permutation groups of degree m. As before we denote a partition of m by $\langle j \rangle = (j_1, j_2, \ldots, j_m)$ where j_i is the number of parts equal to i. Then the cycle indices of G_1 and G_2 can be written in terms of the indeterminates $s_1, s_2, s_3, \ldots$ as follows:

$$Z(G_1) = \sum C_{\langle j \rangle} s_1^{j_1} s_2^{j_2} \ldots s_m^{j_m} \tag{5.1}$$

and

$$Z(G_2) = \sum D_{\langle j \rangle} s_1^{j_1} s_2^{j_2} \ldots s_m^{j_m}, \tag{5.2}$$

where the sums are over all partitions $\langle j \rangle$ of m. Now we denote by

$$Z(G_1) * Z(G_2) \tag{5.3}$$

the polynomial

$$\sum C_{\langle j \rangle} D_{\langle j \rangle} h(\langle j \rangle)^{-1} s_1^{j_1} s_2^{j_2} \ldots s_m^{j_m} \tag{5.4}$$

where again the sum is over all partitions $\langle j \rangle$ of m, and where $h(\langle j \rangle)$ is defined by equation (3.5). The *superposition operator* $*$ is commutative and associative and can be extended by associativity to any number of operands; thus

$$\begin{aligned} Z(G_1) * Z(G_2) * \ldots * Z(G_n) \;\; &= \\ &\sum (C_{\langle j \rangle} D_{\langle j \rangle} \ldots) h(\langle j \rangle)^{1-n} (\prod_{i=1}^{m} s_i^{j_i}). \end{aligned} \tag{5.5}$$

If $P(s_1, s_2, \ldots, s_m)$ is a polynomial in the variables $s_1, s_2, s_3, \ldots, s_m$, we denote by $N(P)$ the sum of the coefficients of P. Thus

$$N(P) = P(1, 1, \ldots, 1). \tag{5.6}$$

Also we use $M(P)$ to stand for the number obtained when each s_i of P is replaced by $(-1)^{i+1}$, that is

$$M(P) = P(1, -1, 1, -1, \ldots). \tag{5.7}$$

Then if P is a cycle index or a cycle index sum, $M(P)$ is similar to $N(P)$ but counts negatively the terms corresponding to odd permutations.

Theorem 5.1. Let $G_1, G_2, \ldots, G_n$ be permutation groups of degree m, and let P be the polynomial

$$P = Z(G_1) * Z(G_2) * \ldots * Z(G_n). \tag{5.8}$$

Then $N(P)$ is the number of orbits of superpositions under $S_m \times G_1 \times \cdots \times G_n$, where S_m permutes columns and G_i permutes the ith row for $i = 1, \ldots, n$. Likewise, $M(P)$ is the number of such orbits in which no representative has a column-odd automorphism, i.e., one in which the column permutation is odd.

Proof. That $N(P)$ is the number of orbits of superpositions under $S_m \times G_1 \times \ldots \times G_n$ is the classical Redfield-Read Superposition Theorem (see [R27], [Re59] and [HP73]). Two superpositions, say α' and α'', are called *equivalent* if they are in the same orbit, and in this case we write $\alpha' \sim \alpha''$.

Recall that the cycle index of the symmetric group S_m can be written as

$$Z(S_m) = \sum h(\langle j \rangle) Z(\langle j \rangle), \tag{5.9}$$

where the sum is over all partitions $\langle j \rangle$ of m and $Z(\langle j \rangle)$ denotes the cycle type of a permutation with disjoint cycle decomposition $\langle j \rangle$, i.e.

$$Z(\langle j \rangle) = \prod_{i=1}^{m} s_i^{j_i}. \tag{5.10}$$

Now consider the polynomial

$$P_1 = Z(S_m) * Z(G_1) * \ldots * Z(G_n), \tag{5.11}$$

and observe that $N(P) = N(P_1)$. In fact, $P = P_1$. The reason is that for the term corresponding to $Z(\langle j \rangle)$ in P_1, for any $\langle j \rangle$, the additional factor of $h(\langle j \rangle)^{-1}$ from the superposition operation is precisely cancelled by the coefficient of $Z(\langle j \rangle)$ in $Z(S_m)$.

Now let $N(P) = A + B$, where A is the number of superpositions that have no odd automorphisms and B is the number of those which do have odd automorphisms. Note that an automorphism of a superposition α is simply an element $(\varphi, g_1, \ldots, g_n)$ of $S_m \times G_1 \times \ldots \times G_n$ such that

$$(\varphi, g_1, \ldots, g_n)(\alpha) = \alpha. \tag{5.12}$$

For this to be the case, we must have

$$Z(\varphi) = Z(g_1) = \ldots = Z(g_n) = Z(\langle j \rangle) \tag{5.13}$$

for some partition $\langle j \rangle$ of m. The automorphism is column-odd if, and only if,

$$a(\langle j \rangle) \equiv 1 \pmod 2, \tag{5.14}$$

in which case we say that $Z(\langle j \rangle)$ is of *odd type*. The function $a(\langle j \rangle)$ is defined by equation (3.4).

Now consider the polynomial

$$P_2 = Z(A_m) * Z(G_1) * \ldots * Z(G_n). \tag{5.15}$$

We claim that

$$N(P_2) = 2A + B. \tag{5.16}$$

Consider the orbit $O(\alpha)$ of some superposition α with respect to the group $S_m \times G_1 \times \ldots \times G_n$. This orbit contributes 1 to $N(P_1)$. Since $A_m \times G_1 \times \ldots \times G_n$ has index 2 in $S_m \times G_1 \times \ldots \times G_n$, $O(\alpha)$ must either consist of a single orbit over $A_m \times G_1 \times \ldots \times G_n$, or else the union of two such orbits. Thus $O(\alpha)$ contributes 1 or 2, respectively, to $N(P_2)$. For each α we now determine which alternative applies.

Variants of α obtained by acting on it with an even permutation φ are equivalent, as φ belongs to A_m. And any two variants obtained by acting on α by odd permutations ψ_1 and ψ_2 are also equivalent, because $\psi_1\psi_2^{-1}$ belongs to A_m. Now let $\alpha' = (\psi, e, \ldots, e)\alpha$ where ψ is odd and e denotes the identity permutation. If α' and α are equivalent under $A_m \times G_1 \times \ldots \times G_n$, then we will have

$$\alpha = (\varphi, g_1, \ldots g_n)\alpha' \tag{5.17}$$

for some φ in A_m, g_1 in $G_1, \ldots, g_n$ in G_n. So

$$\alpha = (\varphi, g_1, \ldots, g_n)(\psi, e, \ldots, e)\alpha, \tag{5.18}$$

from which it follows that $(\varphi\psi, g_1, \ldots, g_n)$ is an element of $\mathrm{Aut}(\alpha)$, the automorphisms of α. This implies that $\mathrm{Aut}(\alpha)$ contains an odd permutation, since $\varphi\psi$ is odd.

Conversely, if $\mathrm{Aut}(\alpha)$ does contain an odd permutation, then $\alpha \sim \psi\alpha$ for any ψ in S_m. This can be seen by reading the above computation in the reverse direction.

Thus $O(\alpha)$ is split into two orbits in $A_m \times G_1 \times \ldots \times G_n$ precisely if α has no odd automorphisms. Hence the claim (5.16).

Now we have

$$\begin{aligned} A &= (2A + B) - (A + B) \\ &= N(P_2) - N(P_1) \\ &= N((Z(A_m) - Z(S_m)) * Z(G_1) * \ldots * Z(G_n)). \end{aligned} \tag{5.19}$$

As seen in (3.11), the coefficient of $Z(\langle j\rangle)$ in $Z(A_m) - Z(S_m)$ is

$$(-1)^{a(\langle j\rangle)} h(\langle j\rangle), \tag{5.20}$$

so the effect of the factor $Z(A_m) - Z(S_m)$ is simply to multiply by -1 for every term $Z(\langle j\rangle)$ of odd type in P of (5.8). So $A = M(P)$. □

We shall now apply the theorem to obtain an alternative derivation of the results previously given and some extensions. Suppose $2k$ vertices of the n-cube are to be colored. Then consider a range set consisting of k $+1$'s and k -1's with the group $S_k \times S_k$ acting to permute the $+1$'s and -1's among themselves in all possible ways. A superposition of n copies of this range set gives a set of $2k$ columns. Each column is an n-vector of $+1$'s and -1's, and thus represents a particular vertex of the n-cube.

If the $2k$ columns of the superposition α are distinct, then the $2k$ vertices which they represent form a balanced coloring of the n-cube. For in any one of the n coordinates there are k columns with value $+1$ and k with value-1. But the number of superpositions giving distinct columns in this case is

$$M(Z(S_k)^2 * \ldots * Z(S_k)^2), \tag{5.21}$$

where the superposition operation has length n. This makes use of the fact that

$$Z(S_k \times S_k) = Z(S_k)^2. \tag{5.22}$$

It also relies on the observation that a sequence of columns admits a column-odd automorphism in $S_{2k} \times (S_k \times S_k) \times \ldots \times (S_k \times S_k)$ if, and only if, some two columns are identical.

Corollary 5.

$$N_{n,2k} = M(Z(S_k)^2 * \ldots * Z(S_k)^2), \tag{5.23}$$

where the superposition operator involves n factors.

Proof. As seen above, each superposition counted corresponds to a balanced coloring of the n-cube with exactly $2k$ black vertices. Conversely, any such balanced coloring of the n-cube can be converted to a unique superposition of the sort counted by regarding the black vertices as a set of column vectors. □

To illustrate, consider the case $n = 3$ and $k = 2$. Then we let

$$P = Z(S_2)^2 * Z(S_2)^2 * Z(S_2)^2, \tag{5.24}$$

and from the definition of the superposition operator we find

$$(5.25)\qquad \begin{aligned} P &= (1/4)^3\{(4!)^2 s_1^4 + 2^3(4)^2 s_1^2 s_2 + (8)^2 s_2^2\} \\ &= 9s_1^4 + 2s_1^2 s_2 + s_2^2. \end{aligned}$$

Hence

$$(5.26)\qquad M(P) = 9 - 2 + 1 = 8,$$

which is, by (4.23), equal to $N_{3,4}$ (see Table 1). Furthermore, it is easy to see from (4.25) that for any $n \geq 3$

$$(5.27)\qquad N_{n,4} = (1/4)^n\{(4!)^{n-1} - 2^n(4)^{n-1} + 8^{n-1}\}.$$

Thus we have the following solution of the recurrence relation in equation (3.16):

$$(5.28)\qquad N_{n,4} = (1/4)^n\{(4!)^{n-1} - 2^{3n-3}\}.$$

Recall that $TN_{n,2k}$ is the number of balanced assignments of weight $2k$, where non-negative integral values are assigned to the coordinates of the n-cube and their sum is the total weight.

Corollary 6.

$$(5.29)\qquad TN_{n,2k} = N(Z(S_k)^2 * \ldots * Z(S_k)^2),$$

where the superposition operator involves n terms.

Proof. Balance simply requires that the values of vertices having i-th coordinate $+1$ sum to k and hence the values of those having i-th coordinate -1 also sum to k, for $i = 1$ to n. Such a balanced assignment corresponds uniquely to a superposition under $S_{2k} \times (S_k \times S_k) \times \ldots \times (S_k \times S_k)$ by forming a set of columns with each vertex of the n-cube represented v times if its value is v. $\Box$

As an example, if $n = 3$ and $k = 2$, we use P from (5.24) and find

$$(5.30)\qquad TN_{3,4} = N(P) = 9 + 2 + 1 = 12.$$

Further, for any $n \geq 1$

$$(5.31)\qquad TN_{n,4} = (1/4)^n\{(4!)^{n-1} + 3(2)^{3n-3}\}.$$

Note that Corollary 5 provides an expression for $N_{n,2k}$ which differs in appearance from that of Theorem 3.1. However the two expressions have the same meaning, in the sense that they lead to the same sequence of computations for evaluating $N_{n,2k}$ once the various definitions are traced through. Likewise, Corollary 6 provides an expression for $TN_{n,2k}$ which is computationally equivalent to that of Theorem 4.1.

6 Related Problems

Several questions arising from the above results remain to be investigated.

The numerical data suggest that for each $n, N_{n,2k}$ is unimodal with the maximum at $k = 2^{n-2}$, and we ask for a proof of this. More generally, we ask for the asymptotic behavior of $N_{n,2k}$ and $A_{n,2k}$ as $n \to \infty$.

The earlier results all assume the n-cube to be fixed in place (labeled). We ask for the number of equivalence classes of balanced colorings under the full automorphism group of the n-cube (order $n!2^n$), the rotation subgroup (order $n!2^{n-1}$), the reflection subgroup (order 2^n), or the permutation subgroup (order $n!$). The complications entailed by these refinements seem to be considerable.

The balance condition for a coloring f could be interpreted as independence of coordinate projections from color. In choosing a random vertex v of Q_n, let F_i denote the event $v \in F_i$ and let B denote the event $f(v) =$ black. Then f is balanced if, and only if, F_i is independent of B for each $i = 1, \ldots, n$. A more stringent notion of balance is obtained by requiring also that $F_i \cap F_j$ be independent of B for $1 \leq i < j \leq n$. We ask for an effective enumeration of such colorings.

Finally, the concept of antiantipodal colorings touches on decompositions of colorings. Define a *proper decomposition* of a balanced coloring f of Q_n to be a set $\{f_1, \ldots, f_m\}$ of balanced colorings of Q_n such that $m \geq 2$, $w(f_i) > 0$ for $1 \leq i \leq m$, and each black vertex of f is assigned to black by exactly one of the factor colorings f_i for $1 \leq i \leq m$. A balanced coloring of positive weight having no proper decomposition is termed *irreducible*, and we ask for an enumeration of such colorings. Note that the smallest irreducible colorings have weight 2, the two black vertices being antipodal. It is seen that $A_{n,2k}$ is simply the number of balanced colorings of weight $2k$ having no decomposition with a factor of weight 2. It follows that the colorings enumerated by $A_{n,4}$ and $A_{n,6}$ are irreducible.

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