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Analytical and Numerical Aspects of Certain Nonlinear Evolution Equations. I. Analytical

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Nonlinear partial difference equations are obtained which have as limiting forms the nonlinear Schrödinger, Korteweg–deVries and modified Korteweg–deVries equations. These difference equations have a number of special properties. They are constructed by methods related to the inverse scattering transform. They can be used as a basis for numerical schemes to the associated nonlinear evolution equations. Experiments have shown that they compare very favorably with other known numerical methods (papers II, III). In paper II, the Ablowitz–Ladik scheme for the nonlinear Schrödinger equation is compared to other known numerical schemes, and generally proved to be faster than all utilized finite difference schemes but somewhat slower than the finite Fourier (pseudospectral) methods. In paper III, a proposed scheme for the Korteweg–deVries equation proved to be faster than both the finite difference and finite Fourier methods already considered.

1. INTRODUCTION

In recent years there has been rapid advancement in the study of physically interesting nonlinear problems. The progress in this field has, in part, been due to the synergetic approach [1a, 1b], which consists of the simultaneous use of conventional analysis and numerical experiments to investigate nonlinear phenomena. In this paper we derive a numerical scheme for the Korteweg–deVries (KdV) equation and the modified Korteweg–deVries (MKdV) equations based on the inverse scattering transform (IST). In papers II, III of this work, we show that the schemes compare favorably with other known methods. Before proceeding, it may be helpful to review some of the recent developments in this area.

The inverse scattering transform (see, for example, a recent review of this subject by Ablowitz and Segur [3]) was first discovered by Gardner, Greene, Kruskal, and Miura [4, 5] in their study of the KdV equation. Subsequently, Lax [6] put the ideas in an alternative form which allows the method to be readily generalized. Zakharov and Shabat [7] found a new eigenvalue problem which led them to the solution of the nonlinear Schrödinger (NLS) equation. Ablowitz, Kaup, Newell and Segur [8] showed that a generalization of the Zakharov–Shabat eigenvalue problem allows one

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to find the solution to a class of interesting evolution equations which, in addition to the above, includes the sine-Gordon, MKdV, self-induced transparency equations, etc.

These ideas also apply to certain classes of nonlinear differential-difference equations. Using discrete scattering procedures developed by Case and Kac [9, 10], Flaschka [11] was able to solve the Toda lattice equations. Similar results were found by Manakov [12]. Subsequently, Ablowitz and Ladik [13] presented a discretized version of the generalized Zakharov–Shabat eigenvalue problem which allowed them to isolate a class of differential-difference equations solvable by inverse scattering.

Ablowitz and Ladik [14, 15] further generalized this theory to cover nonlinear partial-difference equations. They found a class of such equations and further introduced an equation which can be used as a numerical scheme for the NLS equation. It has the following advantages [16]: (see also the following section)

(i) This scheme maintains many of the important properties of the original problem. One can associate with this scheme an infinite set of conservation laws, just as in the case of the corresponding partial difference equation. This scheme has traveling wave solutions, with special properties, these are the solitons ([1, 4, 5]).

(ii) The associated linear scheme is always neutrally stable.

(iii) This scheme maintains a certain joint x, t symmetry of the original equation.

(iv) The order of accuracy is the same for both the linear and nonlinear schemes.

(v) This scheme depends globally on the mesh points, but it does suggest others which are local.

These nice properties motivate us to look for a numerical scheme for the MKdV and the KdV equations in an analogous way.

In the next section we review the procedure of finding the partial-difference equations together with the results for the NLS equation, which has already been discussed by Ablowitz and Ladik [16], We then develop and introduce a new scheme for the MKdV and the KdV equations based on the above theory.

2. Nonlinear Partial Difference Equations

The key step in obtaining partial difference equations which can be solved by inverse scattering is to make an association between the nonlinear evolution equation and a linear eigenvalue (scattering) problem. In this discussion all the difference equations are related to the eigenvalue problem [13] (see [3, 13] for the continuous version):

$$\begin{aligned} V_{1n+1}^m &= zV_{1n}^m + Q_n^m V_{2n}^m + S_n^m V_{2n+1}^m, \\ V_{2n+1}^m &= \frac{1}{z} V_{2n}^m + R_n^m V_{1n}^m + T_n^m V_{1n+1}^m, \end{aligned} \tag{2.1}$$

where z is the eigenvalue and the potentials R_n^m, Q_n^m, S_n^m, T_n^m are defined on the spacelike interval $|n| < \infty$ and the timelike interval $m > 0$. The various evolution equations are distinguished by the associated time (m) dependence of the eigenfunctions

$$\Delta^m V_{1n}^m = A_n^m(z)\, V_{1n}^m + B_n^m(z)\, V_{2n}^m,$$
$$\Delta^m V_{2n}^m = C_n^m(z)\, V_{1n}^m + D_n^m(z)\, V_{2n}^m,$$

where $\Delta^m V_{in}^m = V_{in}^{m+1} - V_{in}^m$ $(i = 1, 2)$. That is to say for each partial difference equation there corresponds a set of functions A_n^m, B_n^m, C_n^m, D_n^m depending in general on the potentials. The equations for determining the sets $A_n^m, \ldots, D_n^m$, and hence the evolution equations are obtained by requiring the eigenvalue z to be invariant with respect to m and by forcing the consistency

$$\Delta^m(E_n V_{in}^m) = E_n(\Delta^m V_{in}^m), \qquad i = 1, 2, \tag{2.3}$$

where E_n is the shift operator in the spatial coordinate defined by $E_n V_{in}^m = V_{in+1}^m$, $i = 1, 2$. Performing the operations indicated in (2.3) results in four equations.

For the special case associated with the NLS equation we let $R_n^m = \mp Q_n^{m*}$, $S_n^m = T_n^m = 0$ (where Q_n^{m*} is the complex conjugate of Q_n^m and $\mp$ refer to choices of multiplications in the usual sense), and the four equations are given by

$$\begin{aligned}
z\Delta_n A_n^m &= Q_n^{m+1} C_n^m \pm B_{n+1}^m Q_n^{m*},\\
\frac{1}{z} B_{n+1}^m - zB_n^m + A_{n+1}^m Q_n^m - D_n^m Q_n^{m+1} &= \Delta^m Q_n^m,\\
zC_{n+1}^m - \frac{1}{z} C_n^m \mp D_{n+1}^m Q_n^{m*} \pm A_n^m Q_n^{m+1} &= \mp\Delta^m Q_n^{m*}\\
\frac{1}{z}\Delta_n D_n^m &= \mp Q_n^{m+1*} B_n^m - Q_n^m C_{n+1}^m,
\end{aligned} \tag{2.4}$$

where $\Delta_n A_n^m = A_{n+1}^m - A_n^m$, etc.

This system can be solved in a deductive way. Using the ideas in [16], expansions in powers of z and $1/z$ are sought. The series

$$\begin{aligned}
A_n^m &= \sum_{k=-1}^{1} A_n^{(2k)} z^{2k}, & B_n^m &= \sum_{\substack{k=-1\\k\neq 0}}^{1} B_n^{(k)} z^k,\\
C_n^m &= \sum_{\substack{k=-1\\k\neq 0}}^{1} C_n^{(k)} z^k, & D_n^m &= \sum_{k=-1}^{1} D_n^{(2k)} z^{2k}
\end{aligned} \tag{2.5}$$

are substituted into (2.4) and the various powers of z^k are set equal to zero. The coefficients of the finite power series (2.5) are assumed to be independent of z. One can find each of the unknowns $A_n^{(-2)}, A_n^{(0)}, \ldots, D_n^{(-2)}$ in terms of the potentials. The condition under which we can solve (2.4) requires that the potential evolve according to the evolution equation

$$\begin{aligned}
\Delta^m Q_n^m = {} & Q_n^m A_-^{(0)} - Q_n^{m+1} A_-^{(0)*} - A_-^{(-2)*}\left(Q_{n+1}^{m+1}(1 \pm Q_n^m Q_n^{m*}) \prod_{k=-\infty}^{n} A_k^m\right) \\
& + A_-^{(-2)}\left(Q_{n-1}^m(1 \pm Q_n^{m+1} Q_n^{m+1*}) \prod_{-\infty}^{n-1} A_k^m\right) \\
& + A_-^{(2)}\left[Q_{n+1}^m \pm \frac{Q_n^m}{2}\left(Q_{n+1}^{m+1} Q_n^{m+1*} + Q_{n+1}^m Q_n^{m*} - \sum_{k=-\infty}^{n} \Delta^m \tilde{S}_k^m\right)\right] \\
& - A_-^{(2)*}\left[Q_{n-1}^{m+1} \pm \frac{Q_n^{m+1}}{2}\left(Q_n^{m+1*} Q_{n-1}^{m+1} + Q_n^{m*} Q_{n-1}^m - \sum_{k=-\infty}^{n-1} \Delta^m \tilde{S}_k^m\right)\right]
\end{aligned}$$

where $\Delta_k^m = (1 \pm Q_k^{m+1} Q_k^{m+1*})/(1 \pm Q_k^m Q_k^{m*})$ and $\tilde{S}_k^m = (Q_{k+1}^m Q_k^{m*} + Q_k^m Q_{k-1}^{m*})$.

The $A_-^{(i)}$ are arbitrary constants of summation (fixed at $n \to -\infty$) and $D_-^{(-i)*} = A_-^{(i)}$, $i = 2, 0, -2$.

So by a suitable choice of the constants, one can obtain a partial difference equation which is consistent with the NLS equation

$$iq_t = q_{xx} \pm 2|q|^2 q. \tag{2.7}$$

In the linear limit if we want

$$i\Delta^m Q_n^m = \frac{\sigma}{2}(Q_{n+1}^{m+1} - 2Q_n^{m+1} + Q_{n-1}^{m+1} + Q_{n+1}^m - 2Q_n^m + Q_{n-1}^m),$$

where $\sigma = \Delta t/(\Delta x)^2$ and $Q_n^m = \Delta x q(n\,\Delta x, m\,\Delta t) = \Delta x q_n^m$ then the constants are chosen according to $A_-^{(2)} = -i(\sigma/2)$, $A_-^{(0)} = i\sigma$, $A_-^{(-2)} = -i\sigma/2$. This particular choice of constants in (2.6) leads to evolution equation

$$\begin{aligned}
i\frac{\Delta^m q_n^m}{\Delta t} = {} & \frac{1}{2(\Delta x)^2}(q_{n+1}^m - 2q_n^m + q_{n-1}^m P_{n-1} + q_{n+1}^{m+1} P_n - 2q_n^{m+1} + q_{n-1}^{m+1}) \\
& \pm \frac{1}{4}[q_n^m(q_n^{m*} q_{n+1}^m + q_n^{m+1*} q_{n+1}^{m+1}) + q_n^{m+1}(q_{n-1}^m q_n^{m*} + q_{n-1}^{m+1} q_n^{m+1*}) \\
& + 2q_n^m q_n^{m*} q_{n+1}^{m+1} P_n + 2q_n^{m+1} q_n^{m+1*} q_{n-1}^m P_{n-1} - q_n^m S_n - q_n^m S_{n-1}^*],
\end{aligned} \tag{2.8$_*$}$$

where

$$P_n = \prod_{k=-\infty}^{n} \Delta_k^m, \qquad S_n = \sum_{k=-\infty}^{n} \Delta^m \sigma_k^m,$$

$$\sigma_k^m = q_k^m q_{k-1}^{m*} + q_{k+1}^m q_k^{m*}, \qquad \Delta_k^m = (1 \pm q_k^{m+1} q_k^{m+1*}(\Delta x)^2)/(1 \pm q_k^m q_k^{m*}(\Delta x)^2).$$

This scheme is implicit and global. However, a local scheme is suggested in which $P_n = 1$ and $S_n = 0$ for all n. Equation (2.8)$_*$ is consistent with the NLS equation

(2.7) with the truncation error of order $O((\Delta t)^2, (\Delta x)^2)$. Similarly the local scheme also has the same truncation error.

For the special case associated with the MKdV equation

$$R_t \mp 6R^2 R_x + R_{xxx} = 0 \tag{2.9}$$

we let $R_n^m = \pm Q_n^m$, $T_n^m = S_n^m = 0$, and the four equations are given by

$$z\Delta_n A_n^m + R_n^m B_{n+1}^m \mp R_n^{m+1} C_n^m = 0,$$

$$\pm\Delta^m R_n^m + zB_n^m \pm R_n^{m+1} D_n^m = \pm A_{n+1}^m R_n^m + \frac{1}{z} B_{n+1}^m, \tag{2.11}$$

$$zC_{n+1}^m - \frac{1}{z} C_n^m + R_n^m D_{n+1}^m - R_n^{m+1} A_n^m = \Delta^m R_n^m, \tag{2.12}$$

$$\frac{1}{z} \Delta_n D_n^m = R_n^{m+1} B_n^m \mp R_n^m C_{n+}^m$$

Using the ideas in [13, 15, 16], the coefficients in the equations for the time dependence of the eigenfunctions are expanded as

$$\begin{aligned} A_n^m &= \sum_{k=-2}^{2} z^{2k} A_n^{(2k)}, & B_n^m &= \sum_{k=-1}^{2} z^{(2k-1)} B_n^{(2k-1)}, \\ C_n^m &= \sum_{k=-1}^{2} z^{(2k-1)} B_n^{(2k-1)}, & D_n^m &= \sum_{k=-2}^{2} z^{2k} A_n^{(2k)}. \end{aligned} \tag{2.14}$$

With the expanded form of A_n^m, B_n^m, C_n^m, D_n^m, Eqs. (2.10), (2.11), (2.12), and (2.13) yield a sequence of twenty equations in eighteen unknowns corresponding to equating powers of z^5, z^{-5}, z^4,..., z, z^{-1}, all of which must be independently satisfied. To solve these equations it is most convenient to solve the resultant equations correponding to z^5 and z^{-5} first, then solve the equations corresponding to z^4, $1/z^4$, etc. Carrying out the algebra we find the values of $A_n^{(4)}$,..., $D_n^{(-4)}$ in terms of the potentials (see Taha [2]). The remaining two equations are consistent under the conditions

$$A_-^{(i)} = D_-^{(-i)}, \qquad i = 4, 2, 0, -2, -4. \tag{2.15}$$

The following evolution equation is

$$\begin{aligned} \Delta^m R_n^m = {} & R_{n+2}^m A_-^{(4)} - R_{n+2}^{m+1} \gamma_{n+1} D_-^{(4)} + R_{n+1}^m S_{n+1} \\ & - R_{n+1}^{m+1} P_n - [R_{n-2}^{m+1} A_-^{(4)} - R_{n-2}^m \gamma_{n-2} D_-^{(4)} + R_{n-1}^{m+1} S, \\ & - R_{n-1}^m P_{n-1}] + R_n^m \Big\{ D_-^{(0)} \pm \sum_{l=-\infty}^{n} [R_l^{m+1} \{R_{l-2}^{m+1} A_-^{(4)} \\ & - R_{l-2}^m \gamma_{l-2} D_-^{(4)} + R_{l-1}^{m+1} S_{l-2} - R_{l-1}^m P_{l-1}\} \\ & - R_l^m (R_{l+2}^m A_-^{(4)} - R_{l+2}^{m+1} \gamma_{l+1} D_-^{(4)} + R_{l+1}^m S_{l+1} - R_{l+1}^{m+1} P_l)] \Big\} \end{aligned}$$

$$
\begin{aligned}
&- R_n^{m+1} \Big\} A_-^{(0)} \pm \sum_{l=-\infty}^{n-1} [R_l^{m+1}\{R_{l-2}^{m+1} A_-^{(4)} - R_{l-2}^m \gamma_{l-2} D_-^{(4)} \\
&+ R_{l-1}^{m+1} S_{l-2} - R_{l-1}^m P_{l-1}\} - R_l^m [R_{l+2}^m A_-^{(4)} - R_{l+2}^{m+1} \gamma_{l+1} D_-^{(4)} \\
&+ R_{l+1}^m S_{l+1} - R_{l+1}^{m+1} P_l]] \Big\}, \qquad (2.16)_*
\end{aligned}
$$

where

$$
\begin{aligned}
&S_n = A_-^{(2)} + A_-^{(4)} F_n + D_-^{(4)} \sum_{j=-\infty}^{n} H_j,\ P_n = \left(D_-^{(2)} + \sum_{j=-\infty}^{n} [A_-^{(4)} E_j + D_-^{(4)} G_j] \eta_j \right) \gamma_n, \\
&\gamma_- = \quad_{=-\infty} (\delta_i^{m+1}/\delta_i^m),\ \delta_i^m = \quad \mp R_i^m, \\
&\eta_n = \gamma_n^{-1}/\delta_n^m,\ H_n = \pm\{R_n^m R_{n+1}^{m+1} \delta_n^{m+1} - R_{n-}^m \; R_n^{m+1} \delta_n^m\} \beta_{n-1}, \\
&\beta_n = \gamma_n/\delta_{n+1}^m,\ F_n = \mp \left| R_{n+1}^{m+1} R_n^{m+1} - \sum_{j=-\infty}^{n} \Delta^m (R_j^m R_{j+1}^m) \right|, \\
&G_n = \mp (R_n^{m+1} R_{n+1}^{m+1} - R_n^m R_{n-1}^m) \gamma_{n-1} \delta_n^{m+1}, \\
&E_n = \mp (R_n^m R_{n-1}^{m+1} \delta_n^{m+1} - R_{n+1}^m R_n^{m+1} \delta_n^m). \qquad (2.17)
\end{aligned}
$$

In the limit as $m + 1 \to m$, Eq. $(2.16)_*$ becomes

$$
\begin{aligned}
R_{nt} = {} & (A_-^{(4)} - D_-^{(4)})(R_{n+2} - R_{n-2}) + (A_-^{(2)} - D_-^{(2)})(R_{n+1} - R_{n-1}) \\
& \pm (A_-^{(4)} - D_-^{(4)})[R_{n-1}^2 (R_n + R_{n-2}) - R_{n+1}^2 (R_n + R_{n+2}) - R_n^2 (R_{n+2} - R_{n-2})] \\
& \pm (A_-^{(2)} - D_-^{(2)}) R_n^2 (R_{n-1} - R_{n+1}) \\
& + (A_-^{(4)} - D_-^{(4)}) R_n [R_{n+1} R_n (R_{n+1} R_n + R_{n+1} R_{n+2} + R_n R_{n-1}) \\
& - R_n R_{n-1} (R_n R_{n-1} + R_n R_{n+1} + R_{n-2} R_{n-1})]. \qquad (2.18)
\end{aligned}
$$

Let $R = \Delta x\, U$, and by a proper choice (see below) of the constants and taking limit as $\Delta x \to 0$ in Eq. (2.18) yield the MKdV equation

$$
U_t \mp 6U^2 U_x + U_{xxx} = 0. \qquad (2.19)
$$

From Eq. $(2.16)_*$ let us consider the linear part which can be written as

$$
\begin{aligned}
U_n^{m+} - U_n^m = {} & (U_n^m - U_n^{m+1}) A_-^{(0)} + (U_{n+1}^m - U_{n-1}^{m+1}) A_-^{(2)} \\
& + (U_{n-1}^m - U_{n+1}^{m+1}) D_-^{(2)} + (U_{n+2}^m - U_{n-2}^{m+1}) A_-^{(4)} \\
& + (U_{n-2}^m - U_{n+2}^{m+1}) D_-^{(4)}. \qquad (2.20)
\end{aligned}
$$

To choose the constants, we require a scheme of order $O((\Delta t)^2, (\Delta x)^2)$, (expanding U_{n+1}^m, U_n^{m+1},... in Taylor series). With this requirement we find

$$\begin{aligned} A_-^{(2)} &= -\tfrac{2}{3}A_-^{(0)} + \tfrac{1}{2}\alpha = D_-^{(-2)}, \\ D_-^{(2)} &= -\tfrac{2}{3}A_-^{(0)} - \tfrac{1}{2}\alpha = A_-^{(-2)}, \\ A_-^{(4)} &= \tfrac{1}{6}A_-^{(0)} - \tfrac{1}{4}\alpha = D_-^{(-4)}, \\ D_-^{(4)} &= \tfrac{1}{6}A_-^{(0)} + \tfrac{1}{4}\alpha = A_-^{(-4)}, \end{aligned} \tag{2.21}$$

$$\alpha = \frac{\Delta t}{(\Delta x)^3}$$

$$A_-^{(0)} = \text{arbitrary constant.}$$

In order to get a local scheme of order $O((\Delta t)^2, (\Delta x)^2)$ for the MKdV equation from $(2.16)_*$, let $R_n^m = \Delta x\, U_n^m$, keep the terms through order $O((\Delta x)^3)$ and then drop the sum terms of the form

$$\sum_{j=-\infty}^{n} A^m(U_j^m U_{j-1}^m), \qquad \sum_{j=-\infty}^{n} [(U_j^m)^2 - (U_j^{m+1})^2]$$

and replace γ_n by 1. Equation $(2.16)_*$ gives the local scheme

$$\begin{aligned} U_n^{m+1} - U_n^m = {} & \{(U_{n+2}^m - U_{n-2}^{m+1}) A_-^{(4)} + (U_{n-2}^m - U_{n+2}^{m+1}) D_-^{(4)} \\ & + (U_{n+1}^m - U_{n-1}^{m+1}) A_-^{(2)} + (U_{n-1}^m - U_{n+1}^{m+1}) D_-^{(2)} + (U_n^m - U_n^{m+1}) A_-^{(0)}\} \\ & \pm (\Delta x)^2 [\{U_n^{m+1}[U_n^m U_{n-2}^m + U_{n-1}^m U_{n+1}^m] \\ & - U_n^m[U_n^{m+1} U_{n+2}^{m+1} + U_{n+1}^{m+1} U_{n-1}^{m+1}] + U_{n-1}^{m+1}[U_{n-1}^m U_{n-2}^m + U_n^m U_{n-1}^m] \\ & - U_{n+1}^m[U_n^{m+1} U_{n+1}^{m+1} + U_{n+1}^{m+1} U_{n+2}^{m+1}]\} A_-^{(4)} \\ & + \{U_{n+1}^m[U_n^m U_{n+1}^{m+1} + U_{n+1}^m U_{n+2}^{m+1}] + U_n^m[U_n^m U_{n+2}^{m+1} + U_{n+1}^{m+1} U_{n-1}^m] \\ & - U_n^{m+1}[U_n^{m+1} U_{n-2}^m + U_{n+1}^{m+1} U_{n-1}^m] \\ & - U_{n-1}^{m+1}[U_{n-1}^{m+1} U_{n-2}^m + U_n^{m+1} U_{n-1}^m]\} D_-^{(4)} \\ & + \{U_n^m U_n^{m+1}[U_{n-1}^m - U_{n+1}^{m+1}] A_-^{(2)} \\ & + [(U_n^m)^2 U_{n+1}^{m+1} - (U_n^{m+1})^2 U_{n-1}^m] D_-^{(2)}\}], \end{aligned} \tag{2.22$_*$}$$

where $A_-^{(4)}, \ldots, D_-^{(4)}$ satisfy Eq. (2.21). Equation $(2.22)_*$ is consistent with the MKdV equation (2.19), with the truncation error of order $O((\Delta t)^2, (\Delta x)^2)$. This truncation error holds also for the full scheme given in Eq. $(2.16)_*$. Since $A_-^{(0)}$ is an arbitrary constant, we have a family of schemes, each one of which satisfies the properties discussed earlier for the NLS equation scheme.

For the special case associated with the KdV equation

$$U_t + 6UU_x + U_{xxx} = 0 \tag{2.23}$$

we let $Q_n^m = R_n^m = 0$, $T_n^m =$ The four compatibility equations for $A_n^m, \ldots, D_n^m$ are given by

$$z\frac{A_{n+1}^m}{1-S_n^m} - z\frac{A_n^m}{1-S_n^{m+1}} + \frac{z}{1-S_n^m} B_{n+1}^m - \frac{S_n^m C_n^m}{z(1-S_n^{m+1})}$$
$$= z\frac{S_n^{m+1} - S_n^m}{z(1-S_n^m)(1-S_n^{m+1})}, \tag{2.24}$$

$$\frac{S_n^m}{z(1-S_n^m)} A_{n+1}^m + \frac{B_{n+1}^m}{z(1-S_n^m)} - z\frac{B_n^m}{\cdot\ S_n^{m+1}} - \frac{S_n^{m+1}}{} \frac{D_n^m}{(\cdot\ S_n^{m+1})}$$
$$= \frac{S_n^{m+1} - S_n^m}{z(1-S_n^{m+1})(1-S_n^m)}, \tag{2.25}$$

$$\frac{z}{-S_n^m} C_{n+}^m + \frac{z}{\cdot - S_n^m} D_{n+1}^m - z\frac{A_n^m}{1-S_n^{m+1}} - \frac{C_n^m}{z(1-S_n^{m+1})}$$
$$= z\frac{(S_n^{m+1} - S_n^m)}{(1-S_n^{m+1})(1-S_n^m)}, \tag{2.26}$$

$$\frac{D_{n+}^m}{z(1-S_n^m)} \cdot + \frac{S_n^m}{z}\frac{C_{n+1}^m}{(1-S_n^m} - \frac{D_n^m}{} \qquad \frac{zB_n^m}{(1-S_n^{m+1})}$$
$$= \frac{S_n^{m+1} - S_n^m}{z(1-S_n^{m+1})(1-S_n^m)}. \tag{2.27}$$

Using the ideas in [13, 15, 16], the coefficients in the equations for the time dependence of the eigenfunctions are expanded as

$$A_n^m = \sum_{k=-2}^{2} z^{2k} A_n^{(2k)}, \qquad B_n^m = \sum_{k=-2}^{2} z^{2k} B_n^{(2k)},$$
$$C_n^m = \sum_{k=-2}^{2} z^{2k} C_n^{(2k)}, \qquad D_n^m = \sum_{k=-2}^{2} z^{2k} D_n^{(2k)}. \tag{2.28}$$

With the expanded form of A_n^m, B_n^m, C_n^m, D_n^m, Eqs. (2.24), (2.25), (2.26), and (2.27) yield a sequence of twenty-four equations in twenty unknowns corresponding to equating powers of z^5, z^3, z, z^{-1}, z^{-3}, z^{-5}, all of which must be independently satisfied. Twenty equations of which give the values of the twenty unknowns (see Taha [2]). The remaining four equations, two of them are trivially satisfied and the third is satisfied under the consistency conditions

$$A_-^{(i)} = D_-^{(-i)}, \qquad i = 4, 2, 0, -2, -4. \tag{2.29}$$

The fourth equation gives the evolution equation

$$
\begin{aligned}
&\frac{S_n^m}{1-S_n^m}\Bigg\{A_-^{(0)} - \sum_{l=-\infty}^{n}\Bigg[E_{l+1} + S_l^{m+1}W_l(A_-^{(2)} + C_{l-2}) \\
&\qquad D_-^{(4)}\gamma_{l-1} + D_-^{(2)} + \sum_{k=-\infty}^{l-1}(H_k+G_k)\Big\{S_l^{m+1}\gamma_l + (\gamma_l-1)\Big|W_l^{-1}\Big\}W_n \\
&\qquad \frac{S_n^{m+1}}{-S_n^{m+1}}\Bigg\{D_-^{(0)} + \sum_{l=-\infty}^{n-1}\Bigg[\frac{-S_l^m}{S_{l+1}^{m+1}}\{\gamma_{l+1}^{-1}N_{l+1} - N_l + M_l \\
&\qquad + S_{l+1}^{m+1}Z_l - S_{l+1}^m\gamma_{l+1}^{-1}N_{l+1}\} + \gamma_l T_{l-2} + (\gamma_l - 1)\Big|W_l^{-1}\Big\}W_{n-} \\
&\qquad + \frac{\cdot}{1-S_n^m}E_{n+} \quad \frac{}{1-S_n^{m+1}}T_{n-2} = \frac{S_n^{m+1}-S_n^m}{(1-S_n^{m+1})(1-S_n^m)},
\end{aligned}
\tag{2.30}
$$

where

$$
\begin{aligned}
E_n &= A_-^{(2)}S_n^m W_{n-1} - S_n^{m+1}D_-^{(2)} + H_n + G_n - S_n^{m+1}\sum_{k=-\infty}^{n}(H_k+G_k) \\
&\quad + S_n^m W_{n-1}C_{n-1} - S_n^m D_-^{(4)},\ C_n = A_-^{(4)} + \sum_{j=-\infty}^{n} P_j W_j^{-1}, \\
T_n &= \gamma_{n+1}M_n + S_{n+1}^{m+1}\gamma_{n+1}Z_n - S_{n+1}^m N_{n+1},\ M_n = S_n^{m+1}W_n A_-^{(4)} - S_n^m D_-^{(4)}, \\
Z_n &= \left(A_-^{(2)} + \sum_{j=-\infty}^{n} Q_j W_j^{-1}\right)W_n,\ N_n = D_-^{(2)} + \sum_{j=-\infty}^{n} F_j, \\
W_n &= \prod_{i=-\infty}^{n}\gamma_i,\ \gamma_i = \left(\frac{1-S_i^m}{1-S_i^{m+1}}\right) \\
H_k &= A_-^{(4)}(S_{k+1}^m\gamma_k - S_k^m)W_{k-1},\ G_k = (S_k^m - S_{k+1}^{m+1})D_-^{(4)}, \\
F_j &= A_-^{(4)}(S_j^{m+1}W_j - S_{j-1}^{m+1}W_{j-1}) + D_-^{(4)}(S_{j-1}^m - S_j^{m+1}), \\
P_j &= A_-^{(4)}(S_j^{m+1} - S_{j+1}^m)W_j + D_-^{(4)}(S_{j+1}^{m+1} - S_j^{m+1}\gamma_j), \\
Q_j &= (S_{j-1}^{m+1} - S_j^m)W_j A_-^{(4)} - (S_{j-1}^m\gamma_j - S_j^m)D_-^{(4)}.
\end{aligned}
\tag{2.31}
$$

In the limit as $m+1 \to m$, Eq. $(2.30)_*$ becomes

$$
\begin{aligned}
&(S_{n+1} - S_{n-1})\alpha + \gamma\{S_{n+2} - S_{n+1}^2 - S_{n+1}S_{n+2} - S_n S_{n+} + S_{n-}^2 \\
&\quad + S_{n-1}S_{n-2} - S_{n-2} + S_n S_{n-1}\} = \frac{S_{nt}}{1-S_n},
\end{aligned}
\tag{2.32}
$$

where

$$
\alpha = A_-^{(2)} - D_-^{(2)} = D_-^{(-2)} - A_-^{(-2)}
$$

and

$$\gamma = A_-^{(4)} - D_-^{(4)} = D_-^{(-4)} - A_-^{(-4)}.$$

Let $S_n^m = 1 - e^{(\Delta x)^2 U_n^m}$, and by a proper choice of the constants and taking limit as $\Delta x \to 0$ in Eq. (2.32) yields the KdV Eq. (2.23).

To determine the constants in Eq. $(2.30)_*$ we apply the same argument as in the MKdV equation case and it turns out that the constants have the same values as given in (2.21).

In order to get a local scheme of order $O((\Delta t)^2, (\Delta x)^2)$ for the KdV equation from Eq. $(2.30)_*$, we follow a similar procedure to that of the MKdV equation. We can establish

$$\begin{aligned} S_n^{m+} - S_n^m = {} & (S_n^m - S_n^{m+1}) A_-^{(0)} + A_-^{(2)}(S_{n+1}^m - S_{n-1}^{m+1}) + D_-^{(2)}(S_{n-1}^m - S_{n+1}^{m+1}) \\ & + A_-^{(4)}(S_{n+2}^m - S_{n-2}^{m+1}) + D_-^{(4)}(S_{n-2}^m - S_{n+2}^{m+1}) \\ & + [(S_n^m)^2 - (S_n^{m+1})^2] A_-^{(0)} + D_-^{(4)}\{S_{n+1}^{m+1}(S_n^{m+1} + S_{n+1}^{m+1} + S_{n+2}^{m+1})\} \\ & - A_-^{(4)}\{S_{n+1}^m(S_n^m + S_{n+1}^m + S_{n+2}^m)\} - D_-^{(4)}\{S_{n-1}^m(S_n^m + S_{n-1}^m + S_{n-2}^m)\} \\ & + A_-^{(4)}\{S_{n-1}^{m+1}(S_n^{m+1} + S_{n-1}^{m+1} + S_{n-2}^{m+1})\} \end{aligned} \qquad (2.33)_*$$

with

$$S_n^m = \quad - e^{(\Delta x)^2 U_n^m}.$$

Equation $(2.33)_*$ is consistent with the KdV equation with truncation error of order $O((\Delta t)^2, (\Delta x)^2)$ as is the full scheme given in Eq. $(2.30)_*$.

As in the case of the MKdV equation, we have a family of schemes for the KdV equation and each one of them satisfies the properties given for the NLS equation scheme.

It is worth mentioning that the partial difference equation for the KdV equation also can be deduced from the discrete Schrödinger equation

$$a_n^m V_{n+1}^m + V_{n-1}^m = \lambda V_n^m \qquad (2.34)$$

with an assumed time dependence of the form

$$\Delta^m V_n^m = \tilde{A}_n^m V_{n+1}^m + \tilde{B}_n^m V_n^m \qquad (2.35)$$

and expanding $\tilde{A}_n^m$, $\tilde{B}_n^m$ in powers of λ as

$$\tilde{A}_n^m = \tilde{A}_n^{(3)} \lambda^3 + \tilde{A}_n^{(1)} \lambda$$

and

$$\tilde{B}_n^m = \lambda^4 \tilde{B}_n^{(4)} + \lambda^2 \tilde{B}_n^{(2)} + \lambda^{(0)} \tilde{A}_n^{(0)}. \qquad (2.36)$$

3. Conclusions

The partial difference equations discussed here are consistent with certain important partial differential equations (NLS, MKdV, KdV). It can be shown that the solutions to the difference equations converge to the solutions of the corresponding partial differential equations. The partial difference equation maintains the joint x, t symmetry of the original partial differential equation. The partial difference equations suggest local schemes which still maintain the joint x, t symmetry of the original equation.

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References

1. a. N. Zabusky, "A Synergetic Approach to Problems of Nonlinear Wave Propagation and Interaction in Nonlinear Partial Differential Equations" (W. F. Ames, Ed.), pp. 223–258, Academic Press, New York, 1967; b. N. Zabusky, Computational synergetics and mathematical innovation, *J. Comp. Phys.* **43** (1981), 195–249.
2. T. Taha, Ph.D. thesis, Clarkson College, Potsdam, New York, 1982.
3. M. Ablowitz and H. Segur, Solitons and the inverse scattering transform, *SIAM*, Philadelphia, 1981.
4. C. Gardner, J. Greene, M. Kruskal, and R. Miura, *Phys. Rev. Lett.* **19** (1967), 1095.
5. C. Gardner, *et al.*, *CPAM* **27** (1974), 97.
6. P. Lax, *CPAM* **21** (1968), 467.
7. V. Zakharov and A. Shabat, *Soviet Phys. JETP* **34** (1972), 62.
8. M. Ablowitz, D. Kaup, A. Newell, and H. Segur, *Stud. Appl. Math.* **53** (1974), 249.
9. K. Case and M. Kac, *J. Math. Phys.* **14** (1973), 594.
10. K. Case, *J. Math. Phys.* **14** (1973), 916.
11. H. Flaschka, *Prog. Theor. Phys.* **51** (1974), 51.
12. S. Manakov, *Soviet Phys. JETP* **40** (1975), 269.
13. M. Ablowitz and J. Ladik, *J. Math. Phys.* **16** (1975), 598.
14. M. Ablowitz and J. Ladik, *Stud. Appl. Math.* **55** (1976), 213.
15. M. Ablowitz and J. Ladik, *J. Math. Phys.* **17** (1976), 1011.
16. M. Ablowitz and J. Ladik, *Stud. Appl. Math.* **57** (1977), 1.

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