Minimum Degree Games for Graphs

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Abstract

Given δ and n, a minimum degree game starts with n disconnected nodes. Two players alternate, each adding a new edge in turn, until the resulting graph has minimum degree at least δ. In the achievement game, the last player to move is the winner; in the avoidance game, the last to play is the loser. We determine a winning strategy for the avoidance game for every δ and n. The achievement game is much harder to analyze. We determine a winning strategy for $\delta \leq 3$ and every n. For arbitrary δ the form of a winning strategy is conjectured, but we have only proved it when $n - \delta$ is odd.

1 Introduction

Various games have been proposed of the following form: beginning with n isolated nodes, two players Alpha and Beta alternate adding edges, with Alpha moving first. The game ends when a certain type of graph has been reached. The game is called an *achievement game* if the last person to move wins; otherwise it is an *avoidance game*.

Winning strategies have been found for some goals, such as diameter-2 graphs by Buckley and Harary [1] and connected graphs by two of the authors [4]. Harary and Plochinski [3] found a winning strategy for the avoidance game where the goal is a graph with a node of degree 3.

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We now investigate a similar problem: Who has a winning strategy when the goal is a graph with all nodes having degree $\geq \delta$? The avoidance game is straightforward; we give a complete analysis in Section 2. The achievement game is similarly amenable to analysis when $n - \delta$ is odd, but seems to be much more difficult when n and δ have the same parity. We solve the achievement game for $\delta = 1$ or 2 in Section 3, and for $\delta = 3$ in Section 4. For $\delta \geq 4$ and $n - \delta$ even we have only a conjecture, which is presented in Section 5. The conjecture is that for $\delta \geq 4$, the winner in the achievement game is the loser in the avoidance game.

A move is called *rational* if it (1) results in an immediate win if possible, (2) avoids an immediate loss if possible, and (3) avoids allowing the opponent an opportunity to win on the next move whenever possible. Clearly, optimal play must be rational, and analysis of winning positions can assume rational play without loss of generality. Since minimum degree δ can only be attained if $n \geq \delta + 1$, we assume this in the sequel.

In general we follow the graph theoretic notation and terminology of [2], except that we use *node* and *edge* rather than *point* and *line*. In a *graph* there are $n \geq 1$ *nodes*, and some (possibly empty) set of unordered pairs of distinct nodes designated as *edges*. Thus, loops and multiple edges are not allowed. The *degree* of a node is the number of edges in which it is contained. An *isolated node* or *isolate* is a node of degree 0. An *end-node* is a node of degree 1. The *complete* graph on n nodes, denoted K_n, has n nodes and all of the $\binom{n}{2}$ possible edges joining distinct pairs of nodes.

2 The Avoidance Game

The avoidance game is much simpler than the achievement game. This is because there is only one final position which is always reached, assuming that both players make rational moves.

We will call nodes with degree δ or more *saturated*, and nodes with degree less than δ *unsaturated*. As long as any node is unsaturated, the game continues, and neither player can force the other to add an edge to the last node until all other possible edges have been put in.

Once there is only one unsaturated node of degree $\delta - 1$ left, any player connecting an edge to that node loses, and so both players will "stall", by adding edges to already-saturated nodes. This continues until the graph consists of one node of degree $\delta - 1$, and K_{n-1} on the other nodes. Figure 1 illustrates this position. In all the figures, a large circle represents the complete graph on the saturated nodes.

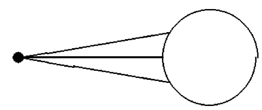


Figure 1. Penultimate position for avoidance game, $\delta = 4$.

At this point, the player whose move it is will lose. Which player that is can be determined from the parity of the number of moves already made. The number of edges in the graph is:

$$\binom{n-1}{2} + \delta - 1.$$

If this number is even, then it is Alpha's move, and he loses. If it is odd, then Beta is the loser. Thus we have settled the avoidance game.

Theorem 1. *For the avoidance game with minimum degree δ, the winner is:*

$$\begin{cases} \text{Alpha} & n \equiv 0,3 \pmod 4 \\ \text{Beta} & n \equiv 1,2 \pmod 4 \end{cases} \quad \text{when } \delta \text{ is odd,}$$

$$\begin{cases} \text{Alpha} & n \equiv 1,2 \pmod 4 \\ \text{Beta} & n \equiv 0,3 \pmod 4 \end{cases} \quad \text{when } \delta \text{ is even.} \qquad \square$$

3 Achievement for $\delta = 1$ and 2

As with the avoidance game, the analysis of the achievement game will consist of determining the set of penultimate positions, in which the next player to move will lose. Then we will see which player wins in each situation, and which player can force the game into a position favorable to him.

The $\delta = 1$ case is simple, since there is only one penultimate position. The winner of the game is the person whose move makes all the nodes degree 1 or higher, i.e., gets rid of the last isolated node. The possible moves connect either two isolated nodes, one isolated node to a saturated node, or two saturated nodes (a stall).

Clearly, graphs with only one or two unsaturated (isolated) nodes are Mover-win positions. Therefore the graph will eventually have three isolated nodes, assuming rational play and $n \geq 5$. At this point, connecting an edge to any of the isolated nodes results in a win for the other player, so both players will stall as long as they can. This will continue until the graph becomes $3K_1 \cup K_{n-3}$, at which point the player who must move loses the game (see Figure 2) as the other player can then eliminate all isolated nodes.

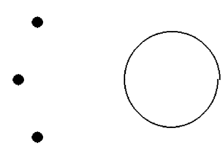


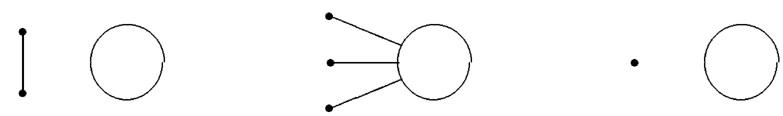
Figure 2. Penultimate position for achievement game, $\delta = 1$.

The number of edges in K_{n-3} is $\binom{n-3}{2}$, which is even if $n \equiv 0, 3 \pmod 4$, in which case Alpha will be the player forced to move. If $n \equiv 1, 2 \pmod 4$, Beta will be forced to move. Of course Alpha is the winner in one step if $n = 2$, and Beta can win with his first move if $n = 3$ or 4. Thus we have easily determined the achievement game winner for degree 1.

Theorem 2. *For the minimum degree* 1 *achievement game,*

$$\textit{the winner is: } \begin{cases} \text{Alpha} & n = 2, \ \textit{or } n \equiv 0, 3 \pmod 4 \ \textit{for } n \geq 7 \\ \text{Beta} & n = 3, 4, \ \textit{or } n \equiv 1, 2 \pmod 4 \ \textit{for } n \geq 5. \end{cases} \qquad \square$$

For $\delta = 2$, the ultimate positions are those with either one endnode or two nonadjacent endnodes, and the complete graph on the remaining nodes. The set of positions that necessarily lead to these positions in one step are the penultimate positions. For $n \geq 5$ there are three such positions: (a) two adjacent endnodes, (b) three independent endnodes, or (c) one isolated node. In each case, both players will stall until one is forced to connect an edge to one of the unsaturated nodes. Figure 3 shows these possibilities.



(a) (b) (c)

Figure 3. Penultimate positions for achievement game, $\delta = 2$.

In the first case, the player to move will be forced to lose when the graph consists of $K_2 \cup K_{n-2}$. The total number of edges is $\binom{n-2}{2} + 1$, which is odd when $n \equiv 2, 3 \pmod 4$, and even when $n \equiv 0, 1 \pmod 4$. The second case consists of K_{n-3} with three extra edges, for a total of $\binom{n-3}{2} + 3$, which is odd when $n \equiv 0, 3 \pmod 4$, and even when $n \equiv 1, 2 \pmod 4$. In the third case, there are $\binom{n-1}{2}$ edges, which is odd when $n \equiv 0, 3 \pmod 4$, and even when $n \equiv 1, 2 \pmod 4$.

From this we can see immediately that for $n \geq 5$ Alpha wins when $n \equiv 3 \pmod 4$, and Beta wins when $n \equiv 1 \pmod 4$, since the number of edges in each of the penultimate graphs has the same parity. For $n = 3$, all three edges must be played in order to achieve minimum degree 2 on all 3 nodes, so Alpha must be the winner. When n is even and greater than 6, the winner's strategy is to steer the game to the position of Figure 3(b) or 3(c). This leads to the following result.

Theorem 3. *For the minimum degree* 2 *achievement game,*

$$\textit{the winner is:} \begin{cases} \text{Alpha} & n = 3,\ 6 \textit{ or } n \equiv 0, 3 \pmod 4 \textit{ for } n \geq 7, \\ \text{Beta} & n = 4,\ 5 \textit{ or } n \equiv 1, 2 \pmod 4 \textit{ for } n \geq 9. \end{cases}$$

Proof. For even n, let W denote Alpha when $n/2$ is even and Beta when $n/2$ is odd, and let L denote the other player. Then W is always the player for whom Figures 3(b) and 3(c) are winning positions. We will show by a general argument that for even $n \geq 8$, W can force the penultimate position to be that of Figure 3(b) or 3(c). The smaller cases $n = 4$ and 6 are considered separately; it turns out that L can win them both.

The simplicity of the theorem leads one to hope that W's strategy for even $n \geq 8$ would be simple and easy to state. Such a strategy may exist, but the best one we have found is involved, and needs some preparation.

type	connect	resulting state
1	C-C	$[a, b+1, c-2]$
2	B-B	$[a+2, b-2, c]$
3	B-C	$[a+2, b-1, c-1]$
4	A-A	$[a-2, b, c]$
5	A-B	$[a, b-1, c]$
6	A-C	$[a, b, c-1]$
7	A-S	$[a-1, b, c]$
8	B-S	$[a+1, b-1, c]$
9	C-S	$[a+1, b, c-1]$
10	S-S	$[a, b, c]$

Table 1. Possible moves for $\delta = 2$.

We will classify positions according to the *state* $[a, b, c]$, where a is the number of endnodes connected to saturated nodes, b is the number of pairs of endnodes connected to each other, and c is the number of isolates. For example, the beginning position is $[0, 0, n]$.

For $\delta = 1$, there were three types of move: connecting two saturated nodes, connecting a saturated node to an isolated one, or connecting two isolated nodes. For $\delta = 2$, we have ten types of move, given in Table 1, where the starting position is $[a, b, c]$. Here A represents an end-node which is adjacent to a saturated node, B represents an end-node which is adjacent to another end-node, C represents an isolated node, and S represents a saturated node.

We will need two measures of how far away a position is from the end of the game. The *weight* of a position will be $a + 2b + c$, the number of non-saturated nodes. The *deficiency* of a position is $a + 2b + 2c$, the amount that the sum of the degrees of the unsaturated vertices needs to be raised by to finish the game, since each A-node needs one more edge to be saturated, B-node pairs need one more edge each, and a C-node needs two new edges. Both functions are nonincreasing during the game, although several moves leave the weight fixed. Any move other than stalling decreases the deficiency.

A state will be denoted *Mover-win* if it is a winning position for whichever player has the next move. It will be denoted *W-win (L-win)* if it is a winning position for W (L) no matter whose move it is. It turns out that every state belongs to one of these three classes, though our proof of the winning strategies will not require this fact explicitly. The status of each state will not be affected by n, since changing n only affects the identity of W, not the

Mover	Type-Result	Type-Result	Type-Result	Type-Result
	[0, 0, 6]			
Alpha	1-[0, 1, 4]			
Beta	1-[0, 2, 2]	3-[2, 0, 3]		
Alpha	1-[0, 3, 0]	4-[0, 0, 3]		
Beta	2-[2, 1, 0]	1-[0, 1, 1]	9-[1, 0, 2]	
Alpha	4-[0, 1, 0]	9-[1, 1, 0]	1-[1, 1, 0]	
Beta		5-[1, 0, 0]	7-[0, 1, 0]	8-[2, 0, 0]

Table 2: Game tree for $n = 6$.

state that a given move results in.

States [1, 0, 0] and [2, 0, 0], the ultimate positions, are Mover-win states, since the mover may connect the remaining A-nodes and end the game. State [0, 1, 0] (Figure 3(a)), is an L-win state, and [0, 0, 1] and [3, 0, 0] (Figures 3(b) and 3(c)), are W-win states, by our definition of W and L.

Case I: $n = 4$. **Winning strategy for Beta.**

Starting with [0, 0, 4] as the initial state, Alpha must make a type 1 move, to [0, 1, 2]. Beta replies with a 1-move, to [0, 2, 0]. All four nodes are of type B, so Alpha can only make a 2-move, giving the ultimate state [2, 0, 0]. Then Beta wins by joining the remaining two unsaturated nodes. This is an exceptional case, since [0, 2, 0] is a penultimate state only when $n = 4$, and the winner in this case, Beta, is the player identified for general n as L.

Case II: $n = 6$. **Winning strategy for Alpha.**

The game tree is developed in Table 2 down to ultimate and penultimate states which have already been analyzed. The table is organized so that the result portion of the (i, j) entry (row i, column j) represents a position reached by a move of the type listed next to it. The previous position is given by the $(i-1, j)$ entry if it is nonempty, or else by the $(i-1, j-1)$ entry.

Here the fact that $n = 6$ has limited the options open to Beta. For instance, in [0, 3, 0] all 6 nodes are of type B so that Beta has no move of type 8 or 10, as he would for larger n. At the end, Alpha can win in one move from either of the ultimate positions [1, 0, 0] or [2, 0, 0], and has a winning strategy for the penultimate position [0, 1, 0] whichever player has the move. Thus Alpha has a winning strategy when $n = 6$. This is the other anomalous case for $\delta = 2$, as Alpha is the player identified as L when $n/2$ is odd.

Case III: $n \geq 8$. **Winning strategy for W.**

We treat the opening sequence, the middle game, and the end game separately. Two sets of states are needed for the discussion;

$$\begin{aligned} X &= \{[0,0,4],[0,2,0],[0,0,3],[1,1,0],[0,1,0]\}, \\ Y &= X \cup \{[3,1,0],[2,1,0],[1,1,1],[0,1,1],[0,1,2],[2,0,4],[2,0,3], \\ &\qquad [2,0,0],[1,0,4],[1,0,3],[1,0,2],[1,0,0],[0,0,2]\}. \end{aligned}$$

The strategy for W is to avoid any move which results in a state in Y or with $b \geq 2$. In the process, L is never able to make a move which results in a state in X or with $b \geq 3$.

The opening sequence is managed by W so as to ensure that there is at least one saturated node. The initial move, by Alpha, results in the state $[0,1,n-2]$ with no saturated nodes. If W is Beta, he replies with a type 3 move to give $[2,0,n-3]$. This marks the end of the opening sequence, for there is a saturated node and $[2,0,n-3] \notin Y$ as $n-3 \geq 5$. If W is Alpha, Beta's first move can be a type 3, giving $[2,0,n-3]$ again, or else type 1, yielding $[0,2,n-4]$. Alpha's second move is of type 1 or 3 respectively, leaving $[2,1,n-5]$ in either case. This has a saturated node and is not in Y as $n-5 \geq 3$.

The middle game starts with L's move directly after the initial sequence, and continues until the deficiency $a+2b+2c$ attains the value 1 or 2. Since it starts at $2n-4$ or $2n-6$ for $n \geq 8$ and decreases by 0, 1, or 2 at each move, a value of 1 or 2 will eventually be reached. Of course type 10 moves are the only ones which leave the deficiency unchanged, and for each state the number of such moves cannot exceed $\binom{s}{2}$, where $s = n-a-2b-c$ is the number of saturated nodes.

Here is W's middle game strategy, depending on the state.

Case (i). $[a,0,c]$ for $a+2c \geq 3$ and $(a,c) \neq (0,4),(0,3)$.

If $c \geq 2$, make a 1-move, to $[a,1,c-2]$, except in the three special cases $[1,0,3],[2,0,2]$, and $[3,0,2]$. For those, a 9-move is made, giving $[2,0,2],[3,0,1]$, and $[4,0,1]$ respectively.

If $c=1$, make a 9-move, to $[a+1,0,0]$, except in the special case of $[1,0,1]$. For the latter a 7-move gives $[0,0,1]$.

If $c = 0$, make a 7-move, to $[a-1,0,0]$, unless $a = 3$. For $[3,0,0]$ a 10-move leaves the state unchanged. Recall that the definition of W ensures that he always has the opportunity to stall when faced with the state $[3,0,0]$ or $[0,0,1]$.

Case (ii). $[a,1,c]$ for $a+2c \geq 1$ and $(a,c) \neq (0,1)$.

If $c \geq 2$, make a 9-move to $[a+1,1,c-1]$ unless the state is $[0,1,2]$. In that case, make the 3-move to $[2,0,1]$.

If $c = 1$, make a 3-move to $[a+2,0,0]$ except that for $[0,1,1,]$ one 8-moves to $[1,0,1]$.

If $c = 0$ make an 8-move to $[a+1,0,0]$. Note that $a \geq 2$ from the hypotheses of Case (ii).

Case (iii). $[a,2,c]$ for $(a,c) \neq (0,0)$.

Make an 8-move to $[a+1,1,c]$ unless the state is $[0,2,1]$, $[1,2,0]$, or $[2,2,0]$. In those cases make the 2-move to, respectively, $[2,0,1]$, $[3,0,0]$, or $[4,0,0]$.

One can now verify by induction that whatever moves L may make, if W follows his strategy then the following two hypotheses are true throughout the middle game.

H_W: If it is W's move then the current state $[a,b,c]$ is not in X and has $b \leq 2$.

H_L: If it is L's move then the current state $[a,b,c]$ is not in Y and has $b \leq 1$.

The first move of the middle game is L's, and as discussed when defining the opening sequence the state faced by L does satisfy H_L. This is the base step of the induction. Of course $n \geq 8$ was required to verify this. Also, the opening sequence ensured the existence of at least one saturated node, which persists for the remainder of the game.

The details of the induction step depend upon which player has the move. First, assume it is L's move, that H_L holds for the current state $[a,b,c]$, and that $a+2b+2c \geq 3$. No move can increase b by more than 1, so $b \leq 2$ in the resulting state. By working backward from each of the five forbidden states in A, one sees that each state which can reach A in one move lies in B. Thus H_L implies that H_W must hold for the state resulting from L's move.

Now assume that it is W's move, that H_W holds for the current state $[a,b,c]$, that $a+2b+2c \geq 3$, and that W follows his strategy as given in cases (i), (ii) and (iii). Because $b \leq 2, a + 2b + 2c \geq 3$, and $[a, b, c] \notin X$ one can check that exactly one of the three cases applies. It is also straightforward to verify in each case that the resulting state has a b-value of at most 1 and does not belong to the set Y of configurations forbidden to L. In this way it can be seen that the move resulting from W's strategy satisfies H_L.

By induction, then, the middle game ends with a configuration $[a, b, c]$ for which H_W or H_L holds, depending on whether it is W's move or L's move, and having $a + 2b + 2c = 1$ or 2. The latter can only hold for $[1, 0, 0], [2, 0, 0], [0, 1, 0]$, and $[0, 0, 1]$. All but $[0, 0, 1]$ are in Y, so that is the state if it is L's move. But then W has a winning strategy, as noted when discussing the penultimate states. Since $[0, 1, 0]$ is in X, this is not the state if it is W's move. But we know that $[1, 0, 0]$ and $[2, 0, 0]$ are ultimate states, from which the mover can win in a single move, and that W can win from the penultimate state $[0, 0, 1]$. So once again W has a winning strategy for the end-game, thus concluding our proof of Theorem 3. □

4 Achievement for $\delta = 3$

For $\delta = 3$, the situation becomes more complicated. Nodes with degree zero, one or two are unsaturated, and our states must include each type of node, as well as each possible pattern of interconnection between these nodes. Moreover, while for $\delta = 2$ unsaturated nodes could occur only separately (A-nodes or C-nodes) or in pairs (B-nodes), now we can have arbitrarily long paths of nodes of degree two. The possible connected configurations of unsaturated nodes are:

l_i: **Light**; a path consisting of i unsaturated nodes, with none connected to saturated nodes.

m_i: **Medium**; a path consisting of i unsaturated nodes, with one adjacency to a saturated node.

h_i: **Heavy**; a path consisting of i unsaturated nodes, with two adjacencies to saturated nodes.

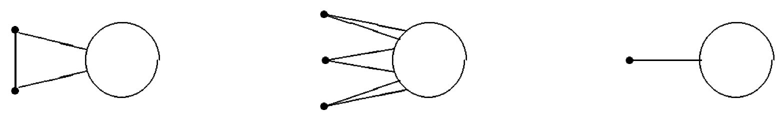
c_i: **Cycle**; a cycle of i unsaturated nodes.

We will call any maximal connected group of unsaturated nodes a *component*. Any combination of these components may occur together. We will describe a state as a quadruple

$$(l_{\lambda_1}, m_{\lambda_2}, h_{\lambda_3}, c_{\lambda_4}),$$

where each λ_j is a numerical partition (possibly empty) into positive integer parts, and each part of λ_4 is at least 3. Partitions will be written as $\langle p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k} \rangle$, where α_i is the number of parts of size p_i. A part of size j in l_{λ_1} denotes a light component l_j, and so on. A state is completely characterized by its four partitions $l_{\lambda_1}, m_{\lambda_2}, h_{\lambda_3}$, and c_{λ_4}. The *weight* of a state is the number of unsaturated nodes, the sum of all the parts in $\lambda_1, \lambda_2, \lambda_3$ and λ_4.

Lemma 1. *For $\delta = 3$, the penultimate configurations are* $h_{\langle 2 \rangle}$, $h_{\langle 1^3 \rangle}$, and $m_{\langle 1 \rangle}$.



(a) (b) (c)

Figure 4. Penultimate positions for achievement game, $\delta = 3$.

Notice that these are the same positions as for $\delta = 2$, except that the degrees are all one higher. The edge totals are $3 + \binom{n-2}{2}$, $6 + \binom{n-3}{2}$, and $1 + \binom{n-1}{2}$ respectively. Much as in the $\delta = 2$ case, these are all even (a win for Beta by stalling) when $n \equiv 0 \pmod 4$ and are all odd (a win for Alpha) when $n \equiv 2 \pmod 4$. In the odd n cases, the parity for Figure 4(a) differs from the others, and for $n \geq 7$ the winner's strategy is to steer the play so that the penultimate position reached is 4(b) or 4(c).

Once again, we can work out who wins all of the low weight states. The number of possible moves is larger, and the number of states to be considered now grows exponentially with n, but it is still feasible to compute the outcome of each state for $n \leq 15$. This was enough to determine the pattern of winners, and to find a winning strategy.

Any move from one state to another involves connecting some path or cycle component to another one, or to a saturated node. In Table 2, a move will be denoted by the two types of components which are joined, and by the components which result. Note that which nodes are joined will be important; joining a node to the end of a path will give a different result than joining it to the middle. Also, we now have the option of joining one component to itself. As before, S denotes a saturated node.

Lemma 2. *The possible moves for $\delta = 3$ are those given in Table 2.*

Proof. By considering every possible pair of components of a configuration, and connecting each type of node in one to each type in the other, we arrive at the given list of moves. □

For example, suppose we want to connect two light configurations (paths), say an l_i to an l_j. Then we may add an edge between them in three ways: connecting endpoints on both, connecting the endpoint of one to a node in the middle of the other, or connecting interior nodes on both. The first case results in an l_{i+j} (move 1), the second in three medium configurations (move 2), and the third in four medium configurations (move 3), as shown in Figure 5.

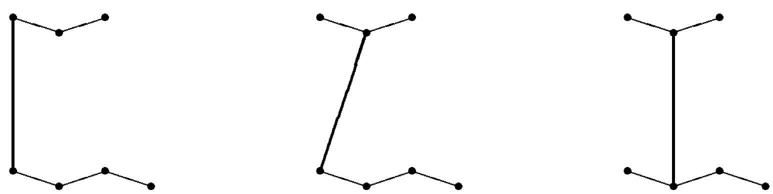


Figure 5. Moves involving an l_i and l_j.

There are several conventions which have been used to simplify the form of Table 2. Necessary inequalities for parameters are listed in the comment column, but is is implicit that all indices are to be non-negative. Resulting components of type l_0 or m_0 will never be possible, but h_0 may be. This is to be interpreted as a saturated node, and so is not represented in the heavy partition of the resultant state. The comment "reverse i and j" means that for completeness one must also consider the move with the roles of i and j reversed. As S represents a saturated node, any move involving S requires that there be at least one saturated node. There are three cases in which there must be at least two saturated nodes in order to perform the move with a particular parameter value; move 15 for $i = 1$, move 16 for $k = 0$, and move 28 for $k = 0$.

Theorem 4. *For $\delta = 3$, the winner is:*

$$\begin{cases} \text{Alpha} & n \equiv 1, 2 \pmod 4 \; for \; n \geq 6 \\ \text{Beta} & n = 4, 5 \; or \; n \equiv 0, 3 \pmod 4 \; for \; n \geq 7 \end{cases}$$

move	connect	resulting component	comment
1	l_i-l_j	l_{i+j}	
2	l_i-l_j	$m_i m_k m_{j-k-1}$	$1 \leq k \leq j-k-1$; reverse i and j
3	l_i-l_j	$m_k m_a m_{j-k-1} m_{i-a-1}$	$1 \leq k \leq j-k-1,\ 1 \leq a \leq i-a-1$
4	l_i-self	c_i	
5	l_i-self	$m_k h_{i-k-1}$	$1 \leq k \leq i-3$
6	l_i-self	$m_k m_a h_{i-k-a-2}$	$1 \leq k \leq i-3,\ 1 \leq a \leq i-k-3$
7	c_i-self	$h_k h_{i-k-2}$	$1 \leq k \leq i-k-2$
8	c_i-c_j	$h_{i-1} h_{j-1}$	
9	l_i-c_j	$m_i h_{j-1}$	
10	l_i-c_j	$m_k m_{i-k-1} h_{j-1}$	$1 \leq k \leq i-k-1$
11	l_i-S	m_i	
12	l_i-S	$m_k m_{i-k-1}$	$1 \leq k \leq i-k-1$
13	c_i-S	h_{i-1}	
14	m_i-S	h_i	
15	m_i-S	$h_k m_{i-k-1}$	$k \leq i-2$
16	m_i-self	$h_k h_{i-k-1}$	$k \leq i-k-1$
17	m_i-self	$h_k m_a h_{i-k-a-2}$	$k \leq i-a-k-2,\ 1 \leq a \leq i-3$
18	m_i-c_j	$h_i h_{j-1}$	
19	m_i-c_j	$h_k m_{i-k-1} h_{j-1}$	$k \leq i-2$
20	m_i-l_j	m_{i+j}	
21	m_i-l_j	$m_j h_k m_{i-k-1}$	$k \leq i-2$
22	m_i-l_j	$m_k m_{j-k-1} h_i$	$1 \leq k \leq j-k-1$
23	m_i-l_j	$h_k m_{i-k-1} m_{j-a-1} m_a$	$k \leq i-2,\ 1 \leq a \leq j-a-1$
24	m_i-m_j	h_{i+j}	
25	m_i-m_j	$h_k h_j m_{i-k-1}$	$k \leq i-2$; reverse i and j
26	m_i-m_j	$m_{i-k-1} m_{j-a-1} h_k h_a$	$k \leq i-2,\ a \leq j-2$
27	h_i-self	$h_k h_a h_{i-k-a-2}$	$k \leq a \leq i-k-3$
28	h_i-S	$h_k h_{i-k-1}$	$k \leq i-k-1$
29	h_i-c_j	$h_k h_{i-k-1} h_{j-1}$	$k \leq i-k-1$
30	h_i-l_j	$m_j h_k h_{i-k-1}$	$k \leq i-k-1$
31	h_i-l_j	$h_k h_{i-k-1} m_a m_{j-a-1}$	$k \leq i-k-1,\ 1 \leq a \leq j-a-1$
32	h_i-m_j	$h_k h_j h_{i-k-1}$	$k \leq i-k-1$
33	h_i-m_j	$h_k h_{i-k-1} h_a m_{j-a-1}$	$k \leq i-k-1, a \leq j-2$
34	h_i-h_j	$h_k h_{i-k-1} h_a h_{j-a-1}$	$k \leq i-k-1,\ a \leq j-a-1$

Table 2. Possible moves for $\delta = 3$.

Proof. Using the list of moves in Table 2, we may check all possible states for the first several values of n, as we did for $\delta = 2$. As we did in the last section, for $n \geq 6$ we will give a simple strategy for W to use until a certain weight is reached, and then show that the possible positions at that point are all W-win states.

Lemma 3. *For odd n, W can win by the following strategy: keep the number of parts (of any kind) of size greater than one to a minimum, until a position of weight 9, 10 or 11 is reached.*

The observation that led to this lemma is that states with many parts of size one tend to be W-win states. Also, if W never creates a part of size greater than one unless forced (such as on the first move, if W is Alpha), then L cannot create many larger parts, since W can "destroy" these parts as fast as the L creates them.

For example, consider a value of n where Alpha is W. The initial position is $l_{\langle 1^n \rangle}$, and after Alpha moves the position becomes $l_{\langle 1^{n-2}2^1 \rangle}$. Then Beta has two possibilities: connect an edge incident to the existing edge, or one separate from it. These leave positions $l_{\langle 1^{n-3}3 \rangle}$ and $l_{\langle 1^{n-4}2^2 \rangle}$, respectively. In the former case Alpha can connect the center node to an isolated node, resulting in the position $l_{\langle 1^{n-4} \rangle} m_{\langle 1^3 \rangle}$. In the latter, he cannot destroy either of the parts of size two (because he has no saturated node to connect one to). The best he can do is connect the two edges, creating $l_{\langle 1^{n-4}3 \rangle}$, with one component of size 3.

If Beta is W, he follows the same strategy. After Alpha's first move, he connects an edge to the first edge, giving $l_{\langle 1^{n-3}3 \rangle}$. If Alpha connects two isolated nodes, Beta can connect the new edge to the center node on the 3-path, resulting in $l_{\langle 1^{n-5} \rangle} m_{\langle 1^2 2 \rangle}$. If Alpha creates a 4-path instead, Beta can connect the first and third nodes, getting $l_{\langle 1^{n-4} \rangle} m_{\langle 1 \rangle} h_{\langle 2 \rangle}$.

From now on, whatever move L makes, W can answer it. If L connects two isolated nodes, then W can attach a node of degree two (every part bigger than one has such a node) to the edge, so that the number of parts of size bigger than one doesn't increase. If L stalls by connecting saturated nodes, or doesn't create a large part, W can connect a saturated node to any unsaturated node.

Following this strategy, the game will eventually reach one of two positions:

1. weight of 9, 10 or 11, with no light nodes, at most one part of size two or three and no larger parts.

2. weight of 9, 10 or 11, possibly with light nodes but at most one part of size two and no larger parts.

By computer investigation, all such positions are W-win states. □

5 Achievement for $\delta \geq 4$

For $\delta \geq 4$, there are three penultimate positions, which are direct generalizations of the states in $\delta = 2$ and $\delta = 3$: two adjacent nodes of degree $\delta - 1$ with K_{n-2}, three independent nodes of degree $\delta - 1$ with K_{n-3}, or one node of degree $\delta - 2$ with K_{n-1}.

The number of edges in the first position is $\binom{n-2}{2} + 2(\delta - 2) + 1$, which is even for $n \equiv 0, 1 \pmod 4$, and odd for $n \equiv 2, 3 \pmod 4$. The number of edges in the second and third positions are, respectively, $\binom{n-3}{2} + 3(\delta - 1)$, and $\binom{n-1}{2} + \delta - 2$. The parity of these is the same as δ for $n \equiv 1, 2 \pmod 4$, and the opposite for $n \equiv 0, 3 \pmod 4$.

This suffices to prove that Alpha can win when δ is even and $n \equiv 3 \pmod 4$, or when δ is odd and $n \equiv 2 \pmod 4$. In either case, the number of edges for each position is odd, so Alpha wins. Similarly, Beta can win when δ is even and $n \equiv 1 \pmod 4$, or when δ is odd and $n \equiv 0 \pmod 4$.

This covers half the cases. For the other cases, by assuming that a player can force the second or third position (as happened for $\delta = 2, 3$), we arrive at the following conjecture:

Conjecture 1. *For the achievement game with $\delta \geq 4$, the winner is:*

$$\left\{ \begin{array}{ll} \text{Alpha} & n \equiv 1, 2 \pmod 4 \\ \text{Beta} & n \equiv 0, 3 \pmod 4 \end{array} \right. \text{when } \delta \text{ is odd,}$$

$$\left\{ \begin{array}{ll} \text{Alpha} & n \equiv 0, 3 \pmod 4 \\ \text{Beta} & n \equiv 1, 2 \pmod 4 \end{array} \right. \text{when } \delta \text{ is even.}$$

The above exhaustive methods for $\delta \leq 3$ become impossibly complicated for larger values of δ, so it appears that a new approach will be needed to settle the conjecture. Note that Conjecture 1 and Theorem 1 differ only in exchanging Alpha and Beta. Thus the conjecture is that in any minimum degree game with $\delta \geq 4$, optimal play results in the same player making the final move whether the objective be achievement or avoidance.

References

[1] F. Buckley and F. Harary, Diameter avoidance games for graphs, *Bull. Malaysian Math. Soc.* **7** (1984) 29-33.

[2] F. Harary, *Graph Theory*, Addison-Wesley, Reading (1969).

[3] F. Harary and K. Plochinski, On degree achievement and avoidance games for graphs, *Mathematics Magazine* **60** (1987) 316-321.

[4] F. Harary and R. Robinson, Connect-it games, *College Math. J.* **15** (1984) 411-419.