Numerically examining the stability of solutions of nonlinear wave equations

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1 Introduction and motivation

I will talk about different numerical methods for examining the stability of solutions of nonlinear wave equations.

- Direct numerical evolution
- Spectral stability: periodic problems
  1. Finite differences
  2. Continuation methods
  3. Hill's method
- Spectral stability: whole line problems

2 Direct numerical evolution

Advantages

- "Nonlinear" stability

Disadvantages

- Stability vs. small growth rates
- How to seed
- Numerical instabilities vs. dynamical instabilities

2.1 An example from Bose-Einstein condensates

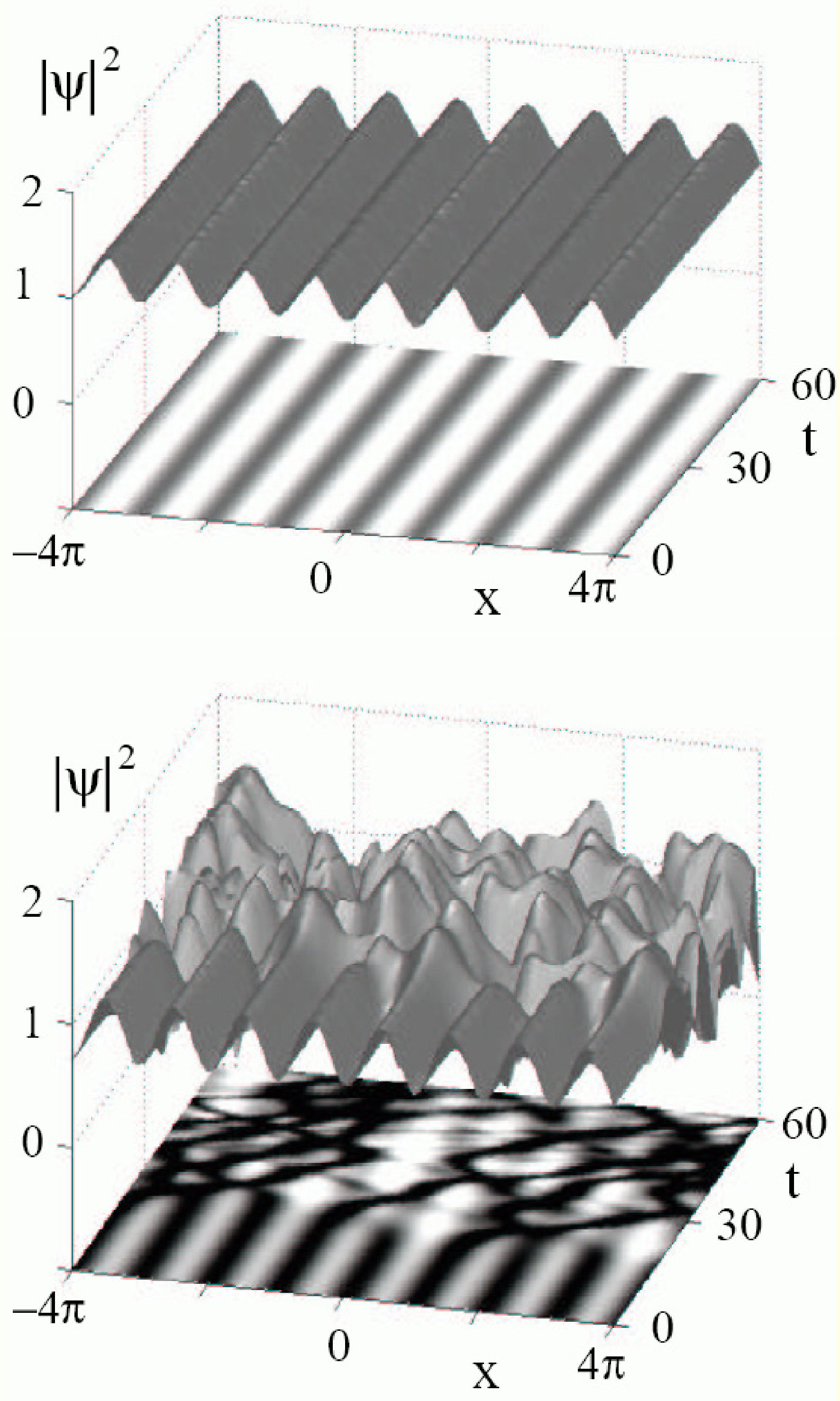
Consider

$$i\psi_t = -\tfrac{1}{2}\psi_{xx} + |\psi|^2\psi + \psi V_0 \sin^2 x.$$

This equation has an exact solution

$$\psi = \left(\sqrt{B}\cos x + i\sqrt{B - V_0}\sin x\right) e^{-i(B+1/2)t}, \quad (*)$$

where B is a free parameter.



The dynamics of the exact solution $(*)$ with $\boldsymbol{V_0 = -1}$, $\boldsymbol{B = 1/2}$ (bottom) and $\boldsymbol{B = 1}$ (top). The top picture appears stable.

Spectral stability

Consider the evolution system

$$\boldsymbol{u}_t = \boldsymbol{N}(\boldsymbol{u}). \qquad (*)$$

with an equilibrium solution $\boldsymbol{u}_e$:

$$\boldsymbol{N}(\boldsymbol{u}_e) = \boldsymbol{0}.$$

Is this solution stable or unstable? Start with linear stability analysis first: let

$$\boldsymbol{u} = \boldsymbol{u}_e + \epsilon\psi + \mathcal{O}(\epsilon^2).$$

Substitute in $(*)$ and retain first-order terms in ϵ:

$$\psi_t = \mathcal{L}[\boldsymbol{u}_e(\boldsymbol{x})]\psi.$$

Separation of variables: $\psi(x,t) = e^{\lambda t}\phi(x)$:

$$\boxed{\mathcal{L}[u_e(x)]\phi = \lambda\phi.}$$

- This is a spectral problem.
- If $\Re(\lambda) \leq 0$ for all bounded $\phi(x)$, then u_e is spectrally stable.

3 Spectra of linear operators with periodic coefficients

Our starting point is

$$\boldsymbol{\mathcal{L}\phi = \lambda\phi},$$

with

$$\boldsymbol{\mathcal{L} = \sum_{k=0}^{M} f_k(x)\partial_x^k, \qquad f_k(x+L) = f_k(x).}$$

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We wish to determine

- The spectrum $\sigma(\mathcal{L}) = \{\lambda \in \mathbb{C} : ||\phi|| < \infty\}$.
- For any $\lambda \in \sigma(\mathcal{L})$: what are the corresponding eigenfunctions $\phi(\lambda, x)$?

What space do the eigenfunctions live in?

There are several natural options:

1. The eigenfunctions are periodic with the same period as the coefficients
2. The period of the eigenfunctions is an integer multiple of that of the coefficients
3. The eigenfunctions are bounded

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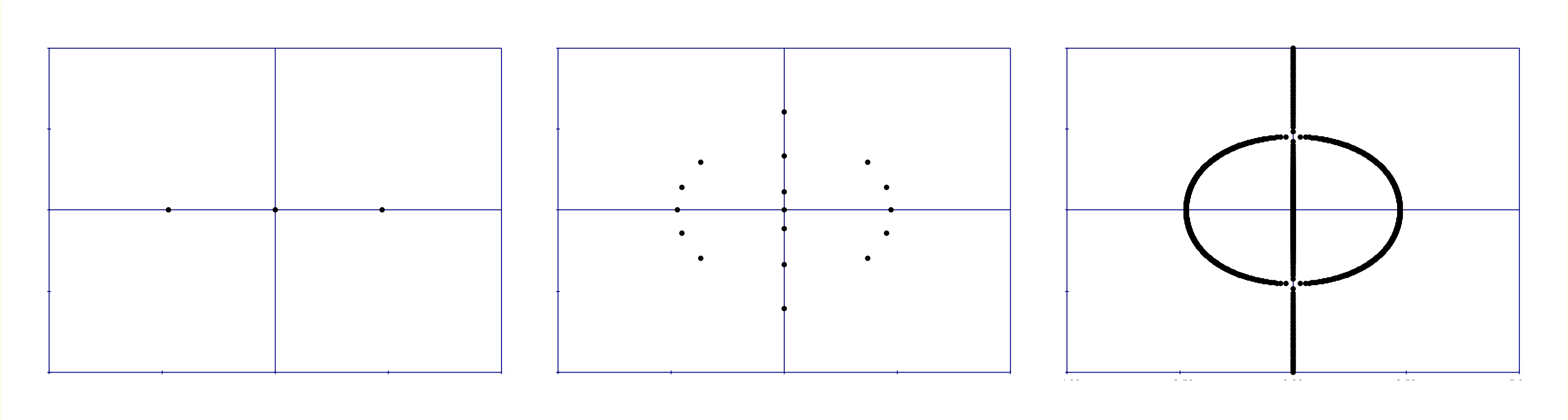

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- (1) or (2) are easy to do numerically, but are not always justified by the applications.
- All three approaches would lead us to conclude "instability" in the previous example. They do not always lead to the same conclusion.

The Floquet/Bloch decomposition

Floquet's theorem: Consider

$$\phi_x = A(x)\phi, \quad A(x+L) = A(x). \qquad (*)$$

Floquet's theorem states that the fundamental matrix Φ for this system has the decomposition

$$\Phi(x) = P(x)e^{Rx},$$

with $P(x+L) = P(x)$ and R constant.

Conclusion: all bounded solutions of $(*)$ are of the form

$$\phi = e^{i\mu x}\Phi(x),$$

with $\mu \in [-\pi/L, \pi/L)$ and $\Phi(x+L) = \Phi(x)$.

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- Different μ, different periods.
- Effectively: $\partial_x \longrightarrow \partial_x + i\mu$. This gives $\mathcal{L} \rightarrow \mathcal{L}_\mu$.
- Warning: symmetry is broken.
- Instead of continuous spectrum: one parameter family of point spectra.

3.1 Finite differences

1. Pick a set of μ values.
2. Discretize $\mathcal{L}_\mu$ using a finite-difference discretization of a certain order $\longrightarrow$ Banded matrices, with corners due to the periodic boundary conditions.
3. Use favorite eigenvalue solver.

3.2 Continuation methods

(Rademacher, Sandstede, Scheel)

Exploit the continuous dependence of the spectrum on μ: the spectrum consists of a set of isola parametrized by μ.

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- Using finite differences, find starting points.
- Using a continuation method (e.g., use AUTO) continue from the starting values to fill out the different isola.
- Fast method, especially powerful for problems with coefficients whose Fourier series converge slowly.

3.3 Hill's method

Let's return to our original spectral problem:

$$\mathcal{L}\phi = \lambda\phi,$$

with

$$\mathcal{L} = \sum_{k=0}^{M} f_k(x)\partial_x^k, \qquad f_k(x+L) = f_k(x).$$

We have

$$f_k(x) = \sum_{j=-\infty}^{\infty} \hat{f}_{k,j}\, e^{i2\pi jx/L},$$

with

$$\hat{f}_{k,j} = \frac{1}{L}\int_{-L/2}^{L/2} f_k(x)\, e^{-i2\pi jx/L}\, dx.$$

The eigenfunctions are expanded as

$$\phi = e^{i\mu x} \sum_{j=-\infty}^{\infty} \hat{\phi}_j \, e^{i2\pi j x/L},$$

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Next:

- Substitute in the equation, cancel $e^{i\mu x}$
- Determine the n-th Fourier coefficient

This gives

$$\Rightarrow \qquad \boxed{\hat{\mathcal{L}}_\mu \hat{\phi} = \lambda \hat{\phi}},$$

with $\hat{\phi} = (\cdots, \hat{\phi}_{-2}, \hat{\phi}_{-1}, \hat{\phi}_0, \hat{\phi}_1, \hat{\phi}_2, \cdots)^T$ and

$$\hat{\mathcal{L}}_{\mu,nm} = \sum_{k=0}^{M} \hat{f}_{k,n-m} \left[i\left(\mu + \frac{2\pi m}{L}\right)\right]^k$$

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More transparently:

$$f_k(x)\partial_x^k \leftrightarrow \begin{pmatrix} * & & & * & & & * & & \\ & * & & & * & & & * & \\ & & * & & & * & & & * \\ * & & & * & & & * & & \\ & * & & & * & & & * & \\ & & * & & & * & & & * \\ * & & & * & & & * & & \\ & * & & & * & & & * & \\ & & * & & & * & & & * \end{pmatrix}.$$

Note:

so far, no approximations have been made. At this point: **Cut off at N modes** $\longrightarrow \boldsymbol{(2N+1)\times(2N+1)}$ matrix.

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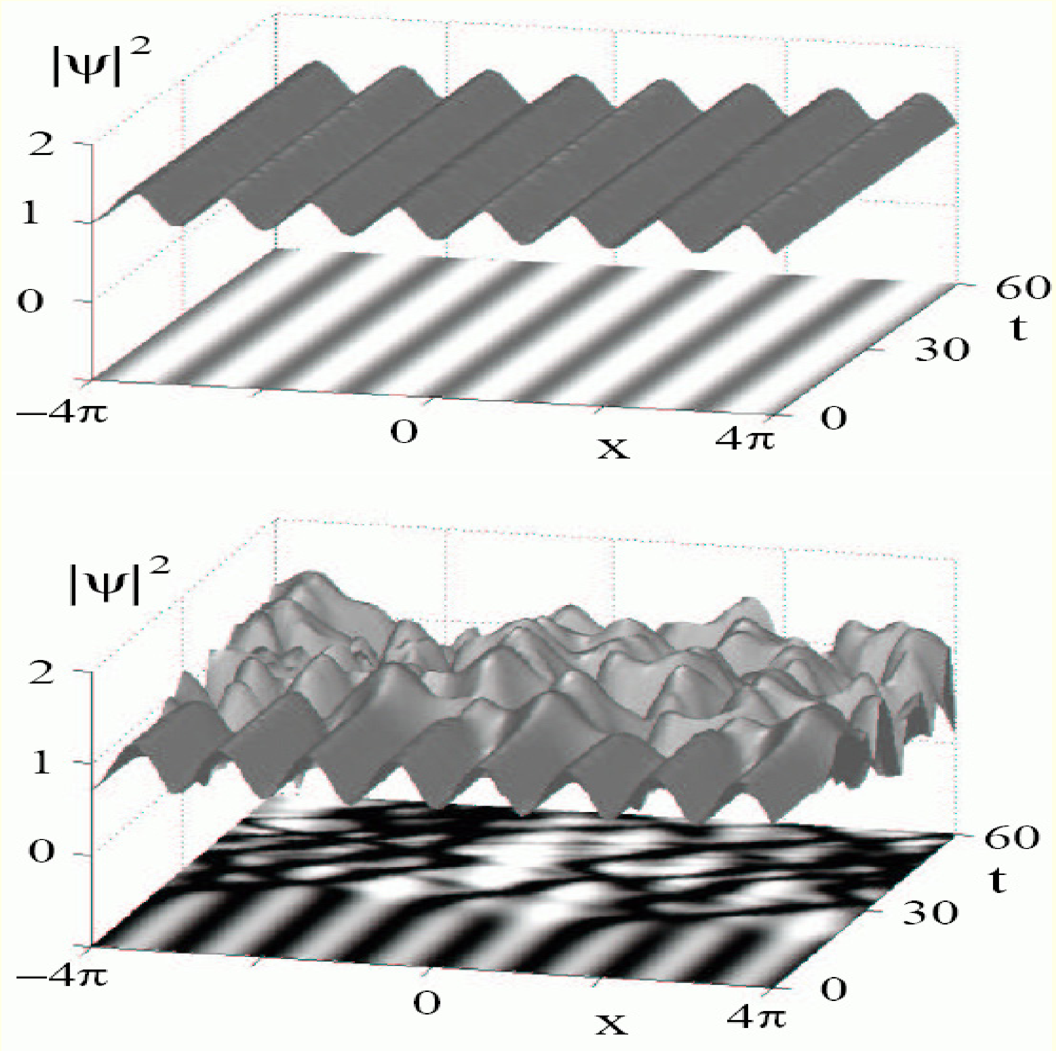
Theoretical results:

(Chris Curtis)

- No spurious modes,
- Eigenfunction approximations converge,
- For self-adjoint problems the entire spectrum is obtained, and
- The convergence is spectrally fast.

4 Periodic problems: examples

4.1 The BEC Example



The spectra corresponding to the linear stability problem of the exact solution $(*)$ for varying $\boldsymbol{B}$ values: $\boldsymbol{B \in [0,1]}$.

4.2 A 2-D NLS equation periodic example

Consider

$$i\psi_t - \psi_{xx} + \psi_{yy} + 2|\psi|^2\psi = 0.$$

This equation has exact 1-D solutions of the form

$$\psi = \phi(x)e^{i\omega t} = k\,\mathrm{sn}(x,k)e^{i(1+k^2)t}.$$

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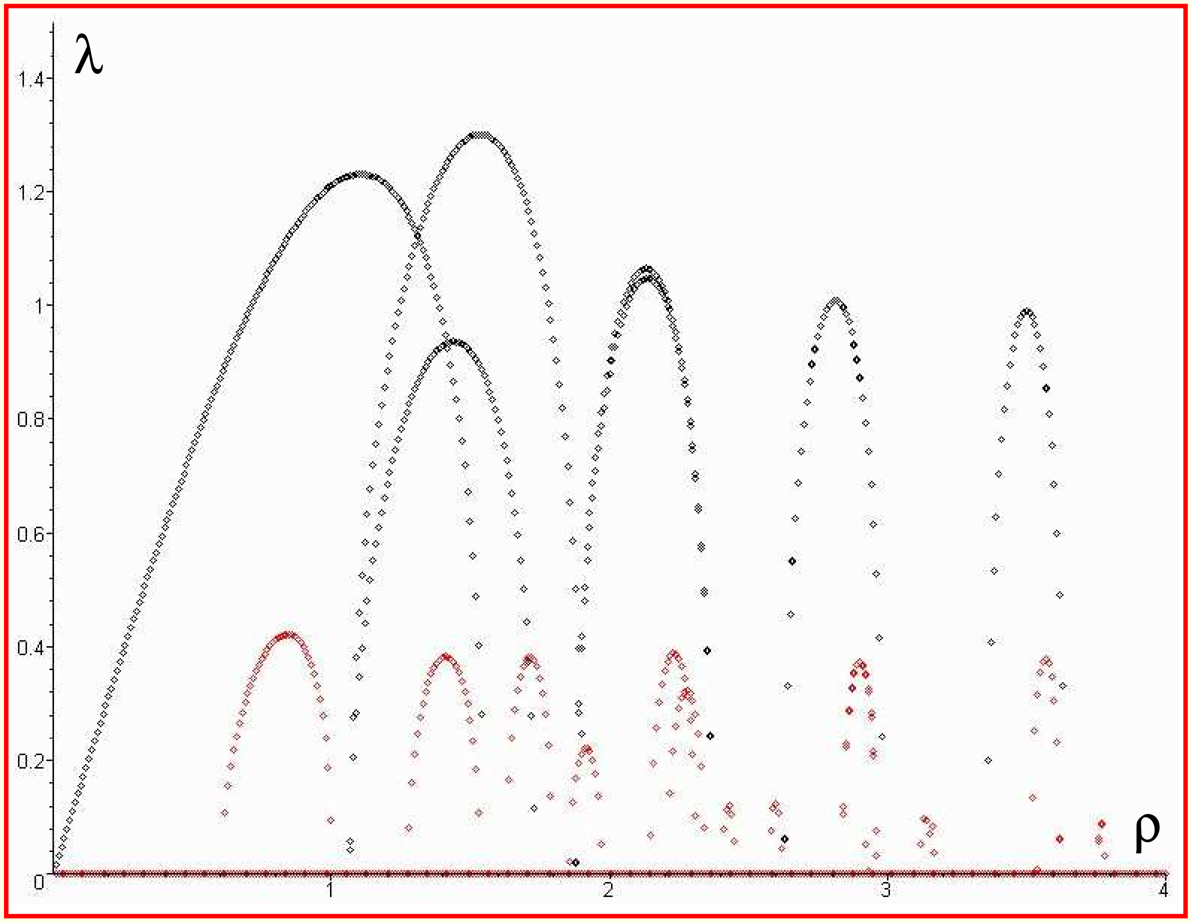
$$\psi = \phi(x)e^{i\omega t} = k\,\mathrm{sn}(x,k)e^{i(1+k^2)t}.$$

The linear stability problem is:

$$(\omega - 6\phi^2 + \rho^2 + \partial_x^2)U = \lambda V$$
$$-(\omega - 2\phi^2 + \rho^2 + \partial_x^2)V = \lambda U.$$

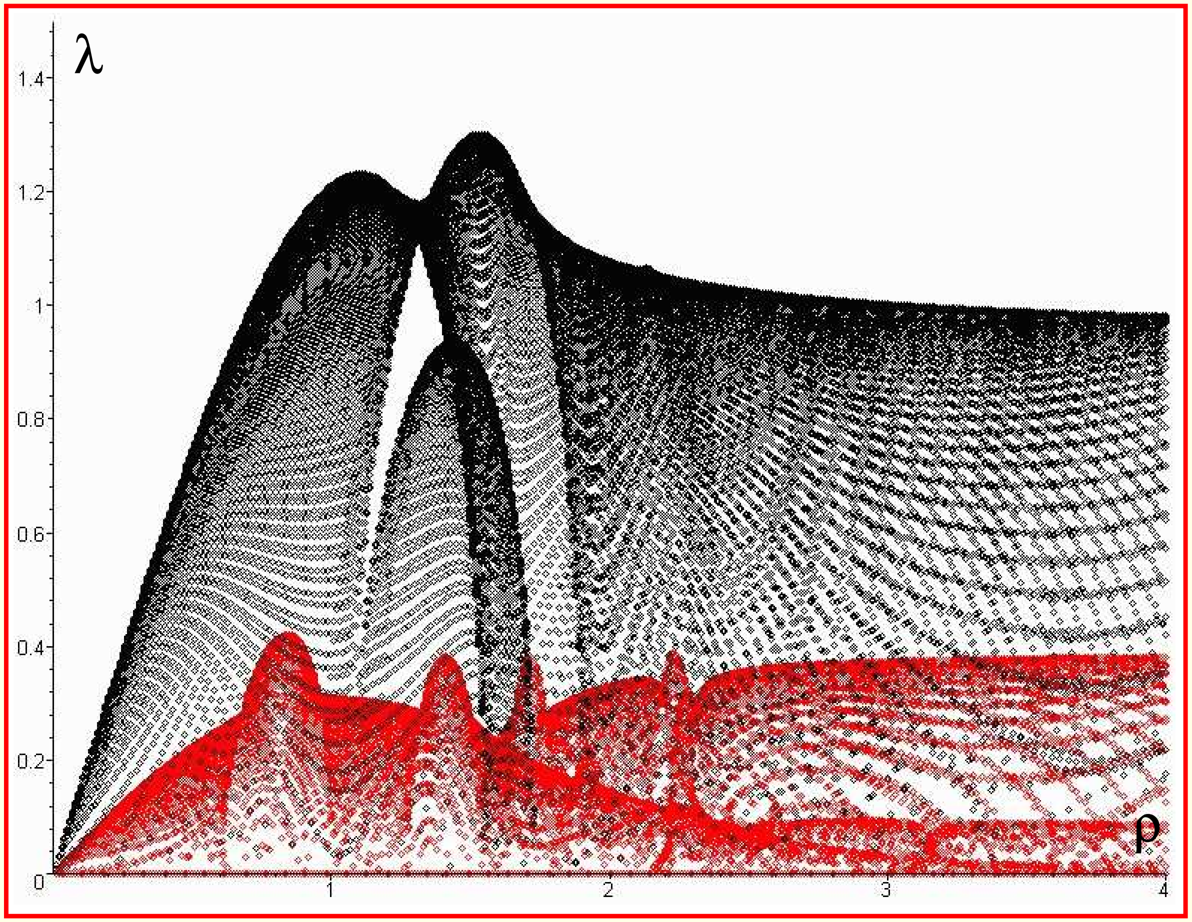
Thus: for a given value of k, compute spectra for a range of ρ values

One finds ($k = \sqrt{0.8}$):



Unstable eigenvalues for the linear stability problem of the $\mathbf{sn}$ solution, using periodic perturbations.

We can now compute “all” unstable modes:



Unstable eigenvalues for the linear stability problem of the **sn** solution ($\boldsymbol{k = \sqrt{0.8}}$).

5 Whole-line problems

The spectrum for a one-dimensional problem on the whole line consists of the essential spectrum (can be computed analytically) and discrete eigenvalues. How do we compute the discrete eigenvalues?

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- The Evans function: see Humpherys, Lyng, Zumbrun.
- Map the whole line to a finite interval.
- Use Hermite functions.
- Easiest: do a periodic problem with a large period. Some isola degenerate to isolated eigenvalues.

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- The Evans function: see Humpherys, Lyng, Zumbrun.
- Map the whole line to a finite interval.
- Use Hermite functions.
- Easiest: do a periodic problem with a large period. Some isola degenerate to isolated eigenvalues.
  1. Truncate the coefficient functions: creates discontinuities in the derivatives.

2. Approximate the coefficient functions by a periodic function that is analytic on the real line.

   Example: $\mathbf{sech}^2 x$ is approximated by $\mathbf{cn}^2(x, k)$ for large elliptic modulus.

6 Whole-line problems: examples

6.1 Two-dimensional NLS

Consider

$$i\psi_t + \psi_{xx} - \psi_{yy} + 2|\psi|^2\psi = 0.$$

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$$\psi = e^{it}\text{sech}x.$$

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This equation has an exact 1-D solution of the form

$$\psi = e^{it}\mathrm{sech}x.$$

It also has an exact periodic solution

$$\psi = ke^{i(2k^2-1)t}\mathrm{cn}(x,k),$$

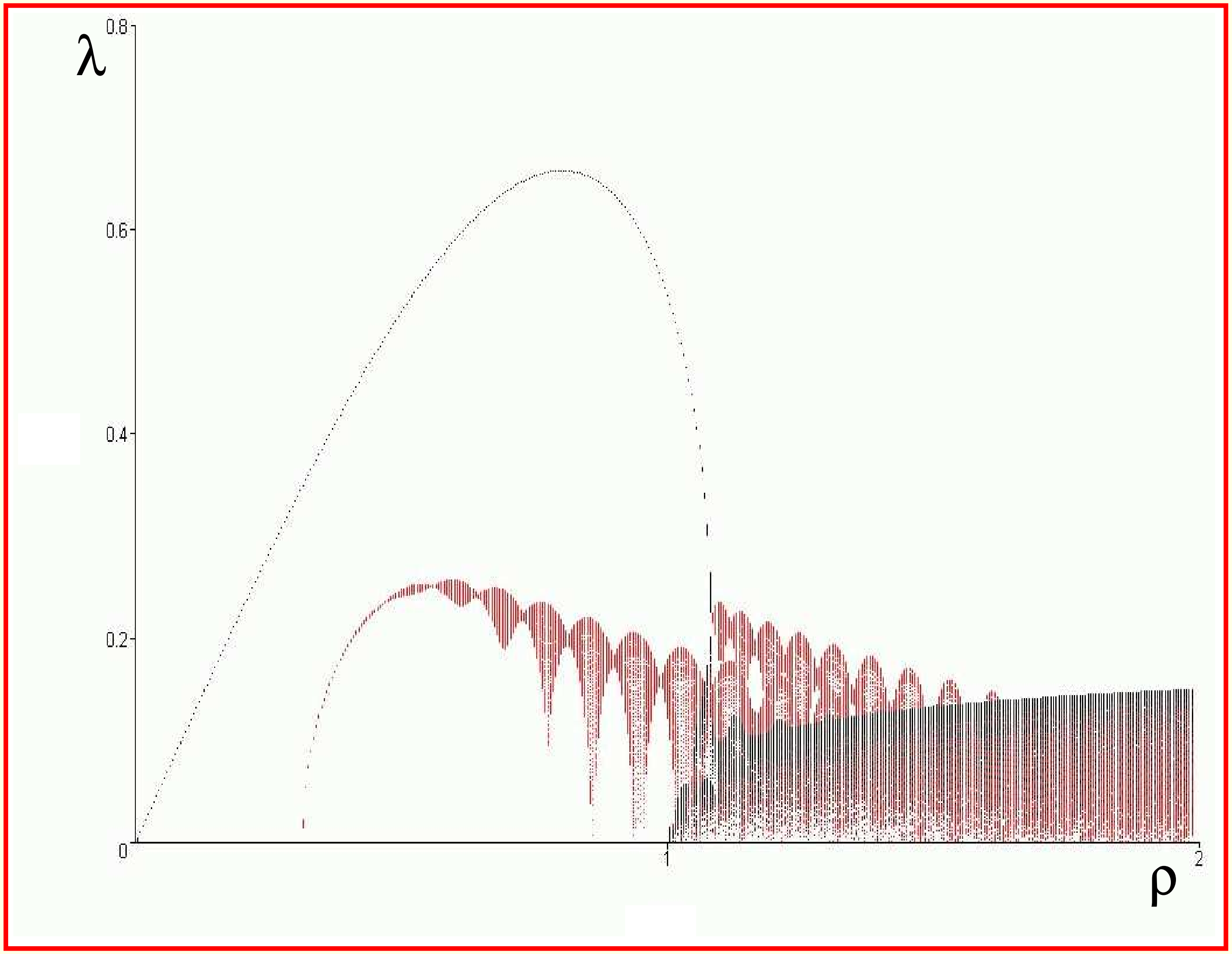
for any $k \in [0,1)$.

The linear stability problem is:

$$-(-\omega + 6\phi^2 + \rho^2 + \partial_x^2)U = \lambda V$$
$$(-\omega + 2\phi^2 + \rho^2 + \partial_x^2)V = \lambda U.$$

Thus: for a given value of k, compute spectra for a range of ρ values

For the cn solution, we find:



Unstable eigenvalues for the linear stability problem of the cn solution ($\boldsymbol{k = 0.999999999}$).

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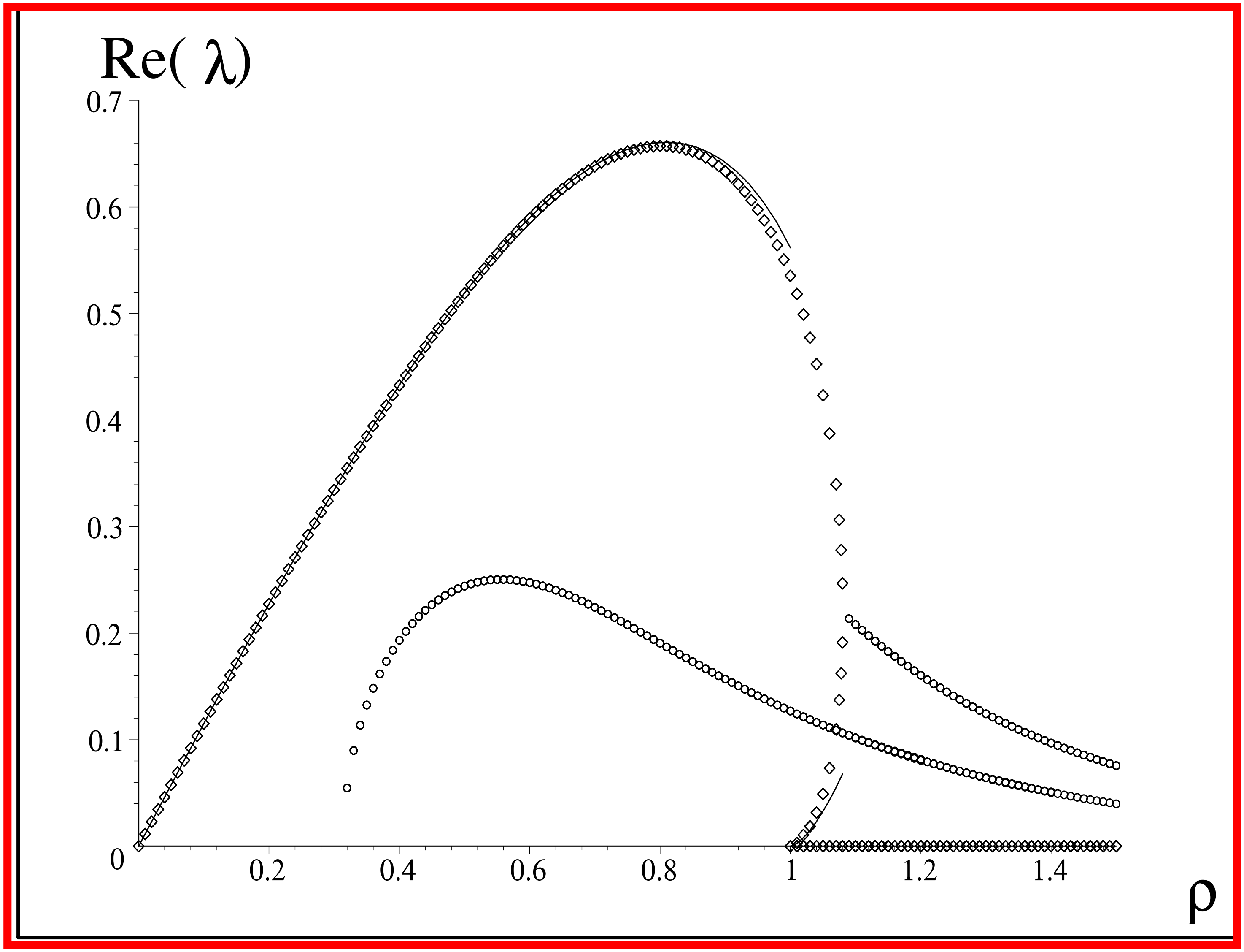

Unstable eigenvalues for the linear stability problem of the sech solution.

6.2 A system of Boussinesq equations

Consider (Bona, Chen, Saut, 2002)

$$\eta_t + \nabla \cdot \mathbf{v} + \nabla \cdot \eta \mathbf{v} + a \nabla \cdot \Delta \mathbf{v} - b \Delta \eta_t = 0,$$
$$\mathbf{v}_t + \nabla \eta + \frac{1}{2} \nabla |\mathbf{v}|^2 + c \nabla (\Delta \eta) - d \Delta \mathbf{v}_t = 0,$$

Here η gives the wave amplitude, $\mathbf{v} = (u, v)$ is the velocity vector, and the constants a, b, c and d satisfy

$$a + b = \frac{1}{2}\left(\theta^2 - \frac{1}{3}\right), \quad c + d = \frac{1}{2}(1 - \theta^2) \geq 0.$$

Large classes of one-dimensional solitary-wave solutions were constructed by Chen, Chen and Nguyen

(2008). The linear stability problem for these solutions is of the form

$$\boldsymbol{L(\rho)\phi = \lambda M(\rho)\phi,}$$

with $\boldsymbol{\phi = (\eta_1, u_1, v_1)^T}$, and $\boldsymbol{\rho}$ is the wave length in the transverse direction.

The elevation solitary wave

Let $\boldsymbol{a = -1/9}$, $\boldsymbol{b = 1/3}$, $\boldsymbol{c = -1/9}$, $d = 2/9$,

$$\boldsymbol{\eta = \eta_0 \mathrm{sech}^2(\lambda x), \quad u = \eta_0 \sqrt{\frac{3}{\eta_0 + 3}} \mathrm{sech}^2(\lambda x),}$$

with

$$\boldsymbol{\lambda = \frac{1}{2} \sqrt{\frac{2\eta_0}{3(a-b) + 2b(3+\eta_0)}}.}$$

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with

$$\boldsymbol{\lambda = \frac{1}{2}\sqrt{\frac{2\eta_0}{3(a-b)+2b(3+\eta_0)}}.}$$

We approximate this with a periodic profile:

$$\boldsymbol{\eta = \eta_0 \mathrm{cn}^2(\lambda x, k), \quad u = \eta_0 \sqrt{\frac{3}{\eta_0+3}} \mathrm{cn}^2(\lambda x, k),}$$

with

$$\boldsymbol{\lambda = \frac{1}{2}\sqrt{\frac{2\eta_0}{3(a-b)+2b(3+\eta_0)}}.}$$

With $\rho = 0$ (one-dimensional perturbations) and $\eta_0 = 0.1$:

k	$\lambda_{\max}$	period	# of modes
0.8	2.97E-3		20
0.9	2.14E-3	20	20
0.99	6.87E-4	28	20
0.999	2.18E-4	37	30
0.999999	7.09E-6	62	70

$\rightarrow$ Tentative conclusion: spectrally stable

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- Conclusion persists for higher amplitude.
- Solitary wave appears to be unstable w.r.t. transverse perturbations $(\boldsymbol{\rho}) = \mathbf{0.2}$: growth rate $\sim \mathbf{0.0376}$.

For larger amplitudes, larger periods and more Fourier modes are necessary.

The depression solitary wave

Let $\boldsymbol{\eta_0 = -1}$. Similar to before,

- the depression solitary wave appears spectrally stable w.r.t. one-dimensional perturbations, but
- is spectrally unstable w.r.t. transverse perturbations $(\boldsymbol{\rho}) = \mathbf{0.3}$: growth rate $\sim \mathbf{0.0718}$.

The multi-pulse solitary wave

Let $\boldsymbol{a = 0 = c}$, $\boldsymbol{b = 1/6 = d}$. Then

$$\boldsymbol{\eta = -\tfrac{45}{4}\mathrm{sech}^4\xi + \tfrac{15}{2}\mathrm{sech}^2\xi, \quad u = \tfrac{6}{5}\mathrm{sech}^2\xi,}$$

with $\boldsymbol{\xi = 3x/\sqrt{10}}$.

k	λ_{max}	period	# of modes
0.9	1.28	5.4	20
0.99	1.25	7.8	30
0.999	1.25	10.2	30
0.9999	1.2495363	12.6	50
0.999999	1.2494668	17.4	70