Modeling of Water Waves: Theory and Simulations

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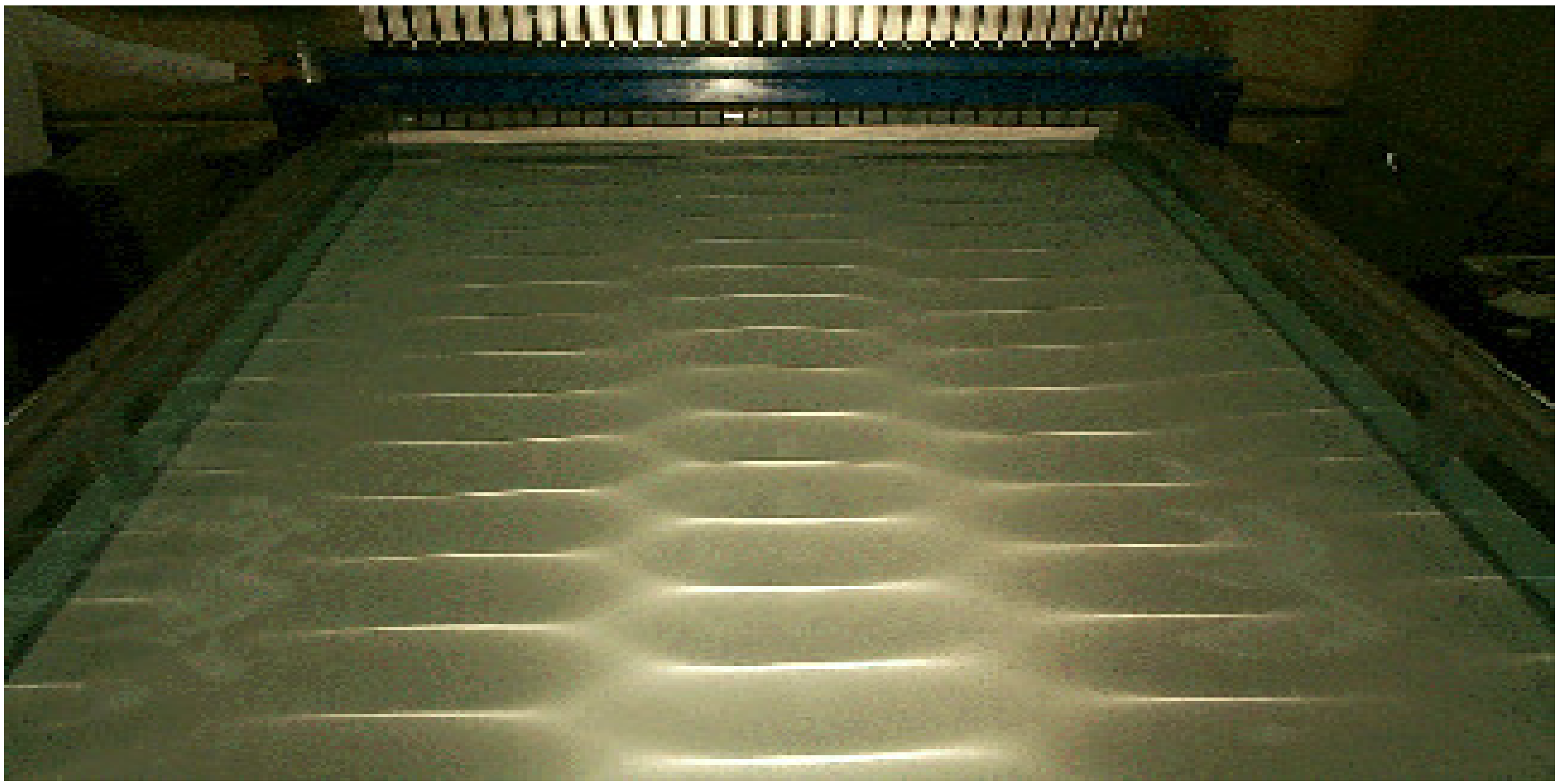
Why do we study water waves?

- **Useful:** approximately **71%** of the earth surface is covered by ocean;
- **Important:** waves affect every aspect of our lives, including beach preservation, the safety of ships, the design of harbor, the prediction and warning of tsunami, the wave electric generators, etc.;
- **Learning:** understanding the most fascinating phenomena of universe;
- **Fun:** deep in mathematics, and involves many different branches of mathematics (PDE, dynamical systems, perturbation theory, topological index theory, harmonic analysis ...), and also involves many branches of engineering, such as civil engineering, oceanography, mechanical engineering, ...;
- **Knowledge:** the study of water waves requiring many tools (modeling, theory, simulations, experiments, field observations).

How do we study water waves?

Unknowns: velocities $(u, v, w)(x, y, z, t)$, pressure $P(x, y, z, t)$, surface $\eta(x, y, t)$, the domain $\Omega(t)$. More variables if the flow is compressible.



Modeling Equations

Assume continuity:

- **Navier-Stokes equation** (mass, momentum and energy conservation) posed on $\Omega(t)$ with free boundary $\eta(x, y, t)$;
- no viscous: **Euler equation** posed on $\Omega(t)$ with free boundary $\eta(x, y, t)$, which has the advantages of flow being potential and tools such as those in harmonic analysis can be used.

Remarks:

- Many recent important achievements on the Navier-Stokes equations and Euler equations. But for **free boundary problems**, namely equations posed on $\Omega(t)$ associated with $\eta(x, y, t)$, results are mainly on Euler equations;
- **Open Million dollar question:** prove or give a counter example on the existence of smooth global (in time and space) solution of Navier-Stokes equation posed in $\mathbb{R}^3$.

Approximate Models with small parameters

Let a be a typical wave amplitude, h_0 be the average water depth and L be a typical wave length.

- **small amplitude (or deep water): $\alpha = \frac{a}{h_0}$ is small, (Stokes 1849, $\cdots$);**
- **long wave (or shallow water): $\epsilon = \frac{h_0^2}{L^2}$ is small, (Airy 1845, $\cdots$);**
- **nonlinear dispersive: $\alpha/\epsilon \approx O(1)$ (Boussinesq 1871, KdV 1895, $\cdots$);**

- **The $O(1)$ terms $\Rightarrow$ linear wave equation ($u_{tt} - u_{xx} = 0$);**
- **The $O(1) + O(\alpha)$ terms $\Rightarrow$ deep water, Nonlinear Schrödinger equation, ...;**
- **The $O(1) + O(\epsilon)$ terms $\Rightarrow$ shallow water, shallow water equations, ...;**
- **The $O(1) + O(\alpha) + O(\epsilon)$ terms $\Rightarrow$ nonlinear dispersive, Boussinesq systems.**

Nonlinear dispersive Models

Under the small amplitude and long wave assumption and assume $\alpha = \epsilon$,

- **The $O(1) + O(\epsilon)$ terms $\Rightarrow$ Boussinesq systems;**
- **The $O(1) + O(\epsilon)$ terms + almost one-dimensional +moving in one-direction $\Rightarrow$KP- equation;**
- **The $O(1) + O(\epsilon)$ terms + one-dimensional+moving in one-direction $\Rightarrow$ KdV, BBM-type equations.**

Road from a Boussinesq-type system to a KP equation

For weakly nonlinear and long waves, with the scaling explicit, a Boussinesq system read:

$$\begin{aligned} \eta_t + \nabla \cdot \boldsymbol{v} + \epsilon \nabla \cdot (\eta \boldsymbol{v}) - \frac{\epsilon}{6} \Delta \eta_t = 0, \\ \boldsymbol{v}_t + \nabla \eta + \epsilon \frac{1}{2} \nabla |\boldsymbol{v}|^2 - \frac{\epsilon}{6} \Delta \boldsymbol{v}_t = 0. \end{aligned} \tag{BBMsys}$$

KP equation:

$$(\eta_t + \eta_x + \frac{2}{3} \epsilon \eta \eta_x - \frac{\epsilon}{6} \eta_{xxt})_x + \frac{\epsilon}{2} \eta_{yy} = 0. \tag{KP}$$

Additional assumptions: weakly three-dimensional, so $\widehat{y} = \epsilon^{\frac{1}{2}} y, \quad \widehat{v} = \epsilon^{-\frac{1}{2}} v$, and propagate predominantly in one direction.

Road from a Boussinesq-type system to a KP equation

For weakly nonlinear and long waves, with the scaling explicit, a Boussinesq system read:

$$
\begin{aligned}
&\eta_t + u_x + v_y + \epsilon(\eta u)_x + \epsilon(\eta v)_y - \frac{\epsilon}{6}\eta_{xxt} - \frac{\epsilon}{6}\eta_{yyt} = 0, \\
&u_t + \eta_x + \frac{\epsilon}{2}(u^2)_x + \frac{\epsilon}{2}(v^2)_x - \frac{\epsilon}{6}u_{xxt} = 0, \\
&v_t + \eta_y + \frac{\epsilon}{2}(u^2)_y + \frac{\epsilon}{2}(v^2)_y - \frac{\epsilon}{6}v_{xxt} = 0.
\end{aligned}
\tag{S1}
$$

KP equation:

$$
(\eta_t + \eta_x + \frac{2}{3}\epsilon\eta\eta_x - \frac{\epsilon}{6}\eta_{xxt})_x + \frac{\epsilon}{2}\eta_{yy} = 0. \tag{KP}
$$

Additional assumptions: weakly three-dimensional, so $\widehat{y} = \epsilon^{\frac{1}{2}} y, \quad \widehat{v} = \epsilon^{-\frac{1}{2}} v$, and propagate predominantly in one direction.

From Boussinesq systems to KP-type equations

Substituting $\widehat{y} = \epsilon^{\frac{1}{2}} y, \quad \widehat{v} = \epsilon^{-\frac{1}{2}} v$ **into (BBMsys) and dropping** $O(\epsilon^2)$ **and the circumflex,**

$$\begin{aligned}
&\boxed{\eta_t + u_x} + \epsilon v_y + \epsilon(\eta u)_x + \epsilon^2(\eta v)_y - \frac{\epsilon}{6}\eta_{xxt} - \frac{\epsilon^2}{6}\eta_{yyt} = 0, \\
&\boxed{u_t + \eta_x} + \frac{\epsilon}{2}(u^2)_x + \frac{\epsilon^2}{2}(v^2)_x - \frac{\epsilon}{6}u_{xxt} = 0, \\
&v_t + \eta_y + \frac{\epsilon}{2}(u^2)_y + \frac{\epsilon^2}{2}(v^2)_y - \frac{\epsilon}{6}v_{xxt} = 0.
\end{aligned} \tag{S1}$$

By considering the O(1) terms of the first two equations and using the wave is moving only to the right,

$$\boxed{\eta(x,t) = u(x,t) + O(\epsilon)} \quad \textbf{and} \quad \partial_x = -\partial_t + O(\epsilon). \tag{xt}$$

For the next order, it is natural to use $u = \eta + \epsilon A(\eta, u, v) + O(\epsilon^2)$.

From Boussinesq systems to KP-type equations

Substituting it into the first two equations in (S1), one obtains

$$\eta_t + \eta_x + \epsilon A_x + \epsilon v_y + \epsilon(\eta^2)_x - \frac{\epsilon}{6}\eta_{xxt} = O(\epsilon^2),$$
$$\eta_t + \eta_x + \epsilon A_t + \frac{1}{2}\epsilon(\eta^2)_x - \frac{\epsilon}{6}\eta_{xxt} = O(\epsilon^2).$$

For this pair of equations to be consistent,

$$A_x + v_y + \frac{1}{2}(\eta^2)_x = A_t + O(\epsilon).$$

By using (xt), one can choose

$$A = -\frac{1}{2}\int v_y dx - \frac{1}{4}\eta^2 \text{ and } \boxed{u = \eta - \frac{\epsilon}{4}\eta^2 - \frac{\epsilon}{4}\int v_y \, dx + O(\epsilon^2)}.$$

From Boussinesq systems to KP-type equations

Substituting this into the first equation of (S1)

$$\eta_t + \eta_x + \frac{2}{3}\epsilon\eta\eta_x - \frac{1}{6}\epsilon\eta_{xxt} = -\frac{\epsilon}{2}v_y + O(\epsilon^2).$$

Differentiating with respect to x,

$$(\eta_t + \eta_x + \frac{2}{3}\epsilon\eta\eta_x - \frac{1}{6}\epsilon\eta_{xxt})_x = -\frac{\epsilon}{2}(v_y)_x + O(\epsilon^2).$$

Using (xt) and the leading order relation from the third equation in (S1),

$$(v_y)_x = -(v_y)_t + O(\epsilon) = \eta_{yy} + O(\epsilon)$$

yields the KP-type equation (KP II) (or KDV)

$$(\eta_t + \eta_x + \frac{2}{3}\epsilon\eta\eta_x - \frac{\epsilon}{6}\eta_{xxt})_x + \frac{\epsilon}{2}\eta_{yy} = 0. \qquad \text{(KPep)}$$

Traveling wave solutions

Traveling wave solution is in the form of $u(x,t) = u(x - \omega t)$. On the traveling frame: $\xi = x - \omega t$

$$\boxed{\textbf{PDE} \Rightarrow \textbf{ODE}}.$$

Let $u(x,t) = u(\lambda\xi)$ and $\eta(x,t) = \eta(\lambda\xi)$ in

$$\begin{aligned} \eta_t + u_x + (u\eta)_x - \frac{1}{6}\eta_{xxt} &= 0, \\ u_t + \eta_x + uu_x - \frac{1}{6}u_{xxt} &= 0. \end{aligned} \qquad \text{(BBMsys1D)}$$

After one integration and taking the constants to be zero,

$$\begin{aligned} -\omega\eta + (u + \eta u) + \frac{\omega}{6}\lambda^2\eta'' &= 0, \\ -\omega u + \eta + \frac{1}{2}u^2 + \frac{\omega}{6}\lambda^2 u'' &= 0. \end{aligned} \qquad \text{(ODE)}$$

It is possible sometimes to get a single ODE equation.

Solve the ODE theoretically and numerically

Starting from the ODE or ODE system:

- **Prove the existence of traveling wave solutions (homoclinic orbit or heteroclinic orbit or center) by using (simple ODE phase space analysis, shooting method, topological index theory, $\cdots$). Examples: (C (2000), C, Chen, Nguyen(2007)) for Boussinesq systems and for the higher-order nonlinear acoustic wave equation (C,Torres,Walsh 2008).**
- **demonstrate numerically (AUTO, $\cdots$)) the existence of traveling wave solutions based on the ODE. This has been accomplished for many Boussinesq-type systems (C (2000)).**

Explicit traveling wave solutions

Explicit traveling wave solutions (solitary waves, cnoidal waves and sinusoidal waves) in the form of Jacobi elliptic function expansion

$$u(\xi) = \sum_{j=0}^{M} a_j \mathbf{sn}^j(\xi, m)$$

- $m = 1$, **solitary waves,**
- $m = 0$, **sinusoidal waves,**
- $m \in (0, 1)$ **cnoidal waves.**

Substitute the ansatz into (ODE), and if the ODE becomes an algebraic equation in terms of sn, then by requiring all the coefficient to be zero,

$$\boxed{\textbf{ODE} \Rightarrow \textbf{Algebraic equation}}$$

in terms of a_j, λ, ω, and m.

When does it work? the derivatives in all the terms are even (or odd).

Examples of explicit traveling wave solutions

Some of the solutions,

- **the solitary wave solution for (BBMsys)**

$$u(x,t) = \pm\frac{15}{2}\textbf{sech}^2\left(\frac{3}{\sqrt{10}}(x \mp \frac{5}{2}t)\right),$$
$$\eta(x,t) = \frac{15}{2}\textbf{sech}^2\left(\frac{3}{\sqrt{10}}(x \mp \frac{5}{2}t)\right) - \frac{45}{4}\textbf{sech}^4\left(\frac{3}{\sqrt{10}}(x \mp \frac{5}{2}t)\right).$$

- **the cnoidal wave solution for (BBMsys) (C, Chen, Nguyen (2007)) in the form of**

$$u(\xi) = a_0 + a_2\textbf{cn}(\lambda\xi, m), \ \textbf{where } \xi = x - \omega t$$

a_0 and a_2 are functions of ω, m, λ, where ω and λ are multi-valued functions of $m^2 \in (\frac{1}{2},\ 1)$, i.e. several one-parameter family of solutions.

Wave patterns: linear plane waves

- the 2D patterns are the oblique interaction of two plane waves;
- parameters involved in describing a plane wave:
  - traveling direction and speed: $\mathbf{c} = c_0(1, 0)$, so the direction is in the y-direction;
  - the angle of the plane wave and the wave length of the plane wave $\mathrm{k}_1 = l_1(1, \tau_1)$;
- to search for this plane wave means to find solutions in the form of

$$\eta(\mathbf{x}) = \eta_{\mathbf{k_1}} e^{i\mathbf{k_1}\cdot(\mathbf{x}-\mathbf{c}t)}, \quad \mathbf{v}(\mathbf{x}) = \mathbf{v}_{\mathbf{k_1}} e^{i\mathbf{k_1}\cdot(\mathbf{x}-\mathbf{c}t)};$$

- substitute this ansatz into the linear part of the equations

$$\begin{aligned} \eta_t + \nabla\cdot\boldsymbol{v} + \nabla\cdot\eta\boldsymbol{v} - \frac{1}{6}\Delta\eta_t &= 0, \\ \boldsymbol{v}_t + \nabla\eta + \frac{1}{2}\nabla|\boldsymbol{v}|^2 - \frac{1}{6}\Delta\boldsymbol{v}_t &= 0, \end{aligned} \qquad (\mathrm{BBM}^2)$$

Wave patterns: linear plane waves

- k_1, c, $\eta_{\mathbf{k_1}}$ and $\mathrm{v}_{\mathbf{k_1}}$ satisfy

$$\begin{aligned}
-(1+\frac{1}{6}|\mathbf{k}|^2)(\mathbf{c}\cdot\mathbf{k})\eta_{\mathbf{k}} + \mathbf{k}\cdot\mathbf{v}_{\mathbf{k}} =& 0,\\
\mathbf{k}\eta_{\mathbf{k}} - (1+\frac{1}{6}|\mathbf{k}|^2)(\mathbf{c}\cdot\mathbf{k})\mathbf{v}_{\mathbf{k}} =& 0.
\end{aligned}$$

- For the nontrivial ($(\eta_{\mathbf{k}}, \mathrm{v}_{\mathbf{k}}) \neq 0$) solutions (plane waves) to exist, k_1 and c are such that the determinant (dispersion relation) is zero, i.e. satisfy

$$\Delta(\mathbf{k},\mathbf{c}) = (1+\frac{1}{6}|\mathbf{k}|^2)^2(\mathbf{c}\cdot\mathbf{k})^2 - |\mathbf{k}|^2 = 0. \tag{Det}$$

- Similarly, for the other plane wave,

$$\eta(\mathbf{x}) = \eta_{\mathbf{k_2}} e^{i\mathbf{k_2}\cdot(\mathbf{x}-\mathbf{c}t)}, \quad \mathbf{v}(\mathbf{x}) = \mathbf{v}_{\mathbf{k_2}} e^{i\mathbf{k_2}\cdot(\mathbf{x}-\mathbf{c}t)},$$

where $\mathrm{k}_2 = l_2(1, -\tau_2)$, k_2 and c have to satisfy (Det).

Sketch of the linear study on wave patterns

Assume k_1 and k_2 are the solutions to (Det), then we have wave patterns with parameters consist of 5 parameters c_0, l_1, l_2, τ_1 and τ_2

- **3 parameter families of patterns because (Det) has to be satisfied by $(\mathbf{k}_1, \mathbf{c})$ and $(\mathbf{k}_2, \mathbf{c})$, and**
- **amplitude $\eta_{\mathbf{k}_1}$, $\eta_{\mathbf{k}_2}$, $\mathbf{v}_{\mathbf{k}_2}$, $\mathbf{v}_{\mathbf{k}_1}$;**
- **symmetric lattice: $\tau_1 = \tau_2$, $l_1 = l_2$;**
- **symmetric pattern: symmetric lattice plus**

$$\eta_{\mathbf{k}_1} = \eta_{\mathbf{k}_2}, \quad \mathbf{v}_{\mathbf{k}_2} = \mathbf{v}_{\mathbf{k}_1}.$$

For symmetric patterns, two parameters for the lattice and half of the number of parameters for the amplitudes.

Sketch of the nonlinear study on wave patterns

Idea:

- add the nonlinear term in, using a perturbation approach (Lyapunov Schmidt);
- invert the linear operator around the kernel and find the bound for the pseudo-inverse;
- perturbation parameter: w in $\mathbf{c} = c_0(1, w)$ and amplitudes of the plane waves;
- study the symmetric patterns, asymmetric patterns with symmetric lattice and asymmetric patterns with asymmetric lattice separately when required.

A little specific:

- To search for 3D traveling waves, assume η and $\mathbf{v}$ are functions of $\mathbf{x} = \mathbf{x}' - \mathbf{c}t$, where $\mathbf{x} = (y, x) \in \mathbb{R}^2$, and $\mathbf{c}$ is the velocity of the traveling wave, in the form of

$$\eta(\mathbf{x}) = \sum_{\mathbf{k}\in\Gamma} \eta_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}, \quad \mathbf{v}(\mathbf{x}) = \sum_{\mathbf{k}\in\Gamma} \mathbf{v}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}},$$

Sketch of the nonlinear study on wave patterns

- substitute the ansatz into the equations, then for every mode, the linear system reads

$$
\begin{aligned}
-(1+\frac{1}{6}|\mathbf{k}|^2)(\mathbf{c}\cdot\mathbf{k})\eta_{\mathbf{k}} + \mathbf{k}\cdot\mathbf{v}_{\mathbf{k}} &= -\, i q_{\mathbf{k}}, \\
\mathbf{k}\eta_{\mathbf{k}} - (1+\frac{1}{6}|\mathbf{k}|^2)(\mathbf{c}\cdot\mathbf{k})\mathbf{v}_{\mathbf{k}} &= -\, i \mathbf{p}_{\mathbf{k}},
\end{aligned}
\qquad \text{(Linear)}
$$

- assume $\pm \mathrm{k}_1$ and $\pm \mathrm{k}_2$ are the only solutions to (Det), namely the kernel is 4-dimensional, one can solve the degenerated equations (Linear) explicitly (2-dimensional for symmetric patterns); Hopefully this is possible with some requirements on k_1, k_2 and c;
- by investigate the pseudo-inverse (hopefully bounded near the kernel), and the bifurcation equation, one can (hopefully) prove the existence of wave patterns.

3D symmetric wave patterns.

For symmetric pattern, we have

Theorem 0. *(C. and Iooss 2006) For almost every* (k, τ), *k represents the wave number in y direction and τ represents the ratio between the periods in y and x directions, let c_0 be the phase velocity corresponding to the dispersion relation, there is a neighborhood of* $(c_0, 0, 0)$ *in* $\mathbb{R}^3$ *such that the bifurcation surface for* (c_b, C, D), *where c_b is the critical velocity of the traveling wave, C and D are the averages of the elevation η and of the horizontal velocity, is analytic. The form of the free surface, even in y, and with $C = D = 0$, is given by*

$$\eta = \varepsilon \cos ky \cos k\tau x - \frac{\varepsilon^2}{2(1+\tau^2)}\{\frac{1-\tau^2}{4}\cos 2k\tau x + c_1 \cos 2ky + d_1 \cos 2ky \cos 2k\tau x\} + O(\varepsilon^3).$$

Proof: Around the kernel, 0 is an isolated eigenvalue, and the pseudo-inverse is bounded for all wave numbers.

Numerical simulations, PDE with BC

Boussinesq system:

$$
\begin{aligned}
&\eta_t + \nabla \cdot \boldsymbol{v} + \nabla \cdot (\eta \boldsymbol{v}) - \frac{}{6}\Delta \eta_t = 0, \\
&\boldsymbol{v}_t + \nabla \eta + \frac{1}{2}\nabla |\boldsymbol{v}|^2 - \frac{1}{6}\Delta \boldsymbol{v}_t = 0.
\end{aligned}
\tag{BBM^2}
$$

Consider a wave tank L unit wide and H unit long with flat bottom, (BBM^2) is valid on the domain $\Omega = (0, L) \times (0, H)$.

- **at one end (wave maker end) where $y = 0$,**
  $\eta(x, 0, t) = h_0(x, t), \quad \boldsymbol{v}(x, 0, t) = \boldsymbol{v}_0(x, t)$**,**
- **at the other end $y = H$, $\eta(x, H, t) = \boldsymbol{v}(x, H, t) = 0$, (other BC can be treated also);**
- **at $x = 0$ and $x = L$, periodic (other type of BC can be treated, see (C, (2008)) for detail):**

$$
\partial_x^m (\eta, \boldsymbol{v})(0, y, t) = \partial_x^m (\eta, \boldsymbol{v})(L, y, t), \quad \textbf{for} \quad m = 0, 1, 2, \cdots, \tag{BC1}
$$

Treatment of the nonzero boundary conditions

- **Let $G(x,y,t)$ and $\boldsymbol{V}(x,y,t)$ satisfy boundary conditions at $y=0$, $y=H$, and (BC1) at $x=0$ and $x=L$. In this case,**

$$G(x,y,t) = h_0(x,t)\frac{H-y}{H}, \quad \boldsymbol{V}(x,y,t) = \boldsymbol{v}_0(x,t)\frac{H-y}{H}.$$

- **Introduce the new variables**

$$\bar{\eta} = \eta - G(x,y,t), \quad \bar{\boldsymbol{v}} = \boldsymbol{v} - \boldsymbol{V}(x,y,t),$$

- **Under the new variables (the same notations) are**

$$\left.\begin{aligned} &\eta_t + \nabla\cdot\boldsymbol{v} + \nabla\cdot(\eta\boldsymbol{v}) - \frac{1}{6}\Delta\eta_t = -\nabla\cdot(\eta\boldsymbol{V} + G\boldsymbol{v}) + F_1(G,\boldsymbol{V}),\\ &\boldsymbol{v}_t + \nabla\eta + \frac{1}{2}\nabla|\boldsymbol{v}|^2 - \frac{1}{6}\Delta\boldsymbol{v}_t = -\nabla(\boldsymbol{v},\boldsymbol{V}) + F_2(G,\boldsymbol{V}),\end{aligned}\right. \quad \textbf{in } \Omega,$$

with $\eta(x,0,t)=\eta(x,H,t)=0$, $\boldsymbol{v}(x,0,t)=\boldsymbol{v}(x,H,t)=0$, and periodic in x.

Time discretization

The second-order semi-implicit Crank-Nicolson-leap-frog scheme (with the first step computed by a semi-implicit backward-Euler scheme) is used. More precisely, denote

$$h^0 = \frac{\eta^1 - \eta^0}{\Delta t}, \ \boldsymbol{W}^0 = \frac{\boldsymbol{v}^1 - \boldsymbol{v}^0}{\Delta t},$$

$$h^n = \frac{\eta^{n+1} - \eta^{n-1}}{2\Delta t}, \ \boldsymbol{W}^n = \frac{\boldsymbol{v}^{n+1} - \boldsymbol{v}^{n-1}}{2\Delta t} \quad \textbf{for } n \geq 1.$$

The scheme reads (unknowns h^n and $\boldsymbol{W}^n$), $n = 0, 1, 2, \cdots$,

$$h^n - \frac{1}{6}\Delta h^n = (-F_1(G, \boldsymbol{V}) - \nabla \cdot (\eta \boldsymbol{V} + G\boldsymbol{v}) - \nabla \cdot \boldsymbol{v} - \nabla \cdot \eta \boldsymbol{v})^n,$$

$$\boldsymbol{W}^n - \frac{1}{6}\Delta \boldsymbol{W}^n = (-F_2(G, \boldsymbol{V}) - \nabla(\boldsymbol{v}, \boldsymbol{V}) - \nabla \eta - \frac{1}{2}\nabla |\boldsymbol{v}|^2)^n,$$

where the superscripts on the right-hand side mean that the expression is to be evaluated at t^n.

Resulting equations from time discretization

Hence, at each time step, we only have to solve a sequence of Poisson type equations of the form

$$u - \alpha \Delta u = f \text{ in } \Omega.$$

Many numerical schemes can be used at this stage. For the spectral method,

- **map the physical domain $[0, L] \times [0, H]$ to computational domain in (r, z) plane $\widehat{\Omega} = (0, 2\pi) \times (-1, 1)$ for (BC1),**

$$u - \alpha \widehat{\Delta} u = f \text{ in } \widehat{\Omega} \qquad \text{(Poisson)}$$

$$\textbf{for (BC1)} \quad \widehat{u}(r, \pm 1) = 0; \qquad \partial_r \widehat{u}(0, z) = \partial_r \widehat{u}(2\pi, z)$$

- **in the periodic direction, use bases $\phi_l(r) = e^{ilr}$, with $l = -N/2, \cdots, N/2$;**

Space discretization for (BC1)

- **in the Dirichlet BC, use bases** $\xi_j(z) = T_j(z) - T_{j+2}(z)$ **for** $j = 0, 1, \cdots, M-2\}$**, with** $T_k(z)$ **being the Chebyshev polynomial of degree** k **(recall** $T_k(\cos(\theta)) = \cos(k\theta)$**). We note that** $\xi_j(z)$ **satisfies the homogeneous Dirichlet BC;**
- **Fourier-Chebyshev Galerkin method: Let**

$$V_{NM} = \textbf{span}\{\phi_l(r)\xi_j(z) : l = -N/2, \cdots, N/2; j = 0, 1, \cdots, M-2\},$$

and look for $\widehat{u}_{NM} \in V_{NM}$ **such that for** $|l| \leq N/2$ **and** $0 \leq j \leq M-2$**,**

$$(\widehat{u}_{NM}, \phi_l(r)\xi_j(z)) - \alpha(\widehat{\Delta}\widehat{u}_{NM}, \phi_l(r)\xi_j(z)) = (\widehat{f}_{NM}, \phi_l(r)\xi_j(z)), \tag{Discrete}$$

where $\widehat{f}_{NM}$ **is an interpolation of** $\widehat{f}$ **at the Fourier-Chebyshev collocation points.**

Operation counts and accuracy

- **In this case, the equations are separable and become $M-1$ one-dimensional problem.**
- **Total operation: $O(NM\log(NM))$ operations (Shen(1995)).**
- **The algorithms are spectrally accurate. More precisely, we have the following error estimates (cf. (CHQZ87)):**

$$\|\widehat{u}-\widehat{u}_{NM}\|_{H^1(\widehat{\Omega})} \lesssim \min(N,M)^{1-s}\|\widehat{u}\|_{H^s(\widehat{\Omega})}+\min(N,M)^{1-\rho}\|\widehat{f}\|_{H^\rho(\widehat{\Omega})}.$$

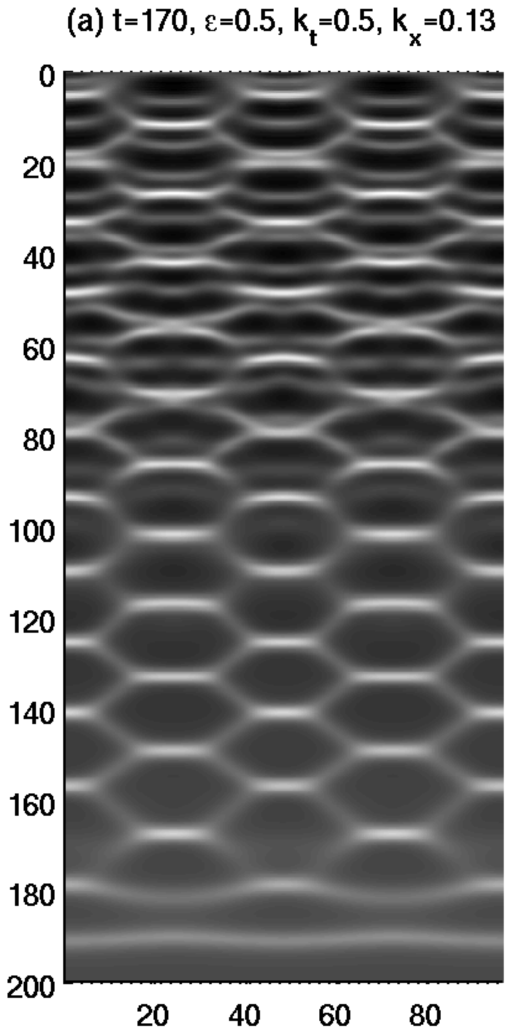
Generate two-dimensional wave patterns–specification

Boundary data at $y = 0$ is taken to be

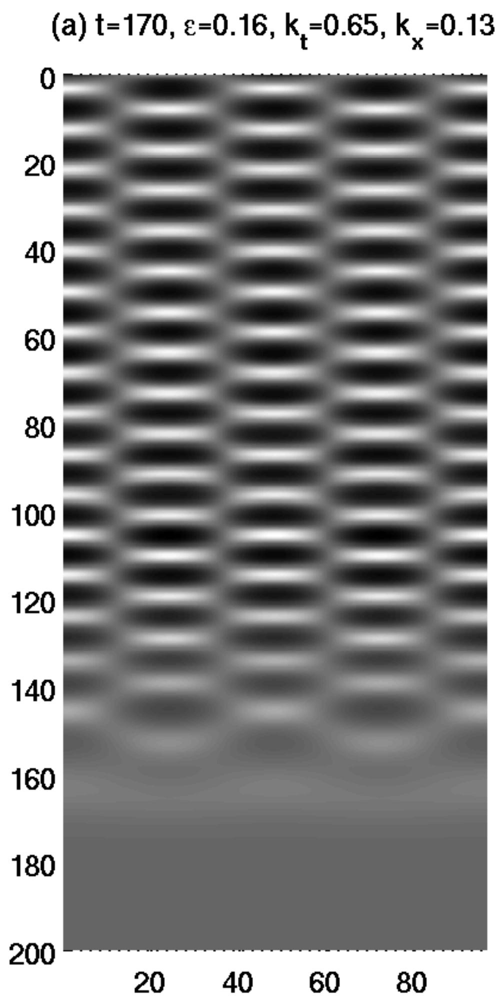
$$\begin{pmatrix} \eta(x,0,t) \\ u(x,0,t) \\ v(x,0,t) \end{pmatrix} = \begin{pmatrix} \epsilon \sin(k_t t)\cos(k_x x) \\ -\frac{\epsilon k_x}{\sqrt{k_x^2+k_y^2}} \cos(k_t t)\sin(k_x x) \\ \frac{\epsilon k_y}{\sqrt{k_x^2+k_y^2}} \sin(k_t t)\cos(k_x x) \end{pmatrix}$$

- **parameters (ϵ, k_t, k_x): the amplitude and frequency of the paddle movement, and the frequency in x;**
- (k_x, k_y, k_t) **satisfies the linear dispersion relation, or solution of the linear equation;**
- **a spectral code is used with 256 mode in x, 1024 mode in y and $\Delta t = 0.04$.**

Numerical and experimental (hexagonal cells)

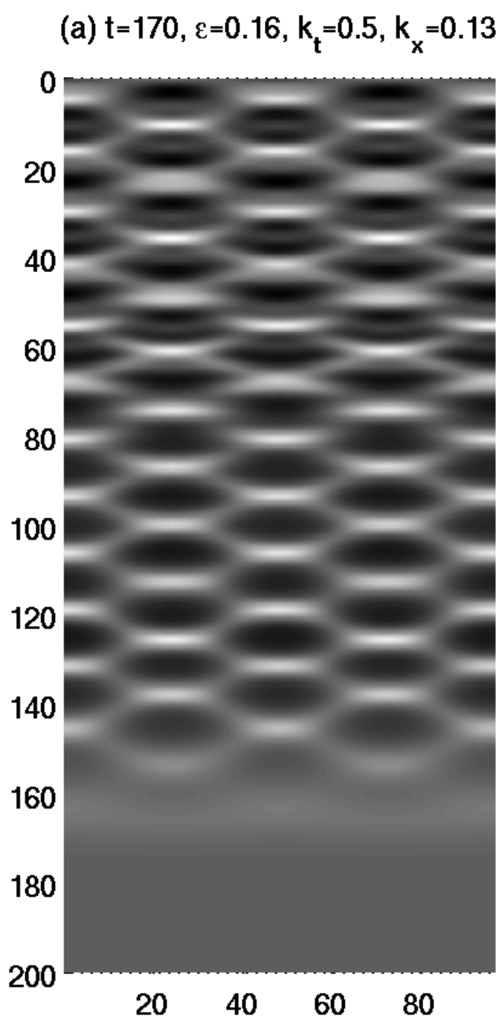


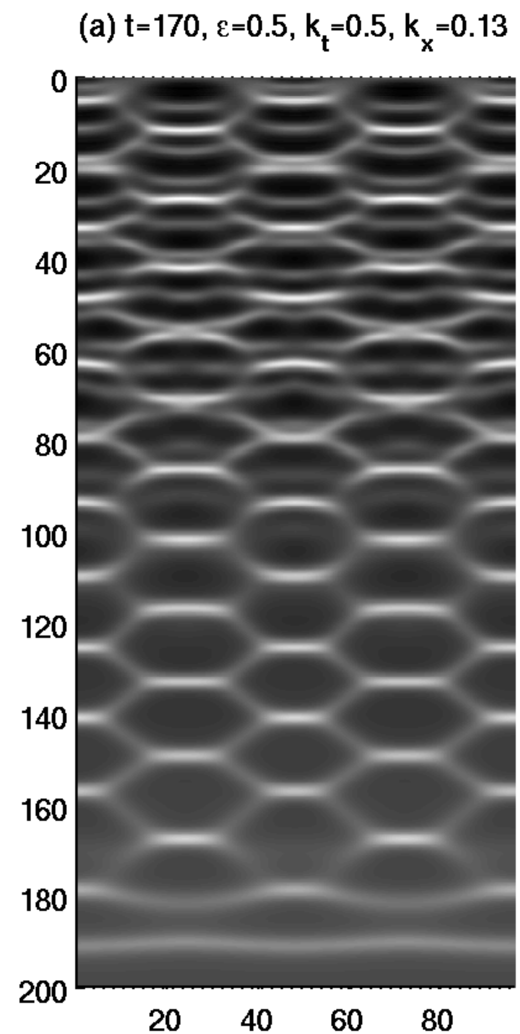
Numerical and experimental (rectangular cells)



Wave patterns with $\epsilon = 0.16$ and $\epsilon = 0.5$

Note: as ϵ (paddle amplitude) increase, L_y (wave number in y) decreases (wider cells).





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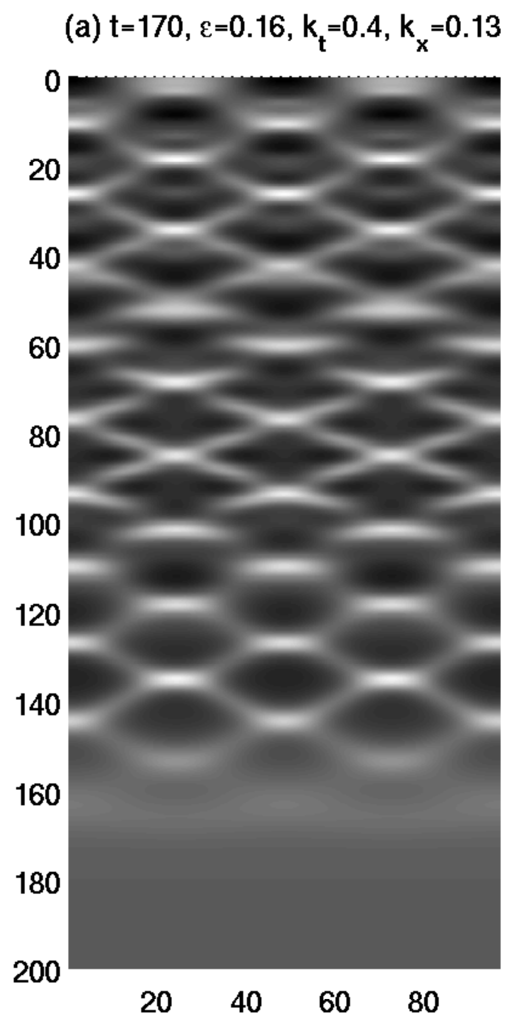
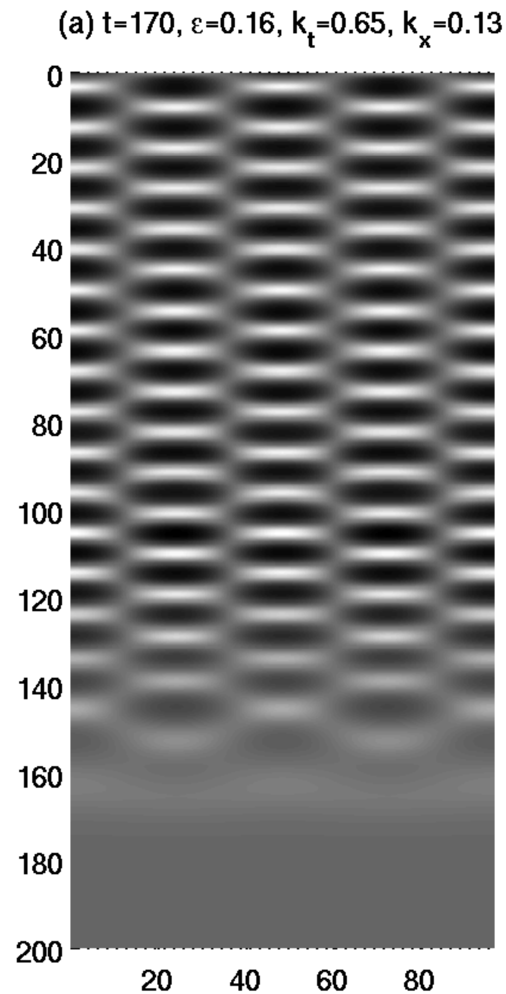
Wave patterns with $k_t = 0.4$ and $k_t = 0.65$

Note: as k_t (frequency of the peddle) increases, L_y (wave number in y) increases (narrow cells).





Observations from numerical experiments

Resulting wave pattern specifications: F_{max}-amplitude, L_y-wave number in y, L_x-wave number in x, L_t wave number in t.
We observe that:

- $L_x = k_x$, L_t **is about** k_t;
- **when** ϵ **increases,** L_y **decreases (wider cell),** F_{max} **is bigger than** ϵ **and increases;**
- **when** k_t **increases (paddle moves faster),** L_y **increases (narrower cell),** F_{max} **decreases;**
- **when** k_x **increases,** L_y **decreases (wider cell) and the change in** F_{max} **is small.**

Open: Analytical relationship (asymptotic relation) between these parameters and quantitative comparison between theoretical, numerical and experimental results.

Ocean waves and Tsunami

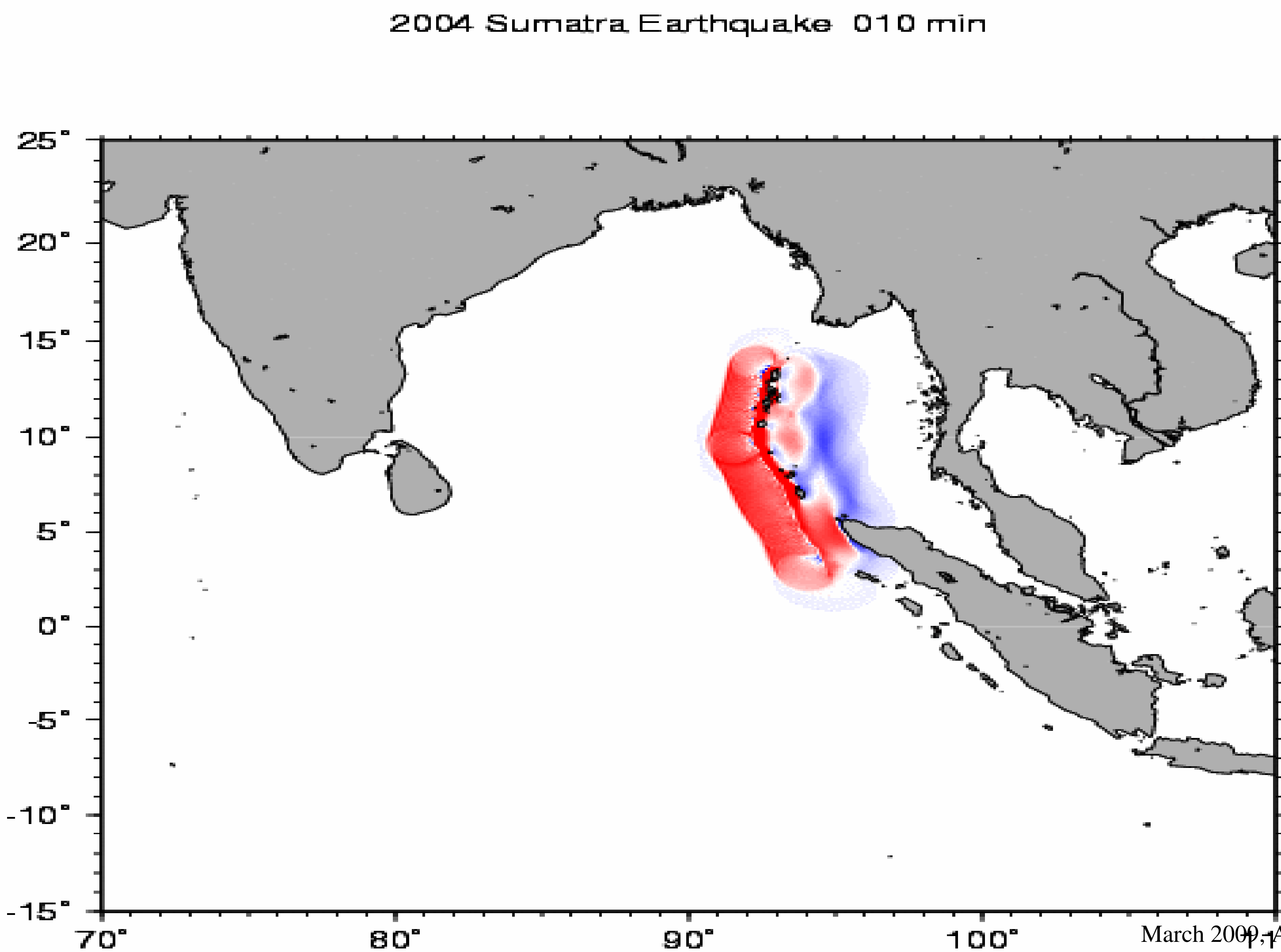
(based mainly on Grilli and al (2007) and wikipedia)

- **During the Dec. 2004 earthquake, an estimated 1200 km (750 mi) of fault line slipped about 15 m (50 ft) along the subduction zone where the India Plate dives under the Burma Plate. The sea bed is estimated to have risen by several meters (4-5 meters), displacing an estimated 30 km^3 of water and triggering devastating tsunami waves. The waves did not originate from a point source, as mistakenly depicted in some illustrations of their spread, but radiated outwards along the entire 1200 km (750 mi) length of the rupture.**
- **Because the 1,200 km (745.6 mi) of fault line affected by the quake was in a nearly north-south orientation, the greatest strength of the tsunami waves was in an east-west direction. Bangladesh, which lies at the northern end of the Bay of Bengal, had very few casualties despite being a low-lying country relatively near the epicenter, Somalia was hit harder than Bangladesh despite being much farther away.**

Ocean waves and Tsunami (based mainly on wikipedia)

Ocean waves and Tsunami (based mainly on wikipedia)

Click here to view the movie



Waves from a rectangular source-IC

The initial data in this sequence of tests are taken to be

$$\eta(x,y,0) = \eta_\sigma(x,y) \equiv 5\alpha^2 e^{-\alpha^{2m}(\sigma^m (x-x_0)^{2m} + \sigma^{-m}(y-y_0)^{2m})}$$
$$u(x,y,0) = 0$$

where

- $\alpha = 0.1$ – **small amplitude and long waves**,
- $m = 8$ – **super Gaussian so IC is similar to having water level raised in a localized area which is approximately** $20/\sqrt{\sigma}$ **times** $20\sqrt{\sigma}$ **(aspect ratio** σ**) in the middle of the wave tank,**
- **the amplitude** $\max(\eta_\sigma(x,y))$ **and the volume** $\int\int \eta_\sigma(x,y)dxdy$ **are independent of** σ**.**

Goal: (a) study the case with $\sigma = 10$**, (b) observe and analyze the effect of aspect ratio** σ **with aspect ratio** $\sigma = 1, 2, 4, 6, 8, 10$ **and (c) show one of the key factors in deciding the heights of the leading waves generated from a rectangular source is the aspect ratio.**

Surface profile at t=0 and t=60 ($\sigma = 10$)

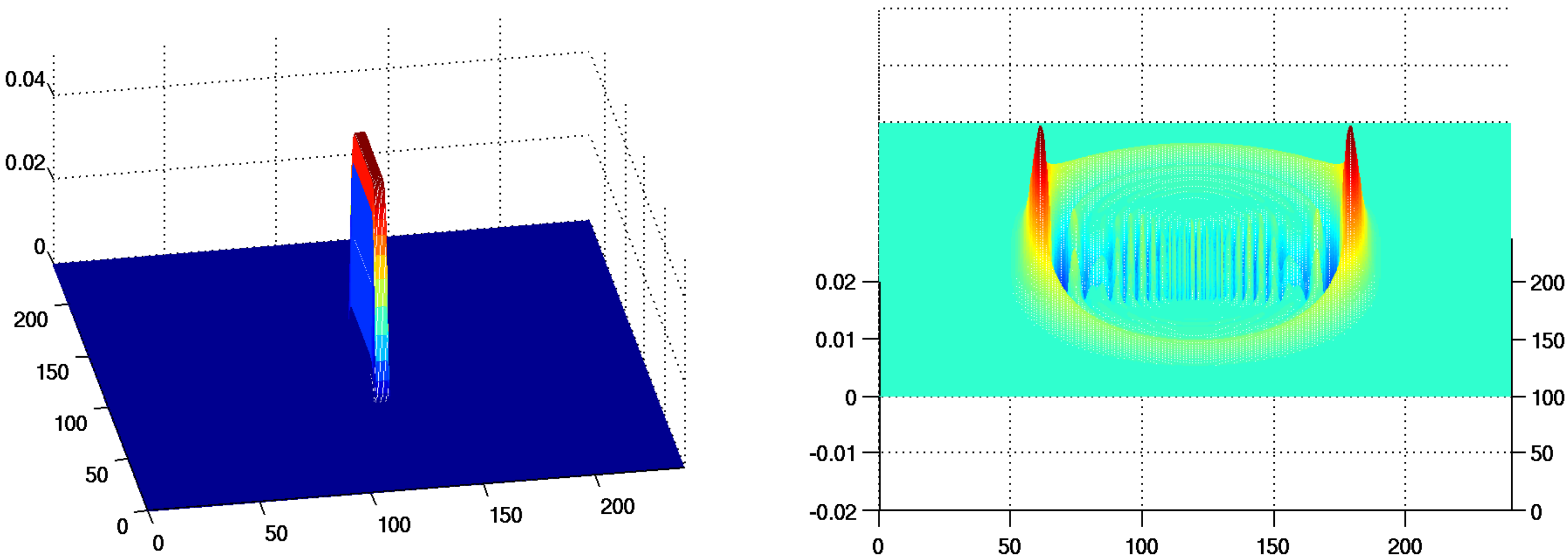


Figure 0: $\eta(x, y, 0)$ **and** $\eta(x, y, 60)$

Plots for x– and y– directional waves($\sigma = 10$)

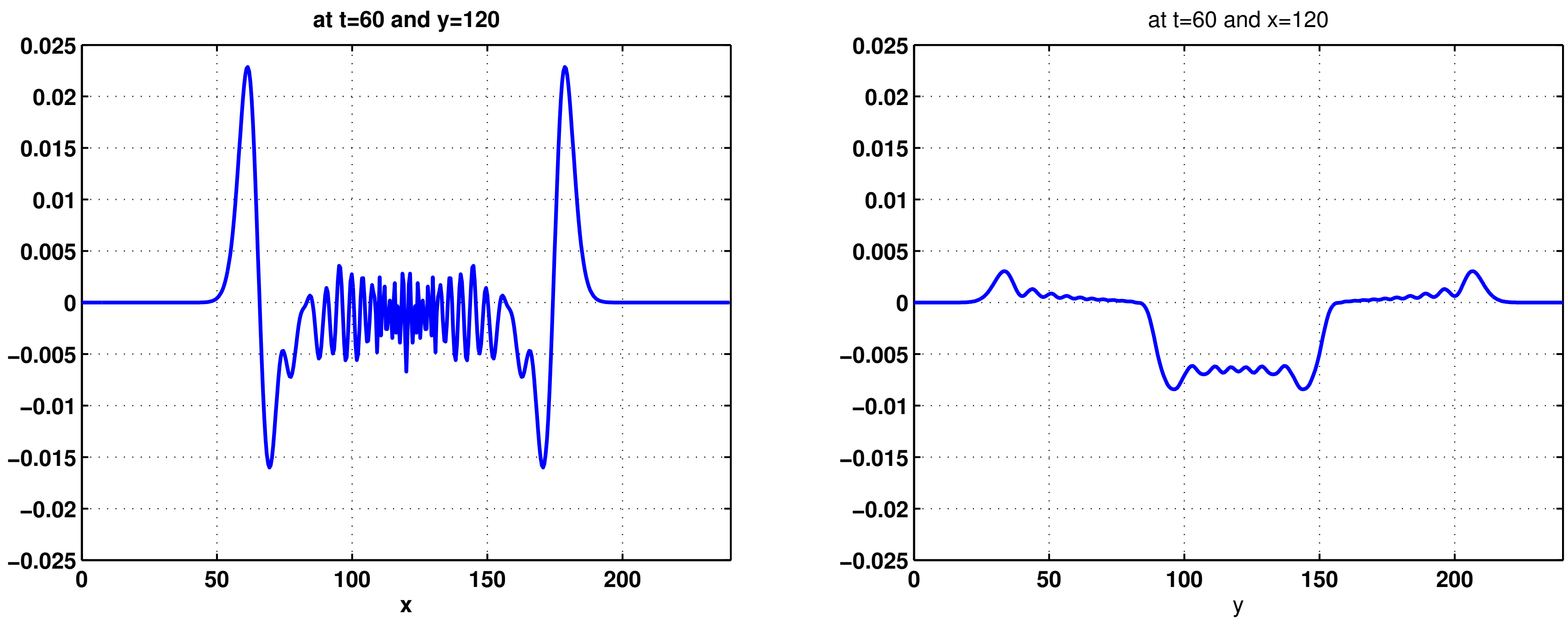


Figure 0: $\eta(x, 120, 60)$ **and** $\eta(120, y, 60)$

Waves heights and location w.r.t. t ($\sigma = 10$)

- $x^*(t)$ **(or $y^*(t)$): the distances between the x (or $y--$) –leading wave and the center (x_0, y_0) at time t;**
- $hx(t)$ **(or $hy(t)$): heights of the x– (or y–) leading waves.**

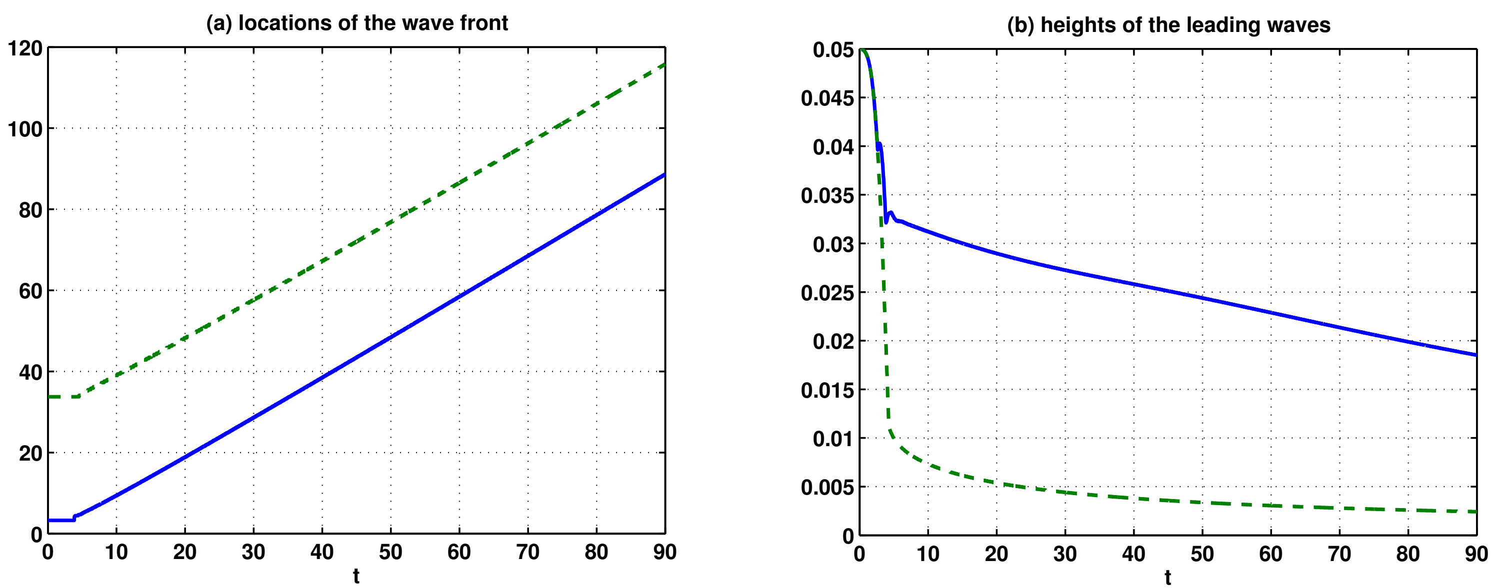


Figure 0: $x^*(t)$, $y^*(t)$, $hx(t)$ **and** $hy(t)$

Conclusion with $\sigma = 10$

- the heights of the x-directional leading wave is much bigger than that of the y-directional leading wave. As time evolves, the ratio ($hx(t)/hy(t)$) is between 7 and 8 for t between 44 and 90;
- the leading waves move with about the same constant speed. The location of the leading wave form an “ellipse” shape. With a least-square linear fitting on the data from $t = 45$ to $t = 90$, one finds that the semi-major axes is approximately $0.97t + 28$ and semi-minor axes is approximately $1.0t - 1.8$ (almost in the linear wave regime);
- after the leading waves have formed, the ratio between the heights of the x– and y–directional leading waves is increasing with respect to r and between 6 and 8 for r between 33 and 58.

Results and conclusions from $\sigma = 1, 2, 4, 6, 8, 10$

	$\sigma = 1$	$\sigma = 2$	$\sigma = 4$	$\sigma = 6$	$\sigma = 8$	$\sigma = 10$
$hx(90)$	$8.8(-3)$	$1.2(-2)$	$1.6(-2)$	$1.8(-2)$	$1.8(-2)$	$1.9(-2)$
$hy(90)$	$8.8(-3)$	$6.1(-3)$	$4.2(-3)$	$3.3(-3)$	$2.8(-3)$	$2.4(-3)$
$hx/hy(90)$	1.0	2.0	3.9	5.4	6.6	7.6

- **as σ increases, the heights of x–directional leading waves increase. As σ increases from 1 to 10, the wave heights more than doubled (see row 2 of Table 1);**
- **as σ increases, the heights of y–directional leading waves decreases. As σ increases from 1 to 10, the wave heights decrease to about 27% (see row 3 of Table 1);**
- **a combination effect of increasing heights in x-directional waves and decreasing heights in y-directional waves is that, as σ increases, the height ratio between x– and y-directional waves increases. At $\sigma = 10$, the ratio is about 7.6 (see row 4 of Table 1).**

Oblique interaction of solitary waves, IC

- **The explicit "exact" 1D solution, with $A = 0.2$ and $K_0 = 1.1$ ($K_0 = 1 + \frac{A}{2}$)**

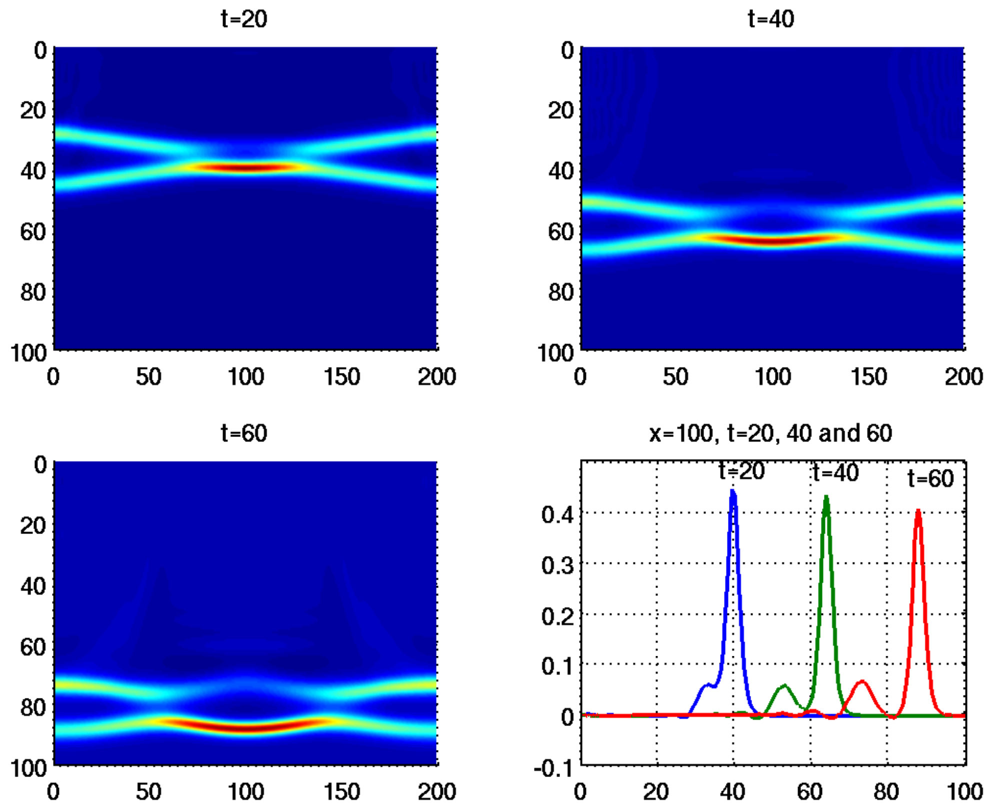
$$\eta_0(x,t) = Asech^2(\sqrt{\frac{3A}{4K_0}}(x - K_0 t)$$

together with the first order approximation for velocity

$$v(x,t) = \eta(x,t) - \frac{\eta(x,t)^2}{4}$$

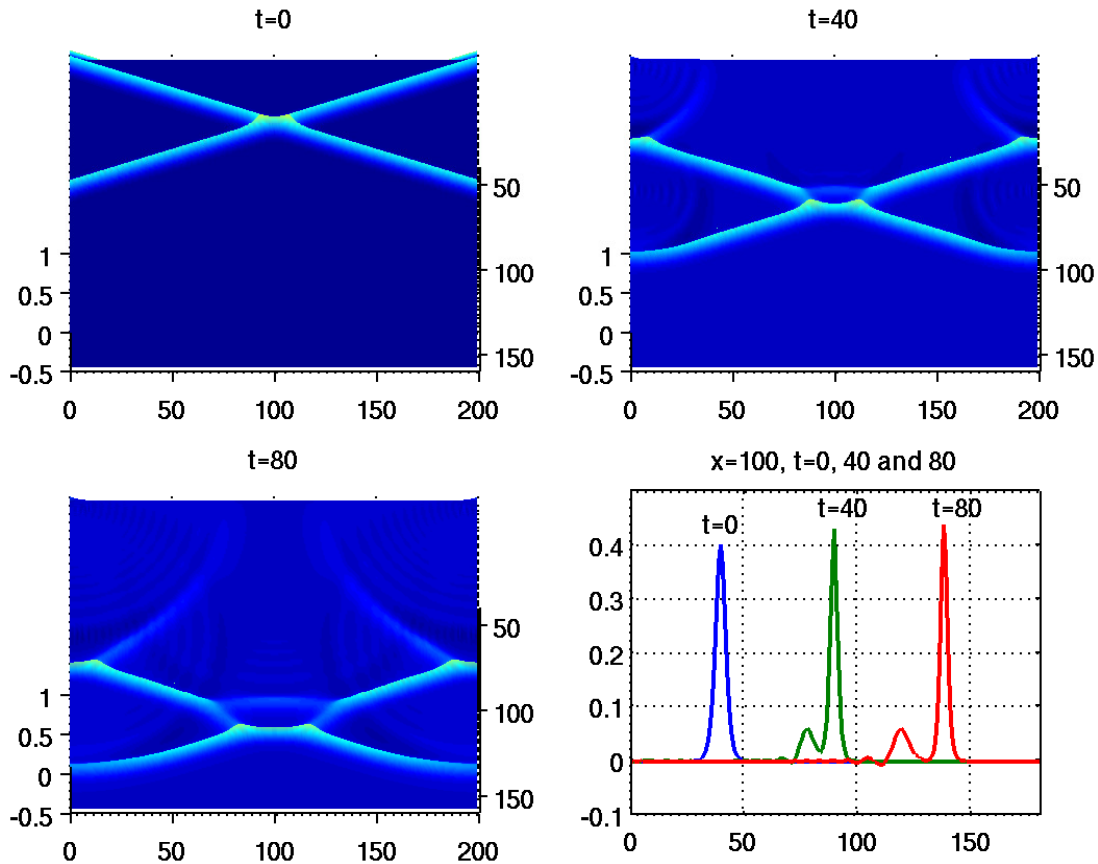
- **Initial condition consists two solitary waves with an attack angle θ;**
- **three cases ($\theta = 10^o, 40^o, 90^o$) will be investigated.**

Oblique interaction, small attack angle $\theta = 10^o$



- **The center of the front wave is much bigger than that of the back. The front wave and the back wave are almost separated;**
- **If there is not boundary, the center is growing.**

Oblique interaction, medium attack angle $\theta = 40^o$



Focus the center, a four-sided cell is formed, where the front stem is bigger and shorter and the back one is smaller and longer. The height of the front stem is more than double the incident wave height.

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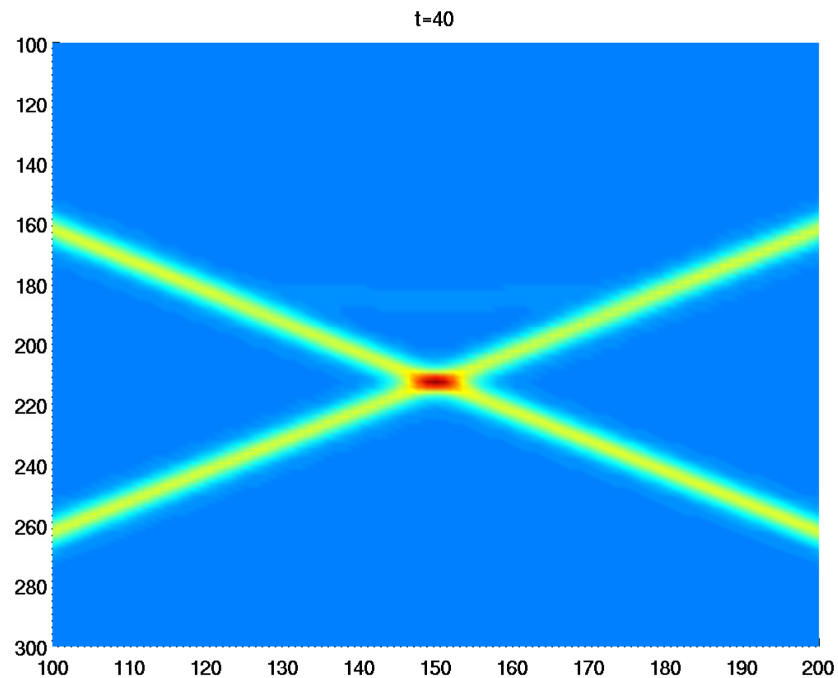
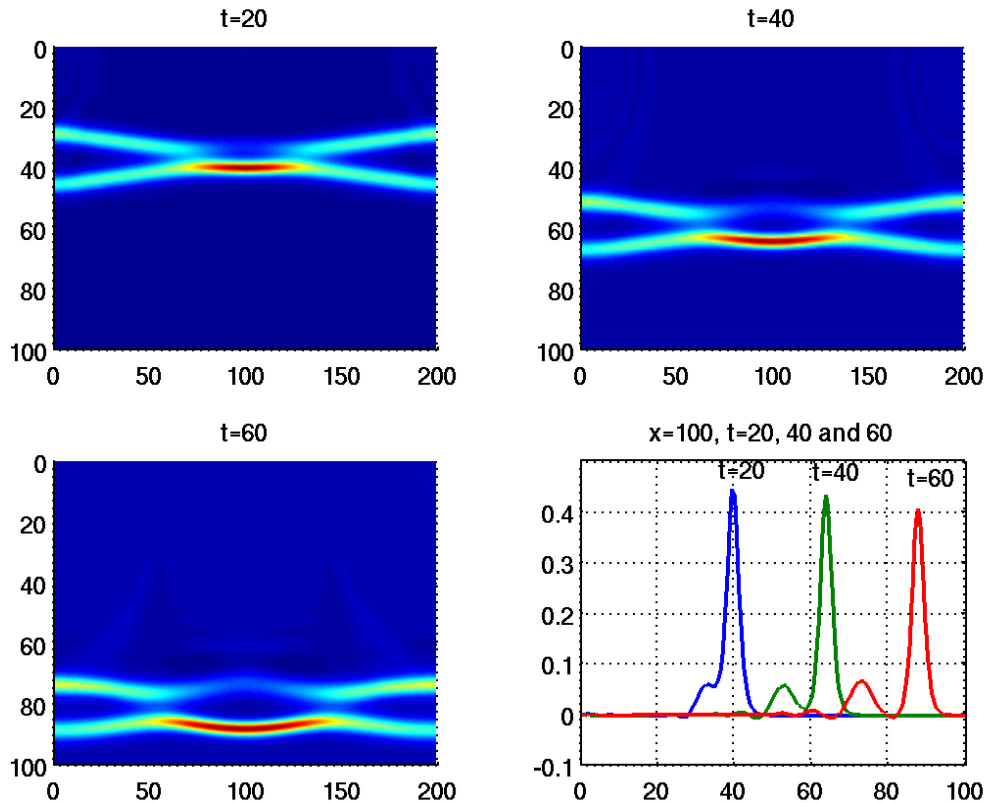
Oblique interaction, large attack angle $\theta = 90^o$



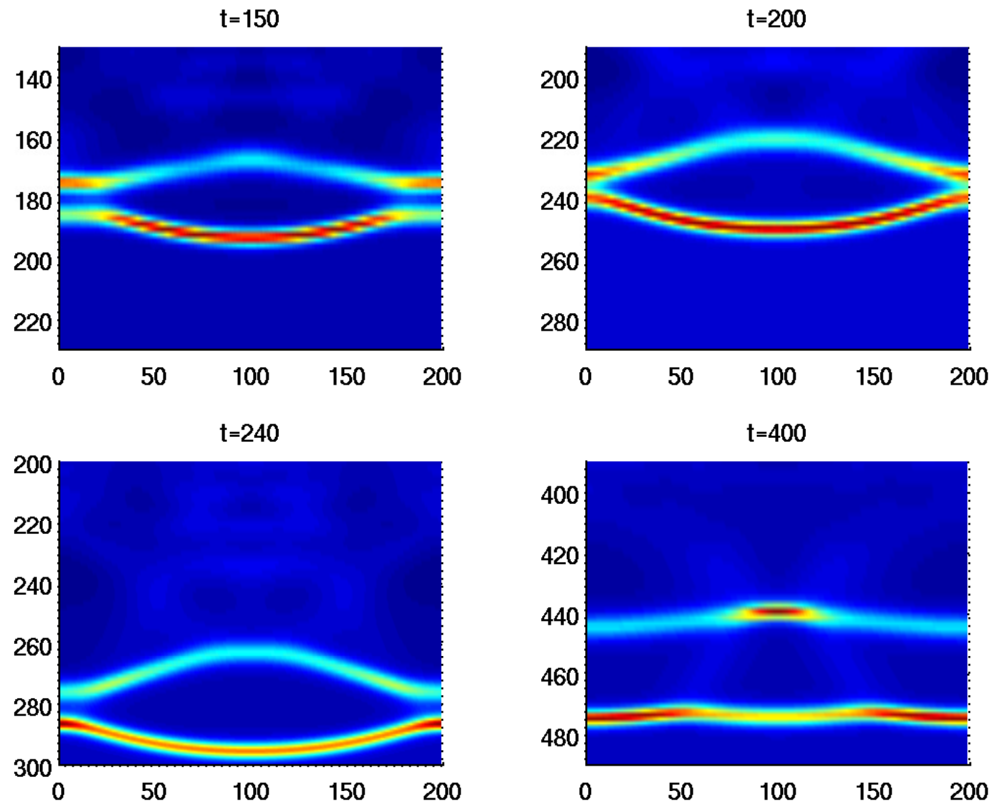
Almost no interaction and travels like a traveling wave.

Oblique interaction (with BC), small attack angle $\theta = 10^o$



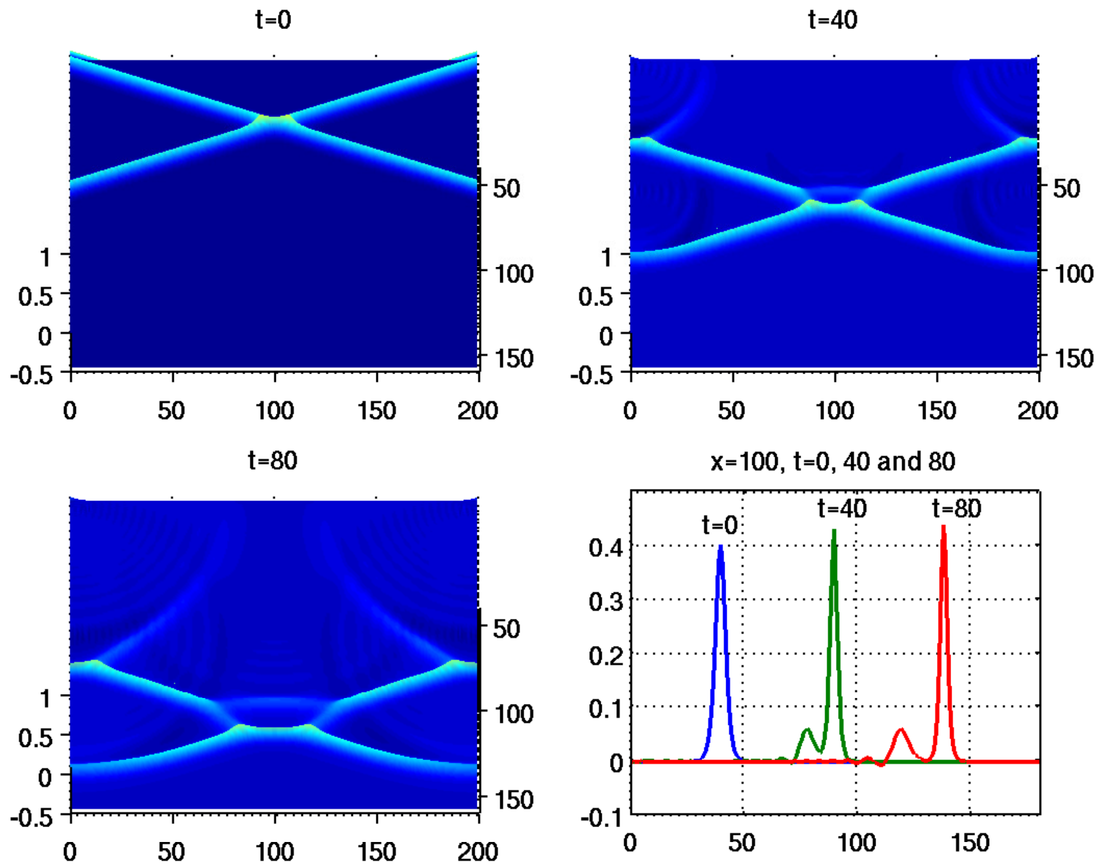
- **If there is not boundary, the center is growing.**
- **With boundary, almost two 1D solitary waves.**

Oblique interaction (with BC), small attack angle $\theta = 10^o$



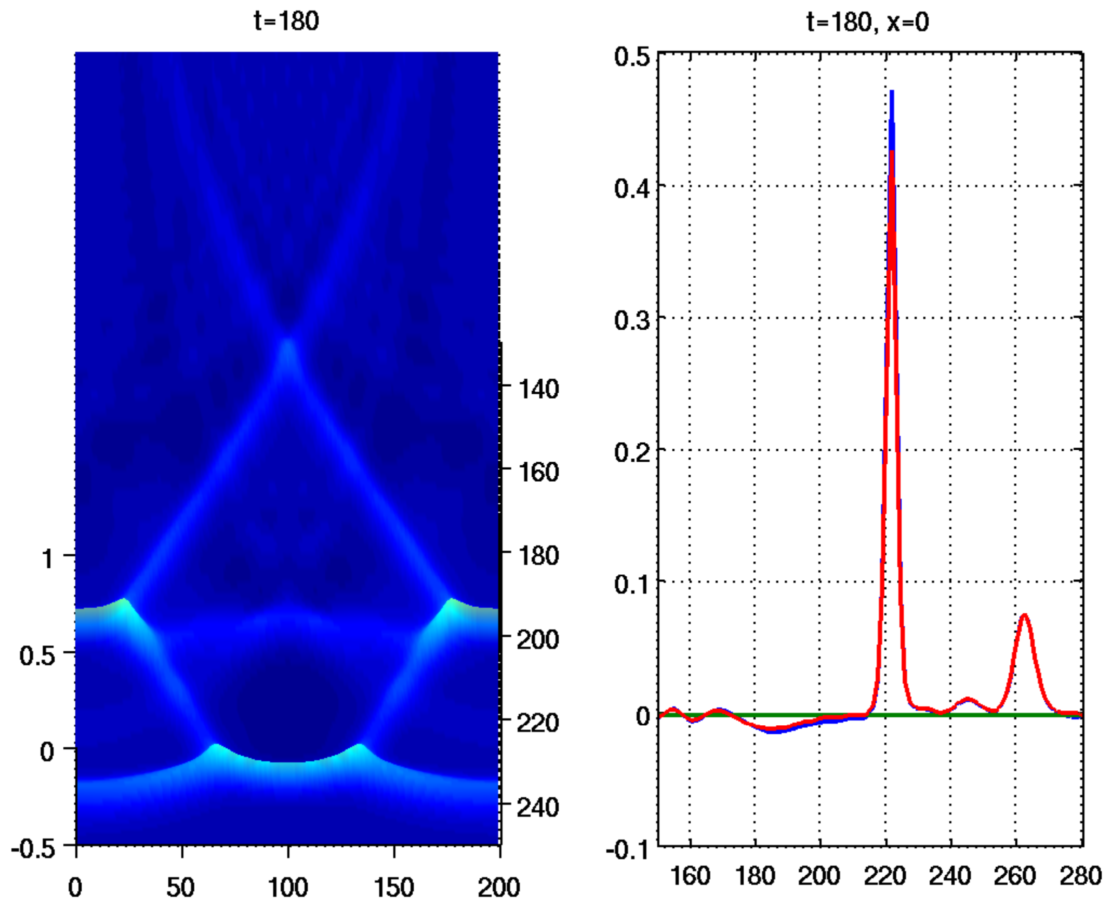
- **If there is not boundary, the center is growing.**
- **With boundary, almost two 1D solitary waves.**

Oblique interaction (with BC), medium attack angle $\theta = 40^o$



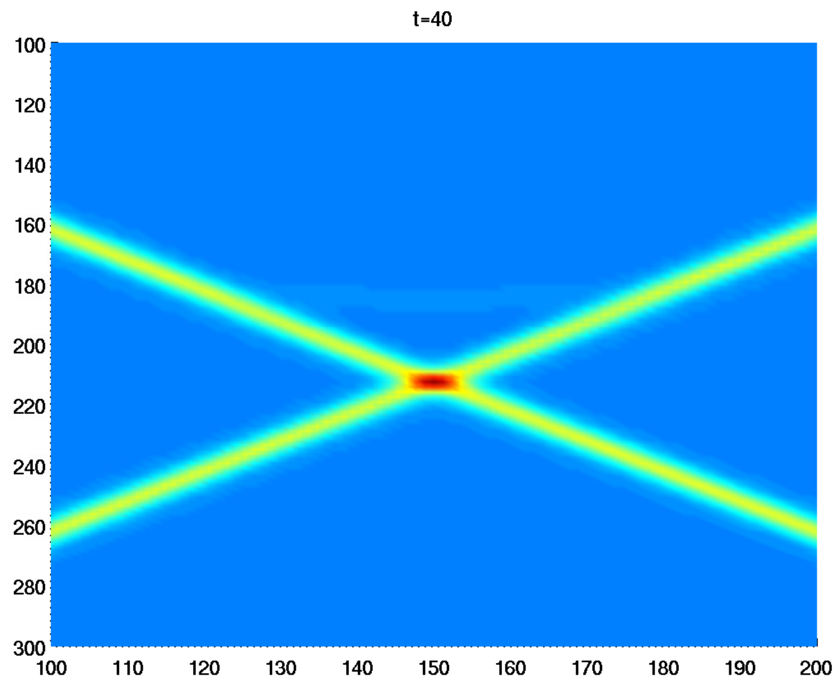
- **Without BC, the center cell;**
- **With BC, interactions.**

Oblique interaction (with BC), medium attack angle $\theta = 40^o$



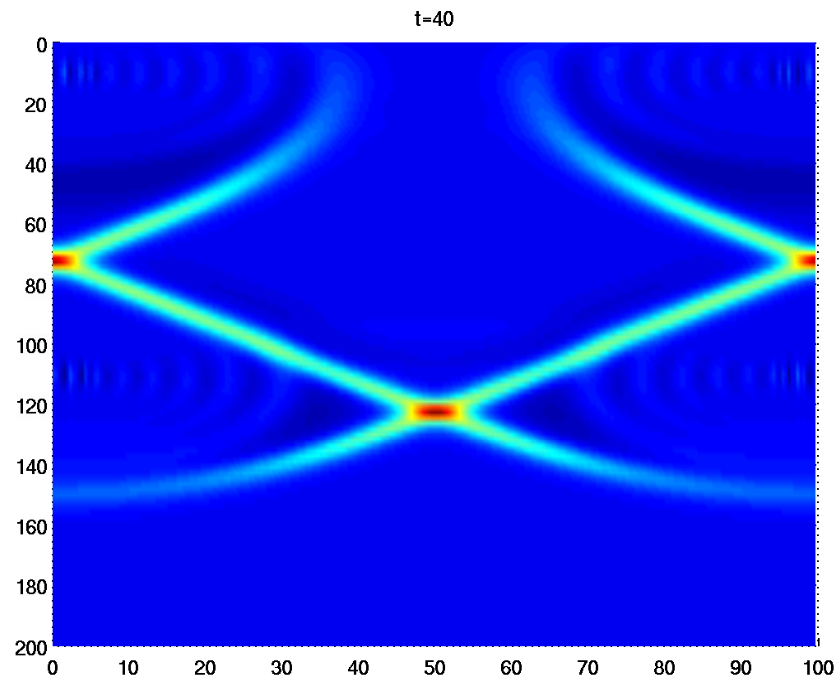
- **Without BC, the center cell;**
- **With BC, interactions.**

Oblique interaction (with BC), large attack angle $\theta = 90^o$



- **Without BC, almost no interaction and travels like a traveling wave.**
- **With BC, the reflective wave is almost the same size, the front wave is almost gone.**

Oblique interaction (with BC), large attack angle $\theta = 90^o$



- **Without BC, almost no interaction and travels like a traveling wave.**
- **With BC, the reflective wave is almost the same size, the front wave is almost gone.**

Thank you!