Reprinted from JOURNAL OF COMPUTATIONAL PHYSICS
All Rights Reserved by Academic Press, New York and London

Vol. 77, No. 2, August 1988
Printed in Belgium

Note

Analytical and Numerical Aspects of Certain Nonlinear Evolution Equations IV. Numerical, Modified Korteweg–de Vries Equation

1. INTRODUCTION

The modified Korteweg–de Vries (MKdV) equation describes a wide class of physical phenomena (e.g., acoustic waves in certain anharmonic lattices [1] and Alfén waves in a collisionless plasma [2]).

In 1984 we derived nonlinear partial difference equations which have as limiting forms the Korteweg–de Vries (KdV) and the MKdV equations [3]. These difference equations have a number of special properties [4] and are constructed by methods related to the inverse scattering transform (IST). We have also implemented similar schemes for the nonlinear Schrödinger (NLS) (Ablowitz–Ladik) and the KdV equations and compared them with known numerical schemes [5, 6]. Experiments have shown that the IST schemes for the NLS and KdV equations compare very favorably with the other known numerical methods. This work aims to implement and compare the proposed schemes which were developed in [3] with certain other known numerical methods for the MKdV equation

$$u_t + 6u^2 u_x + u_{xxx} = 0. \tag{1.1}$$

The MKdV, KdV, and NLS equations are essentially classical in the literature in nonlinear phenomena.

The following numerical methods are applied to the MKdV equation: (i) a proposed global scheme, (ii) a proposed local scheme, (iii) an implicit scheme, (iv) a split step Fourier method (Tappert), and (v) a pseudospectral method (Fornberg and Whitham).

Our approach for comparison is: (a) fix the accuracy (L_∞) for computations beginning at $t=0$ and ending at $t=T$; (b) leave other parameters free (e.g., Δt, or Δx), and compare the computing time required to attain such accuracy for various choices of the parameters.

To obtain the optimal computing time for each scheme, the following procedure was used: Let $T_M(\Delta X) \equiv$ the computing time involved when the problem was solved by method M using a spatial step of Δx, with Δt chosen to be the largest Δt which makes the $L_\infty < \text{Tol}$ (if no such Δt exists then $T(\Delta x) \equiv \infty$) and choose $T_M \equiv \min_{\Delta x} [T_M(\Delta x)]$ as the optimal computing time for method M.

0021-9991/88 $3.00

The above methods are applied to the MKdV equation (1.1) subject to the conditions:

(a) The initial conditions. (i) 1-soliton solution. The exact solution of (1.1) on the infinite interval is

$$u(x, t) = k \operatorname{sech}(kx - k^3 t + \eta_0), \tag{1.2}$$

where $\eta_0 =$ constant.

For initial conditions, Eq. (1.2) is used at $t = 0$, and different values of k are tested and η_0 is chosen to be zero.

(ii) Collisions of two solitons. The exact solution of (1.1) on the infinite interval is

$$u(x, t) = i(\log(f^*/f))_x, \tag{1.3}$$

where * denotes a complex conjugate,

$$f = 1 + ie^{\eta_1} + ie^{\eta_2} - e^{\eta_1 + \eta_2 + A_{12}},$$

$$\eta_i = k_i x - k_i^3 t + \eta_i^{(0)},$$

and

$$e^{A_{ij}} = \left(\frac{k_i - k_j}{k_i + k_j}\right)^2.$$

For initial conditions, Eq. (1.3) is used at $t = 0$, and three different sets of values of the parameters are studied, namely,

$k_1 = 0.5,$	$k_2 = 1.0,$	$\eta_1^{(0)} = 1.1,$	$\eta_2^{(0)} = 3.33;$
$k_1 = 0.5,$	$k_2 = 2.0,$	$\eta_1^{(0)} = 0.625,$	$\eta_2^{(0)} = 8.75;$
$k_1 = -0.5,$	$k_2 = 1.0,$	$\eta_1^{(0)} = -1.1,$	$\eta_2^{(0)} = 1.0.$

The solitons are allowed to interact and return to their original shapes.

(b) The boundary conditions. Periodic boundary conditions on the interval $[-20, 20]$ are imposed. The numerical solution is compared to the exact solution. In addition two of the conserved quantities are computed; namely; $\int u^2\,dx$, and $\int [u^4 - (u_x)^2]\,dx$.

Recently strong numerical schemes have been proposed for solving nonlinear evolution equations, including a scheme using finite element techniques introduced by Bona, Dougalis, and Karakashian [12], and an adaptive numerical scheme introduced by Sanz-Serna [13]. In the future it would be useful to compare our schemes to these newer ones as well as consider initial conditions more general than those yielding interacting solitary waves.

2. Representation of the MKdV Equation Using Numerical Methods

(i) The proposed global scheme which is based on the IST is (Taha and Ablowitz, [3])

$$\begin{aligned}\Delta^m R_n^m &= R_{n+2}^m A_-^{(4)} - R_{n+2}^{m+1}\gamma_{n+1} D_-^{(4)} + R_{n+1}^m S_{n+1} - R_{n+1}^{m+1} P_n \\ &\quad - [R_{n-2}^{m+1} A_-^{(4)} - R_{n-2}^m \gamma_{n-2} D_-^{(4)} + R_{n-1}^{m+1} S_{n-2} - R_{n-1}^m P_{n-1}] \\ &\quad + R_n^m \left(D_-^{(0)} - \sum_{l=-\infty}^{n} T_l \right) - R_n^{m+1} \left(A_-^{(0)} - \sum_{l=-\infty}^{n-1} T_l \right), \end{aligned} \tag{2.1}$$

where

$$\Delta^m R_n^m = R_n^{m+1} - R_n^m,$$

$$\begin{aligned} T_l &= R_l^{m+1}\{R_{l-2}^{m+1} A_-^{(4)} - R_{l-2}^m \gamma_{l-2} D_-^{(4)} + R_{l-1}^{m+1} S_{l-2} - R_{l-1}^m P_{l-1}\} \\ &\quad - R_l^m \{R_{l+2}^m A_-^{(4)} - R_{l+2}^{m+1}\gamma_{l+1} D_-^{(4)} + R_{l+1}^m S_{l+1} - R_{l+1}^{m+1} P_l\}, \end{aligned}$$

$$S_n = A_-^{(2)} + A_-^{(4)} F_n + D_-^{(4)} \sum_{j=-\infty}^{n} H_j,$$

$$P_n = \left(D_-^{(2)} + \sum_{j=-\infty}^{n} [A_-^{(4)} E_j + D_-^{(4)} G_j]\eta_j \right) \gamma_n,$$

$$\gamma_n = \prod_{i=-\infty}^{n} (\delta_i^{m+1}/\delta_i^m), \qquad \delta_i^m = 1 + R_i^m,$$

$$\eta_n = \gamma_n^{-1}/\delta_n^m, \qquad H_n = -\{R_n^m R_{n+1}^{m+1} \delta_n^{m+1} - R_{n-1}^m R_n^{m+1} \delta_n^m\}\beta_{n-1},$$

$$\beta_n = \gamma_n/\delta_{n+1}^m, \qquad F_n = \left[R_{n+1}^{m+1} R_n^{m+1} - \sum_{j=-\infty}^{n} \Delta^m (R_j^m R_{j+1}^m) \right],$$

$$G_n = (R_n^{m+1} R_{n+1}^{m+1} - R_n^m R_{n-1}^m)\gamma_{n-1}\delta_n^{m+1},$$

$$E_n = (R_n^m R_{n-1}^{m+1}\delta_n^{m+1} - R_{n+1}^m R_n^{m+1}\delta_n^m),$$

$$A_-^{(2)} = -\tfrac{2}{3}A_-^{(0)} + \tfrac{1}{2}\alpha, \qquad D_-^{(2)} = -\tfrac{2}{3}A_-^{(0)} - \tfrac{1}{2}\alpha,$$

$$A_-^{(4)} = \tfrac{1}{6}A^{(0)} - \tfrac{1}{4}\alpha, \qquad D_-^{(4)} = \tfrac{1}{6}A_-^{(0)} + \tfrac{1}{4}\alpha,$$

$$\alpha = \frac{\Delta t}{(\Delta x)^3}, \qquad A_-^{(0)} = \text{arbitrary constant},$$

$$R = \Delta x\, u, \qquad |n| < p \text{ (half the length of the interval of interest)}, \qquad m > 0.$$

This scheme is implemented with the value of $A^{(0)} = \frac{3}{2}\alpha$. This proposed scheme is unconditionally stable, and has a truncation error of order $O((\Delta t)^2) + O((\Delta x)^2)$. This scheme is implemented using the sweeping/iteration technique presented by the authors [5, 6].

(ii) The proposed local scheme which is derived from Eq. (2.1) with $A^{(0)} = \frac{3}{2}\alpha$ is

$$\frac{u_n^{m+1}-u_n^m}{\Delta t} = \frac{u_{n-1}^{m+1}-3u_n^{m+1}+3u_{n+1}^{m+1}-u_{n+2}^{m+1}}{2(\Delta x)^3} + \frac{u_{n-2}^m-3u_{n-1}^m+3u_n^m-u_{n+1}^m}{2(\Delta x)^3}$$

$$-\frac{1}{2\Delta x}\Bigg[u_{n+2}^{m+1}\{(u_{n+1}^m)^2+(u_n^m)^2\} - u_{n-2}^m\{(u_{n-1}^{m+1})^2+(u_n^{m+1})^2\}$$

$$+\frac{u_{n+1}^{m+1}}{2}[u_n^m u_{n+1}^m + u_n^{m+1}u_{n+1}^{m+1} + 2u_{n-1}^m u_n^m]$$

$$-\frac{u_{n-1}^m}{2}[u_n^m u_{n-1}^m + u_n^{m+1}u_{n-1}^{m+1} + 2u_n^{m+1}u_{n+1}^{m+1}]$$

$$+\frac{u_n^m}{2}(u_n^{m+1}u_{n+1}^{m+1} + u_n^m u_{n+1}^m)$$

$$-\frac{u_n^{m+1}}{2}(u_{n-1}^{m+1}u_n^{m+1} + u_{n-1}^m u_n^m)$$

$$-3[(u_n^m)^2 u_{n+1}^{m+1} - (u_n^{m+1})^2 u_{n-1}^m]\Bigg]. \tag{2.2}$$

This scheme is unconditionally stable according to linear analysis, and has a truncation error of order $O((\Delta t)^2)+O((\Delta x)^2)$. This scheme is implemented using the sweeping/iteration technique.

(iii) An implicit scheme [7]:

$$\frac{u_n^{m+1}-u_n^m}{\Delta t} = \frac{u_{n-1}^{m+1}-3u_n^{m+1}+3u_{n+1}^{m+1}-u_{n+2}^{m+1}}{2(\Delta x)^3} + \frac{u_{n-2}^m-3u_{n-1}^m+3u_n^m-u_{n+1}^m}{2(\Delta x)^3}$$

$$-\frac{1}{2(\Delta x)}\{\theta[(u^3)_{n+1}^{m+1} - (u^3)_{n-1}^{m+1} + (u^3)_{n+1}^m - (u^3)_{n-1}^m]$$

$$+3(1-\theta)[(u^2)_n^{m+1}(u_{n+1}^{m+1}-u_{n-1}^{m+1}) + (u^2)_n^m(u_{n+1}^m - u_{n-1}^m)]\}. \tag{2.3}$$

This scheme is unconditionally stable according to linear analysis and has a truncation error of order $O((\Delta t)^2)+O((\Delta x)^2)$. This scheme is implemented using the sweeping/iteration technique. Several values of θ were employed and experimentally we found that $\theta = \frac{2}{3}$ gave the best results.

(iv) Split step Fourier method (Tappert [8]). For convenience the spatial period was normalized to $[0, 2\pi]$, then Eq. (1.1) becomes

$$u_t + 6\frac{\pi}{p}u^2 u_X + \frac{\pi^3}{p^3}u_{XXX} = 0, \tag{2.4}$$

where p is half the length of the interval of interest, and $X = (x+p)\,\pi/p$.

To implement this method for the MKdV equation (2.4), as the first step, one first approximates,

$$u_t + 6\frac{\pi}{p}u^2 u_X = 0. \tag{2.5}$$

Equation (2.5) can be approximated by using an implicit scheme such as

$$\tilde{u}_n^{m+1} = u_n^m - \frac{\Delta t}{12\Delta X}\frac{\pi}{p}\{[8(\tilde{u}^3)_{n+1}^{m+1} - 8(\tilde{u}^3)_{n-1}^{m+1} - (\tilde{u}^3)_{n+2}^{m+1} + (\tilde{u}^3)_{n-2}^{m+1}] + [8(u^3)_{n+1}^m - 8(u^3)_{n-1}^m - (u^3)_{n+2}^m + (u^3)_{n-2}^m]\}, \tag{2.6}$$

where $\tilde{u}$ is a solution of Eq. (2.5) and u is the solution of Eq. (2.4). For the second step, one approximates

$$u_t + \frac{\pi^3}{p^3}u_{XXX} = 0 \tag{2.7}$$

by means of the discrete Fourier transform such as

$$u(X_j, t+\Delta t) = F^{-1}(e^{(ik^3\pi^3/p^3)\Delta t}F(\tilde{u}(X_j, t))), \tag{2.8}$$

where F denotes discrete Fourier transform and F^{-1} its inverse. This scheme is unconditionally stable according to linear analysis, and has a truncation error of order $O((\Delta t)^2) + O((\Delta X)^4)$.

(vi) Pseudospectral method by Fornberg and Whitham [9]. The pseudospectral method for Eq. (2.4) is

$$u(X, t+\Delta t) - u(X, t-\Delta t) + 2i\frac{6\pi}{p}\Delta t\, u^2(X, t)\, F^{-1}(kF(u)) - 2iF^{-1}\left\{\sin\left(\frac{\pi^3 k^3}{p^3}\Delta t\right)F(u)\right\} = 0. \tag{2.9}$$

The linear stability requirement for this scheme is $\Delta t/(\Delta x)^3 < 3/2\pi^2$.

3. Conclusions

According to our numerical experiments we have made the following conclusions (see Tables I and II as examples):

(1) The proposed global scheme, based on IST, proved to be faster than all of the methods we considered. It is worth noting that this proposed global scheme behaves much better than the other utilized schemes either when better accuracy is required or for large amplitudes. This result is similar to that for the NLS equation

TABLE I

Comparison of the Computing Time (E) Which Is Required to Attain an Accuracy (L_∞) < 0.005 for Computations Beginning at $t = 0$ and Ending at $t = 1.0$, for the Numerical Methods Utilized in Solving the MKdV *Equation*[a]

No.	Method	Mesh size	Time min	Time s	$t = T$	L_∞	v_1	v_2	Normalized E
1.	Proposed global scheme	$\Delta x = 0.1$, $\Delta t = 0.25$	E: 0	6	0.25	0.00187	0.00009	0.00486	1
					0.5	0.00279	0.00017	0.00508	
					1.0	0.00448	0.00033	0.00556	
2.	Proposed local scheme	$\Delta x = 0.06$, $\Delta t = 0.12$	E: 0	10	0.24	0.00232	0.00002	0.00168	1.7
					0.48	0.00319	0.00003	0.00171	
					0.96	0.00466	0.00006	0.00177	
3.	Implicit scheme	$\Delta x = 0.08$, $\Delta t = 0.1$	E: 0	8	0.34	0.00231	0.00002	0.00297	1.3
					0.5	0.00310	0.00003	0.00298	
					1.0	0.00453	0.00005	0.00303	
4.	Pseudospectral scheme by Fornberg and Whitham	$\Delta x = 0.625$, $\Delta t = 0.0055$	E: 0	12	0.25	0.00258	−0.00120	−0.02976	2.0
					0.5	0.00407	0.00218	0.07897	
					1.0	0.00457	−0.00143	−0.03534	
5.	Tappert	$\Delta x = 0.3125$, $\Delta t = 0.0041$	E: 0	36	0.25	0.00361	0.00000	−0.00010	6
					0.5	0.00407	0.00000	−0.00013	
					1.0	0.00473	0.00000	0.00001	

[a] 1-*soliton* as an initial condition with amplitude = 1 on the interval $[-20, 20]$.

Δx = the increment in x.
Δt = the increment in t.
$v_1 = (u_1 - u_{10})/u_{10}$, u_{10} = the exact value of $\int u^2\,dx$.
$v_2 = (u_2 - u_{20})/u_{20}$, u_{20} = the exact value of $\int [u^4 - (u_x)^2]\,dx$.
u_1 = the calculated value of the conserved quantity of the MKdV equation which is $\int u^2\,dx$.
u_2 = the calculated value of the conserved quantity of the MKdV equation which is $\int [u^4 - (u_x)^2]\,dx$.
$L_\infty = \max |\tilde{u}_n^m - u_n^m|$, $\tilde{u}_n^m$ is the numerical solution and u_n^m is the exact solution at the point $(n\Delta x, m\Delta t)$ for all n, m.

TABLE II

Comparison of the Computing Time (E) Which Is Required to Attain an Accuracy (L_∞) < 0.02 for Computations Beginning at $t = 0$ and Ending at $t = 3.0$, for the Numerical Methods Utilized in Solving the MKdV *Equation*[a]

No.	Method	Mesh size	Time min	Time s	$t = T$	L_∞	ν_1	ν_2	Normalized E
1.	Proposed global scheme	$\Delta x = 0.0565$ $\Delta t = 0.0565$	E: 3	38	0.5	0.00457	−0.00053	−0.00127	
					1.0	0.00708	−0.00099	−0.00492	
					1.5	0.00995	−0.00073	−0.00228	1
					2.0	0.01332	−0.00031	0.00189	
					2.5	0.01541	0.00016	0.00561	
					3.0	0.01983	0.00056	0.00801	
2.	Proposed local scheme	$\Delta x = 0.012$ $\Delta t = 0.00925$	E: 24	23	0.5	0.00258	0.00007	−0.00003	
					1.0	0.00259	0.00012	−0.00012	
					1.5	0.00477	0.00008	−0.00006	6.7
					2.0	0.00910	0.00002	0.00006	
					2.5	0.01492	−0.00000	0.00019	
					3.0	0.01988	−0.00001	0.00027	
3.	Implicit scheme	$\Delta x = 0.012$ $\Delta t = 0.009$	E: 23	23	0.5	0.00273	0.00016	0.00019	
					1.0	0.00273	0.00026	0.00025	
					1.5	0.00417	0.00018	0.00020	6.4
					2.0	0.00814	0.00003	0.00010	
					2.5	0.01425	−0.00004	0.00009	
					3.0	0.01989	−0.00006	0.00013	
4.	Pseudospectral scheme by Fornberg and Whitham	$\Delta x = 0.3125$ $\Delta t = 0.00111$	E: 6	4	0.5	0.00193	−0.00030	−0.01587	
					1.0	0.00206	−0.00015	−0.00045	
					1.5	0.00317	0.00020	0.01008	1.7
					2.0	0.00446	0.00060	0.02389	
					2.5	0.01727	−0.00074	−0.04232	
					3.0	0.01996	−0.00113	−0.07964	
5.	Tappert	$\Delta x = 0.078125$ $\Delta t = 0.00125$	E: 25	49	0.5	0.00560	−0.00001	−0.00061	
					1.0	0.00246	−0.00001	−0.00028	
					1.5	0.00304	−0.00002	0.00018	7.1
					2.0	0.00901	−0.00006	0.00003	
					2.5	0.01449	−0.00006	−0.00016	
					3.0	0.01990	−0.00007	−0.00031	

[a] *Two-solitons* as an initial condition with amplitudes $\frac{1}{2}$ and 2, respectively, and they are allowed to interact on the interval $[-20, 20]$.

[5] and it suggests that eventually a similar conclusion will be drawn from the implementation of the KdV schemes.

(2) The pseudospectral method becomes competitive with the IST global scheme when both high accuracy and large amplitudes are involved.

(3) The implicit scheme behaves better than the proposed local scheme and the pseudospectral method for low amplitudes, and it is much better than the split step (Tappert) method.

(4) The proposed local scheme behaves better than the pseudospectral method for small amplitudes for the 1-soliton case and becomes competitive with the implicit scheme for large amplitudes.

(5) The split step Fourier method behaves much slower than all of the methods we considered.

We note that since the proposed local scheme did not perform as well as its global version, it will be under further investigation. All the numerical calculations were inspected at every step by using the conserved quantities $\int u^2\,dx$, and $\int (u^4-(u_x)^2)\,dx$ (Table I–II). The two conserved quantities are calculated by means of Simpson's rule. In the finite difference schemes we have discretized u_x using a central difference approximation. For the Fourier based schemes the discrete Fourier transform was used to estimate u_x in the computation of the conserved quantity $\int (u^4-(u_x)^2)\,dx$. The proposed global scheme is the only utilized scheme which has an infinite number of conserved quantities, and true soliton solutions. It is worth mentioning that these IST schemes can also be used in combination with other numerical schemes to study a wider class of physically important nonlinear evolution equations. For example, they can be used to study perturbed forms of the KdV, MKdV, and NLS equations [11].

Acknowledgments

This research has been supported in part by the U.S. Army Research Office and by the Research Foundation of the University of Georgia (by way of a Faculty Research Grant and a Michael Award), and by AFSOR Grant 84.0005, NSF Grant DMS-8501325.

References

1. N. J. Zabusky, *Nonlinear Partial Differential Equations*; *A symposium on Methods of Solution*, edited by W. F. Ames (Academic Press, New York, 1967), p. 223; N. Zabusky, *J. Comput. Phys.* **43**, 195 (1981).
2. A. Scott, F. Chu, and D. McLaughlin, *Proc. IEEE* **61**, No. 10, 1443 (1973).
3. T. R. Taha and M. J. Ablowitz, *J. Comput. Phys.* **55**, 1982 (1984).
4. M. Ablowitz and H. Segur, *Solitons and the inverse scattering transform*, SIAM, Philadelphia, (1981), p. 114.
5. T. R. Taha and M. J. Ablowitz, *J. Comput. Phys.* **55**, 203 (1984).

6. T. R. TAHA AND M. J. ABLOWITZ, *J. Comput. Phys.* **55**, 231 (1984).
7. M. D. KRUSKAL, Princeton University, private communication, 1981.
8. F. TAPPERT, "Numerical solutions of the Korteweg–de Vries equation and its generalizations by the split-step Fourier method," in *Nonlinear Wave Motion*, edited by A. C. Newell, Lecture in Appl. Math. Vol. 15 (Amer. Math. Soc., Providence, RI, 1974), p. 215.
9. B. FORNBERG AND G. B. WHITHAM, *Philos. Trans. R. Soc. London A* **289**, 373 (1978).
10. J. W. COOLEY, P. A. W. LEWIS, AND P. D. WELCH, *IEEE Trans. Educ.* **E-121**, 27 (1969).
11. Y. KODAMA AND M. J. ABLOWITZ, *Stud. Appl. Math.* **64**, 255 (1981).
12. J. BONA, V. DOUGALIS, AND O. KARAKASHIAN, *Comp. & Maths. with Appls.* **12A**, No. 7, 859 (1986).
13. J. M. SANZ-SERNA, *J. Comput. Phys.* **47**, 199 (1982).

RECEIVED: February 13, 1987; REVISED: October 27, 1987

THIAB R. TAHA
Department of Computer Science
The University of Georgia,
Athens, Georgia 30602

MARK J. ABLOWITZ
Department of Mathematics and Computer Science,
Clarkson University,
Potsdam, New York 13676

Printed by Catherine Press, Ltd., Tempelhof 41, B-8000 Brugge, Belgium