# Numerical Simulations of the Complex Modified Korteweg-de Vries Equation

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**Abstract**

In this paper implementations of three numerical schemes for the numerical simulation of the complex modified Korteweg-de Vries (CMKdV) equation are reported. The first is an integrable scheme derived by methods related to the Inverse Scattering Transform (IST). The second is derived from the first and is called the local IST scheme. The third is a standard finite difference scheme for the CMKdV equation. Travelling-wave solution as well as a double homoclinic orbit are used as initial conditions. Numerical experiments have shown that the standard scheme is subject to instability and the numerical solution becomes unbounded in finite time. In contrast the integrable IST scheme does not suffer from any instabilities. The main difference among the three schemes is in the discretization of the nonlinear term in the CMKdV equation. This demonstrates the importance of proper discretization of nonlinear terms when a numerical method is designed for solving a nonlinear differential equation.

# 1 Introduction

In 1991 Herbst *et al.* [1] derived an integrable differential difference equation, based on the IST, that has as its limiting form the CMKdV equation. Also, they derived analytical expressions for the homoclinic orbits associated with the above equation, and investigated the effect of discretization of the equation in the vicinity of these orbits. They showed that a standard finite difference scheme is subject to an instability. On the other hand, they showed that the integrable differential difference scheme of the CMKdV equation does not suffer from any instabilities. Recently Taha derived an integrable partial difference equation, based on the IST, that has as its limiting form the CMKdV equation [2]

$$q_t + 6|q|^2 q_x + q_{xxx} = 0 \tag{1.1}$$

Here $q$ is a complex valued function, and $|\cdot|$ denotes the modulus. In the present paper this partial difference equation is used as a numerical scheme for solving Eq. (1.1). This scheme, call it the integrable IST scheme, as well as its local version, are implemented and compared to a standard finite difference scheme for the numerical simulation of Eq. (1.1). Our numerical experiments have shown that the standard scheme suffers from instability, and the numerical solution becomes unbounded in finite time. On the other hand, our numerical experiments have shown that the integrable partial difference scheme does not suffer from any instabilities. This result is in agreement of the result found by [1] for the integrable differential difference scheme. Also, in this paper an integrable partial difference equation that has as its limiting form the defocusing CMKdV equation

$$q_t - 6|q|^2 q_x + q_{xxx} = 0 \tag{1.2}$$

is derived. The method of derivation is similar to the one given in [2]. In section 2 the partial difference equations for (1.1) and (1.2) are given.

## 2 The Representation of the CMKdV Equations Using Numerical Methods

**(i)** The integrable IST scheme is

$$\begin{aligned} \Delta^m Q_n^m &= Q_{n+2}^m A_-^{(4)} - Q_{n+2}^{m+1}\,\gamma_{n+1} D_-^{(4)} + Q_{n+1}^m S_{n+1} - Q_{n+1}^{m+1} P_n \\ &- Q_{n-2}^{m+1} A_-^{(4)} + Q_{n-2}^m \gamma_{n-2} D_-^{(4)} - Q_{n-1}^{m+1} S_{n-2}^* + Q_{n-1}^m P_{n-1}^* \\ &+ Q_n^m (A_-^{(0)} \mp \sum_{l=-\infty}^{n} T_l) - Q_n^{m+1} (A_-^{(0)} \mp \sum_{l=-\infty}^{n-1} T_l^*), \end{aligned} \tag{2.1}$$

where * denotes the complex conjugate, and

$$\begin{aligned} \Delta^m Q_n^m &= Q_n^{m+1} - Q_n^m, \gamma_n = \prod_{i=-\infty}^{n} \frac{\delta_i^{m+1}}{\delta_i^m}, \delta_i^m = 1 \pm |Q_i^m|^2, \\ \beta_n &= \frac{\gamma_n}{\delta_{n+1}^m}, S_n = A_-^{(2)} + A_-^{(4)} F_n + D_-^{(4)} \sum_{j=-\infty}^{n} H_j, \\ F_n &= \pm[Q_{n+1}^{m+1} (Q^*)_n^{m+1} - \sum_{j=-\infty}^{n} \Delta^m ((Q^*)_j^m Q_{j+1}^m)] \\ H_n &= \mp\{(Q^*)_n^m Q_{n+1}^{m+1} \delta_n^{m+1} - (Q^*)_{n-1}^m Q_n^{m+1} \delta_n^m\} \beta_{n-1}, \\ P_n &= (D_-^{(2)} + \sum_{j=-\infty}^{n} [A_-^{(4)} E_j + D_-^{(4)} G_j] \eta_j) \gamma_n, \end{aligned}$$

$$
\begin{aligned}
\eta_j &= \frac{\gamma_j^{-1}}{\delta_j^m}, E_n = \pm\left[Q_n^m(Q^*)_{n-1}^{m+1}\delta_n^{m+1} - Q_{n+1}^m(Q^*)_n^{m+1}\delta_n^m\right] \\
G_j &= \pm(Q_{j+1}^{m+1}(Q^*)_j^{m+1} - Q_j^m(Q^*)_{j-1}^m)\delta_j^{m+1}\gamma_{j-1}, \\
T_l &= Q_l^{m+1}[(Q^*)_{l-2}^{m+1}A_-^{(4)} - (Q^*)_{l-2}^m\gamma_{l-2}D_-^{(4)} + (Q^*)_{l-1}^{m+1}S_{l-2} \\
&- (Q^*)_{l-1}^m P_{l-1}] - (Q^*)_l^m[Q_{l+2}^m A_-^{(4)} - Q_{l+2}^{m+1}\gamma_{l+1}D_-^{(4)} \\
&+ Q_{l+1}^m S_{l+1} - Q_{l+1}^{m+1}P_l],
\end{aligned}
$$

$$
\begin{aligned}
A_-^{(2)} &= -\frac{2}{3}A_-^{(0)} + \frac{1}{2}\sigma, \quad D_-^{(2)} = -\frac{2}{3}A_-^{(0)} - \frac{1}{2}\sigma, \\
A_-^{(4)} &= \frac{1}{6}A_-^{(0)} - \frac{1}{4}\sigma, \quad D_-^{(4)} = \frac{1}{6}A_-^{(0)} + \frac{1}{4}\sigma,
\end{aligned}
$$

where $\sigma = \frac{\Delta t}{(\Delta x)^3}$ , $A_-^{(0)}$ is an arbitrary constant, $Q_n^m = \Delta x q_n^m$, $|n| < p$ ($p$ is half the length of the interval of interest), and $m > 0$. This scheme is consistent with the CMKdV equations (1.1) and (1.2) and has a truncation error of order $0((\Delta t)^2) + 0((\Delta x)^2)$. It is implemented with the value of $A_-^{(0)} = \frac{3}{2}\sigma$.

**(ii)** A local IST scheme which is derived from the integrable IST scheme (2.1) with $A_-^{(0)} = \frac{3}{2}\sigma$ is

$$
\begin{aligned}
\frac{q_n^{m+1} - q_n^m}{\Delta t} &= \frac{q_{n-1}^{m+1} - 3q_n^{m+1} + 3q_{n+1}^{m+1} - q_{n+2}^{m+1}}{2(\Delta x)^3} \\
&+ \frac{q_{n-2}^m - 3q_{n-1}^m + 3q_n^m - q_{n+1}^m}{2(\Delta x)^3} \\
&\mp \frac{1}{2(\Delta x)}\Big[q_{n+2}^{m+1}\{|q_{n+1}^m|^2 + |q_n^m|^2\} - q_{n-2}^m\{|q_n^{m+1}|^2 + |q_{n-1}^{m+1}|^2\} \\
&+ \frac{q_{n+1}^{m+1}}{2}\{(q^*)_n^m q_{n+1}^m + (q^*)_n^{m+1}q_{n+1}^{m+1} + 2q_n^m(q^*)_{n-1}^m\} \\
&- \frac{q_{n-1}^m}{2}\{q_{n-1}^{m+1}(q^*)_n^{m+1} + q_{n-1}^m(q^*)_n^m + 2q_n^{m+1}(q^*)_{n+1}^{m+1}\} \\
&+ \frac{q_n^m}{2}\{(q^*)_n^{m+1}q_{n+1}^{m+1} + (q^*)_n^m q_{n+1}^m\} - \frac{q_n^{m+1}}{2}\{(q^*)_n^{m+1}q_{n-1}^{m+1} + (q^*)_n^m q_{n-1}^m\} \\
&- 3\{|q_n^m|^2 q_{n+1}^{m+1} - |q_n^{m+1}|^2 q_{n-1}^m\}\Big] \qquad (2.2)
\end{aligned}
$$

This scheme has a truncation error of order $0((\Delta t)^2) + 0((\Delta x)^2)$.

**(iii)** A standard numerical scheme is

$$\begin{aligned}\frac{q_n^{m+1}-q_n^m}{\Delta t} &= \frac{(q_{n-1}^{m+1}-3q_n^{m+1}+3q_{n+1}^{m+1}-q_{n+2}^{m+1})}{2(\Delta x)^3}\\ &+ \frac{(q_{n-2}^{m}-3q_{n-1}^{m}+3q_n^{m}-q_{n+1}^{m})}{2(\Delta x)^3}\\ &\mp \frac{3}{2(\Delta x)}\left[|q_n^{m+1}|^2(q_{n+1}^{m+1}-q_{n-1}^{m+1})+|q_n^m|^2(q_{n+1}^m-q_{n-1}^m)\right]\end{aligned} \tag{2.3}$$

This scheme has a truncation error of $0((\Delta t)^2)+0((\Delta x)^2)$.

# 3 Numerical Implementation

The numerical methods described in the previous section are applied to the CMKdV equation (1.1) subject to the following conditions:

**(a)** Travelling-wave solution [1]. The exact solution of (1.1) is

$$q(x,t)=a\exp[i(kx-\omega t)], \tag{3.1}$$

where $\omega$ satisfies the dispersion relation, $\omega=6|a|^2k-k^3$ and $a$ is the complex amplitude. For initial conditions, Eq. (3.1) is used at $t=0$, with $a=\frac{1}{2}, k=1$. Periodic boundary conditions on the interval $[0,4\pi]$ are imposed.

**(b)** A double homoclinic orbit [1]. The initial condition

$$q(x,0)=a\exp(ikx)(1+\epsilon_0 i\cos\mu_n x), \tag{3.2}$$

with $a=\frac{1}{2}, \epsilon_0=0.1, k=\mu_n=\frac{2\pi}{L}$, and $L=4\sqrt{2}\pi$. Periodic boundary condition on the interval [0,L] are imposed. The above parameters allow two unstable modes.

The three schemes are implemented using several values of $\Delta x$ and $\Delta t$. One way to implement the above schemes is to solve banded circulant Toeplitz systems of equations of the form:

$$
\begin{bmatrix}
\alpha & -3 & 1 & & & & & & -1 \\
-1 & \alpha & -3 & 1 & & & & & \\
 & -1 & \alpha & -3 & 1 & & & & \\
 & & . & . & . & . & & & \\
 & & & . & . & . & . & & \\
 & & & & . & . & . & . & \\
 & & & & & -1 & \alpha & -3 & 1 \\
1 & & & & & & -1 & \alpha & -3 \\
-3 & 1 & & & & & & -1 & \alpha
\end{bmatrix}
\begin{bmatrix}
q_{-N+1} \\ q_{-N+2} \\ q_{-N+3} \\ . \\ . \\ . \\ q_{N-2} \\ q_{N-1} \\ q_N
\end{bmatrix}
= \begin{bmatrix}
B_{-N+1} \\ B_{-N+2} \\ B_{-N+3} \\ . \\ . \\ . \\ B_{N-2} \\ B_{N-1} \\ B_N
\end{bmatrix} \tag{3.3}
$$

where $\alpha = 3 + \epsilon, \epsilon = \frac{2(\Delta x)^3}{\Delta t}$, at each time level. Note that the only difference in the three schemes is in the right hand side of Eq. (3.3). There are several algorithms to efficiently solve the above system. A modified Gaussian elimination method [3] is used in this paper. Also, it is worth noting that the above system is very suitable for parallel implementation [4].

# 4 Conclusions

Based on numerical experiments, we have drawn the following conclusions:

1. For the Travelling-wave solution, it is found that in the case of a coarse discretization of Eq. (1.1) by the standard numerical scheme Eq. (2.3), the numerical solution becomes unbounded at $t = 16$ with $N = 40, \Delta x = \frac{L}{N}$. If the discretization is refined, the blow up is postponed. The solution becomes unbounded at $t = 42.15$ with $N = 48$. On the other hand, the numerical solution did not blow up when the integrable IST scheme Eq. (2.1) or its local IST version Eq. (2.2) are used in the discretization of Eq. (1.1) (see Table I, and Figs. 1-3).

Table I shows the values of $L_\infty$ and $L_2$ at $t = 5, 10$, and 20 for the numerical schemes utilized in solving Eq. (1.1), $\Delta t = 0.0125, N = 40$[1]

Table 1

| | | Integrable IST | Local IST | Standard |
|---|---|---|---|---|
| $t = 5$ | $L_\infty$ | 0.00001 | 0.00001 | 0.00001 |
| | $L_2$ | 0.00003 | 0.00004 | 0.00004 |
| $t = 10$ | $L_\infty$ | 0.00313 | 0.00305 | 0.00434 |
| | $L_2$ | 0.01416 | 0.01357 | 0.01909 |
| $t = 20$ | $L_\infty$ | 0.00297 | 0.42809 | blow up at |
| | $L_2$ | 0.01326 | 1.26366 | t = 16. |

2. For the double homoclinic orbit, it is found that in the case of the discretization of Eq. (1.1) by the standard numerical scheme Eq. (2.3), the numerical solution becomes unbounded at $t = 8.74$ with $N = 40$. If the discretization is refined, the blow up is postponed. The solution becomes unbounded at $t = 42.32$ with $N = 56$. On the other hand, the numerical solution did not blow up when the integrable IST scheme Eq. (2.1) or its local version Eq. (2.2) are used in the discretization of Eq. (1.1). (see Figs. 4-6) It is to be noted that the only difference among the three numerical schemes used to solve Eq. (1.1) is in the discretization of the nonlinear term of this equation. This shows that the discretization of a nonlinear term in a nonlinear differential equation is crucial. Also, it shows that the integrable IST schemes, which have the same qualitative properties as the associated continuous equations, will play an important role in the proper discretization of nonlinear terms in nonlinear differential equations.

## References


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3. T. Taha and M. Ablowitz, "Analytical and Numerical Aspects of Certain Nonlinear Evolution Equations II, Numerical Nonlinear Schrödinger Equation", *J. Comput. Phys.* Vol. 55, 2, pp. 203-230, 1984.


[1] $L_\infty = \max(||\tilde{q}| - |q_n||)$, $\tilde{q}_n$ is the numerical solution and $q_n$ is the exact solution at the point $(n\Delta x, t)$ for all $n$, $\Delta x =$ the increment in $x$. $L_2 = \sqrt{\Sigma((|\tilde{q}_n| - |q_n|)^2}$ for all $n$.

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