Almost All Regular Graphs are Hamiltonian

R. W. Robinson
Computer Science Department
University of Georgia
Athens, GA 30602, U.S.A.

N. C. Wormald*
Department of Mathematics
University of Melbourne
Parkville, VIC 3052, Australia

Abstract

In a previous paper the authors showed that almost all labelled cubic graphs are hamiltonian. In the present paper, this result is used to show that almost all r-regular graphs are hamiltonian for any fixed $r \geq 3$, by an analysis of the distribution of 1-factors in random regular graphs. Moreover, almost all such graphs are r-edge-colourable if they have an even number of vertices. Similarly, almost all r-regular bipartite graphs are hamiltonian and r-edge-colourable for fixed $r \geq 3$.

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1 Introduction

Turn the set of labelled r-regular graphs on n vertices into a probability space $\Omega_{n,r} = \Omega_n = \Omega$ with the uniform distribution. Throughout this paper, $r \geq 3$ is fixed, and asymptotics are for $n \to \infty$, unless otherwise specified, where n is restricted to even integers if r is odd. It was first proved by Bender and Canfield [2], and independently by the first author [7] that

$$|\Omega_n| \sim \frac{(rn)! \exp((1-r^2)/4)}{(rn/2)!(r!)^n 2^{rn/2}}. \tag{1.1}$$

By saying that almost all r-regular graphs have a property P, we mean that $\lim_{n\to\infty} \Pr(P) = 1$.

It has been known for some time that for fixed r sufficiently large, almost all r-regular graphs are hamiltonian. The best result to date in this direction is that of Frieze [4] showing it for all constant $r \geq 85$, using an algorithmic approach. The obvious conjecture has been that this can be reduced to $r \geq 3$. Previous early evidence for this was the result of the authors [5] that asymptotically at least $2 - 3e^{-13/12}$ of all cubic (3-regular) graphs are hamiltonian, using the standard second moment method applied to the number of Hamilton cycles of a triangle-free cubic graph. Then very recently, we showed [6] that almost all cubic graphs are hamiltonian. The method involved partitioning the set of cubic graphs into groups characterised by the numbers of short odd cycles, and working with the expected group variance. In this paper we prove the full result which has been sought.

Theorem 1. *For fixed $r \geq 3$, almost all r-regular graphs are hamiltonian.*

We could presumably use the method in [6] to prove Theorem 1 directly. In this, it would be reasonable to expect that due to greater edge density, the case $r \geq 4$ would be easier than $r = 3$. However, there is a quantum jump in difficulty when passing from $r = 3$ to $r \geq 4$ when computing the variance of the number of Hamilton cycles, due to the fact that the subgraph induced by the union of two Hamilton cycles in G must have all vertices of degree 2 or 3 in cubic graphs G, but can also have vertices of degree 4 when G is regular of higher degree. We have been able to verify that for $G \in \Omega_n$, with H denoting the number of Hamilton cycles in G,

$$\mathrm{Exp} H \sim \left(\frac{(r-2)^{(r-2)/2}(r-1)}{r^{(r-2)/2}}\right)^n e\sqrt{\frac{\pi}{2n}}$$

and, after extensive analysis,

$$\frac{\mathrm{Var} H}{(\mathrm{Exp} H)^2} \sim \frac{re^{-2/(r-1)}}{r-2} - 1.$$

However, this proposed course is a technically difficult one and so we turn aside from it without proving these results here. Instead we use the distribution of perfect matchings to achieve our goals. Let M denote the number of perfect matchings of a graph $G \in \Omega$, and , since $M = 0$ if n is odd, restrict n to even integers for all asymptotic statements involving M. From Bollobás and McKay [3, Theorem 3], a calculation much simpler than that for $\mathrm{Var} H$ above yields

$$\frac{\mathrm{Exp} M^2}{(\mathrm{Exp} M)^2} \sim e^{-(2r-1)/4(r-1)^2} \sqrt{\frac{r-1}{r-2}}. \tag{1.2}$$

For $G \in \Omega$ let $X_i(G)$ denote the number of cycles of length i in G. The approach of the present paper begins by following that in [6], but with regard to perfect matchings instead of Hamilton cycles: divide the graphs G in Ω into groups according to the values of the variables X_i for $i = 3, \ldots, b$, where $b \geq 3$, then refine the second moment method for M using these groups. This analysis is restricted to graphs with an even number of vertices, and shows that M is unlikely to be different from its mean by more than a constant factor. Hence, roughly speaking, the removal of a random perfect matching from G produces a reasonably random $(r-1)$-regular graph. Hamiltonicity of this graph implies hamiltonicity of G. The graphs with an odd number of vertices are handled by a similar argument with an extra twist.

The main result we actually obtain concerns the edge decomposition of a graph G into a set of perfect matchings and the edges of a hamilton cycle. For our present purposes, we call this a *complete* decomposition of G. Naturally, for such a decomposition the number of vertices has to be even.

Theorem 2. *For fixed $r \geq 3$, almost all r-regular graphs with an even number of vertices have a complete decomposition.*

Slight modifications of the proof of Theorem 2 to handle an odd number of vertices will give Theorem 1. Immediately from Theorem 2 we also have the following.

Corollary. *For fixed $r \geq 3$, almost all r-regular graphs with an even number of vertices are r-edge-colourable.*

In addition, the same proof works easily to give the analogous results for bipartite r-regular graphs. The case $r = 3$ was already proved in [5].

Theorem 3. *For fixed $r \geq 3$, almost all r-regular bipartite graphs have a complete decomposition.*

Corollary. *For fixed $r \geq 3$, almost all r-regular bipartite graphs are hamiltonian and r-edge-colourable.*

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2 Proof of Theorem 2

The proof is by induction on r. A cubic graph with a Hamilton cycle must have a complete decomposition, and so the validity of Theorem 2 for $r = 3$ comes from the main result of [6]. We can thus take $r \geq 4$. Our main objective is to show that for $G \in \Omega_{2n}$, M is unlikely to be very small compared with its expectation. Indeed, once we have shown that there is a sequence $w(y) > 0$ such that

$$\liminf \Pr(M \geq w(y)\mathrm{Exp}M) \to 1 \quad \text{as} \quad y \to \infty, \tag{2.1}$$

the following argument, linking the spaces $\Omega_{2n,r}$ and $\Omega_{2n,r-1}$, completes the proof of Theorem 2. Apart from the use of (2.1), the asymptotics are for y and b fixed as n goes to ∞.

Let R denote the event $\{M \geq w(y)\mathrm{Exp}M\}$. Define a bicoloured graph B, in which the blue vertices are the elements of $\Omega_{2n,r}$ and the red ones are the elements of $\Omega_{2n,r-1}$, with an edge from a blue vertex v_1 to a red vertex v_2 if and only if v_2 can be obtained from v_1 by deleting the edges of a perfect matching.

Let T_r be the event that $G \in \Omega_{2n,r}$ does not have a complete decomposition, and use $\Pr_r$ for probability in the space $\Omega_{2n,r}$. Choose an edge v_1v_2 of B uniformly at random, with $v_1 \in \Omega_{2n,r}$ and $v_2 \in \Omega_{2n,r-1}$. For this selection of v_1 and v_2, let P_1 be the probability that $v_1 \in R \cap T_r$, and let P_2 be the probability that $v_2 \in T_{r-1}$. From the definition of R we have

$$P_1 \geq w(y)\Pr_r(R \cap T_r). \tag{2.2}$$

On the other hand, by [2, Theorem 1] the maximum and minimum degrees of the red vertices in B are asymptotically equal (in fact, the asymptotic value is $(2n)!/(n!2^n e^{(r-1)/2})$) uniformly as $n \to \infty$. Hence

$$P_2 \leq \Pr_{r-1}(T_{r-1})(1+o(1)). \tag{2.3}$$

If v_2 has a complete decomposition then so does v_1, and so $P_1 \leq P_2$.

Hence (2.2) and (2.3), together with the inductive hypothesis that $\Pr_{r-1}(T_{r-1}) = o(1)$, imply that $\Pr_r(R \cap T_r) = o(1)$ for y fixed. Thus, by (2.1), since y can be chosen arbitrarily large, the theorem follows.

We only have to establish (2.1) for an appropriate sequence $w(y)$. Define

$$\lambda_i = \frac{(r-1)^i}{2i}, \qquad \mu_i = \frac{(-1)^i}{2i}.$$

We note that [8, Corollary 4] implies the following.

Lemma 1. *For any fixed k the variables X_i, $3 \leq i \leq k$, are asymptotically independent Poisson random variables, with* $\mathrm{Exp}X_i \sim \lambda_i$.

For $b \geq 3$ and $y > 0$, let

$$S(y,b) = \{(c_3,\ldots,c_b) | 0 \leq c_i < \lambda_i + y\lambda_i^{2/3}, c_i \text{ an integer, for } i = 3,\ldots,b\}.$$

We will later set y equal to $\sqrt{b}$. By an abuse of notation we also use $S(y,b)$ to denote the event that the sequence $X_3,\ldots,X_b$ is in $S(y,b)$.

We will show

$$\Pr(\bar{S}(y,b)) = O(e^{-y/4}) + o(1) \tag{2.4}$$

where $\bar{S}(y,b)$ denotes the complement of $S(y,b)$. Here the constant implicit in $O()$ is uniform over all b and y, whereas the constant function implicit in $o()$ may depend on both y and b. Let W be arbitrary. We will bound the probability of M being small by way of the obvious inequality

$$\Pr(M < W) \leq \Pr(\{M < W\} \wedge S(y,b)) + \Pr(\bar{S}(y,b)). \tag{2.5}$$

Define the group mean

$$E_{c_3,\ldots,c_b} = \mathrm{Exp}(M | X_3 = c_3, \ldots, X_b = c_b),$$

and the group variance

$$V_{c_3,\ldots,c_b} = \mathrm{Exp}(M^2 | X_3 = c_3, \ldots, X_b = c_b) - E^2_{c_3,\ldots,c_b}.$$

One of our main steps in the proof is to establish the following.

Lemma 2. *For fixed $b \geq 3$,*

$$E_{c_3,\ldots,c_b} \sim \mathrm{Exp}M \prod_{i=3}^{b} \left(1 + \frac{\mu_i}{\lambda_i}\right)^{c_i} e^{-\mu_i}.$$

The proof of Lemma 2 is deferred until later in this section.

Define

$$W(y,b) = \min\{E_{c_3,\ldots,c_b} \,|\, (c_3,\ldots,c_b) \in S(y,b)\}.$$

Lemma 2 will help us to prove the inequality

$$\mathrm{Exp}V_{X_3,\ldots,X_b} \geq (1+o(1))(\mathrm{Exp}M)^2 \Pr(\{M < W(y,b)/2\} \wedge S(y,b)) e^{-O(y)} b^{-1}, \tag{2.6}$$

for $y \geq 1$ where the constant implicit in $O()$ is independent of y and b.

We have

$$\begin{aligned} \mathrm{Var}M &= \mathrm{Exp}V_{X_3,\ldots,X_b} + \mathrm{Var}E_{X_3,\ldots,X_b} \\ &= \mathrm{Exp}V_{X_3,\ldots,X_b} + \mathrm{Exp}E^2_{X_3,\ldots,X_b} - (\mathrm{Exp}M)^2, \end{aligned}$$

and we will show that

$$\mathrm{Exp}E^2_{X_3,\ldots,X_b} \;\geq\; \frac{(\mathrm{Exp}M)^2\sqrt{r-1}(1 - O(e^{-b}) + o(1))}{e^{(2r-1)/4(r-1)^2}\sqrt{r-2}} \tag{2.7}$$

where the constant implicit in $O()$ is uniform over all $b \geq 3$. Hence (1.2) implies that

$$\mathrm{Exp}V_{X_3,\ldots,X_b} = (O(e^{-b}) + o(1))(\mathrm{Exp}M)^2, \tag{2.8}$$

where the constant implicit in $O()$ is uniform over all $b \geq 3$.

From (2.4) – (2.6) it follows that for $y \geq 1$ and $b \geq 1$,

$$\Pr(M < W(y,b)/2) = O(e^{-y/4}) + e^{O(y)} b \,\mathrm{Exp}V_{X_3,\ldots,X_b}/(\mathrm{Exp}M)^2 + o(1)$$

where the implied constants in each $O()$ are independent of y and b. Hence by (2.8),

$$\Pr(M < W(y,b)/2) = O(e^{-y/4} + be^{O(y)-b}) + o(1),$$

where each $O()$ is independent of y and b but $o(1)$ is not.

On choosing, say, $b = y^2$, we obtain

$$\liminf \Pr(M \geq W(y,y^2)/2) = 1 - O(e^{-y/4}),$$

where R denotes the event $\{M \geq W(y,y^2)/2\}$. Thus, setting $w(y) = \sup(W(y,y^2)/\mathrm{Exp}M)$, we get (2.1) as required. By Lemma 2, $w(y) \neq 0$.

To complete the proof, it is required to establish Lemma 2 as well as (2.4), (2.6) and (2.7).

Proof of Lemma 2. Define the uniform probability space $\Phi = \Phi_{2n}$ of all labelled r-regular graphs on $2n$ vertices with a distinguished perfect matching. Note that the elements of Φ can be identified with the edges of the bicoloured graph B defined above, and they are $|\Omega_{2n}|\mathrm{Exp}M$ in number. We use the subscript Φ on Pr and Exp to distinguish references to this space from those to Ω, which remain unsubscripted. Extend the definition of X_i in the obvious way to Φ.

We first show that

$$\mathrm{Exp}_\Phi X_m \sim (\lambda_m + \mu_m). \tag{2.9}$$

For this, we compute $\mathrm{Exp}_\Phi X_m$ in the form of the ratio $\mathrm{Exp}(MX_m)/\mathrm{Exp}M$ by counting r-regular graphs G on $2n$ vertices with a given perfect matching D once for every m-cycle C that they contain, and dividing by the total number of r-regular G on $2n$ vertices with the perfect matching D. This calculation resembles that giving equation (2.6) in [6], so we give only the crucial details.

For $m \geq 3$, the number of m-cycles in the complete graph which use precisely w edges of D can be computed asymptotically by tracing out such cycles edge by edge. First give the cycle a distinguished vertex and direction, such that the last edge in the cycle is not in D. This has the effect of multiplying the number of cycles by $2(m-w)$. Each edge of the cycle is either an edge of D, in which case that edge is determined uniquely and the next edge is not in D, or is not an edge of D and has asymptotically $2n$ choices. In a generating function we can represent these two options by the two terms x and yx^2 respectively, where x marks all the edges of the cycle chosen and y marks the edges of the cycle chosen in D. From this the number of m-cycles without distinguished vertex or direction is seen to be asymptotic to

$$\begin{aligned} &(2n)^{m-w}[x^m y^w]\left(\frac{1}{2(m-w)}(x+yx^2)^{m-w}\right) \\ = \quad &(2n)^{m-w}[x^m y^w]\left(-(1/2)\log(1-(x+yx^2))\right). \end{aligned} \tag{2.10}$$

Here square brackets denote the extraction of coefficients.

Given D and a m-cycle using precisely w edges of D, we can compute asymptotically the number of ways to determine the rest of an r-regular graph G, and also the number of ways to determine an r-regular graph which contains D. By [2, Theorem 1], the asymptotic ratio of these two numbers can be computed and put into the form

$$\left(\frac{r-1}{(r-2)^2}\right)^w (r-2)^m(2n)^{w-m}. \tag{2.11}$$

Multiplying this by (2.10) and summing over $w \geq 0$ gives

$$\begin{aligned} \mathrm{Exp}(MX_m)/\mathrm{Exp}M &\sim -[x^m]\frac{1}{2}\log(1-x(r-2)-(r-1)x^2) \\ &= -\frac{1}{2}[x^m]\log(1-(r-1)x)-\frac{1}{2}[x^m]\log(1+x). \end{aligned}$$

This gives (2.9).

Similar calculations, following the template of the derivation of [6, equation (2.9)], yield

$$\Pr_\Phi(X_3=c_3,\ldots,X_b=c_b) \sim \prod_{i=3}^{b}(\lambda_i+\mu_i)^{c_i}e^{-(\lambda_i+\mu_i)}/c_i!.$$

The rest of the proof follows that in [6]. ∎

The following is proved in [6].

Lemma 3. *Let $\eta_1,\eta_2,\ldots$ be given. Suppose that $\eta_1>0$ and that for some $c>1$, $\eta_{i+1}/\eta_i>c$ for all $i\geq 1$. Then uniformly over $x\geq 1$,*

$$R(x)=\sum_{i=1}^{\infty}\sum_{t=\eta_i(1+y_i)}^{\infty}\frac{\eta_i^t}{t!e^{\eta_i}}=O(e^{-c_0x})$$

where $y_i=x\eta_i^{-1/3}$ and $c_0=\min\{\eta_1^{1/3},\eta_1^{2/3}\}/4$.

Proof of (2.4). This follows immediately from Lemmas 1 and 3 with $\eta_i=\lambda_{i+2}$ and $x=y$. ∎

Proof of (2.7). For any real x and non-negative integer b we have

$$\begin{aligned} \mathrm{Exp}E^2_{X_3,\ldots,X_b} &= E_S+\Pr(\bar{S}(x,b))\mathrm{Exp}(E^2_{X_3,\ldots,X_b}|\bar{S}(x,b)) \\ &\geq E_S \end{aligned}$$

where

$$\begin{aligned} E_S &= \sum_{(c_3,\ldots,c_b)\in S(x,b)} E^2_{c_3,\ldots,c_b} \Pr(X_3 = c_3, \ldots, X_b = c_b) \\ &\sim (\mathrm{Exp}M)^2 \prod_{i=3}^{b} (1 - Z_i) e^{\mu_i^2/\lambda_i} \end{aligned}$$

by Lemmas 1 and 2, with

$$Z_i = \sum_{t=\lambda_i + x\lambda_i^{2/3}}^{\infty} \frac{(\lambda_i + \mu_i)^{2t}}{t!\lambda_i^t e^{(\lambda_i+\mu_i)^2/\lambda_i}}.$$

Put

$$\eta_i = (\lambda_i + \mu_i)^2/\lambda_i.$$

It is easily verified that for $i \geq 3$ and $x \geq 2$,

$$\lambda_i + x\lambda_i^{2/3} > \eta_i + x\eta_i^{2/3}/2.$$

Hence, replacing the value of x in Lemma 3 by $x/2$ and noting that $c_0 > 0.41$ in all cases, we get

$$\sum_{i=3}^{b} Z_i = O(e^{-x/5}).$$

One also has

$$\prod_{i=3}^{b} e^{\mu_i^2/\lambda_i} = e^{-(2r-1)/4(r-1)^2} \sqrt{(r-1)/(r-2)} (1 - O((r-1)^{-b} b^{-1})).$$

Thus, taking $x \geq 5b$, we obtain (2.7). ∎

Proof of (2.6). Let Ind_S denote the indicator function for an event S. Let S denote $S(y,b)$, and note that for $G \in S$,

$$\begin{aligned} \prod_{i=3}^{b} \left(1 + \frac{\mu_i}{\lambda_i}\right)^{X_i} e^{-\mu_i} &\geq \prod_{i=3}^{b} e^{(-1)^{i+1}/2i} (1 - (r-1)^{-i})^{\lambda_i + y\lambda_i^{2/3}} \\ &\geq e^{-O(y) - O(1)}/\sqrt{b} \end{aligned}$$

by routine calculations using $\log(1 + \mu_i/\lambda_i) \geq -(r-1)^{-i} - (r-1)^{-2i}$. Since $y \geq 1$ the $O(1)$ term can be dropped.

Define the event

$$T = \{M < E_{X_3,\ldots,X_b}/2\}.$$

The expected group variance can be written as

$$\begin{aligned}
&\mathrm{Exp} V_{X_3,\ldots,X_b} \\
&\quad = \mathrm{Exp}\Big(\mathrm{Exp}\Big((M - E_{X_3,\ldots,X_b})^2 \mid X_3,\ldots,X_b\Big)\Big) \\
&\quad \geq \mathrm{Exp}\Big(\mathrm{Exp}\Big((M - E_{X_3,\ldots,X_b})^2 \mathrm{Ind}_{T\wedge S} \mid X_3,\ldots,X_b\Big)\Big) \\
&\quad \geq \mathrm{Exp}\Big(\frac{1}{4}\mathrm{Exp}\Big(E^2_{X_3,\ldots,X_b}\mathrm{Ind}_{T\wedge S} \mid X_3,\ldots,X_b\Big)\Big) \\
&\quad \sim \frac{1}{4}(\mathrm{Exp} M)^2 \mathrm{Exp}\Big(\mathrm{Exp}\Big(\mathrm{Ind}_{T\wedge S}\prod_{i=3}^{b}\Big(1+\frac{\mu_i}{\lambda_i}\Big)^{2X_i} e^{-2\mu_i} \mid X_3,\ldots,X_b\Big)\Big) \\
&\quad \geq \frac{1}{4}(\mathrm{Exp} M)^2 \mathrm{Exp}\Big(\mathrm{Exp}\Big(\mathrm{Ind}_{T\wedge S}\, e^{-O(y)} b^{-1} \mid X_3,\ldots,X_b\Big)\Big) \\
&\quad \geq (\mathrm{Exp} M)^2 \Pr(T\wedge S) e^{-O(y)} b^{-1},
\end{aligned}$$

where the third-last step uses Lemma 2, and the second-last uses the bound derived above. This yields (2.6). ∎

3 Proof of Theorem 1

The basic idea here is to slightly modify any graph with $2n+1$ vertices to one with $2n$ vertices, obtain a Hamilton cycle in the modified graph using Theorem 2, and lift this up to one in the original graph.

Fix $j \geq 1$. Let $\Lambda_{n,r} = \Lambda_n = \Lambda$ denote the uniform probability space whose points are the r-regular graphs on n vertices with an ordered set of j distinguished edges, so that $|\Lambda_{2n}| = rn(rn-1)\cdots(rn-j+1)|\Omega_{2n}|$. Use the subscript Λ on Pr and Exp to distinguish references to this space from those to Ω. We require a minor modification of Theorem 2 so as to apply to Λ.

Theorem 2′. *The probability that $G \in \Lambda_{2n}$ has a complete decomposition in which the Hamilton cycle contains the first distinguished edge but no other distinguished edge tends to 1 as $n \to \infty$.*

Proof. The first step is to modify the Theorem of [6]. So for the moment, take $r = 3$, so that all graphs are cubic. Let $\hat{H}(G)$ denote the number of

Hamilton cycles using the first distinguished edge of $G \in \Lambda_{2n,3}$, but no others. Clearly

$$\mathrm{Exp}_\Lambda \hat{H} \sim \frac{2}{3^j}\mathrm{Exp} H.$$

Select a graph uniformly at random from the labelled cubic graphs with 2n vertices and with an ordered pair of distinct Hamilton cycles distinguished, and let $2n - k$ be the number of edges in the intersection of the two distinguished Hamilton cycles. From [5, proofs of Theorems 2.3 and 2.4], $k = 2n/3 + o(n)$ almost surely as $n \to \infty$. More loosely expressed, this says that the two distinguished Hamilton cycles almost surely have roughly $4n/3$ edges in common. Hence

$$\mathrm{Exp}_\Lambda(\hat{H}(\hat{H}-1)) \sim \frac{4}{9^j}\mathrm{Exp}(H(H-1)).$$

Thus from [6, equation (1.2)],

$$\mathrm{Var}_\Lambda \hat{H} \sim \left(\frac{3}{e} - 1\right)(\mathrm{Exp}_\Lambda \hat{H})^2. \tag{3.1}$$

Thus [6, equation (1.2)] for Ω translates directly over to Λ without alteration. The rest of the proof of the Theorem of [6] now carries over to Λ without change other than the obvious ones required to make the definitions fit: for example, the space Φ_n defined in the proof of Lemma 2 will now be the space of labelled cubic graphs on $2n$ vertices and j distinguished edges and a distinguished Hamilton cycle containing the first distinguished edge but none of the others. Thus we obtain Theorem $2'$ for $r = 3$, which launches the induction.

Now take $r \geq 4$. Let $\hat{M}$ denote the number of perfect matchings of a graph $G \in \Lambda$ which do not contain any of the j distinguished edges of G. Follow the lines of the above modification of the Theorem of [6]. Firstly,

$$\mathrm{Exp}_\Lambda \hat{M} \sim \frac{(r-1)^j}{r^j}\mathrm{Exp} M.$$

Secondly, from [3, proof of Theorem 3], for a random labelled r-regular graph with $2n$ vertices and two distinguished perfect matchings, the number of edges in the intersection of the matchings is almost surely asymptotic to n/r. It follows that (1.2) holds with Exp replaced by Exp_Λ and M replaced by $\hat{M}$.

Now rework the proof of Theorem 2, with Ω replaced by Λ and M replaced by $\hat{M}$. The space Φ_{2n} defined in the proof of Lemma 2 will now be the space of labelled r-regular graphs on $2n$ vertices and j distinguished edges and a distinguished perfect matching not containing any distinguished edge. In this way, we obtain

$$\liminf \Pr_\Lambda(\hat{M} \geq W(y)\mathrm{Exp}_\Lambda \hat{M}) = 1 - O(e^{-y/4})$$

and so the analogue of (2.1) is established.

This time, define B, so that the blue vertices are the elements of $\Lambda_{2n,r}$ and the red ones are the elements of $\Lambda_{2n,r-1}$, with an edge from a blue vertex v_1 to a red vertex v_2 if and only if v_2 can be obtained from v_1 by deleting the edges of a perfect matching which does not contain any distinguished edge.

Let T_r be the event that $G \in \Lambda_{2n,r}$ does not have a complete decomposition of the type required in the theorem, and use $\Pr_r$ for probability in the space $\Lambda_{2n,r}$. Then the remainder of the proof of Theorem 2 applies, and the theorem follows. ∎

To complete the proof of Theorem 1, recall that by Theorem 2 we only need to consider $\Omega_{2n+1,r}$, and hence can assume that r is even. Consider linking the spaces $\Omega_{2n+1,r}$ and $\Lambda_{2n,r}$ by the following operation. Delete the vertex labelled $2n+1$ from $G \in \Omega_{2n+1,r}$, and randomly add $r/2$ distinguished edges amongst the resulting vertices of degree $r-1$ so as to obtain an r-regular graph G'. This lies in the space $\Lambda_{2n,r}$ almost surely. (The only problem occurs when a multiple edge is created in G', but the probability of this happening is $O(1/n)$ as can be seen by applying Lemma 1 to find the expected number of 3-cycles.) Conversely, $G' \in \Lambda_{2n,r}$ can almost surely be created in a unique way by this operation. For G' will be uniquely created if its distinguished edges are pairwise non-incident, and not creatable at all if some two distinguished edges are incident. It is easy to see that the expectation of the latter is $O(1/n)$. If G' has a complete decomposition of the type described in Theorem 2$'$ then G has a Hamilton cycle. Hence Theorem 1 follows for n odd from Theorem 2$'$ applied with $j = r/2$. (A more rigorous description of this argument would require definition of another bipartite graph like B.)

4 Proof of Theorem 3

This only requires a very small number of straightforward modifications of the proof of Theorem 2. Redefine Ω_{2n} to be the probability space of r-regular bipartite graphs on $2n$ vertices with the uniform distribution, and then redefine M, X_i and all subsequent notation accordingly. We only need to prove Theorem 2 in the present context. For this proof define

$$\lambda_i = \frac{(r-1)^i}{i}, \qquad \mu_i = \frac{(-1)^i}{i}.$$

Then for fixed even $i > 3$, it is easy to verify the analogue of Lemma 1 with $\mathrm{Exp}X_i \sim \lambda_i$. (The basic calculations required for this appeared in [7], and are simpler than the corresponding ones for graphs.) In place of (1.2) we now have from [3, Theorem 6]

$$\frac{\mathrm{Exp}M^2}{(\mathrm{Exp}M)^2} \sim e^{-1/2(r-1)^2} \frac{r-1}{\sqrt{r(r-2)}}. \tag{4.1}$$

Consider the proof of Lemma 2. In place of (2.10) we now have

$$n^{m-w}[x^m y^w](-\log(1-(x+yx^2))).$$

Also, from the main theorem of [1] the expression to replace (2.11) is

$$\left(\frac{r-1}{(r-2)^2}\right)^w (r-2)^m n^{w-m}.$$

Thus (2.9) and Lemma 2 hold with the new definitions, except that in Lemma 2, b and the subscripts on the c_j become even, and in the product i is restricted to ranging over even integers. A similar restriction applies in the proof of (2.7), and we note that

$$\prod_{i=4}^{b} e^{\mu_i^2/\lambda_i} = e^{-1/2(r-1)^2} \frac{r-1}{\sqrt{r(r-2)}}(1 - O((r-1)^{-b} b^{-1})).$$

Thus the rest of the proof of Theorem 2 applies, with the appropriate modification to (2.7). The result is Theorem 3.

5 Conjectures

Conjecture 1. *Let $r \geq 4$. The probability that a random r-regular graph on $2n$ vertices has $\lfloor r/2 \rfloor$ pairwise edge-disjoint Hamilton cycles tends to 1 as $n \to \infty$.*

Conjecture 2. *Let $r \geq 4$. The probability that a random r-regular bipartite graph on $2n$ vertices has $\lfloor r/2 \rfloor$ pairwise edge-disjoint Hamilton cycles tends to 1 as $n \to \infty$.*

The methods of the present paper are probably sufficient to prove these conjectures for r odd, but this would require showing that H does not usually vary too much for $G \in \Omega$, including the establishment of the relations claimed for $\mathrm{Exp}H$ and $\mathrm{Var}H$ in the Introduction. For r even the inductive step could also be done in this way. Something new would be required however to launch the induction at $r = 4$.

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